Second-Order Approximation for Variance Reduction in Multiple Importance Sampling

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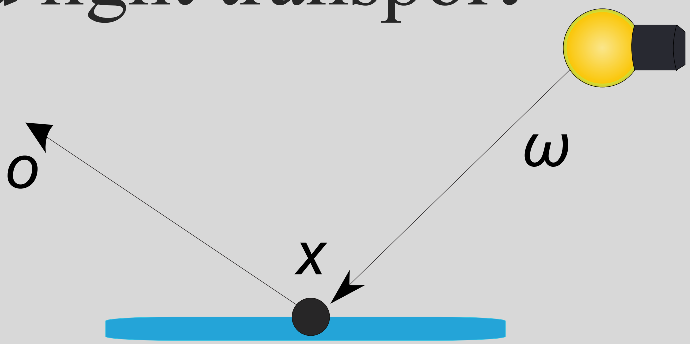
Motivation

- Realistic rendering and light transport



- Reflected radiance [Kajiya1986]

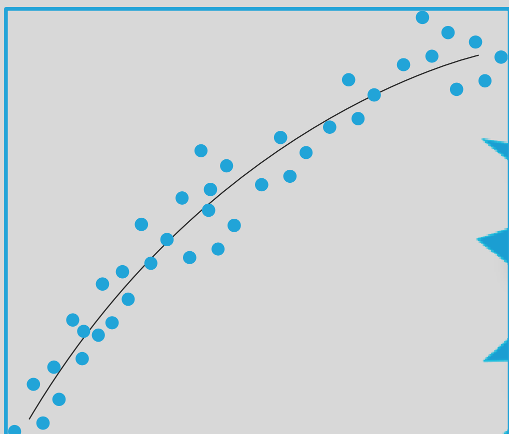
$$L = \int f(x, o, \omega)$$

- Monte Carlo Estimation

$$\tilde{L}_N = \frac{1}{N} \sum_{n=1}^{N} \tilde{L}(n)$$

Convergence

- Variance
  - Estimation of the convergence
  - Estimation of the noise



Sampling Techniques

- Light-based Importance Sampling
- BRDF-based Importance Sampling
- Multiple Importance Sampling
  - [Veach98] [Pajot11] [Georgiev12]

$$\tilde{L}_{(N_l+N_b)} = \frac{1}{N_l}\sum_{n=1}^{N_l} w_l(n)\tilde{L}_l(n) + \frac{1}{N_b}\sum_{n=1}^{N_b} w_b(n)\tilde{L}_b(n)$$

$$w_l(n) + w_b(n) = 1$$

$$w_l(n) > 0 \quad w_b(n) > 0$$

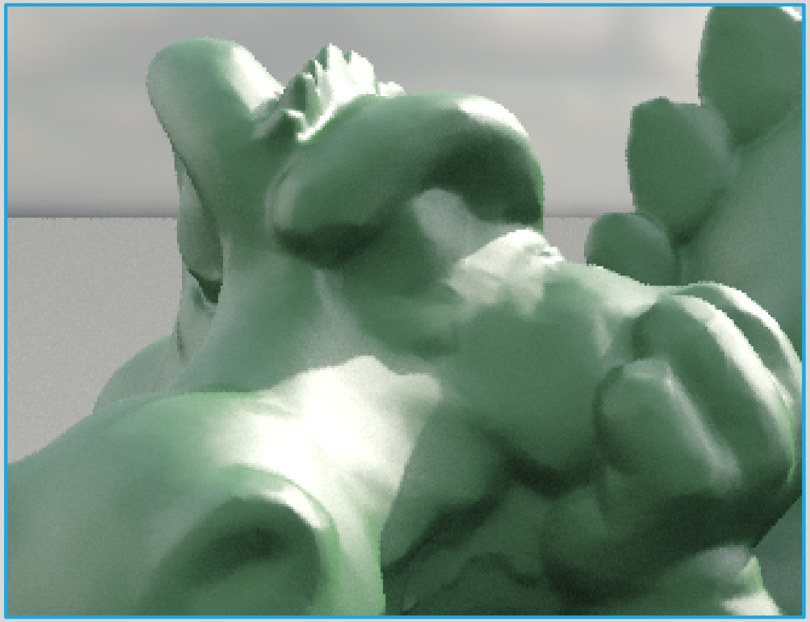
PG
2013

Previous Work

- Heuristic strategies for weights [Veach98]
  - Balance Heuristic $$w_l(n) = \frac{\rho_l(n)}{\rho_b(n) + \rho_l(n)}$$
  - Power Heuristic
  - Maximum Heuristic $$\rho_s(n) = p(s)p_s(n)$$
  - α-max heuristic [Georgiev12]
- The problem: how to compute $p(s)$
- means
  - Balancing scheme: $$p(l) : p(b) = N_l : N_b$$

The problem

- Good $p(s)$ for each pixel
  - Not enough knowledge about sampling strategy
  - No prior knowledge about the scene
  - Previously, $p(s) = 0.5$ (default balancing)



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Contribution

- Good balancing for each pixel [per-pixel $p(s)$]
  - **Without prior knowledge** about
    - Scene characteristics
    - Sampling strategies
- Lighter solution than [Pajot11]
  - Low computation cost
  - No additional storage
  - Independent of any specific techniques

Main Idea

- Minimize the variance to get $p(s)$
  - Small set of samples N_m
    - with balance heuristic
    - with cheap and approximated minimization
- Use the $p(s)$ for the rest of the samples N_n
- Keep the samples for the Minimization

$$\tilde{L}_N = \frac{N_m}{N}\tilde{L}_{N_m} + \frac{N_n}{N}\tilde{L}_{N_n}$$

Minimization

- Minimize the sampling variance.

$$V[\tilde{L}_N] = \frac{1}{N} V[\tilde{L}_1] \qquad V[\tilde{L}_1] = E[\tilde{L}_1^2] - L^2$$

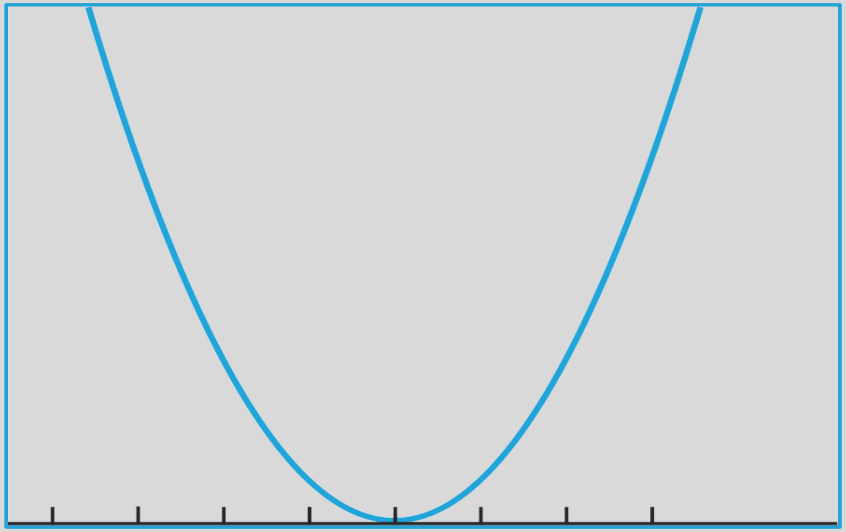
- Minimize the $E[\tilde{L}_1^2] \iff \frac{d}{d\rho(s)} E[\tilde{L}_1^2] = 0$

- Taylor Expansion at $p(s) = 0.5$

$$\frac{d}{dp(s)} E[\tilde{L}_1^2] \approx \sum_{d=0}^{D} (-1)^d (2p(s) - 1)^d \int \frac{f(x, o, \omega)}{(p_l + p_b)^{(d+1)}}$$

Second-order Approximation

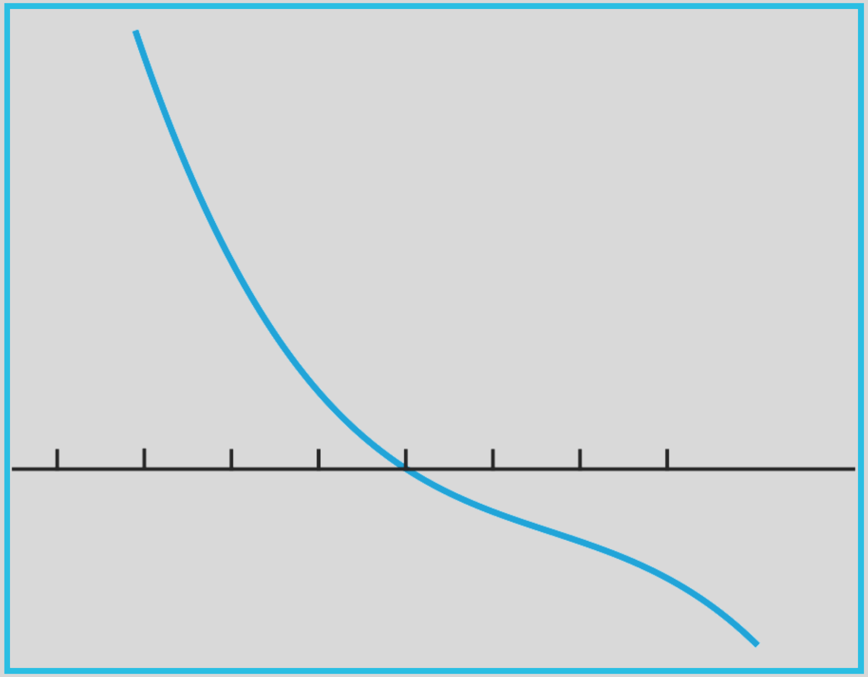
- Second-order Taylor Expansion



$$\frac{d}{d\rho(s)} E[\tilde{L}_1^2] = 0$$

Third-order Approximation

- No extreme value in [0,1]



$$\frac{d}{d\rho(s)} E[\tilde{L}_1^2] = 0$$

Fourth-order Approximation

- Why not 4th-order?

$$\left(\frac{1}{12}\frac{3^{1/3}\left(G\,p\left(-36\,G\,p^4\,Gp^4+72\,Gp^8+9\,G^2\,p^8+4\sqrt{6}\sqrt{-38\,G\,p^4\,Gp^4+9\,G^2\,p^8+54\,Gp^8\,Gp^4}\right)\right)^{1/3}}{Gp^4}+\frac{1}{12}\frac{G\,p^2\left(-8\,Gp^4+3\,G\,p^4\right)3^{2/3}}{Gp^4\left(G\,p\left(-36\,G\,p^4\,Gp^4+72\,Gp^8+9\,G^2\,p^8+4\sqrt{6}\sqrt{-38\,G\,p^4\,Gp^4+9\,G^2\,p^8+54\,Gp^8\,Gp^4}\right)\right)^{1/3}}\right.$$

$$\left.+\frac{1}{4}\frac{G\,p^3}{Gp^4}\right)q\left(-\frac{1}{24}\frac{3^{1/3}\left(G\,p\left(-36\,G\,p^4\,Gp^4+72\,Gp^8+9\,G^2\,p^8+4\sqrt{6}\sqrt{-38\,G\,p^4\,Gp^4+9\,G^2\,p^8+54\,Gp^8\,Gp^4}\right)\right)^{1/3}}{Gp^4}\right.$$

$$-\frac{1}{24}\frac{G\,p^2\left(-8\,Gp^4+3\,G\,p^4\right)3^{2/3}}{Gp^4\left(G\,p\left(-36\,G\,p^4\,Gp^4+72\,Gp^8+9\,G^2\,p^8+4\sqrt{6}\sqrt{-38\,G\,p^4\,Gp^4+9\,G^2\,p^8+54\,Gp^8\,Gp^4}\right)\right)^{1/3}}+\frac{1}{4}\frac{G\,p^3}{Gp^4}+\frac{1}{2}$$

$$I\sqrt{3}\left(\frac{1}{12}\frac{3^{1/3}\left(G\,p\left(-36\,G\,p^4\,Gp^4+72\,Gp^8+9\,G^2\,p^8+4\sqrt{6}\sqrt{-38\,G\,p^4\,Gp^4+9\,G^2\,p^8+54\,Gp^8\,Gp^4}\right)\right)^{1/3}}{Gp^4}\right.$$

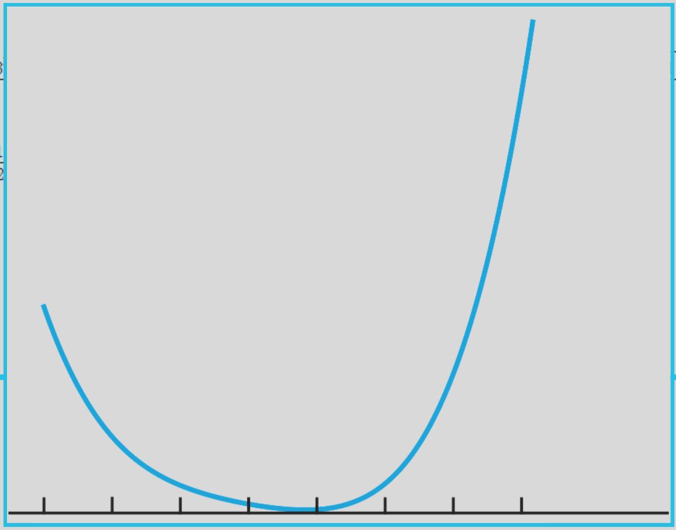
$$\left.\left.-\frac{1}{12}\frac{G\,p^2\left(-8\,Gp^4+3\,G\,p^4\right)3^{2/3}}{Gp^4\left(G\,p\left(-36\,G\,p^4\,Gp^4+72\,Gp^8+9\,G^2\,p^8+4\sqrt{6}\sqrt{-38\,G\,p^4\,Gp^4+9\,G^2\,p^8+54\,Gp^8\,Gp^4}\right)\right)^{1/3}}\right)\right)q\left(-\frac{1}{24}\,3 \ldots Gp^8\,Gp^4))^{1/3}\right.$$

$$-\frac{1}{24}\frac{G\,p^2\left(-8\,Gp^4+3\,G\,p^4\right)3^{2/3}}{Gp^4\left(G\,p\left(-36\,G\,p^4\,Gp^4+72\,Gp^8+9\,G^2\,p^8+4\sqrt{6}\sqrt{-38\,G\,p^4\,Gp^4+9\,G^2\,p^8+54\,Gp^8\,Gp^4}\right)\right)^{1/3}}+\frac{1}{4}\frac{G\,p^3}{Gp^4}-\frac{1}{2}\ldots$$

$$I\sqrt{3}\left(\frac{1}{12}\frac{3^{1/3}\left(G\,p\left(-36\,G\,p^4\,Gp^4+72\,Gp^8+9\,G^2\,p^8+4\sqrt{6}\sqrt{-38\,G\,p^4\,Gp^4+9\,G^2\,p^8+54\,Gp^8\,Gp^4}\right)\right)^{1/3}}{Gp^4}\right.$$

$$\left.\left.-\frac{1}{12}\frac{G\,p^2\left(-8\,Gp^4+3\,G\,p^4\right)3^{2/3}}{Gp^4\left(G\,p\left(-36\,G\,p^4\,Gp^4+72\,Gp^8+9\,G^2\,p^8+4\sqrt{6}\sqrt{-38\,G\,p^4\,Gp^4+9\,G^2\,p^8+54\,Gp^8\,Gp^4}\right)\right)^{1/3}}\right)\right)q$$

$$\frac{d}{d\rho(s)}E[\tilde{L}_1^2]=0$$



Estimation of balancing

- May converge slowly

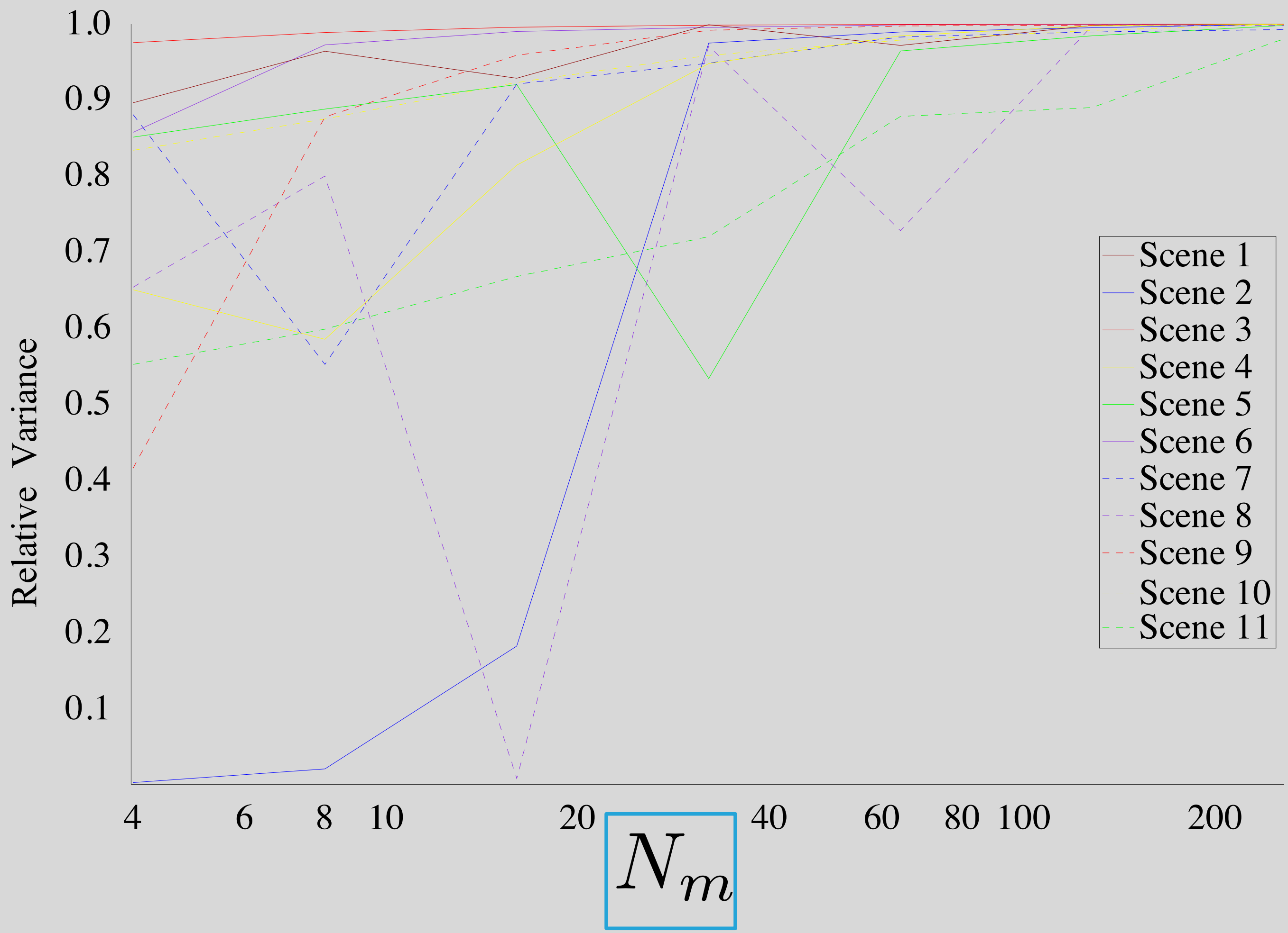
$$\frac{d}{dp(s)} E[\tilde{L}_1^2] \approx \sum_{d=0}^{D} (-1)^d (2p(s) - 1)^d \int \frac{f(x, o, \omega)}{(p_l + p_b)^{(d+1)}}$$

$$L = \int f(x, o, \omega)$$

- Non-converged $p(s)$ still gives good hint
- Advantage
  - Estimate $\tilde{L}_{N_m}$ and $p(s)$ at the same time

How much is enough?

- The convergence of the minimization

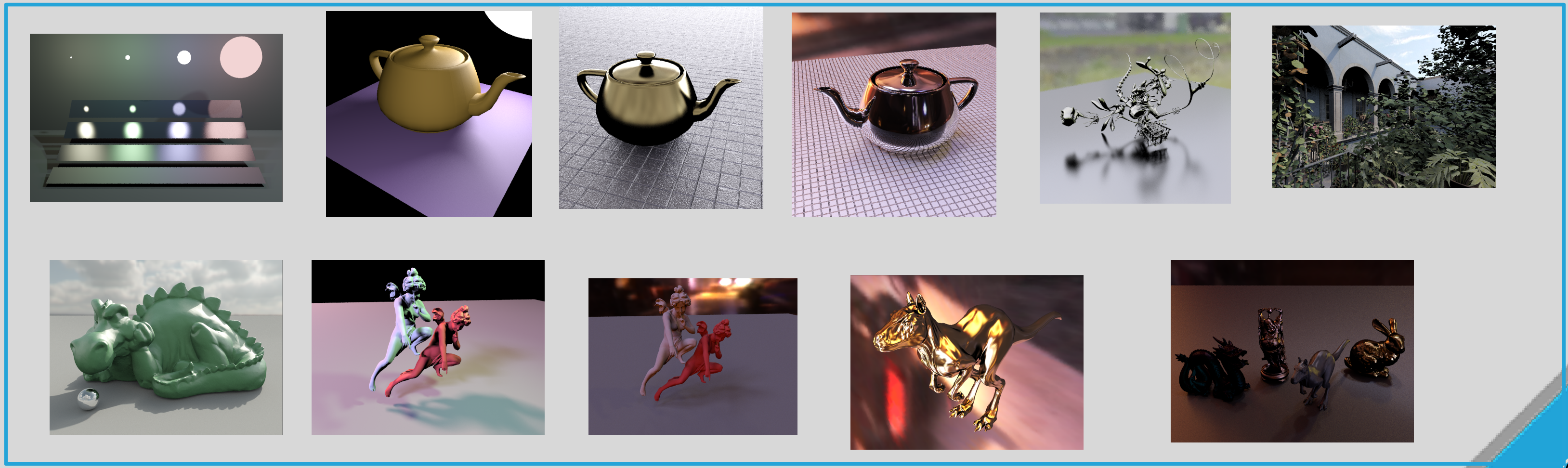


Results

- Variance comparison

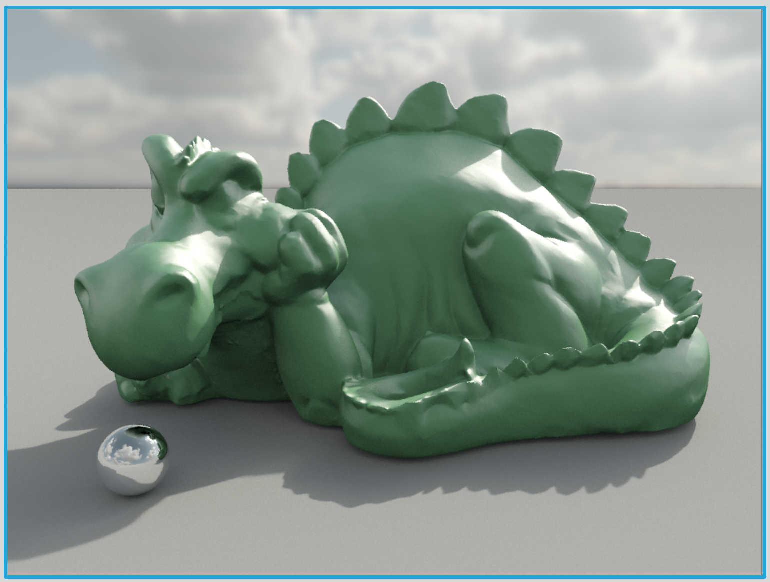
The smaller the better

Method	sc1	sc2	sc3	sc4	sc5	sc6	sc7	sc8	sc9	Sc10	Sc11
Balance 0.5	1	1	1	1	1	1	1	1	1	1	1
Power 0.5	1.1	1.0	1.0	1.1	1.1	1.2	1.1	1.1	1.0	1.1	1.1
Max 0.5	1.4	1.1	1.0	1.3	1.3	1.3	1.7	1.4	1.0	1.5	1.4
Ours	0.47	0.62	0.36	0.53	0.66	0.39	0.88	0.77	0.38	0.75	0.82

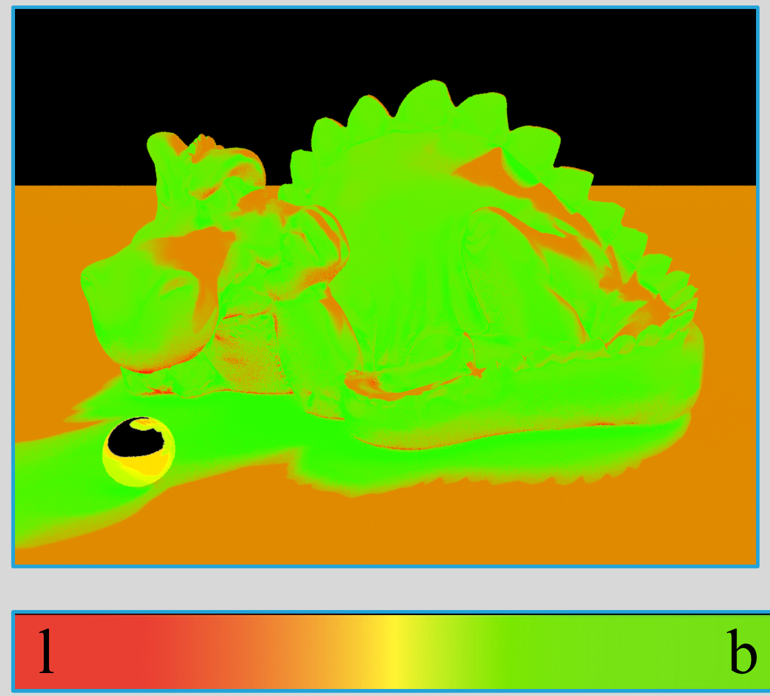


Results

- Different $p(s)$ for each pixel



specular dragon + diffuse ground



Coherent with the original paper [Clarberg08]

Results

- Efficiency comparison [Veach98]
  - –Variance & Time

$$\epsilon = \frac{1}{V[\tilde{L}_N]T[\tilde{L}_N]}$$

The bigger the better

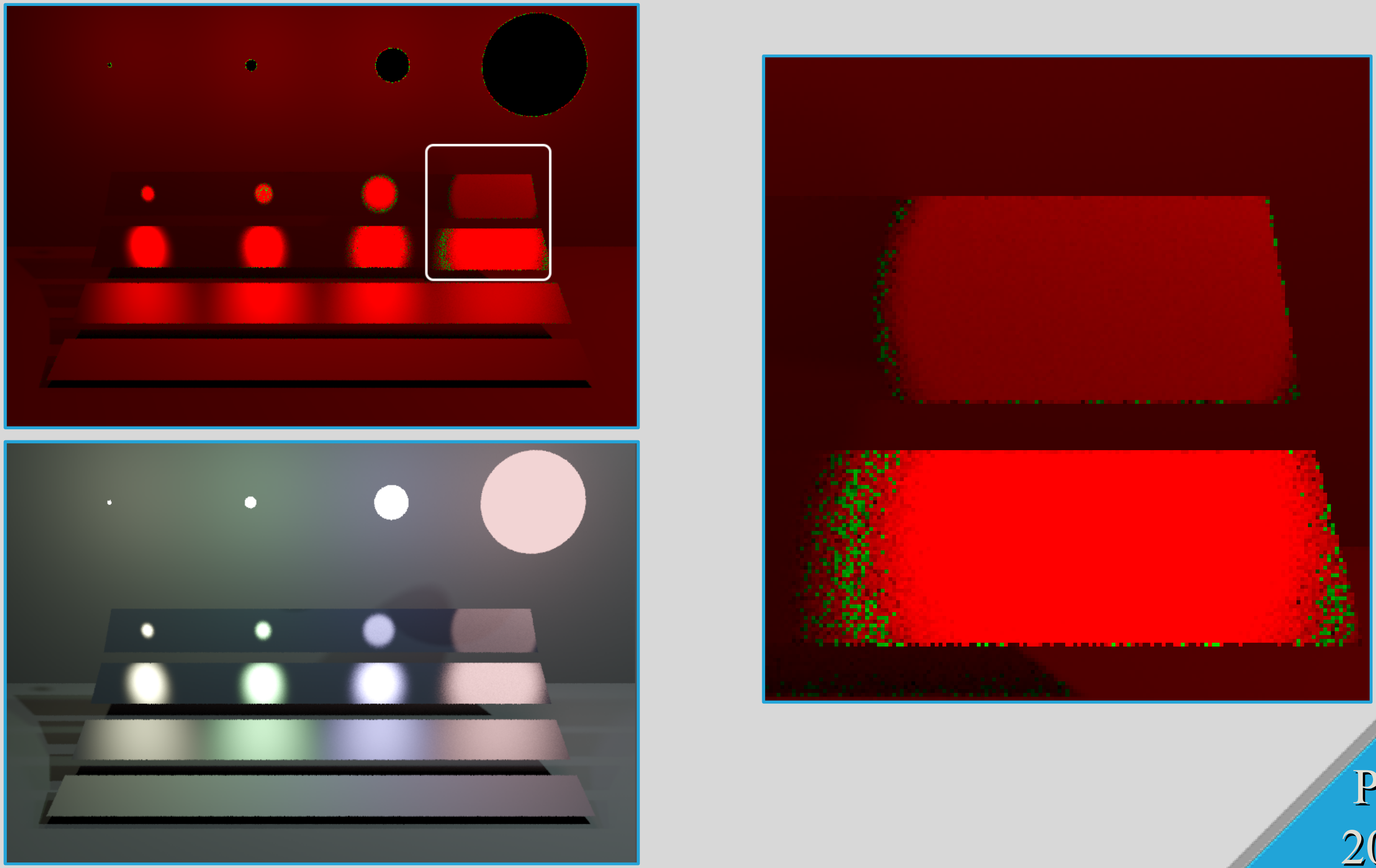
Method	sc1	sc2	sc3	sc4	sc5	sc6	sc7	sc8	sc9	Sc10	Sc11
Balance 0.5	1	1	1	1	1	1	1	1	1	1	1
Power 0.5	0.89	1.0	0.99	0.90	0.94	0.89	0.94	0.94	1.0	0.93	0.93
Max 0.5	0.72	0.94	0.99	0.78	0.80	0.80	0.63	0.73	1.0	0.69	0.76
Ours	1.28	1.17	1.32	1.30	1.16	1.32	1.03	1.22	1.44	1.11	1.15

Ours: 128 samples for minimization, 128 samples for using the p(S)

Others: 256 samples for rendering using p(S) = 0.5

Limitation

- Average variance reduction only



Conclusion

- Contribution
  - Variance reduction without prior knowledge
  - Efficient and simple solution
  - Good balancing for most of pixels
- Future work
  - Variance reduction for all the pixels
  - More sampling strategies support
  - Multi-pass sampling techniques integration

Thank you for your attention