Self Interacting Markov chains

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Some self-references :
↪ *A pair of joint works with Laurent Miclo :* `Stoch. Analysis and Appl. (2006)` + `Proc. Royal Soc. London A. (2003)` ⊂ *Toulouse Univ. (2002).*
↪ `Interacting MCMC`, *joint work with Brockwell & Doucet, UBC (2008).*
↪ `Fluctuations of Interacting MCMC`, *a series of joint works with Bercu & Doucet, HAL-INRIA preprints (2008).*

Summary

Self interacting processes in biology and engineering

- **Biology :**
  - Genetic type reinforcement : "beneficial" interactions with the past.
  - Historical and spatial interactions processes : ethology models, ant-and-bee systems, the particle swarm methods, and ant colony models, and others.
- **Stochastic engineering :**
  - Re-initialization of random search models and stochastic algorithms.
  - $\rightsquigarrow$ *New class of stochastic algorithms* $\subset$ Global optimization problems, interacting MCMC models for complex distribution flows, bayesian learning, filtering, and spectral analysis of Feynman-Kac-Schroedinger operators.

Some important questions

- Stochastic models & Rigorous stochastic analysis.
- $\uparrow$ Application model areas.

Standard notation

E measurable state space, $\mathcal{P}(E)$ proba. on E, $\mathcal{B}(E)$ bounded meas. functions.

- $(\mu, f) \in \mathcal{P}(E) \times \mathcal{B}(E) \quad \longrightarrow \quad \mu(f) = \int \mu(dx)\ f(x)$
- $M(x, dy)$ **integral operator on E**

$$\begin{aligned} M(f)(x) &= \int M(x, dy) f(y) \\ [\mu M](dy) &= \int \mu(dx) M(x, dy) \qquad (\Longrightarrow [\mu M](f) = \mu[M(f)]\ ) \end{aligned}$$

- **Bayes-Boltzmann-Gibbs transformation :** $G : E \rightarrow [0, \infty[$ with $\mu(G) > 0$

$$\Psi_G(\mu)(dx) = \frac{1}{\mu(G)}\ G(x)\ \mu(dx)$$

Self interacting sequences

E-valued random sequence $(X_n)_{n\geq 0}$ s.t.

$$\mathbb{P}\left(X_n \in dx_n \mid X_0, \ldots, X_{n-1}\right) = \Phi\left(\frac{1}{n}\sum_{0\leq p<n} \delta_{X_p}\right)(dx_n)$$

with $\Phi \; : \; \nu \in \mathcal{P}(E) \rightarrow \Phi(\nu) \in \mathcal{P}(E)$ & $\mathrm{Law}(X_0) = \eta \in \mathcal{P}(E)$.

$\subset$ Self Interacting Markov Chain models

E-valued random sequence $(X_n)_{n\geq 0}$ s.t.

$$\mathbb{P}\left(X_n \in dx_n \mid X_0, \ldots, X_{n-1}\right) = K_{\frac{1}{n}\sum_{0\leq p<n} \delta_{X_p}}(x_{n-1}, dx_n)$$

with

- A collection of Markov transitions $K_\eta(x, dy)$ with $\eta \in \mathcal{P}(E)$
- An initial distribution $\mathrm{Law}(X_0) = \eta \in \mathcal{P}(E)$.

Some questions :

1. Motivations, application areas ?
2. Toy examples ?
3. $\exists$ Conditons ? s.t. $\frac{1}{n}\sum_{0\leq p<n}\delta_{X_p} \rightarrow_{n\rightarrow\infty}$?
4. If any convergence $\rightsquigarrow$ decay rates ?
5. Links with Vertex/Edge-Reinforced type Random Walks on $\mathbb{Z}^d$?

Links with Vertex/Edge-Reinforced type Random Walks on $E = \mathbb{Z}^d$

- Expert in Oxford = Pierre Tarrès

  $\oplus$ *M. Benaim, P. Diaconis, R. Pemantle, S. Volkov,...*

- **Non compact & "degenerate" integer lattice models**

$$K_{\frac{1}{n}\sum_{0\leq p<n}\delta_{X_p}}(x_{n-1},x_n) =\propto M(x_{n-1},x_n) \times \left(a_n + \sum_{0\leq p<n} 1_{X_p}(x_n)\right)$$

- Our framewework :
  - More regular class of transitions.
  - Evolutionary & genetic mutation-selection models.
  - $\rightsquigarrow$ Self-interacting quantum and diffusion Monte Carlo.
  - Stochastic sampling and MCMC application domain.

A Toy model [μ a fixed probability meas.]

$$\Phi(\eta) := \epsilon\ \eta + (1-\epsilon)\ \mu \quad \text{with} \quad \epsilon \in [0,1]$$

$$\Downarrow$$

$$\mathbb{P}\left(X_n \in dx_n \mid X_0, \ldots, X_{n-1}\right) \;=\; \epsilon\ \frac{1}{n} \sum_{0 \leq p < n} \delta_{X_p}(dy) + (1-\epsilon)\ \mu(dy)$$

$$\Downarrow$$

μ-i.i.d. sequences with ϵ-repetitions

$$\Downarrow$$

We expect : $\frac{1}{n}\sum_{0\leq p<n} \delta_{X_p} \to_{n\to\infty} \mu$, *as soon as* $\epsilon \neq 1$

Mean value elementary estimate

$$S_n := \frac{1}{n+1} \sum_{0 \leq p \leq n} \delta_{X_p} = \frac{n}{n+1}\; S_{n-1} + \frac{1}{n+1}\; \delta_{X_{n+1}}$$

$$\Downarrow \quad ([\mu(f) = 0]$$

$$\overline{S}_n(f) \quad := \quad \mathbb{E}\left(S_n(f)\right) = \left[\frac{n+\epsilon}{n+1}\right]\; \overline{S}_{n-1}(f) = a(n)\; \eta(f)$$

with

$$\frac{1}{2e\;\, n^{1-\epsilon}} \leq \; a(n) := \left(\prod_{0<p\leq n} \frac{p+\epsilon}{p+1}\right) \; \leq \frac{2}{n^{1-\epsilon}}$$

Explicit analytic expansion

$$\mathbb{E}\left(\left[S_n(f) - \overline{S}_n(f)\right]^2\right)^{\frac{1}{2}} \simeq \begin{cases} 1/\sqrt{n} & \text{if} \quad \epsilon \in \left[0, \frac{1}{2}\right[\\ \sqrt{(\log n)/n} & \text{if} \quad \epsilon = \frac{1}{2} \\ 1/n^{(1-\epsilon)} & \text{if} \quad \epsilon \in \left]\frac{1}{2}, 1\right] \end{cases}$$

and

$$\left\|\overline{S}_n - \mu\right\|_{tv} \simeq 1/n^{(1-\epsilon)}$$

Note

$$\Phi(\eta) := \epsilon\ \eta + (1-\epsilon)\ \mu \quad \text{with} \quad \epsilon \in [0,1]$$

$$\Downarrow$$

$\epsilon = \Phi$-contraction coeff. :

$$[\Phi(\eta_1) - \Phi(\eta_2)] = \epsilon\ [\eta_1 - \eta_2]$$

Summary

Self Interacting sequences

$$\mathbb{P}\left(X_n \in dx_n \mid X_0, \ldots, X_{n-1}\right) = \Phi\left(\frac{1}{n}\sum_{0\leq p<n} \delta_{X_p}\right)(dx_n)$$

Genetic interaction model

$M(x, dy)$ Markov transition on E & a potential function $G : E \to [0, \infty[$

$$\Phi(\mu) := \Psi_G(\mu)M \quad \left(\text{with} \quad \Psi_G(\mu)(dx) = \frac{1}{\mu(G)}\; G(x)\; \mu(dx)\right)$$

$$\Downarrow$$

$$\Phi\left(\frac{1}{n}\sum_{0\leq p<n} \delta_{X_p}\right) := \sum_{0\leq p<n} \frac{G(X_p)}{\sum_{0\leq q<n} G(X_q)}\; M\left(X_p, .\right)$$

$$\Downarrow$$

Genetic reinforced model : Past history selection $\oplus$ M-free exploration

(H) : $\mathcal{B}(E) \supset \mathcal{F} \ni f$, $\|f\| \leq 1$, $d_{\mathcal{F}}(\mu,\eta) = \sup_{f\in\mathcal{F}} \|[\mu-\eta](f)\|$ complete and

$$|[\Phi(\eta) - \Phi(\mu)](f)| \leq \int |[\eta-\mu](g)| \ \Gamma_{\mu}(f, dg)$$

with

$$\beta(\Phi) := \sup_{\mu\in\mathcal{P}(E), f\in\mathcal{F}} \Gamma_{\mu}(f,1) < 1$$

Example

Genetic type (G, M)-transformations :

$$\Phi(\mu) := \Psi_G(\mu)M \quad \text{with} \quad \mathcal{F} := \{f \ : \ \mathrm{osc}(f) \leq 1\}$$

$$\beta(\Phi) \leq \left[2\left(\frac{G_{max}}{G_{min}}\right) - 1\right] \ \beta(M)$$

with the Dobrushin coefficient :

$$\beta(M) := \sup_{x,y} \|M(x,.) - M(y,.)\|_{tv} := \sup_{f\in\mathcal{F}} \mathrm{osc}(M(f))$$

Th 1 : [DM & Miclo [2002]]

(H) :

$$\exists ! \mu = \Phi(\mu)$$

and the three types of decay rates :

$$\mathbb{E}\left(\left[S_n(f) - \mu\right]^2\right)^{\frac{1}{2}} \leq \begin{cases} \frac{1}{\sqrt{n}} & \text{if} \quad \beta(\Phi) \in \left[0, \frac{1}{2}\right[\\ \sqrt{\frac{(\log n)}{n}} & \text{if} \quad \beta(\Phi) = \frac{1}{2} \\ \frac{1}{n^{(1-\beta(\Phi))}} & \text{if} \quad \beta(\Phi) \in \left]\frac{1}{2}, 1\right] \end{cases}$$

Remarks :

- $\rightsquigarrow \mathbb{L}_p$-convergence rates (without the log scales) in DM & Miclo [2003].
- **Alternative approach** $\rightsquigarrow$ Measure valued processes

$$\eta_n = \Phi(\eta_{n-1}) \longrightarrow_{n\to\infty} \mu = \Phi(\mu)$$

$$\Downarrow$$

Mean Field particle approximations = Spatial interaction processes

$$(X_n^{(1,N)}, X_n^{(2,N)}, \ldots, X_n^{(N,N)}) \text{ "iid" } \sim \Phi\left(\frac{1}{N}\sum_{1\leq i\leq N} \delta_{X_{n-1}^{(i,N)}}\right)$$

[Top eigenvalues and ground states] (ν-reversible $M \oplus$ some reg.)

$$\Phi(\eta)(f) = \frac{\eta(Q(f))}{\eta(Q(1))} \quad \text{with} \quad Q(f) = G\ M(f)$$

$$\Downarrow$$

$$\text{Top eigenvalues } \lambda \ \oplus \text{Top eigenvector } Q(h) = \lambda h$$

$$\mu(dx) = \Psi_h(\nu)M \quad \text{and} \quad \Psi_G(\mu) = \Psi_h(\nu)$$

Proof :
$$\begin{aligned} \Phi\left(\mu\right)\left(f\right) \propto \mu(GM(f)) &= \nu(hM[GM(f)]) \\ &= \nu(Q(h)M(f)) \propto \nu(hM(f)) \propto \mu(f) \end{aligned}$$

$\rightsquigarrow$ Note : $\supset$ Mean field particle model = QMC & DMC approximations

$\rightsquigarrow$ Self interacting genetic model $\simeq$ Self-interacting QMC & DMC

Summary

Self interacting Markov chains

$$\mathbb{P}(X_n \in dx_n \mid X_0, \ldots, X_{n-1}) = K_{\frac{1}{n}\sum_{0\leq p<n}\delta_{X_p}}(x_{n-1}, dx_n)$$

with

- A collection of Markov transitions $K_\eta(x, dy)$ with $\eta \in \mathcal{P}(E)$‘
- An initial distribution $\mathrm{Law}(X_0) = \eta \in \mathcal{P}(E)$.

Example : Genetic type mutation/selection transitions

$$K_\eta(x, dy) := \epsilon \; M(x, dy) + (1-\epsilon) \; \eta$$

or

$$K_\eta(x, dy) := \epsilon \; M(x, dy) + (1-\epsilon) \; \Psi_G(\eta)$$

and more generally

$$K_\eta(x, dy) := \epsilon(\eta) \; M(x, dy) + (1-\epsilon(\eta)) \; \Phi(\eta)$$

(H') : $\mathcal{F} \subset \mathcal{B}(E)$ as above s.t. :

$$|[\eta_1 - \eta_2]\, K_\mu(f)| \leq \int\ |[\eta_1 - \eta_2]\,(g)|\ \Gamma^{(1)}_\mu(f, dg)\quad \textbf{[Contraction prop.]}$$

$$|\mu\,[K_{\eta_1} - K_{\eta_2}]\,(f))| \leq \int\ |[\eta_1 - \eta_2]\,(g)|\ \Gamma^{(2)}_{\eta_1}(f, dg)\quad \textbf{[Regularity prop.]}$$

with

$$\lambda^{(1)} + \lambda^{(2)} < 1 \quad \text{and} \quad \lambda^{(i)} := \sup_{\mu \in \mathcal{P}(E), f \in \mathcal{F}} \Gamma^{(i)}_\mu(f, 1)\ , i = 1, 2$$

Th 2 : [weaker version of DM & Miclo [2003]]

(H') :

$$\exists ! \mu = \mu K_\mu$$

and the three types of decay rates : $\Lambda := \lambda_2/(1-\lambda_1)$

$$\mathbb{E}\left([S_n(f)-\mu]^2\right)^{\frac{1}{2}} \leq \begin{cases} \frac{1}{\sqrt{n}} & \text{if} \quad \Lambda \in \left[0, \frac{1}{2}\right[\\ \sqrt{\frac{(\log n)}{n}} & \text{if} \quad \Lambda = \frac{1}{2} \\ \frac{1}{n^{(1-\Lambda)}} & \text{if} \quad \Lambda \in \left]\frac{1}{2}, 1\right] \end{cases}$$

Extensions/Proof ideas :

- Under very weak conditions *($\oplus$ Resolvent estimates $\oplus$ martingales $\leq$)* :

$$\sup_{n\geq 1} \sqrt{n} \times \mathbb{E}\left([S_n(f) - S_n^\omega(f)]^2\right)^{\frac{1}{2}} < \infty \quad \text{with} \quad S_n^\omega := \frac{1}{n}\sum_{0\leq p<n} \omega(S_p)$$

- $\oplus$ Weaker conditions $\rightsquigarrow$ Fixed point mapping $\omega(\eta) = \omega(\eta)K_\eta$ analysis.
- $\mathbb{L}_p$-mean error estimates with the same decay rates.

Summary

i-MCMC sampling of nonlinear m.v.p.

- **Problem :** sample series of random variables w.r.t. η_n with

$$\eta_n = \Phi_n(\eta_{n-1})$$

$\sim$ *Cooling schemes, temp. variations, constraints sequences, subset restrictions, observation data, conditional events, Feynman-Kac flows:*

$$\eta_n(dx) \propto e^{-\beta_n V(x)} \lambda(dx),\ 1_{A_n}(x)\ \lambda(dx),\ \mathbb{P}(X_n \in dx \mid Y_0, \ldots, Y_n), \ldots$$

- **Natural interacting sampling idea :**

Use η_{n-1} or <u>its empirical approx.</u> to sample w.r.t. η_n

$\rightsquigarrow$ **Interacting MCMC models :**

$$\left\{ \begin{array}{l} \text{Use the occupation measures} \\ \text{of an MCMC with target } \eta_{n-1} \end{array} \right\} \rightsquigarrow \text{MCMC target } \eta_n$$

The key natural idea

Find a series of MCMC models $X^{(n)} := (X_k^{(n)})_{k\geq 0}$ s.t.

$$\frac{1}{k+1}\sum_{0\leq l\leq k}\delta_{X_l^{(n)}} \quad := \quad \eta_k^{(n)} \simeq_{k\uparrow\infty} \; \eta_n$$

$$\Downarrow$$

Use $\eta_k^{(n)} \simeq \eta_n$ to define $X^{(n+1)}$ with target η_{n+1}

$$
\begin{array}{ccc}
((n-1)\text{-th chain}) & & (n\text{-th chain}) \\
X_0^{(n-1)} & & X_0^{(n)} \\
\downarrow & & \downarrow \\
X_1^{(n-1)} & & \vdots \\
\downarrow & & \vdots \\
\vdots & & \downarrow \\
\downarrow & & X_k^{(n)} \\
 & & \downarrow \\
X_k^{(n-1)} & \xrightarrow{\eta_k^{(n-1)} \simeq \eta_{n-1}} & M_{n,\eta_k^{(n-1)}} \simeq M_{n,\eta_{n-1}} \\
\downarrow & & \downarrow \\
\vdots & & X_{k+1}^{(n)}
\end{array}
$$

Key condition : $M_{n,\eta}(x, dy)$ with invariant measure $\Phi_n(\eta)$

$$\Rightarrow \Phi_n(\eta_k^{(n-1)}) \simeq \Phi_n(\eta_{n-1}) = \eta_n$$

Advantages

- Using η_n the sampling η_{n+1} is often easier.
- Improve the proposition step in any Metropolis type model with target η_{n+1} ($\rightsquigarrow$ enters the stability prop. of the flow η_n)
- Increases the precision at every time step.

  But CLT variance often $\geq$ CLT variance mean field models.
- Easy to combine with mean field stochastic algorithms :

  $\rightsquigarrow$ a natural technique to refine and improve the convergence of a given sampled mean field particle algorithm.

Interacting Markov chain Monte Carlo models

- Find M_0 and a collection of transitions $M_{n,\mu}$ s.t.

$$\eta_0 = \eta_0 M_0 \quad \text{and} \quad \Phi_n(\mu) = \Phi_n(\mu) M_{n,\mu}$$

- $(X_k^{(0)})_{k\geq 0}$ Markov chain $\sim M_0$.
- Given $X^{(n)}$, we let $X_k^{(n+1)}$ with Markov transtions $M_{n+1,\eta_k^{(n)}}$

Rationale :

$$\eta_k^{(n)} \simeq \eta_n \quad \Rightarrow \quad \left(M_{n+1,\eta_k^{(n)}} \simeq M_{n+1,\eta_n}\right) \Longrightarrow \eta_k^{(n+1)} \simeq \Phi_n(\eta_n) = \eta_{n+1}$$

Example : $M_{n,\mu}(x,.) = \Phi_n(\mu) \quad \rightsquigarrow \quad X_k^{(n+1)} \text{ random var. } \sim \Phi_{n+1}\left(\eta_k^{(n)}\right)$

Functional Central Limit Theorem :

$$\sqrt{k}\ \left[\eta_k^{(n)} - \eta_n\right] \longrightarrow_{k\uparrow\infty} \sum_{q=0}^{n}\ \alpha_n(q)\ \ V_q D_{q,n}$$

- The parameters $\alpha_n(n-q) = \sqrt{(2q)!}/q!$ **(Mean field models** $\alpha_n(q) = 1$**)**
- $\perp$ Centered Gauss. fields $(V_q)_{q\geq 0} \sim$ Fluctuations of the q-th MCMC sampler around the limiting measure η_q

$$\mathbb{E}\left(V_q(f)^2\right)$$

$$= \eta_q\left[(f-\eta_q(f))^2\right] + 2\sum_{m\geq 1}\eta_q\left[(f-\eta_q(f))\ M^m_{q,\eta_{q-1}}(f-\eta_q(f))\right]$$

- First order decomposition of the (nonlinear) sg $\rightarrow \Phi_{p,n}(\eta_p) = \eta_n$

$$\Phi_{p,n}(\eta) - \Phi_{p,n}(\mu) \simeq (\eta-\mu)D_{p,n} + (\eta-\mu)^{\otimes 2}\ldots$$

$\rightsquigarrow$ **Extensions :** Non asymptotic $\mathbb{L}_p$-estimates $\oplus$ exponential concentration $\subset$ Bercu, DM, Doucet (2008).