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FEYNMAN-KAC FORMULAE
GENEALOGICAL AND INTERACTING PARTICLE MODELS

Applications:

non linear filtering, smoothing/path estimation, directed polymers, rare event analysis and spectral analysis of Feynman-Kac-Schrödinger semigroups.

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1. Evolutionary particle methods
2. Feynman-Kac modeling
3. Motivations
   - Killing
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5. Applications
   - Non linear filtering
   - Rare event analysis
   - Particle polymerizations
   - Spectral analysis

6. Stochastic analysis
    - Contraction and stability
    - Annealed properties
7. Asymptotic behavior
    - Uniform estimates (space/time)
    - Increasing propagation of chaos
    - Central limit theorems
    - Large deviations principles

Evolutionary particle methods

$$\Downarrow$$

- **Signal-Processing/Nonlinear filtering**
  - Branching and interacting particle systems.
  - Monte-Carlo optimal filters.
  - Genetic algorithms.
  - Bootstrap or particle filters.
  - Sampling-Importance-Resampling.
  - Condensation filters.
- **Rare event analysis**: Restart method.

- **Physics and biology**:
  - Moran particle systems.
  - Weighted genetic algorithm.
  - Interacting particle systems.
  - Particle polymerizations.
- **Numerical function optimization**:
  - Genetic algorithms.
  - Evolutionary random search algorithms.

$$\Downarrow$$

Same particle model!!

$\neq$ Interpretations

- **Mathematical physics**:
  particle microscopic interpretation

- **Pure mathematics**:
  stochastic linearization technique

- **Probability and Statistics**:
  approximating simulation technique (recursive rejections)

- **Evolutionary mathematics and engineering sciences**:
  adaptative stochastic search method,

  $\Rightarrow$ **Physical and biological intuitions** $\in$ **Engineering problems**

Common features

- **Objectives**
  - Approximation/Simulation of (conditional) distributions.
  - Global optimization of a numerical fitness function.
  - Sampling a rare event/flexible polymer chain.
  - Evolving a particle in an aborbing medium.
  - ../...
- **Particle Methods**
  - Random explorations of the state space
    $\longrightarrow$ Markov evolution
  - Random selections of the proposal
    $\longrightarrow$ potential/likelihood/fitness

2 ingredients:

(Potential function, Markov exploration kernel)$=(G_n(x), \mathcal{M}_n(x, dy))$

$$\Downarrow$$

Feynman-Kac or Boltzmann-Gibbs' formula

Physics $\longleftrightarrow$ Biology $\longleftrightarrow$ Engineering

$$\mathbb{Q}_n = \frac{1}{Z_n} \quad \exp\left\{-\sum_{p=0}^{n} V_p(\mathcal{X}_p)\right\} \ \mathbb{P}_n = \frac{1}{Z_n} \ \left\{ \ \prod_{p=0}^{n} \ G_p(\mathcal{X}_p) \ \right\} \ \mathbb{P}_n$$

Feynman-Kac distribution flows

$\longrightarrow$ Markov $\quad \mathcal{X}_n \in (E_n, \mathcal{E}_n), \quad \mathcal{M}_n(x_{n-1}, dx_n)$

Potential $\quad G_n : E_n \longrightarrow (0,1)$

$$\eta_n(f_n) \; = \; \gamma_n(f_n)/\gamma_n(1)\,, \quad \gamma_n(f_n) = \mathbb{E}(f_n(\mathcal{X}_n) \prod_{p=0}^{n-1} G_p(\mathcal{X}_p))$$

$$\gamma_n(f_n) \; = \; \int_{E_0 \times \ldots \times E_n} f_n(x_n) \; [\prod_{p=0}^{n-1} G_p(x_p)] \mathbb{P}_n(d(x_0, \ldots, x_n))$$

with

$$\mathbb{P}_n(d(x_0, \ldots, x_n)) = \; \eta_0(dx_0)\mathcal{M}_1(x_0, dx_1) \ldots \mathcal{M}_n(x_{n-1}, dx_n)$$

Example : $\mathcal{X}'_n$ non anticipative

$$\mathcal{X}_n = (\mathcal{X}'_0, \ldots, \mathcal{X}'_n) \in E_n = E'_0 \times \ldots \times E'_n$$

$$\Downarrow$$

$$\gamma_n(f_n) = \mathbb{E}(f_n(\mathcal{X}'_0, \ldots, \mathcal{X}'_n) \prod_{p=0}^{n-1} G_p(\mathcal{X}'_0, \ldots, \mathcal{X}'_p))$$

Stability property ($\hookrightarrow$ genealogical models)

-Historical model, $\mathcal{X}'_n$ **Markov**

$$\mathcal{X}_n \;=\; (\mathcal{X}'_0, \ldots, \mathcal{X}_n) \in E_n = E'_0 \times \ldots \times E'_n$$

$$\mathcal{X}'_n \;\in\; E'_n\,, \;\; \sim \mathcal{M}'_n(\mathcal{X}'_{n-1}, \centerdot)$$

-"marginal potentials" $G_n(x'_0, \ldots, x'_n) = G'_n(x'_n)$

$$\begin{aligned}\gamma_n(f_n) &= \mathbb{E}(f_n(\mathcal{X}_n) \prod_{p=0}^{n-1} G_p(\mathcal{X}_p)) \\ &= \mathbb{E}(f_n(\mathcal{X}'_0,\ldots,\mathcal{X}'_n) \prod_{p=0}^{n-1} G_p(\mathcal{X}'_0,\ldots,\mathcal{X}'_p))\end{aligned}$$

n-time marginals

$$\gamma'_n(f'_n) = \mathbb{E}(f'_n(\mathcal{X}'_n) \prod_{p=0}^{n-1} G'_p(\mathcal{X}'_p))$$

Feynman-Kac structural stability

$$(G_n, \mathcal{M}_n) \longrightarrow (G'_n, \mathcal{M}'_n)$$

Updated Feynman-Kac measures

$$\mu_n(f_n) = \lambda_n(f_n)/\lambda_n(1)$$

with

$$\begin{aligned}\lambda_n(f_n) &= \gamma_n(f_n G_n) \\ &= \mathbb{E}(f_n(\mathcal{X}_n) \prod_{p=0}^{\mathbf{n}} G_p(\mathcal{X}_p))\end{aligned}$$

Note

$$\mu_n(f_n) = \frac{\lambda_n(f_n)}{\lambda_n(1)} = \frac{\gamma_n(f_n G_n)}{\gamma_n(G_n)} = \frac{\eta_n(f_n G_n)}{\eta_n(G_n)}$$

$$\mu_n(f_n) = \frac{\eta_n(f_n G_n)}{\eta_n(G_n)}$$

$$\Downarrow$$

Gibbs' updating transf.

$$\eta_n \longrightarrow \mu_n = \Psi_n(\eta_n)$$

with

$$\Psi_n(\eta_n)(dy_n) = \frac{G_n(y_n)}{\eta_n(G_n)}\ \eta_n(dy_n)$$

Motivating ex.

- Path estimation

$\rightarrow$ observation $\mathcal{Y}_n = H_n(\mathcal{X}_n, \underbrace{\mathcal{V}_n}_{indp.noise})$

$$\mathbb{P}(H_n(x_n, \mathcal{V}_n) \in dy_n) = \underbrace{g_n(y_n, x_n)}_{=G_n(x_n)} \; q_n(dy_n)$$

$$\Downarrow$$

$$\begin{aligned} \eta_n &= \text{Law}(\mathcal{X}_n \mid Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1}) \\ \mu_n &= \text{Law}(\mathcal{X}_n \mid Y_0 = y_0, \ldots, Y_n = y_n) \end{aligned}$$

Hist. proc. $\mathcal{X}_n = (\mathcal{X}'_0, \ldots, \mathcal{X}'_n)$

$$\Downarrow$$

Filtering/smoothing/path est.

$$\mu_n = \mathsf{Law}(\mathcal{X}'_0, \ldots, \mathcal{X}'_n \mid Y_0 = y_0, \ldots, Y_n = y_n)$$

- .../...

- Unormalized$\longleftrightarrow$normalized

$$\gamma_n(f_n) = \eta_n(f_n) \quad \prod_{p=0}^{n-1} \eta_p(G_p)$$

- Unnormalized meas. $\gamma_n = \gamma_{n-1}\mathcal{Q}_n$ with

$$\mathcal{Q}_n = \mathcal{G}_{n-1}\mathcal{M}_n$$

Boltzmann-Schrödinger's $\times$-sg

$$\mathcal{G}_n(f_n)(x_n) = G_n(x_n)f(x_n)$$

Linear evolution

$$\gamma_n = \gamma_p \mathcal{Q}_{p,n} \quad \text{with} \quad \mathcal{Q}_{p,n} = \mathcal{Q}_{p+1}\mathcal{Q}_{p+2}\cdots\mathcal{Q}_n$$

$$\mathcal{Q}_{n-1}(f_n) = G_{n-1} \ \times \ \mathcal{M}_n(f_n) \Longrightarrow \gamma_n(f_n) = \gamma_{n-1}(G_{n-1}\mathcal{M}_n(f_n))$$

$$\Downarrow$$

Normalized distributions

$$\begin{aligned}\eta_n(f_n) &= \frac{\gamma_n(f_n)}{\gamma_n(1)} = \frac{\gamma_{n-1}(G_{n-1}\mathcal{M}_n(f_n))}{\gamma_{n-1}(G_{n-1})} \\ &= \Psi_{n-1}(\eta_{n-1})\mathcal{M}_n(f_n)\end{aligned}$$

with the Gibbs' measure

$$\Psi_n(\eta_n)(dy_n) = \frac{G_n(y_n)}{\eta_n(G_n)} \ \eta_n(dy_n)$$

Linear meas. val. proc.

$$\gamma_n = \gamma_{n-1}\mathcal{Q}_n \quad \text{with} \quad \mathcal{Q}_n = \mathcal{G}_{n-1}\mathcal{M}_n$$

Non linear meas. val. proc.

$$\eta_n = \Phi_n(\eta_{n-1}) \quad \text{with} \quad \Phi_n(\eta) = \Psi_{n-1}(\eta)\mathcal{M}_n$$

Sub-Markov$\longrightarrow$ Markov

$$\mathcal{G}_n(f_n)(x_n) = G_n(x_n)f(x_n)$$

$\longrightarrow$ Killing random motions, $E_n^c = E_n \cup \{c\}$

$$\mathcal{G}_n^c(x_n, dy_n) = G_n(x_n)\ \delta_{x_n}(dy_n) + (1 - G_n(x_n))\ \delta_c(dy_n)$$

$$\Downarrow$$

$$\mathcal{X}_n^c \longrightarrow \widehat{\mathcal{X}_n^c} \longrightarrow \mathcal{X}_{n+1}^c$$

Feynman-Kac killing interpretation

$$\gamma_n(1) \;=\; \mathbb{P}^c(T \geq n) \quad \text{and} \quad \eta_n(f_n) = \mathbb{E}^c(f(\mathcal{X}_n^c) \mid T \geq n)$$

Life time

$$T = \inf\{n \geq 0\ ;\ \widehat{\mathcal{X}_n^c} = c\}$$

Boltzmann-Gibbs' transformations

$$\Psi_n(\eta_n)(f_n) = \frac{\eta_n(f_n G_n)}{\eta_n(G_n)} \Rightarrow \eta_n(G_n)\ \Psi_n(\eta_n)(f_n) = \eta_n(G_n f_n)$$

$$\Downarrow$$

$$\Psi_n(\eta_n)(f_n) = \eta_n(\ [G_n f_n + \eta_n(1 - G_n)\ \Psi_n(\eta_n)(f_n)])$$

$$\Downarrow$$

$$\eta_n = \eta_n \mathcal{S}_{n,\eta_n}$$

with Markov kernel

$$\mathcal{S}_{n,\eta_n}(x_n, dy_n) = G_n(x_n)\ \delta_{x_n}(dy_n) + (1 - G_n(x_n))\ \Psi_n(\eta_n)(dy_n)$$

Sub-Markov$\longrightarrow$ Markov

$$\mathcal{G}_n(f_n)(x_n) = G_n(x_n) f(x_n)$$

$\longrightarrow$ interacting process E_n

$$\mathcal{S}_{n,\eta_n}(x_n, dy_n) = G_n(x_n)\ \delta_{x_n}(dy_n) + (1 - G_n(x_n))\ \Psi_n(\eta_n)(dy_n)$$

Gibbs' measure

$$\Psi_n(\eta_n)(dy_n) = \frac{G_n(y_n)}{\eta_n(G_n)}\ \eta_n(dy_n)$$

Non homogeneous/Non linear Markov chain

$$\mathcal{X}_n \longrightarrow \widehat{\mathcal{X}}_n \longrightarrow \mathcal{X}_{n+1}$$

$$\Downarrow$$

Non linear meas. val. proc.

$$\eta_n = \mathsf{Law}(\mathcal{X}_n) \;\rightarrow\; \eta_{n+1} = \eta_n \mathcal{K}_{n+1,\eta_n}$$

with

$$\mathcal{K}_{n+1,\eta_n} = \mathcal{S}_{n,\eta_n} \mathcal{M}_{n+1}$$

Measure valued processes

$$\eta_{n+1} = \Phi_{n+1}(\eta_n) = \Psi_n(\eta_n)\mathcal{M}_{n+1} = \eta_n \mathcal{S}_{n,\eta_n}\mathcal{M}_{n+1}$$

$$\downarrow$$

$$\neq \text{ Markov interpretation}$$

I.: $\forall G_n(E_n) \subset (0,\infty)$

$$\eta_n \overset{updating}{\longrightarrow} \mu_n = \Psi_n(\eta_n) \overset{prediction}{\longrightarrow} \eta_{n+1} = \mu_n \mathcal{M}_{n+1}$$

II.: $\forall G_n(E_n) \subset (0,1)$ ($\sim \|G_n\| < \infty$

$$\eta_n \stackrel{updating}{\longrightarrow} \mu_n = \eta_n \mathcal{S}_{n,\eta_n} \stackrel{prediction}{\longrightarrow} \eta_{n+1} = \mu_n \mathcal{M}_{n+1}$$

III.: $G_n(E_n) \subset (1,\infty)$ ($\sim G_n(E_n) \subset (\epsilon_n, \infty)$)

$$G_n = G_n^+ + G_n^- \quad \text{and} \quad G_n(x_n) > G^-(y_n) \geq 1$$

$$\mathcal{S}_{n,\eta}(x_n, dy_n) = \frac{G_n^-(x_n)}{\eta(G_n)}\ \delta_{x_n}(dy_n) + \left(1 - \frac{G_n^-(x_n)}{\eta(G_n)}\right)\ \Psi_n^+(\eta)(dy_n)$$

Updated flows

$$\mu_n = \Psi_n(\mu_{n-1}\mathcal{M}_n)$$

NOTE

$$\begin{aligned}\mu_n(f) &= \frac{\mu_{n-1}\mathcal{M}_n(fG_n)}{\mu_{n-1}\mathcal{M}_n(G_n)} \\ &= \frac{\mu_{n-1}(\widehat{G}_{n-1}\widehat{\mathcal{M}}_n(f))}{\mu_{n-1}(\widehat{G}_{n-1})}\end{aligned}$$

updated (potential-kernel)

$$\widehat{G}_{n-1} = \mathcal{M}_n(G_n) \quad \text{and} \quad \widehat{\mathcal{M}}_n(f) = \frac{\mathcal{M}(G_n f)}{\mathcal{M}(G_n)}$$

$$\Downarrow$$

$$\mu_{n+1} = \widehat{\Phi}_{n+1}(\mu_n) = \widehat{\Psi}_n(\mu_n)\widehat{\mathcal{M}}_{n+1} = \mu_n \hat{\mathcal{S}}_{n,\mu_n}\widehat{\mathcal{M}}_{n+1}$$

$$\Downarrow$$

(idem)

$$(G_n, \mathcal{M}_n) \longleftrightarrow (\widehat{G}_n, \widehat{\mathcal{M}}_n)$$

BUT $\neq$ **particle interpretations!**

IV.: $G_n(E_n) \subset [\mathbf{0}, \infty)$

Ex.: $G_n(x_n) = 1_{A_n}(x_n),\ A_n \in \mathcal{E}_n$

$$\begin{aligned} \lambda_n(1) &= \mathbb{E}(\prod_{p=0}^{n} G_n(\mathcal{X}_p)) \\ &= \mathbb{P}(\mathcal{X}_0 \in A_0, \ldots, \mathcal{X}_n \in A_n) \end{aligned}$$

$$\mu_n = \mathbb{P}(\mathcal{X}_0 \in dx_0, \ldots, \mathcal{X}_n \in dx_n | \mathcal{X}_0 \in A_0, \ldots, \mathcal{X}_n \in A_n)$$

As before:

$$\eta_n \xrightarrow{upd.} \mu_n = \left\{ \begin{array}{l} \Psi_n(\eta_n) \\ \eta_n \mathcal{S}_{n,\eta_n} \end{array} \right. \xrightarrow{pred.} \eta_{n+1} = \mu_n \mathcal{M}_{n+1}$$

but

$$\Psi_n(\eta_n)(f_n) = \frac{\eta_n(f_n G_n)}{\eta_n(G_n)} \qquad \forall \eta_n \in \mathcal{P}_n(E_n) = \{\eta \in \mathcal{P}(E_n)\ ;\ \eta(G_n) > 0\}$$

$\neq$Markov int.$\Rightarrow\neq$ McKean measure

$$\eta_{n+1} = \Phi_{n+1}(\eta_n) = \eta_n \mathcal{K}_{n+1,\eta_n}$$

$$\downarrow$$

McKean measures (canonical space) $\sim (\eta_0, \mathcal{K}_{n,\eta})$

$$\mathbb{K}_n(d(x_0,\ldots,x_n)) = \eta_0(dx_0)\ \mathcal{K}_{1,\eta_0}(x_0,dx_1)\ \ldots\ \mathcal{K}_{n,\eta_{n-1}}(x_{n-1},dx_n)$$

Ex.:

$$\mathcal{K}_{n,\eta}(x,dy) = \Phi_n(\eta)(dy) \Rightarrow \mathbb{K}_n = \eta_0 \otimes \ldots \otimes \eta_n$$

Interacting Particle methods $\Longrightarrow$ Solve meas.val. proc.

$$\eta_n = \Phi_{n+1}(\eta_n)$$

Idea $\xi_n = (\xi_n^1, \ldots, \xi_n^N) \in E_n^N$ Markov s.t. $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \delta_{\xi_n^i} = \eta_n$

bips interpretations

- stochastic linearization
- simulation technique
- microscopic particle interpretation
- stochastic and adaptative grid

[$\neq$ Markov interpretation] $\Longrightarrow$ [$\neq$ Particle interpretation]

$$\eta_{n+1} = \Phi_{n+1}(\eta_n) = \eta_n \mathcal{K}_{n+1,\eta_n}$$

$$\downarrow$$

$$\left\{ \begin{array}{rcl} m(\xi_{n-1}) = \frac{1}{N}\sum_{i=1}^N \delta_{\xi^i_{n-1}} & \simeq & \eta_{n-1} \\ & \Downarrow & \\ \Phi_n(m(\xi_{n-1})) & \simeq & \Phi_n(\eta_{n-1}) = \eta_n \end{array} \right.$$

-$\Phi_n \Rightarrow$ conditional **iid** particles

$$\xi_n = (\xi_n^1, \ldots, \xi_n^N) \text{ iid } \sim \ \Phi_n(m(\xi_{n-1}))$$

-$\mathcal{K}_{n,\eta} \Rightarrow$ conditional **indp** particles

$$\xi_n^j \sim \mathcal{K}_{n,m(\xi_{n-1})}(\xi_{n-1}^j, dy)$$

Boltzmann-Gibbs' transf. $m(x) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x^i}$

$$m(x) \to \Psi_n(m(x)) = \sum_{i=1}^{N} \frac{G_n(x^i)}{\sum_{j=1}^{N} G_n(x^j)}\ \delta_{x^i}$$

$G_n(E_n) \subset (0,1) \Longrightarrow$ Markov interpt.

$$\mathcal{S}_{n,m(x)}(x^i, dy) = G_n(x^i)\ \delta_{x^i}(dy) + (1 - G_n(x^i))\ \Psi_n(m(x))(dy)$$

$$\Downarrow$$

$$\begin{array}{rcl} \Phi_{n+1}(m(x)) & = & \Psi_n(m(x))\mathcal{M}_{n+1} \\ \Phi_{n+1}(m(x)) & = & m(x)\mathcal{K}_{n+1,m(x)} \quad \text{with} \quad \mathcal{K}_{n+1,m(x)} = \mathcal{S}_{n,m(x)}\mathcal{M}_{n+1} \end{array}$$

$\Phi_n \Rightarrow$ conditional **iid** particles

$$(\xi_{n+1}^1, \ldots, \xi_{n+1}^N) \text{ iid } \sim \Phi_{n+1}(m(\xi_n))$$

$$\Phi_{n+1}(m(\xi_n)) \;=\; \Psi_n(m(\xi_n))\mathcal{M}_{n+1} = \sum_{i=1}^N \frac{G_n(x^i)}{\sum_{j=1}^N G_n(x^j)} \, \mathcal{M}_{n+1}(x^i, \textbf{.})$$

$$\Downarrow$$

$$\eta_n \;\xrightarrow{upd.}\; \Psi_n(\eta_n) \;\xrightarrow{predict.}\; \eta_{n+1}$$

$$\xi_n \;\xrightarrow{selection}\; \widehat{\xi}_n \;\xrightarrow{mutation}\; \xi_{n+1}$$

Genetic adaptative stoch. grid

$$\xi_n \in E_n^N \xrightarrow{\quad selection \quad} \widehat{\xi}_n \in E_n^N \xrightarrow{\quad mutation \quad} \xi_{n+1} \in E_{n+1}^N$$

-selection

$$\widehat{\xi}_n = (\widehat{\xi}_n^1, \ldots, \widehat{\xi}_n^N) \quad \textbf{iid} \quad \Psi_n(m(\xi_n))$$

-mutation

$$\widehat{\xi}_n^i \longrightarrow \xi_{n+1}^i \sim \mathcal{M}_{n+1}(\widehat{\xi}_n^i, \cdot)$$

$\mathcal{K}_{n,\eta} \Rightarrow$ conditional **indp** particles

$$\xi^i_{n+1} \sim \mathcal{K}_{n+1,m(\xi_n)}(\xi^i_n, .) = \mathcal{S}_{n,m(\xi_n)}\mathcal{M}_{n+1}(\xi^i_n, .)$$

$$\Phi_{n+1}(m(\xi_n)) = m(\xi_n)\mathcal{S}_{n,m(\xi_n)}\mathcal{M}_{n+1}$$

$$\Downarrow$$

$$\eta_n \xrightarrow{upd.} \eta_n\mathcal{S}_{n,\eta_n} \xrightarrow{predict.} \eta_{n+1}$$

$$\xi_n \xrightarrow{selection} \widehat{\xi}_n \xrightarrow{mutation} \xi_{n+1}$$

Genetic adaptative stoch. grid $(G_n(E_n) \subset (0,1))$

$$\xi_n \in E_n^N \xrightarrow{\ selection\ } \widehat{\xi}_n \in E_n^N \xrightarrow{\ mutation\ } \xi_{n+1} \in E_{n+1}^N$$

-**selection** = cond. indpt $\xi_n^i \longrightarrow \widehat{\xi}_n^i$

$$\sim \mathcal{S}_{n,m(\xi_n)}(\xi_n^i, dy) = G_n(\xi_n^i)\ \delta_{\xi_n^i}(dy) + (1 - G_n(\xi_n^i))\ \Psi_n(m(\xi_n))(dy)$$

-**mutation** *[idem]*

$$\widehat{\xi}_n^i \longrightarrow \xi_{n+1}^i \sim \mathcal{M}_{n+1}(\widehat{\xi}_n^i, \textbf{.})$$

Genealogical trees and historical models

$$\text{Selection} = \text{birth-death} \longrightarrow \text{ancestral lines}$$

$$\text{(selection/mutation)} \longleftrightarrow (G_n, \mathcal{M}_n)$$

Historical model

$$\begin{array}{rcl} \mathcal{X}_n & = & (\mathcal{X}'_0, \ldots, \mathcal{X}_n) \in E_n = E'_0 \times \ldots \times E'_n \\ \mathcal{X}'_n & \in & E'_n, \ \sim \mathcal{M}'_n(\mathcal{X}_{n-1}, \cdot) \end{array}$$

$$\Downarrow$$

Path particles

$$\left\{ \begin{array}{rcl} \xi^i_n & = & (\xi'^i_{0,n}, \ldots, \xi'^i_{n,n}) \in E_n = E'_0 \times \cdots \times E'_n \\ \widehat{\xi}^i_n & = & (\widehat{\xi}'^i_{0,n}, \ldots, \widehat{\xi}'^i_{n,n}) \in E_n = E'_0 \times \cdots \times E'_n \end{array} \right.$$

Genetic-path model

path-selection

$$\sum_{i=1}^{N} \frac{G_n(\xi'^i_{0,n},\ldots,\xi'^i_{n,n})}{\sum_{j=1}^{N} G_n(\xi'^i_{0,n},\ldots,\xi'^i_{n,n})}\ \delta_{(\xi'^i_{0,n},\ldots,\xi'^i_{n,n})}$$

path-mutation

$$\xi^i_{n+1} = (\underbrace{[\widehat{\xi}'^i_{0,n},\ldots\ldots,\widehat{\xi}'^i_{n,n}]}_{selected-path}, \quad \underbrace{\xi'^i_{n+1,n+1}}_{\substack{path\,-\,extension \\ \sim \mathcal{M}'_{n+1}(\widehat{\xi}^i_n,\,\cdot)}} \quad)$$

Genealogies

–Historical model, $\mathcal{X}'_n$ Markov

$$\mathcal{X}_n = (\mathcal{X}'_0, \ldots, \mathcal{X}_n) \in E_n = E'_0 \times \ldots \times E'_n$$

$$\mathcal{X}'_n \in E'_n, \sim \mathcal{M}'_n(\mathcal{X}'_{n-1}, \cdot)$$

–"marginal potentials" $G_n(x'_0, \ldots, x'_n) = G'_n(x'_n)$

Genealogical particle tree model

path-selection

$$\sum_{i=1}^{N} \frac{G'_n(\xi'^i_{n,n})}{\sum_{j=1}^{N} G'_n(\xi'^j_{n,n})} \; \delta_{(\xi'^i_{0,n},\ldots,\xi'^i_{n,n})}$$

path-mutation

$$\xi^i_{n+1} = (\underbrace{[\widehat{\xi}'^i_{0,n},\ldots\ldots,\widehat{\xi}'^i_{n,n}]}_{selected-path}, \quad \underbrace{\xi'^i_{n+1,n+1}}_{\substack{path\,-\,extension \\ \sim\, \mathcal{M}'_{n+1}(\widehat{\xi}^i_{n,n},\,\centerdot)}} \quad)$$

Marginal genetic model

$$\xi'_n \xrightarrow{\quad G'_n \quad} \widehat{\xi}'_n \xrightarrow{\quad \mathcal{M}'_{n+1} \quad} \xi'_{n+1}$$

Historical process and genealogical Markov model

$$\xi_n \xrightarrow{\quad G_n \quad} \widehat{\xi}_n \xrightarrow{\quad \mathcal{M}_{n+1} \quad} \xi_{n+1}$$

Particle approximating measures

Limiting Feynman-Kac path measures

-<u>path-measures</u>

$$\begin{aligned}\eta_n(f_n) &= \gamma_n(f_n)/\gamma_n(1)\\ \gamma_n(f_n) &= \mathbb{E}(f_n(\mathcal{X}'_0,\ldots,\mathcal{X}'_n)\ \prod_{p=0}^{n-1} G'_p(\mathcal{X}'_p))\end{aligned}$$

-<u>n-time marginals</u>

$$\begin{aligned}\eta'_n(f'_n) &= \gamma'_n(f'_n)/\gamma'_n(1)\\ \gamma'_n(f'_n) &= \mathbb{E}(f'_n(\mathcal{X}'_n)\ \prod_{p=0}^{n-1} G'_p(\mathcal{X}'_p))\end{aligned}$$

Particle measures

-<u>path-measures</u>

$$\eta_n^N \;=\; \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_n^i} = \frac{1}{N}\sum_{i=1}^{N} \delta_{(\xi_{0,n}'^i,\ldots,\xi_{n,n}'^i)}$$

$$\gamma_n^N(f_n) \;=\; [\prod_{p=0}^{n-1} \eta_p^N(G_p)] \;\times\; \eta_n^N(f_n)$$

-<u>n-time marginals</u> $\xi_n'^i = \xi_{n,n}'^i$

$$\eta_n'^N \;=\; \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_n'^i}$$

$$\gamma_n'^N(f_n') \;=\; [\prod_{p=0}^{n-1} \eta_p^N(G_p)] \;\times\; \eta_n'^N(f_n')$$

Particle approx. measures *(continued)*

Updated FK measures

$$\begin{aligned}\mu_n(f_n) &= \eta_n(f_n G_n)/\eta_n(G_n)\\ \lambda_n(f_n) &= \gamma_n(f_n G_n) = \mathbb{E}(f_n(\mathcal{X}_n)\ \prod_{p=0}^{n} G_p(X_p))\end{aligned}$$

Updated particle measures

-<u>Boltzmann-Gibbs' updating</u>

$$\mu_n^N(f_n) = \eta_n^N(f_n G_n)/\eta_n^N(G_n)$$

(or) **updated particle profiles**

$$\mu_n^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\widehat{\xi}_n^i} \qquad (\widehat{\xi}_n^i \ iid \ \sim \ \Psi_n(\eta_n^N))$$

Un-normalized particle measures

$$\begin{aligned}\lambda_n^N(f_n) = \gamma_n^N(f_n G_n) &= [\prod_{p=0}^{n-1} \eta_p^N(G_p)] \times \eta_n^N(f_n G_n) \\ &= [\prod_{p=0}^{n} \eta_p^N(G_p)] \times \Psi_n(\eta_n^N)(f_n)\end{aligned}$$

Particle approx. measures *(continued)*

McKean measures

$$\mathbb{K}_n(d(x_0,\ldots,x_n)) = \eta_0(dx_0)\ \mathcal{K}_{1,\eta_0}(x_0,dx_1)\ \ \ldots\ \ \mathcal{K}_{n,\eta_{n-1}}(x_{n-1},dx_n)$$

Particle McKean measures

$$\mathbb{K}_n^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{(\xi_0^i,\ldots,\xi_n^i)}$$

Ex.:

$\Phi_n \Rightarrow$ conditional **iid** particles

$$(\xi_n^1, \ldots, \xi_n^N) \text{ iid } \Phi_n(m(\xi_{n-1})) \Rightarrow \lim_{N\to\infty} \mathbb{K}_n^N = \mathbb{K}_n = \eta_0 \otimes \ldots \otimes \eta_n$$

$\mathcal{K}_{n,\eta} \Rightarrow$ conditional **indp** particles

$$\xi_{n+1}^i \sim \mathcal{S}_{n,m(\xi_n)}\mathcal{M}_{n+1}(\xi_n^i, \textbf{.}) \Rightarrow \lim_{N\to\infty} \mathbb{K}_n^N = \mathbb{K}_n \neq \eta_0 \otimes \ldots \otimes \eta_n$$

Some asympt. results (mild assumpt.)

$\mathbb{L}_p$**-unif. estimates** ($\hookrightarrow$*(CLT)*)

$$\mathbb{E}(\sup_{f\in\mathcal{F}} |\eta_n^N(f) - \eta_n(f)|^p)^{1/p} \leq I(\mathcal{F})/\sqrt{N}$$

Φ_n**-exp. asympt. stable**

$$\exists \alpha \in (0,1] \ : \quad \sup_{n\geq 0} \mathbb{E}(\sup_{f\in\mathcal{F}} |\eta_n^N(f) - \eta_n(f)|^p)^{1/p} \leq I(\mathcal{F})/N^{\alpha/2}$$

$$\mathcal{M}_n \text{ **mixing** } \Longrightarrow \ \alpha = 1$$

Exp. estimates ($\hookrightarrow$*(LDP)*)

$$\mathbb{P}(\sup_{f\in\mathcal{F}} |\eta_n^N(f) - \eta_n(f)| \geq \epsilon) \leq c(n)\ \mathcal{N}(\epsilon, \mathcal{F})\ e^{-N\epsilon^2/c(n)}$$

$\uparrow$ **prop. chaos**

$$\mathbb{P}_n^{(q,N)} = \mathrm{Law}((\xi_0^i, \ldots, \xi_n^i)_{1\leq i\leq q}) \quad \text{and} \quad \mathbb{P}_{[n]}^{(q,N)} = \mathrm{Law}((\xi_n^i)_{1\leq i\leq q})$$

$$\Downarrow$$

$$\|\mathbb{P}_{[n]}^{(N,q)} - \eta_n^{\otimes q}\|_{tv} \leq \frac{\exp\left(q\, c(n)\right)}{N} \quad \text{and} \quad \mathrm{Ent}(\mathbb{P}_n^{(N,q)} \mid \mathbb{K}_n^{\otimes q}) \leq\ c\ \frac{q\ n}{N}$$

IV./bips: $G_n(E_n) \subset [\mathbf{0}, \infty)$

Ex.: $G_n = 1_{A_n} \longrightarrow$ **bips-rejection**

$$\mathcal{S}_{n,m(x)}(x^i, \textbf{.}) = 1_{A_n}(x^i)\ \delta_{x^i} + (1 - 1_{A_n}(x^i))\ \Psi_n(m(x))$$

$A_n(x) = \{i\ ;\ x^i \in A_n\}$

$$\Psi_n(m(x)) = \frac{1}{|A_n(x)|} \sum_{i \in A_n(x)} \delta_{x^i}$$

Stopping time

$$\tau^N = \inf\{n \geq 0\ ;\ \eta_n^N(G_n) = 0\}$$

$$\mathbb{P}(\tau^N \leq n) \leq c(n) \exp\left(-N\ (\gamma_n^2(1)/(8n))^2\right)$$

Applications

- **Signal processing**
  - Non linear filtering
  - Smoothing and path estimation
  - Interacting Kalman-Bucy filters
- **Rare event analysis**
  - Multilevel Feynman-Kac models
  - Genealogical tree based algorithms
- **Directed polymers simulation**
  - Directed polymers models
  - Particle polymerizations
  - Self-avoiding and reinforced Markov motions

- **Spectral analysis of FKS semigroups**
  - Lyapunov exponents and spectral radii
  - Particle exponents

Signal Processing

$(\mathcal{X}_n, \mathcal{Y}_n) \in (E_n, F_n)$ **Markov model**

–initial dist.

$$\nu_0(d(x_0, y_0)) = g_n(x_0, y_0)\ \eta_0(dx_0)\ q_0(dy_0)$$

–Markov kernels

$$P_n((x_{n-1}, y_{n-1}), d(x_n, y_n)) = g_n(x_n, y_n)\ \mathcal{M}_n(x_{n-1}, dx_n)\ q_n(dy_n)$$

Interpret. (engineer/Bayes/probab)'s tribes

$\hookrightarrow$ $\mathcal{Y}_n = H_n(\mathcal{X}_n, \mathcal{V}_n)$

$$\mathbb{P}\left(H_n(x, \mathcal{V}_n) \in dy\right) = g_n(x_n, y)\ q_n(dy)$$

$\hookrightarrow$ $"p(x_n, y_n | x_{n-1}, y_{n-1}) = p(y_n | x_n)\ p(x_n | x_{n-1})"$

$\hookrightarrow$ *change ref. proba*

$$\frac{d\mathbb{P}^1}{d\mathbb{P}^0} / \mathcal{F}_n = \prod_{p=0}^{n} g_n(\mathcal{X}_p, \mathcal{Y}_p)$$

Interacting particle filters

$$Y = y \quad \text{fixed} \quad \Longrightarrow \quad \underbrace{G_n(x_n) = g_n(x_n, y_n)}_{likelihood-functions}$$

$$\Downarrow$$

$$\begin{aligned}
\eta_n &= \text{Law}(\mathcal{X}_n \mid Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1}) \\
\mu_n &= \text{Law}(\mathcal{X}_n \mid Y_0 = y_0, \ldots, Y_n = y_n)
\end{aligned}$$

Genetic bips $\longrightarrow (G_n, \mathcal{M}_n)$

Particle filters

$$\lim_{N\to\infty} \eta_n^N = \eta_n \quad \text{and} \quad \lim_{N\to\infty} \mu_n^N = \mu_n$$

Particle densities

$$\lim_{N\to\infty} \prod_{p=0}^{N} \eta_p^N(G_p) \;=\; \lim_{N\to\infty} \lambda_n^N(1) = \frac{d\mathbb{P}((Y_0,\ldots,Y_n) \in \,.\,)}{d(q_0 \otimes \ldots \otimes q_n)}$$

Interacting path-particle filters

-Historical model, $\mathcal{X}'_n$ Markov

$$\begin{aligned} \mathcal{X}_n &= (\mathcal{X}'_0, \ldots, \mathcal{X}_n) \in E_n = E'_0 \times \ldots \times E'_n \\ \mathcal{X}'_n &\in E'_n, \ \sim \mathcal{M}'_n(\mathcal{X}'_{n-1}, \boldsymbol{.}) \end{aligned}$$

-Historical observations,

$$\mathcal{Y}_n = H_n(\mathcal{X}_n, \mathcal{V}_n) = H_n((\mathcal{X}'_0, \ldots, \mathcal{X}'_n), \mathcal{V}_n)$$

same model but *path*-**(likelihood/kernels)**

$$\begin{aligned} G_n(x_n) &= G_n(x'_0, \ldots, x'_n) \\ \mathcal{M}_{n+1}(x_n, dz_n) &= \mathcal{M}_{n+1}((x'_0, \ldots, x'_n), d(z'_0, \ldots, z'_{n+1})) \\ &= \delta_{(x'_0,\ldots,x'_n)}(d(z'_0, \ldots, z'_n))\ \mathcal{M}'_{n+1}(z'_n, dz_{n+1}) \end{aligned}$$

$$\Downarrow$$

Genetic bips with path-particles $\longrightarrow (G_n, \mathcal{M}_n)$

$$\lim_{N\to\infty} \mu_n^N = \mathrm{Law}((\mathcal{X}'_0, \ldots, \mathcal{X}'_n) \mid Y_0 = y_0, \ldots, Y_n = y_n)$$

Genealogical tree based algorithms

–**Historical model,** $\mathcal{X}'_n$ **Markov**

$$
\begin{aligned}
\mathcal{X}_n &= (\mathcal{X}'_0, \ldots, \mathcal{X}_n) \in E_n = E'_0 \times \ldots \times E'_n \\
\mathcal{X}'_n &\in E'_n, \ \sim \mathcal{M}'_n(\mathcal{X}'_{n-1}, \cdot)
\end{aligned}
$$

–**<u>Marginal</u> observations,**

$$\mathcal{Y}_n = H_n((\mathcal{X}'_0, \ldots, \mathcal{X}'_n), \mathcal{V}_n) = H'(\mathcal{X}'_n, \mathcal{V}_n)$$

$$\Downarrow$$

same model but (likelihood/kernels)

$$G_n(x'_0, \ldots, x'_n) = G'_n(x'_n)$$

Genealogical model $\longrightarrow (G'_n, \mathcal{M}'_n)$

$$\mathsf{Law}((\mathcal{X}'_0,\ldots,\mathcal{X}'_n) \mid Y_0 = y_0,\ldots,Y_n = y_n) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{(\widehat{\xi}'^i_{0,n},\ldots,\widehat{\xi}'^i_{n,n})}$$

$$i-\text{th ancestral line} \longrightarrow (\widehat{\xi}'^i_{0,n},\ldots,\widehat{\xi}'^i_{n,n})$$

Interacting Kalman-Bucy filters

piecewise (linear/Gauss) $\mathcal{X}_n^1 \in E_n^{(1)}$ Markov $\mathcal{M}_n^{(1)}$

$$\begin{cases} \mathcal{X}_n^2 &= A(\mathcal{X}_n^1)\ \mathcal{X}_{n-1}^2 + B(\mathcal{X}_n^1)\ \mathcal{W}_n \\ \mathcal{Y}_n &= C(\mathcal{X}_n^1)\ \mathcal{X}_n^2 + \mathcal{V}_n \end{cases}$$

Optimal filter of $\mathcal{X}^2$ given $\mathcal{X}_1 = x$

$$\begin{aligned} \mu^{(2)}_{[x],n} &= \lambda^{(2)}_{[x],n}(f)/\lambda^{(2)}_{[x],n}(1) \\ \lambda^{(2)}_{[x],n}(f) &= \mathbb{E}^{(2)}_{[x,n]}(f(\mathcal{X}^2_n) \prod_{p=0}^{n} G_p(x_p, \mathcal{X}^2_p)) \end{aligned}$$

"quenched likelihood"

$$G_n(x_n, z_n) = \frac{d\mathcal{N}(C(x_n)\ z_n, R^v_n)}{d\mathcal{N}(0, R^v_n)}(y_n)$$

$$\Updownarrow$$

Quenched Kalman-Bucy filters Gauss=(mean, cov.) recursion

$$\mu^{(2)}_{[x],n-1} \xrightarrow{prediction} \eta^{(2)}_{[x],n} \xrightarrow{updating} \mu^{(2)}_{[x],n}$$

Quenched Kalman-Bucy filter

$$\eta^{(2)}_{[x],n} = \Phi^{(2)}_n((x_{n-1}, x_n), \eta^{(2)}_{[\mathcal{X}^1],n-1})$$

$$\Downarrow$$

$$\Phi^{(2)}_n((u,v), \mathcal{N}(m,P)) = \mathcal{N}(\ \text{Mean}_{n+1}((u,v),(m,P)),\ \ \text{Cov}_{n+1}(u,v)\ )$$

$$
\begin{array}{lcl}
(m,P) & \xrightarrow{updating} & (\widehat{m}_n, \widehat{P}_n) \\
 & \xrightarrow{prediction} & (\mathsf{Mean}_{n+1}((u,v),(m,P)), \mathsf{Cov}_{n+1}((u,v),(m,P)))
\end{array}
$$

<u>*Updating:*</u>

$$
\left\{
\begin{array}{rcl}
\widehat{m}_n & = & m + \mathsf{Gain}_n(u)\ (y_n - C(u)\ m) \\
\widehat{P}_n & = & (I - \mathsf{Gain}_n(u)\ C(u))\ P
\end{array}
\right.
$$

$$
\mathsf{Gain}_n(u) = P\ C(u)'\ [C(u)\ P\ C(u)' + R_n^v]^{-1}
$$

<u>*Prediction:*</u>

$$
\left\{
\begin{array}{l}
\mathsf{Mean}_{n+1}((u,v),(m,P)) = A(v)\ \widehat{m}_n \\
\mathsf{Cov}_{n+1}((u,v),(m,P)) = A(v)\ \widehat{P}_n\ A(v)' + B(v)\ R_{n+1}^w\ B(v)'
\end{array}
\right.
$$

Annealed models

$$\mu_n^{(1)}(f) = \mathbb{E}(f(\mathcal{X}_n^1)\ |\mathcal{Y}_0 = y_0, \ldots, \mathcal{Y}_n = y_n\ )$$

$$\Downarrow$$

$$\begin{aligned} \mu_n^{(1)}(f) &= \lambda_n^{(1)}(f)/\lambda_n^{(1)}(1) \\ \lambda_n^{(1)}(f) &= \mathbb{E}(f(\mathcal{X}_n^1)\ \prod_{p=0}^{n} G_p(\mathcal{X}_p^1, \mathcal{X}_p^2)\ ) \end{aligned}$$

Important observations

Annealed potentials

$$\begin{aligned}\lambda^{(2)}_{[x],n}(1) &= \mathbb{E}\left(\prod_{p=0}^{n} G_p(x_p, \mathcal{X}^2_p)\right) \\ &= \prod_{p=0}^{n} \eta^{(2)}_{[x],n}(G_p(x_p, \,.\,))\end{aligned}$$

$$\Downarrow$$

$$\lambda^{(2)}_{[x],n}(1) = \prod_{p=0}^{n} G'_n(\eta^{(2)}_{[x],p}, G_p(x_p, \,.\,)) \quad \text{with} \quad G'_n(\eta, g) = \int \eta(du)\ g(u)$$

$$
\begin{aligned}
\lambda_n^{(1)}(f) &= \mathbb{E}(f(\mathcal{X}_n^1) \ \mathbb{E}(\prod_{p=0}^{n} G_p(\mathcal{X}_p^1, \mathcal{X}_p^2) \mid \mathcal{X}^1) \ ) \\
&= \mathbb{E}(f(\mathcal{X}_n^1) \ \prod_{p=0}^{n} \ G'_n \underbrace{(\eta^{(2)}_{[\mathcal{X}^1],p}, G_p(\mathcal{X}_p^1, \centerdot))}_{=\mathcal{X}'_p} \ )
\end{aligned}
$$

<u>Markov chain (point/measure)-valued</u>

$$
\mathcal{X}'_n = (\mathcal{X}_n^1, \eta^{(2)}_{[\mathcal{X}^1],n})
$$

FK in distribution space

–**Markov chain in distribution space,**

$$\mathcal{X}'_n = (\mathcal{X}^1_n, \eta^{(2)}_{[\mathcal{X}^1],n}) \in E' = E^{(1)} \times \mathcal{P}(\mathbb{R}^{dim(\mathcal{X}^2)})$$

evolution:

$$\mathcal{M}'_n = \mathcal{M}^{(1)}_n + \text{Kalman-Bucy recursion}$$

–**Annealed potential,**

$$G'_n : (\eta, g) \in E' \longrightarrow G'_n(\eta, g) = \int \eta(du)\ g(u)$$

$$\Downarrow$$

FK-models

$$\begin{aligned}\mu'_n(f') &= \lambda'_n(f')/\lambda'_n(1)\\ \lambda'_n(f') &= \mathbb{E}(f'(\mathcal{X}'_n)\ \prod_{p=0}^{n} G'_p(\mathcal{X}'_p)\ )\end{aligned}$$

$$\Downarrow$$

bips model (state,measure)-particles

$$\xi'^i_n = (\zeta^i_n, \nu^i_n) \in E' = E^{(1)} \times \mathcal{P}(\mathbb{R}^{dim(\mathcal{X}^2)})$$

(state,measure)-valued bips model

$$\xi_n^i = (\zeta_n^i, \nu_n^i) \xrightarrow{select.} \widehat{\xi}_n^i = (\widehat{\zeta}_n^i, \widehat{\nu}_n^i) \xrightarrow{pred.} \xi_{n+1}^i = (\zeta_{n+1}^i, \nu_{n+1}^i)$$

$$\Downarrow$$

<u>*selection*</u> $\widehat{\xi}_n^i = (\widehat{\zeta}_n^i, \widehat{\nu}_n^i)$ **iid**

$$\sum_{i=1}^N \frac{G'_n(\zeta_n^i, \nu_n^i)}{\sum_{j=1}^N G'_n(\zeta_n^j, \nu_n^j)} \delta_{(\zeta_n^i, \nu_n^i)}$$

note

$$G'_n(u, \mathcal{N}(m, P)) = \frac{d\mathcal{N}(C(u)\ m, C(u)\ P\ C(u)' + R_n^v)}{d\mathcal{N}(0, R_n^v)}(y_n)$$

prediction

$$(\widehat{\zeta}_n^i, \widehat{\nu}_n^i) \longrightarrow (\zeta_{n+1}^i, \nu_{n+1}^i)$$

two steps:

$$\widehat{\zeta}_n^i \quad \longrightarrow \quad \zeta_{n+1}^i \sim \mathcal{M}_{n+1}^{(1)}(\widehat{\zeta}_n^i, \,.\,)$$

$$\widehat{\nu}_n^i \quad \longrightarrow \quad \nu_{n+1}^i = \underbrace{\Phi_{n+1}^{(2)}((\widehat{\zeta}_n^i, \zeta_{n+1}^i), \widehat{\nu}_n^i)}_{Kalman-Bucy\ recursions}$$

Rare event analysis

-<u>**Markov motion**</u>

$$X_t \in (S, d)\,, \quad t \in I = (\mathbb{R}_+ \text{or } \mathbb{N})$$

-<u>**Obstacles**</u> $A_\infty \subset S$

$$R = \inf\{t \geq 0\ ;\ X_t \in A_\infty\}\,, \quad \mathbb{P}(R < \infty) = 1$$

-<u>**Initial set and obstacles**</u>

$$X_0 \in A_0 \subset S - A_\infty$$

-<u>**(rare) target set**</u> $B \subset S$

Objective$\rightarrow$ compute

$$\mathbb{P}(T < R) \quad \text{and} \quad \mathsf{Law}(X_t\,;\,0 \le t \le T \mid T < R)$$

$$T = \inf\{0 \le t < R \;:\; X_t \in B\}$$

Multilevel analysis

$$B = B_{m+1} \subset B_m \subset \ldots \subset B_1 \subset B_0$$

$$\text{s.t.} \quad A_0 = B_0 - B_1 \quad \text{and} \quad B_0 \cap A_\infty = \emptyset$$

hyp. X **cannot enter** B_n **before visiting** B_{n-1}

$$\Downarrow$$

Level entrance times

$$T_0 = 0 \le T_1 \le \ldots \le T_m \le T_{m+1} = T$$

$$T_n = \inf \{0 \le t < R \ : \ X_t \in B_n\}$$

Multilevel model

-**Sub-level times**

$$\begin{aligned}(T < R) &= (T_{m+1} < R) \\ &= (T_1 < R, \ldots, T_{m+1} < R)\end{aligned}$$

-**Level set Markov chain**

$$\mathcal{X}_n = (X_t \; ; \; T_{n-1} \leq t \leq T_n) \in E = \cup_{s \leq t} D([s,t], S)$$

Path-model

$$(\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_n) = [X_t \;;\; 0 \le t \le T_n]$$

$$\Downarrow$$

Potential functions

$$G_n : x = (x_r)_{s \le r \le t} \longrightarrow G_n(x) = 1_{B_n}(x_t)$$

$$\begin{aligned} (T_n < R) &= (T_1 < R, \ldots, T_n < R) \\ &= (G_1(\mathcal{X}_1) = 1 \ldots, G_n(\mathcal{X}_n) = 1) \end{aligned}$$

Multilevel Feynman-Kac model

$$\downarrow$$

$$\begin{aligned}
\mu_n(f) &= \mathbb{E}(f(X_t\ ;\ T_{n-1} \le t \le T_n) \mid T_n < R) \\
&= \mathbb{E}(f(\mathcal{X}_n) \mid T_0 < R, \ldots, T_n < R) \\
&= \lambda_n(f)/\lambda_n(1)
\end{aligned}$$

<u>**FK measures**</u>

$$\begin{aligned}
\lambda_n(f) &= \mathbb{E}\left(\ f(\mathcal{X}_n)\ \prod_{p=0}^{n} G_p(\mathcal{X}_p)\ \right) \\
\lambda_n(1) &= \mathbb{P}(T_n < R) \\
&= \mathbb{E}\left(\prod_{p=0}^{n} G_p(\mathcal{X}_p)\right) = \prod_{p=0}^{n}\ \eta_p(G_p)
\end{aligned}$$

FK path measures

$$\mathbb{E}(f[X_t \,;\, 0 \leq t \leq T_n] \mid T_n < R)$$

$$= \mathbb{E}(f(\mathcal{X}_0, \ldots, \mathcal{X}_n) \mid T_0 < R, \ldots, T_n < R)$$

bips models

-**Mutation** *Markov kernels* $\mathcal{M}_n$

$$\mathcal{X}_n = (X_t \; ; \; T_{n-1} \leq t \leq T_n)$$

-**Boltzmann-Gibb's selection** *Potentials* G_n

$$\Psi_n(\eta) = \eta \mathcal{S}_{n,\eta}$$

$$\mathcal{S}_{n,\eta}(u, dv) = 1_{G_n^{-1}\{1\}}(u) \; \delta_u(dv) + (1 - 1_{G_n^{-1}\{1\}}(u)) \; \Psi_n(\eta)(dv)$$

Rare event particle estimate

$$\begin{aligned}\lambda_n(1) &= \mathbb{P}(T_n < R) = \prod_{p=0}^{n} \eta_p(G_p) \\ \lambda_n^N(1) &= \prod_{p=0}^{n} \eta_p^N(G_p) \\ &= \frac{1}{N^n} \prod_{p=1}^{n} \mathrm{Card}(I^N(p))\end{aligned}$$

$I^N(p)$ =**predicted** particles at level p.

Genealogical tree based model

sub-levels path-particles

$$\begin{aligned}\zeta_n^i &= (\xi_{0,n}^i, \ldots, \xi_{n,n}^i) \\ \widehat{\zeta}_n^i &= (\widehat{\xi}_{0,n}^i, \ldots, \widehat{\xi}_{n,n}^i) \in (\cup_{s \leq t} D([s,t], S))^n\end{aligned}$$

ancestor level lines

$$\begin{aligned}\nu_n^N &= \frac{1}{|I^N(n)|} \sum_{i \in I^N(n)} \delta_{(\xi_{0,n}^i, \ldots, \xi_{n,n}^i)} \\ &\downarrow \\ \nu_n &= \text{Law}\ ([X_t\ ,\ \ 0 \leq p \leq T_n] \mid T_n < R)\end{aligned}$$

Directed polymers simulation

polymer chain= chemical cinetic structure macromolecules $\in$ solvent.

$$\mathcal{X}_n = (X_0, \ldots, X_n) \in E_n = \underbrace{E \times \ldots \times E}_{(n+1)-\text{times}}$$

$n =$ degree of polymerization

Polymerization sequence

$$\mathcal{X}_0 = X_0 \longrightarrow \mathcal{X}_1 = (X_0, X_1) \longrightarrow \mathcal{X}_2 = (X_0, X_1, X_2)$$

$$\downarrow$$

Energy (chemical interactions)

$$G_n(X_0, \ldots, X_n) = \exp -\frac{1}{\epsilon} \sum_{p=0}^{n} V_p(X_0, \ldots, X_p)$$

FK polymer distribution

$$\mu_n(f_n) = \lambda_n(f_n)/\lambda_n(1)$$

$$\lambda_n(f_n) = \mathbb{E}(f_n(\mathcal{X}_n) \quad \prod_{p=0}^{n} G_p(\mathcal{X}_p))$$

$$\Updownarrow$$

Boltzmann-Gibbs' path-form

$$\mu_n \quad = \quad \frac{1}{\lambda_n(1)} \quad \exp\left\{-\frac{1}{\epsilon}\sum_{p=0}^{n} V_p(X_0,\ldots,X_p)\right\} \quad \mathbb{P}_n$$

Repulsive interaction

$$V_p(x_0, \ldots, x_p) = 0 \Longleftrightarrow x_p \notin \{x_0, \ldots, x_{p-1}\}$$

$$V_n(x_0, \ldots, x_n) = \sum_{p=0}^{n-1} 1_{\{x_p\}}(x_n)$$

self-avoiding random motion

$$G_n(x_0, \ldots, x_n) = 1_{E-\{x_0, \ldots, x_{n-1}\}}(x_n)$$

$$\Downarrow$$

$$\mu_n = \mathsf{Law}(X_0, \ldots, X_n | \forall 0 \leq p < q \leq n \,, \quad X_p \neq X_q)$$

Attractive interaction

$$G_n(x_0,\ldots,x_n) = \frac{1}{n}\sum_{p=0}^{n-1} 1_{x_p}(x_n)$$

$\hookrightarrow$ self-reinforced random motions

Particle polymerization,

N-**polymers with degree** n

$$\left\{\begin{array}{rcl} \xi_n^i & = & (\zeta_{0,n}^i, \ldots, \zeta_{n,n}^i) \\ \widehat{\xi}_n^i & = & (\widehat{\zeta}_{0,n}^i, \ldots, \widehat{\zeta}_{n,n}^i) \end{array}\right.$$

$$\xi_n^i \xrightarrow{select.} \widehat{\xi}_n^i \xrightarrow{pred.} \underbrace{\xi_{n+1}^i}_{\substack{elementary \\ extensions}}$$

selection

iid $G_n(x_0, \ldots, x_n) = e^{-\frac{1}{\epsilon} V_n(x_0,\ldots,x_n)}$

$$\Psi_n(m(\xi_n)) = \sum_{i=1}^{N} \frac{G_n(\xi_n^i)}{\sum_{j=1}^{N} G_n(\xi_n^i)} \; \delta_{\xi_n^i}$$

indpt $G_n(x_0, \ldots, x_n) = 1_{E-\{x_0,\ldots,x_{n-1}\}}(x_n)$

$$\mathcal{S}_{n,m(\xi_n)}((\xi_n^i, \boldsymbol{.}) = G_n(\xi_n^i) \; \delta_{\xi_n^i} + (1 - G_n(\xi_n)) \; \Psi_n(m(\xi_n))$$

$$\Psi_n(m(\xi_n)) = \frac{1}{|I^N(n)|} \sum_{i \in I^N(n)} \delta_{\xi_n^i}$$

Spectral analysis (potential $V \in \mathcal{B}_b(E)$)

$$Q(x,dy) = e^{V(x)}\ \mathcal{M}(x,dy) \quad \textit{op. on} \ (\mathcal{B}_b(E), \|\,.\,\|)$$

norm-semigroup:

$$Q^n(f)(x) = \mathbb{E}_x(\ f(X_n)\ \exp\{\sum_{p=0}^{n-1} V(X_p)\}\ )$$

$$|||Q^n||| = \sup_{\|f\|=1} \|Q^n(f)\| = \|Q^n(1)\|$$

Log-Lyapunov expo

$$\begin{aligned} \Lambda(V) &= \log \mathrm{Lyap}(Q) = \log \mathrm{Spr}(Q) \\ &= \lim_{n\to\infty} \frac{1}{n} \sup_{x\in E} \log Q^n(1)(x) \end{aligned}$$

Continuous time version$\rightarrow$ Schrödinger op.

$$L^v = L + V$$

$$Q_t(f)(x) = \mathbb{E}_x\left(\ f(X_t) \ \exp\{\int_0^t V(X_s)ds\} \ \right)$$

Top Lyapunov exponent

$$\Lambda(V) = \lim_{t\rightarrow\infty} \frac{1}{t} \sup_{x\in E} \log Q_t(1)(x)$$

Mean-time average formula

$$\gamma_n^{(x)} =_{def.} \delta_x Q^n \text{and} \quad \eta_n^{(x)} =_{def.} \Phi^n(\delta_x)$$

$$\Downarrow$$

$$\begin{aligned} Q^n(1)(x) &= \gamma_n^{(x)}(1) =_{def.} \delta_x Q^n(1) \\ &= \prod_{p=0}^{n-1} \eta_p^{(x)}(\exp V) \end{aligned}$$

$$\Downarrow$$

$$\Lambda(V) = \lim_{n\to\infty} \sup_{x\in E} \underbrace{\frac{1}{n}\sum_{p=0}^{n-1} \log\left(\eta_p^{(x)}[\exp V]\right)}_{=\Lambda_n^{(x)}(V)}$$

Mild hyp. $\exists ! \eta_\infty = \Phi(\eta_\infty)$

$$\Lambda(V) = \log\left(\eta_\infty[\exp V)\right]$$

$\mathcal{M}$ μ-reversible

$$Q(h_v) = e^{\Lambda(V)}\ h_v \Rightarrow \eta_\infty(f) = \frac{\mu(h_v \mathcal{M}(f))}{\mu(h_v)}$$

Particle Lyapunov exponents

$$\eta_n^{(N,x)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\xi_n^i}$$

$$\Downarrow$$

$$\Lambda_n^{(N,x)}(V) = \frac{1}{n} \sum_{p=0}^{n-1} \log \eta_p^{(N,x)}[\exp(V)]$$

mild assmp.

$$\sup_{x \in E,\ n \geq \sqrt{N}} \sqrt{N}\ \mathbb{E}\left(|\Lambda_n^{(N,x)}(V) - \Lambda(V)|^p\right)^{1/p} < \infty$$

Stochastic analysis

- Stability properties
  - Dobrushin ergodic coefficent
  - Contraction estimates
- Annealing properties
  - Annealed Feynman-Kac models
  - Long time behavior
  - Concentration properties

Dobrushin's ergodic coefficent

-Markov op.

$$\mu \in \mathcal{P}(E) \longrightarrow \mu\mathcal{M} \in \mathcal{P}(E)$$

-Lipschitz-contraction

$$\beta(\mathcal{M}) = \sup_{\mu,\nu\in\mathcal{P}(E)} \frac{\|\mu\mathcal{M} - \nu\mathcal{M}\|_{tv}}{\|\mu - \nu\|_{tv}}$$

-Dobrushin's coef.

$$\alpha(\mathcal{M}) + \beta(\mathcal{M}) = 1$$

$$\alpha(\mathcal{M}) = \inf \sum_{i=1}^{m} \min\left(\mathcal{M}(x, A_i), \mathcal{M}(z, A_i)\right)$$

Ex.: $\mathcal{M}(x, dy) \geq \epsilon \ \mathcal{M}(x', dy)$

$$\|\mu\mathcal{M} - \nu\mathcal{M}\|_{tv} \leq (1 - \epsilon) \ \|\mu - \nu\|_{tv}$$

Non linear contractions

semigroups

$$\gamma_n = \gamma_p \mathcal{Q}_{p,n} \quad \text{and} \quad \eta_n = \Phi_{p,n}(\eta_p)$$

$$\Downarrow$$

$$\begin{aligned}\Phi_{p,n}(\eta)(f) &= \frac{\eta\mathcal{Q}_{p,n}(f)}{\eta\mathcal{Q}_{p,n}(1)} \\ &= \frac{\eta(G_{p,n}\mathcal{M}_{p,n}(f))}{\eta(G_{p,n}(1))}\end{aligned}$$

$$G_{p,n} = \mathcal{Q}_{p,n}(1) \quad \text{and} \quad \mathcal{M}_{p,n}(f) = \frac{\mathcal{Q}_{p,n}(f)}{\mathcal{Q}_{p,n}(1)}$$

Structural stab.

$$\Phi_{p,n}(\eta) = \Psi_{p,n}(\eta)\mathcal{M}_{p,n}$$

Boltzmann-Gibbs transf.

$$\Psi_{p,n}(\eta)(f) = \frac{\eta(fG_{p,n})}{\eta(G_{p,n})}$$

$\Rightarrow$ **Asympt. stab. estimate**

$$\sup_{\mu,\eta\in\mathcal{P}(E)} \|\Phi_{p,n}(\mu)-\Phi_{p,n}(\eta)\|_{tv} = \beta(\mathcal{M}_{p,n})$$

Regularity conditions ((G)+$(\mathcal{M})_m$)

$$G(x)\geq r\ \ G(y) \quad \text{and} \quad \mathcal{M}^m(x,\,.\,)\geq \epsilon\ \ \mathcal{M}^m(y,\,.\,)$$

Counting potential terms for $p+m \leq n$ $(f \geq 0)$

$$\begin{aligned}
\mathcal{M}_{p,n}(f)(x) = \frac{\mathcal{Q}_{p,p+m}\mathcal{Q}_{p+m,n}(f)(x)}{\mathcal{Q}_{p,p+m}\mathcal{Q}_{p+m,n}(1)(x)} \quad &\geq \quad \epsilon^2 \; r^{(m-1)} \quad \frac{\mathcal{M}^m\mathcal{Q}_{p+m,n}(f)(y)}{\mathcal{M}^m\mathcal{Q}_{p+m,n}(1)(y)} \\
G_{p,n}(x) \quad &\geq \quad \epsilon^2 \; r^m \quad G_{p,n}(y)
\end{aligned}$$

$$\Downarrow \quad (\ldots/\ldots)$$

$$\begin{aligned}
\beta(\mathcal{M}_{p,n}) \quad &\leq \quad \left(1 - \epsilon^2 \; r^{m-1}\right)^{[(n-p)/m]} \\
&\stackrel{(m=1)}{=} \left(1 - \epsilon^2\right)^{(n-p)}
\end{aligned}$$

"potential independent!!"

Contraction estimates non linear sg

–strong contractions

$$\|\Psi_{p,n}(\mu)-\Psi_{p,n}(\eta)\|_{tv} \leq 2\sup_{x,y}\frac{G_{p,n}(x)}{G_{p,n}(y)}\ \|\mu-\eta\|_{tv}$$

$$\Downarrow$$

$$\|\Phi_{p,n}(\mu)-\Phi_{p,n}(\eta)\|_{tv} \leq \beta(\Phi_{p,n})\ \|\mu-\eta\|_{tv}$$

contract. coef.

$$\begin{aligned}\beta(\Phi_{p,n}) \quad &\leq \quad \frac{2}{\epsilon^2\ r^m}\ \left(1-\epsilon^2\ r^{m-1}\right)^{[(n-p)/m]}\\ &\stackrel{(m=1)}{=} \frac{2}{r\epsilon^2}\ \left(1-\epsilon^2\right)^{(n-p)}\end{aligned}$$

-Weak contractions

$$\Psi_{p,n}(\eta)(f) - \Psi_{p,n}(\mu)(f) = \frac{\mu(G_{p,n})}{\eta(G_{p,n})}\ \eta\left(\frac{G_{p,n}}{\mu(G_{p,n})}\ (f - \Psi_{p,n}(\mu)(f))\right)$$

$$\Downarrow$$

$\exists f_{p,n}^{(\mu)} \in \mathrm{Osc}_1(E)$

$$|\Phi_{p,n}(\mu)f - \Phi_{p,n}(\eta)f| \leq \beta(\Phi_{p,n})\ |\mu(f_{p,n}^{(\mu)}) - \eta(f_{p,n}^{(\mu)})|$$

Annealing Properties

$$0 \leq V\,, \quad \mathrm{Osc}(V) < \infty\,, \quad \mathcal{M}\,, \quad \theta(n) \in \mathbb{R}_+$$

-**Annealed FK models** *(updated)*

$$\mu_n(f) = \lambda_n(f)/\lambda_n(1)$$

$$\lambda_n(f) = \mathbb{E}(\ f(\mathcal{X}_n)\ \ \exp\{-\sum_{p=1}^{n} \theta(p)\ V(\mathcal{X}_p)\ \})$$

two interpretations

-$\sim$ Markov motion in an absorbing freezing medium

-$\sim$ Nonlinear FK simulated annealing

$$\mu_{n-1} \overset{mutation}{\longrightarrow} \eta_n = \mu_{n-1}\mathcal{M}_n \overset{selection}{\longrightarrow} \mu_n = \eta_n \mathcal{S}_{n,\eta_n}$$

$$\mathcal{S}_{n,\eta}(x, .) = e^{-\theta(n)V(x)}\ \delta_x + (1 - e^{-\theta(n)V(x)})\ \Psi^{\theta(n)}(\eta)$$

Boltzmann-Gibbs' dist. $\sim \eta$ *and* $\theta(n) = \beta$

$$\Psi^{\beta}(\eta)(dy) = \frac{e^{-\beta V(y)}}{\eta(e^{-\beta V})}\ \eta(dy)$$

-objectives

- long time estimates
- concentration regions

Semigroup-variational Method

(sketched lines)

-**nonlinear sg** $(\theta(n) = \beta \Rightarrow$ homogenous$)$

$$\mu_n(f) = \Phi^\beta(\mu_{n-1}) = \frac{\mu\mathcal{M}(e^{-\beta V} f)}{\mu\mathcal{M}(e^{-\beta})}$$

-**Mixing hyp.** $(\mathcal{M})_m$:

$$\mathcal{M}^m(x, \textbf{.}) \geq \epsilon\ \mathcal{M}^m(y, \textbf{.})$$

$$\Downarrow$$

- contraction estimates
- $\exists!$ fixed points homog. models
- cooling schedules $\sim$ mixing parameter m
- concentration fixed point $\hookrightarrow$ log-Lyap. expo.
- variational formula=competition (mutation,selection)

Sketch of proof

-unif contraction $\alpha(n) = \beta \Rightarrow \exists n(\beta)$

$$\|\Phi^{\beta}_{n(\beta)}(\mu) - \Phi^{\beta}_{n(\beta)}(\eta)\|_{tv} \;\leq \frac{1}{e}\; \|\mu - \eta\|_{tv}$$

$$\Downarrow$$

$$\exists \mu^{\beta} = \Phi^{\beta}(\mu^{\beta})$$

-variational formula

$$L_n = \frac{1}{n} \sum_{p=1}^{n} \ \delta_{\mathcal{X}_p}$$

LDP

$$I(\mu) = \inf_{\mu\mathcal{K}=\mu} \int_E \ \mu(dx) \ \mathrm{Ent}[\mathcal{K}(x,\textbf{.})|\mathcal{M}(x,\textbf{.})]$$

$$\Downarrow$$

$$-\Lambda(-\beta V) = \log \mu^{\beta}(e^{\beta V}) = \inf_{\eta} \, [\beta\eta(V) + I(\eta)]$$

$$\Downarrow$$

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log \mu^{\beta}(V \geq V_I + \delta) \leq -\delta \qquad V_I = \inf\{\eta(V) \ ; \ I(\eta) < \infty\}$$

Annealed Feynman-Kac models

$\hookrightarrow$ **Find judicious** $t(n), \beta(n)$ s.t.

$$t(n) \leq p < t(n+1)$$

$$\Downarrow$$

$$\mu_{p+1} = \Phi^{\beta(n)}(\mu_p)$$

i.e.

$$\mu_{p+1}(f) = \lambda_p(f)/\lambda_p(1)$$

with

$$\lambda_p(f) = \mathbb{E}_{t(n),\mu_{t(n)}}(f(\mathcal{X}_p)\ e^{-\beta(n)\sum_{q=t(n)+1}^{p} V(\mathcal{X}_q)})$$

$\nu = \nu\mathcal{M}$**-ess.infimum**

$$V_I \geq V_\nu = \sup\{v \geq 0\ ;\ V \geq v \quad \nu - \text{a.e.}\}$$

Theorem

Mixing param. $m = 1 \Rightarrow V_I = V_\nu$,

$$\begin{aligned} t(n) &= n\Delta \\ \beta(n) &= \beta(0)\ (n+1)^a, \quad \forall a \in (0,1), \quad \alpha(0) < \infty \end{aligned}$$

$$\Downarrow$$

$$\|\mu_{t(n)} - \mu^{\beta(n)}\|_{tv} \leq \frac{Cte}{(n+1)^{1-a}}$$

Mixing param. $m > 1$,

$$\begin{aligned} t(n) &= O(n^{1+b} \log n) \\ \beta(n) &= \beta(0)\ \log(n+e) \end{aligned}$$

$(b = (m-1)\text{osc}(V)\beta(0) < 1)$

$$\Downarrow$$

$$\|\mu_{t(n)} - \mu^{\beta(n)}\|_{tv} \leq \ Cte\ \frac{\log(n+e)}{(n+1)^{1-b}}$$

Proof:

-Contract. est.$\Rightarrow \alpha(m) = (m-1)\mathrm{osc}(V)$

$$\|\Phi_n^\beta(\mu) - \Phi_n^\beta(\eta)\|_{tv} \leq 2\epsilon^{-2}\ e^{2\alpha(m)\beta}\ (1-\epsilon^2\ e^{-\alpha(m)\beta})^{[n/m]}\ \|\mu-\eta\|_{tv}$$

$$\Delta_m(\beta) = \Delta(0)\ (1+[e^{\alpha(m)\beta}])\ (1+[\alpha(m)\beta])$$

$$\Downarrow$$

$$\|\Phi^\beta_{\Delta_m(\beta)}(\mu) - \Phi^\beta_{\Delta_m(\beta)}(\eta)\|_{tv}\ <\ \frac{1}{e}\ \|\mu-\eta\|_{tv}$$

$$\Downarrow$$

-fixed point existence

$$\exists ! \mu^{\beta} = \Phi^{\beta}(\mu^{\beta}) \in \mathcal{P}(E)$$

-fixed point oscillations $(\beta_1 \leq \beta_2)$

$$\|\mu^{\beta_1} - \mu^{\beta_2}\|_{tv} \leq \frac{e \ osc(V)}{e-1} \quad \Delta_m(\beta_1) \ (\beta_2 - \beta_1)$$

Annealed FK models

$$\beta(n) \uparrow, \qquad \alpha(m) = (m-1)osc(V)$$

-time mesh

$$t_m(n+1) = t_m(n) + \Delta_m(\beta(n))$$

$$\Delta_m(\beta) = \Delta(0)\ (1 + [e^{\alpha(m)\beta}])\ (1 + [\alpha(m)\beta])$$

-for $t_m(n) \leq p < t_m(n+1)$

$$\mu_{p+1} = \Phi^{\beta(n)}(\mu_p)$$

$$\mu_{p+1}(f) = \lambda_p(f)/\lambda_p(1)$$

$$\lambda_p(f) = \mathbb{E}_{t_m(n),\mu_{t_m(n)}}\left(f(\mathcal{X}_p)\ e^{-\beta(n)\sum_{q=t_m(n)+1}^{p} V(\mathcal{X}_q)}\right)$$

$$\Downarrow$$

-contraction+oscillations

$$\|\mu_{t_m(n)} - \mu^{\beta(n)}\|_{tv}$$

$$\leq \frac{1}{e}\ \ \|\mu_{t_m(n-1)} - \mu^{\beta(n-1)}\|_{tv} + 2osc(V)\ \ \Delta_m(\beta(n-1))\ (\beta(n) - \beta(n-1))$$

two cases

$$m = 1 \Rightarrow t_1(n) = \Delta(0).n\,, \quad \Delta_1(\beta) = \Delta(0)$$

$$\beta(n) = (n+1)^a\,, \quad a \in (0,1)$$

$$\Downarrow$$

$$e^n \ \|\mu_{t_1(n)} - \mu^{\beta(n)}\|_{tv}$$

$$\leq \|\mu_0 - \mu^{\beta(0)}\|_{tv} + 2osc(V)\ \Delta(0)\ \textstyle\sum_{p=1}^n\ e^p\ ((p+1)^a - p^a)$$

$$\sum_{p=1}^n\ e^p\ ((p+1)^a - p^a) \leq a \sum_{p=1}^n\ \frac{e^p}{p^{1-a}} \Rightarrow \|\mu_{t_1(n)} - \mu^{\beta(n)}\|_{tv} \leq c(a)/n^{1-a}$$

case $m > 1$,

$$\begin{aligned} \beta(n) &= \beta(0) \log(n+e) \\ \Delta_m(\beta(n)) &= 4^2\ \Delta(0)\ (e+n)^b\ \log(n+e) \end{aligned}$$

$$b = \alpha(m)\beta(0) < 1$$

$$\Downarrow$$

$$t_m(n) \leq Cte\ (n+1)^{b+1}\ \log(n+e)$$

$$e^n\ \|\mu_{t_m(n)} - \mu^{\beta(n)}\|_{tv} \leq 1 + Cte\ \log(n+e)\ \sum_{p=1}^{n}\ \frac{e^p}{p^{1-b}}$$

$$\Downarrow$$

$$\|\mu_{t_m(n)} - \mu^{\beta(n)}\|_{tv} \leq c(b)\ \log(n+e)\ \frac{1}{n^{1-b}}$$

Concentration properties

–fixed point formula $\mathcal{Q}(f) = \mathcal{M}(e^{-\beta V} f)$

$$\mu^\beta(f) = \mu^\beta \mathcal{Q}(f)/\mu^\beta \mathcal{Q}(1)$$

$$\Downarrow$$

$$\begin{aligned}
\mu^\beta(Q^n(1)) &= \mu^\beta(\mathcal{Q}^{n-1}(1))\ \mu^\beta(\mathcal{Q}(1)) \\
&= \mathbb{E}_{\mu^\beta}(e^{-\beta \sum_{p=1}^n V(\mathcal{X}_p)}) \\
&= [\mu^\beta(\mathcal{Q}(1))]^n
\end{aligned}$$

and

$$\mu^\beta(e^{\beta V})\ \mu^\beta(Q(1)) = 1$$

Laplace-Varadhan's lem.$\Rightarrow$Variational form.

Lemma

$$
\begin{aligned}
-\frac{\Lambda(-\beta V)}{\beta} &= \frac{1}{\beta}\log \mu^{\beta}(e^{\beta V}) \\
&= \inf\,[\eta(V) + \frac{1}{\beta} I(\eta)] \rightarrow V_I = \inf_{I(\eta)<\infty} \eta(V)
\end{aligned}
$$

-Markov expo. $\leq$

$$
\lim_{\beta\to\infty} \frac{1}{\beta} \log \mu^{\beta}(V \geq V_I + \delta) \leq -\delta
$$

Concentration levels

$m = 1 \Longrightarrow V_I = V_\nu$ and for $f \geq 0$

$$\epsilon \, \frac{\nu(e^{-\beta V} f)}{\nu(e^{-\beta V})} \leq \mu^\beta(f) \leq \frac{1}{\epsilon} \, \frac{\nu(e^{-\beta V} f)}{\nu(e^{-\beta V})}$$

$m > 1$, **-reg. hyp.:**

$$\nu(dx)\mathcal{M}(x, dy) << \nu(dy)\mathcal{M}(y, dx)$$

$$\Downarrow$$

$$\mathcal{M}(x, x) > 0 \text{ and } \int_E \frac{\nu(dx)}{\mathcal{M}(x, x)} < \infty \Rightarrow V_I = V_\nu$$

Proof $\hookrightarrow$ **Metropolis-Hasting**

$$\nu^u(dx) = \frac{1}{\nu(e^{-u})}\ e^{-u(x)}\ \nu(dx) = \nu^u \mathcal{K}^u(dx)$$

$\Rightarrow \quad I(\nu^u) < \infty$ ▪

E **finite** $\nu(x) > 0 (\Rightarrow V_\nu = \inf_E V)$

-$\mathcal{M}$-boundary

$$\partial_{\mathcal{M}}(A) = \{y \in E - A \,; \quad \exists x \in A \quad \mathcal{M}(x,y) > 0\}$$

-n-paths

$$\Gamma^{(n)}_{x,y} = \{\mathcal{X} = (\chi_p)_{0 \leq p \leq n} \in E^{n+1} \,; \text{ from x to y } \}$$

-mean potential along $\mathcal{X}$

$$\chi(V) = \frac{1}{n} \sum_{p=1}^{n} V(\chi_p)$$

-virtual n-communication cost

$$V^{(n)}(x,y) \; = \; \inf\{\chi(V)\,,\; \chi \in \Gamma^{(n)}_{x,y}\}$$

Prop.:

$$V_I = V_c =_{def.} \inf_{x \in E} V^{(n)}(x,x) = \inf_{1 \leq n \leq |E|} V_c^{(n)}$$

$$V_I = V_\nu \Leftrightarrow \exists n \geq 1,\ x \in V^{-1}(V_\nu) \ : \ \Gamma_{x,x}^{(n)} \cap V^{-1}(V_\nu) \neq \emptyset$$

Variational approach

$$-\frac{1}{\beta}\log \mu^{\beta} \in \mathbb{R}_{+}^{E} \to \textbf{accumulation point } U \in \mathbb{R}_{+}^{E}$$

$$\Downarrow \textit{(fixed point formulae)}$$

$$\begin{aligned} U(y) &= \inf_{y\in E} (U(x) + nV^{(n)}(x,y)) - nV_I \\ V_I &= \inf_{x,y\in E} (U(x) + V(x,y)) \end{aligned}$$

Prop.

$$U^{-1}(0) \subset (V \leq V_I) \quad \text{and} \quad \partial_{\mathcal{M}} U^{-1}(0) \subset (V > V_I)$$

$$V_I = V_c = \lim_{n\to\infty} \inf_{x,y} V^{(n)}(x,y)$$

3-points example

$$E = \{0,1,2\} \qquad \mathcal{M} = \begin{pmatrix} p & 1-p & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$V(0) > \frac{V(0)+V(1)+V(2)}{3} > V(2) > V(1) = 0$$

$$\Downarrow$$

$$V_I = \frac{V(0)+V(1)+V(2)}{3} > V_\nu = V(1) = 0$$

$$\left\{ \begin{array}{lcl} U(0) & = & V(0) - V_I \\ U(1) & = & V_I - V(2) \\ U(2) & = & 0 \end{array} \right.$$

Asymptotic behavior

- Weak law of large numbers
- Convergence of empirical processes
- Time uniform estimates
- Propagation of chaos
  - total variation norm
  - relative entropy estimates
- Central limit theorems
  - Particle density profiles
  - Donsker Theorem
  - Fluctuations on path space
- Large deviations principles

Asymptotic behavior bips models

-Measure valued process

$$\eta_n = \Phi_n(\eta_{n-1})$$

-bips approx model (Φ_n)

$$(\xi_n^1, \ldots, \xi_n^N) \quad \text{"iid"} \quad \sim \ \Phi_n\left(\frac{1}{N}\sum_{j=1}^{N} \delta_{\xi_{n-1}^j}\right)(du)$$

<u>Theorem</u> $(0 \leq G_n \leq 1)$

$$\mathbb{P}(T^N \leq n) \leq c(n) \exp\left(-N \ (\gamma_n^2(1)/(8n))^2\right)$$

$$\mathbb{E}\left(|\eta_n^N(f) \ 1_{T^N \geq n} - \eta_n(f)|^p\right)^{\frac{1}{p}} \leq \frac{1}{\sqrt{N}}\, c_p(n,f)$$

proof:(*induction*)

$$\left|1_{T^N \geq n}\Phi_n(\eta_{n-1}^N) - \Phi_n(\eta_{n-1})\right|$$

$$\leq \left|\Phi_n(\eta_{n-1}^N) - \Phi_n(\eta_{n-1})\right| \ 1_{T^N \geq n} + \eta_n \ 1_{T^N < n}$$

$\blacksquare$

Convergence of empirical processes

-**Regular potentials** (G)

$$G_n(x) \;\geq\; r_n \; G_n(y)$$

- $\mathcal{F}$-**empirical process** $f \in \mathcal{F}$, $\|f\| \leq 1$

$$\eta_n^N : f \in \mathcal{F} \longrightarrow \eta_n^N(f) \in \mathbb{R}.$$

$\mathcal{F}$ not too large

-$L_2(\mu)$-covering numbers

$$N(\varepsilon, \mathcal{F}, L_2(\mu)) = \text{min. numb. } (\epsilon, L_2(\mu)) - \text{balls needed to cover } \mathcal{F}$$

-unif. cov.+entropy number

$$\mathcal{N}(\varepsilon, \mathcal{F}) = \sup_{\mu \in \mathcal{P}(E)} N(\varepsilon, \mathcal{F}, L_2(\mu)) \quad \text{and} \quad I(\mathcal{F}) = \int_0^1 \sqrt{\log \mathcal{N}(\varepsilon, \mathcal{F})}\, d\varepsilon$$

Key decomposition

$$\eta_n^N - \eta_n = \sum_{p=0}^{n} \left[\Phi_{p,n}(\eta_p^N) - \Phi_{p,n}\left(\Phi_p(\eta_{p-1}^N)\right)\right]$$

Key lemma $X^{i,N}$ *iid* $\sim P^{(N)}$, $m^N(X) = \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$

-unif. $\mathbb{L}_p$-estimate

$$\mathbb{E}\left(\|m^N(X) - P^{(N)}\|_{\mathcal{F}}^p\right)^{\frac{1}{p}} \leq \frac{b_p}{\sqrt{N}}\, I(\mathcal{F})$$

-unif. expo. estimate

$$P\left(\left\|m^N(X) - P^{(N)}\right)\|_{\mathcal{F}} > 8\,\varepsilon\right) \leq 8\,\mathcal{N}(\varepsilon, \mathcal{F})\, e^{-N\varepsilon^2/2}$$

[(Glivenko-Cantelli)+(Talagrand ineg.) for bips models]

Theorem

$$\mathbb{E}\left(\|\eta_n^N - \eta_n\|_{\mathcal{F}}^p\right)^{\frac{1}{p}} \leq \frac{b_p}{\sqrt{N}}\, c(n)\ I(\mathcal{F})$$

in particular

$$I(\mathcal{F}) \Longrightarrow \lim_{N\to\infty} \|\eta_n^N - \eta_n\|_{\mathcal{F}} = 0 \quad \text{a.e.}$$

for N suff. large

$$P\left(\left\|\eta_n^N - \eta_n\right\|_{\mathcal{F}} > \varepsilon\right) \leq 8(n+1)\,\mathcal{N}\left(\epsilon_n, \mathcal{F}\right) \quad e^{-N\epsilon_n^2/2}$$

with $\epsilon_n = \frac{\varepsilon r_{0,n}}{16(n+1)}$.

For polynomial classes $\exists C, V$:

$$\forall 0 < \varepsilon < C\,, \qquad \mathcal{N}(\varepsilon, \mathcal{F}) \leq \left(\frac{C}{\varepsilon}\right)^V$$

we have for any $N \geq 1$

$$P\left(\sqrt{N}\left\|\eta_n^N - \eta_n\right\|_{\mathcal{F}} > \delta\sigma_n\right) \leq (n+1)\left(\frac{D\ \delta}{\sqrt{V}}\right)^V e^{-2\,\delta^2}$$

$(D \sim C,\ \sigma_n = 2(n+1)/r_{0,n})$

Regular mutations

Theorem

$$\mathcal{M}_n(u, dv) = m_n(u, v)\ p_n(dv)$$

expo. reg.

$$\sup_{u \in E} |\log m_n(u, v)| \leq a_n(v) \quad \text{with} \quad e^{a_n} \in \mathbb{L}_3(p_n)$$

$$\Downarrow$$

$$\sqrt{N}\mathbb{E}\left(\|\eta_n^N - \eta_n\|_{\mathcal{F}}^p\right)^{\frac{1}{p}} \leq b_p\ [I(\mathcal{F}) + c(n)]$$

proof:

$$\|\eta_n^N - \eta_n\|_{\mathcal{F}} = \|\eta_n^N - \Phi_n(\eta_{n-1}^N)\|_{\mathcal{F}} + \int \; |\frac{d\Phi_n(\eta_{n-1}^N)}{d\Phi_n(\eta_{n-1})} - 1| \; d\eta_n$$

$$\frac{d\Phi_n(\mu)}{dp_n}(y) = \frac{\mu(G_{n-1} \; m_n(\,.\,, y))}{\mu(G_{n-1})}$$

$\blacksquare$

Time uniform estimates

Lemma

$$\sqrt{N}\ \mathbb{E}\left(|\eta_n^N f - \eta_n f|^p\right)^{\frac{1}{p}} \leq b_p \quad \sum_{q=0}^{n} \ \sup_{x,y}[\frac{G_{q,n}(x)}{G_{q,n}(y)}]^2\ \beta(\mathcal{M}_{q,n})$$

proof: $\sim$ two key decompositions

$$\eta_n^N - \eta_n = \sum_{q=0}^{n} \ [\Phi_{q,n}(\eta_q^N) - \Phi_{q,n}(\Phi_q(\eta_{q-1}^N))]$$

$$\Phi_{q,n}(\eta_q^N)([f - \Phi_{q,n}(\Phi_q(\eta_{q-1}^N))(f)])$$

$$= \eta_q^N\left(\frac{G_{q,n}}{\eta_q^N(G_{q,n})}\ \mathcal{M}_{q,n}[f - \Phi_{q,n}(\Phi_q(\eta_{q-1}^N))(f)]\right)$$

$\oplus$ **Mixing hyp.**

$(\mathcal{M})_m : \quad \mathcal{M}^m(x, \textbf{.}) \geq \epsilon\ \mathcal{M}^m(y, \textbf{.})$

$$\Downarrow$$

<u>**Theorem**</u> $r = \inf_n r_n > 0$

$$\sqrt{N} \ \sup_{n \geq 0} \mathbb{E}\left(|\eta_n^N f - \eta_n f|^p\right)^{\frac{1}{p}} \leq \frac{b_p}{\sigma(m)}$$

$(\sigma(m) > 0,\ \sigma(1) = r^2(1 - \epsilon^2)\epsilon^2)$

Corollary ($r = \inf_n r_n > 0$)

$$\mathcal{M}_n(u, dv) = m_n(u, v)\ p_n(dv)$$

unif. expo. reg.

$$\sup_{u \in E} |\log m_n(u, v)| \leq a_n(v)\,, \quad \sup_n \|e^{a_n}\|_{3,p_n} < \infty$$

$$\Downarrow$$

$$\sqrt{N} \sup_{n \geq 0} \mathbb{E}\left(\|\eta_n^N - \eta_n\|_{\mathcal{F}}^p\right)^{\frac{1}{p}} \leq b_p\ [I(\mathcal{F}) + c(m)]$$

Φ_n-Stability $\Longrightarrow$ Uniform estimates

<u>Theorem</u> ($\inf_{n\geq 1} r_n = r > 0$)

$$\lim_{n\to\infty} \sup_{\mu,\eta,p\geq 0} \left\| \Phi_{p,p+n}(\mu) - \Phi_{p,p+n}(\eta) \right\|_{\mathcal{F}} = 0$$

$$\Downarrow$$

$$\lim_{N\to\infty} \sup_{n\geq 0} E\left(\left\| \eta_n^N - \eta_n \right\|_{\mathcal{F}} \right) = 0$$

$\oplus$ **Expo. stab. hyp**

$$\|\Phi_{q,q+n}(\mu) - \Phi_{q,q+n}(\eta)\|_{\mathcal{F}} \leq e^{-\gamma\, n}$$

$$\Downarrow$$

$$\sup_{n\geq 0} \mathbb{E}\left(\left\|\eta_n^N - \eta_n\right\|_{\mathcal{F}}^p\right)^{\frac{1}{p}} \leq \frac{b_p\ e^{\gamma'}}{N^{\frac{\alpha}{2}}}\ \ I(\mathcal{F})$$

with $\quad \alpha \geq \dfrac{\gamma}{\gamma+\gamma'} \quad$ and $\quad \gamma' = 1 + \log\left(1/r\right)$

Propagation of chaos

-Particle distributions

$$\begin{array}{rcl} \mathbb{P}_n^{(N)} & = & \mathsf{Law}((\xi_0^i,\ldots,\xi_n^i)_{1\le i\le N}) \\ \mathbb{P}_n^{(N,q)} & = & \mathsf{Law}((\xi_0^i,\ldots,\xi_n^i)_{1\le i\le q}) \\ \mathbb{P}_{[n]}^{(N,q)} & = & \mathsf{Law}((\xi_n^i)_{1\le i\le q}) \end{array}$$

-McKean measure (Φ_n)

$$\mathbb{K}_n(d(x_0,\ldots,x_n)) = \eta_0(dx_0)\ \eta_1(dx_1)\ \ldots\ \eta_n(dx_n)$$

-Regular potentials (G)

$$G_n(x) \;\geq\; r_n\; G_n(y)$$

-Mixing Markov kernels $(\mathcal{M})_m$

$$\mathcal{M}^m(x,\,.\,) \;\geq\; \epsilon\; \mathcal{M}^m(y,\,.\,)$$

Total variation estimates

<u>**Theorem 1**</u>

$$(G) \Longrightarrow \|\mathbb{P}^{(N,q)}_{[n]} - \eta_n^{\otimes q}\|_{tv} \leq \frac{e^{q\,c(n)}}{N} \tag{1}$$

In addition (G)+$(\mathcal{M})_m$

$$\inf_n r_n > 0 \Longrightarrow \sup_n \|\mathbb{P}^{(N,q)}_{[n]} - \eta_n^{\otimes q}\|_{tv} \leq \frac{e^{q\,c}}{N}$$

Note :

(1) applies for genealogical bips models!

$$\xi_n^i = (\xi_{0,n}^i, \ldots, \xi_{n,n}^i)$$

Proof: [*technical*]

$$\eta_{n+1}^{\otimes q} \;=\; \Phi_{n+1}^{(q)}(\eta_n^{\otimes q})$$

same form

$$\Phi_{n+1}^{(q)}(\eta) = \Psi_{n+1}^{(q)}(\eta)\mathcal{M}_{n+1}^{(q)}$$

with Boltzmann-Gibbs' transf.

$$\Psi_{n+1}^{(q)} \hookrightarrow G_n^{(q)} = G_n^{\otimes q}$$

and

$$\mathcal{M}_{n+1}^{(q)}((x_1,\ldots,x_q),d(y_1,\ldots,y_q))$$

$$= \mathcal{M}_{n+1}(x_1,dy_1)\ldots\mathcal{M}_{n+1}(x_q,dy_q)$$

■

$\mathbb{L}_p$-estimates

-Regular potentials (G)

$$G_n(x) \ \geq \ r_n \ G_n(y)$$

-Mixing Markov kernels

$$(\mathcal{M})^{(p)} \ : \qquad\qquad \mathcal{M}_n(x,dy) = k_n(x,y) \ \eta_n(dy)$$

with $\overline{k}_n = \sup_{x\in E} k_n(x,\,.\,) \in \mathbb{L}_p(\eta_n)$

<u>**Theorem**</u>

$$\|\frac{d\mathbb{P}^{(N,q)}_{[n]}}{d\eta_n^{\otimes q}} - 1\|_{p,\eta_n^{\otimes q}} \leq \frac{e^{qc(n)}}{N} \times \|\sup_x k_n(x,\,.\,)\|^q_{p,\eta_n}$$

In addition $(+(\mathcal{M})_m)$

$$\inf_n r_n > 0 \quad \text{and} \quad \sup_n \|\overline{k}\|_{p,\eta_n} \Rightarrow \sup_n \|\frac{d\mathbb{P}^{(N,q)}_{[n]}}{d\eta_n^{\otimes q}} - 1\|_{p,\eta_n^{\otimes q}} \leq \frac{\exp{(qc)}}{N}$$

Relative entropy

-Regular potentials (G)

$$G_n(x) \;\; \geq \;\; r_n \; G_n(y)$$

-Mixing Markov kernels

$$(\mathcal{M})^{(2)} \; : \qquad\qquad \mathcal{M}_n(x,dy) = k_n(x,y) \;\eta_n(dy)$$

with $\overline{k}_n = \sup_{x\in E} k_n(x,\,.\,) \in \mathbb{L}_2(\eta_n)$

<u>**Theorem**</u>

$$\mathrm{Ent}(\mathbb{P}_n^{(N,q)} \mid \mathbb{K}_n^{\otimes q}) \leq \; \frac{q \;\, c(n)}{N}$$

In addition $((\mathcal{M})_m, (\mathcal{M})^{(2)})$

$$\sup_n \|\overline{k}_n\|_{2,\eta_n} < \infty \Longrightarrow \mathrm{Ent}(\mathbb{P}_n^{(N,q)} \mid \mathbb{K}_n^{\otimes q}) \leq \quad c\ \frac{q\ n}{N}$$

$$\Downarrow$$

$\uparrow$ **propagation of chaos** $(q(N)n(N) = o(N))$

$$\lim_{N\to\infty} \mathrm{Ent}(\mathbb{P}_{n(N)}^{(N,q(N))} \mid \mathbb{K}_{n(N)}^{\otimes q(N)}) = 0$$

Fluctuations (G)

Lemma $((p,x) \mapsto f_p(x) \in \mathbb{R}^d)$

$$\sqrt{N}\ M_n^{(N)}(f) = \sqrt{N} \sum_{p=0}^{n} \left[\eta_p^N(f_p) - \Phi_p(\eta_{p-1}^N)(f_p)\right]$$

$$\downarrow \quad (\textit{in law})$$

Gaussian mart. $M_n(f)$

$$\langle M(f^i), M(f^j)\rangle_n = \sum_{p=0}^{n} \eta_p \left(\left(f_p^i - \eta_p(f_p^i)\right)\left(f_p^j - \eta_p(f_p^j)\right)\right)$$

Theorem [**unnormalized flows**]

$$\begin{aligned}\Gamma_n^N(f) &= \gamma_n^N(\mathcal{Q}_{n,T}f) - \gamma_n(\mathcal{Q}_{n,T}f)\\ &= \sum_{p=0}^{n} \gamma_p^N(1)[\eta_p^N \mathcal{Q}_{p,T} - \Phi_p(\eta_{p-1}^N)\mathcal{Q}_{p,T}](f)\end{aligned}$$

martingale $n \leq T$

$$\langle \Gamma^N(f^i), \Gamma^N(f^j)\rangle_n = \frac{1}{N}\sum_{p=0}^{n} \; (\ldots)^2 \; [(i)(j)]$$

$\sqrt{N}\Gamma_n^N(f) \stackrel{in-law}{\longrightarrow} \Gamma_n(f)$ (*Gauss-mart.*)

$$\eta_n^N(f) - \eta_n(f) = \frac{\gamma_n(1)}{\gamma_n^N(1)}\ \gamma_n^N\left(\frac{1}{\gamma_n(1)}\,(f - \eta_n(f))\right)$$

$$\Downarrow$$

<u>**Theorem**</u> **[normalized flows]**

$$W_n^{\eta,N}(f) \ = \ \sqrt{N}\left(\eta_n^N(f) - \eta_n(f)\right)$$

$$\downarrow \qquad (\textit{in-law})$$

$$W_n^{\eta}(f) \ = \ W_n^{\gamma}\left(\frac{1}{\gamma_n(1)}\,(f - \eta_n(f))\right)$$

with

$$\mathbb{E}\left(W_n^\eta(f)W_n^\eta(h)\right) = \sum_{p=0}^{n} \int \left(\frac{G_{p,n}}{\eta_p(G_{p,n})}\right)^2 \left(\mathcal{M}_{p,n}f - \eta_n f\right)\left(\mathcal{M}_{p,n}h - \eta_n h\right) d\eta_p$$

-Functional cv th [Donsker]

$$I(\mathcal{F}) < \infty \Rightarrow \quad \textit{cv in law in } l^\infty(\mathcal{F})$$

Fluctuations on path space

$$(G) + [\mathcal{M}_n(x,dy) = k_n(x,y)\ p_n(dy)]$$

$$\Downarrow$$

$$d\mathbb{P}_n^{(N)} = \exp H_n^{(N)} \quad d\mathbb{K}_n^{\otimes N}$$

interaction potential

$$H_n^{(N)}((x_0^i,\ldots,x_n^i)_{1\leq i\leq N}) = N \sum_{p=1}^{n} \int \ \log \frac{d\Phi_p(m(x_{p-1}))}{d\Phi_p(\eta_{p-1})}\ dm(x_p)$$

Characteristic functions

$$\mathbb{E}_N(\exp{(iW_n^N(\varphi))}) = \overline{\mathbb{E}}_N(\exp{(iW_n^N(\varphi) + H_n^{(N)})}),$$

with

$$W_n^N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\delta_{(\xi_0^i,\ldots,\xi_n^i)} - \mathbb{K}_n) = \sqrt{N}(\mathbb{K}^N - \mathbb{K}_n)$$

$H_n^{(N)}$ (symmetric stat.)$+$($\mathbb{L}_2$-spectral formulae)

$$\Downarrow$$

$$\{W_n^N(\varphi)\ ;\ \varphi \in \mathbb{L}_2(\mathbb{P}')\}$$

cv in law to a Gaussian field.

Large deviations (E,d) $[\mathcal{K}_{n,\eta},\ \Omega = E^{n+1}]$

$$\pi^N : x \in \Omega^N \to \pi^N(x) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x^i} \in M = \mathcal{P}(\Omega)$$

McKean measure

$$\mathbb{P} = \eta_0 \times \ \mathcal{K}_{1,\eta_0} \times \ \ \ldots \ \ \times \mathcal{K}_{n,\eta_{n-1}}$$

Notations $\nu = (\nu_p)_{0\leq p\leq n} \in \mathcal{P}(E)^{n+1}$

$$\mathbb{Q}_\nu = \eta_0 \times K_{1,\nu_0} \times \ldots \times K_{n,\nu_{n-1}}$$

marginals mapping

$$\phi : \mu \in \mathcal{P}(\Omega) \to \phi(\mu) = (\mu_p)_{0\leq p\leq n} \in \mathcal{P}(E)^{n+1}$$

$\Rightarrow$ *fixed point* $\mathbb{Q}_{\phi(\mathbb{P})} = \mathbb{P}$

Ref. product meas.

$$\mathbb{Q}_\eta^N = (\mathbb{Q}_{\phi(\eta)})^{\otimes N}$$

$$\Downarrow \ (\textit{Sanov's theo})$$

under $\mathbb{Q}_\eta^N$, LDP with rate

$$H_\eta(\mu) = \mathsf{Ent}(\mu \mid \mathbb{Q}_{\phi(\eta)})$$

$\mathcal{K}_{p,\mu}(x, \centerdot) \sim \mathcal{K}_{p,\nu}(x, \centerdot)$,

$\forall \alpha \in \mathbb{R}$, $\forall \eta \in \mathcal{P}(\Omega)$

$$\mathcal{K}^{(\alpha)}_{p,\mu}(x,dy) = \frac{1}{Z^{\alpha}_{p}(\mu,\eta_{p-1})(x)} \quad [\frac{d\mathcal{K}_{p,\mu}(x, \centerdot)}{d\mathcal{K}_{p,\eta_{n-1}}(x, \centerdot)}(y)]^{\alpha} \quad \mathcal{K}_{p,\eta_{p-1}}(x,dy)$$

$$\Downarrow$$

N-**bips model**

$$\mathbb{P}^{(N)}_{\alpha,\eta} = \mathrm{Law}((\xi^i_0,\ldots,\xi^i_n)_{1\leq i\leq N})$$

$\alpha = 0 \Rightarrow \mathbb{P}^{(N)}_{0,\eta} = \mathbb{Q}^{N}_{\eta} = (\mathbb{Q}_{\phi(\eta)})^{\otimes N}$

$$\frac{d\mathbb{P}^N_{\alpha,\eta}}{d\mathbb{Q}^N_\eta}(\omega) = \exp\left(N[\alpha S_N(\omega,\eta) + C_\alpha(\pi_N(\omega),\eta)]\right)$$

with

$$S_N(\omega,\eta) = \sum_{p=0}^{n-1} \int \ m(\omega_p,\omega_{p+1})(d(u,v)) \ \log\left[\frac{d\mathcal{K}_{p+1,m(\omega_p)}(u,\,\cdot\,)}{d\mathcal{K}_{p+1,\eta_p}(u,\,\cdot\,)}(v)\right]$$

and

$$C_\alpha(\mu,\eta) = -\sum_{p=0}^{n-1} \ \int \ \mu_p(du) \ \log\left[Z^\alpha_{p+1}(\mu_p,\eta_p)(u)\right]$$

Note: $(\alpha = 1) \to N$**-bips law**

$$\mathbb{P}^N_{1,\eta} = \mathbb{P}^N_{1,\eta'} =_{def.} \mathbb{P}^N_1$$

$$\Downarrow \quad (\textit{mild reg. } C_\alpha)$$

Under $\mathbb{P}^N_1$ **LDP with rate**

$$I(\mu) = H_\mu(\mu) = \mathsf{Ent}(\mu \mid \mathbb{Q}_{\phi(\mu)})$$

On Laplace-Varadhan's integral Lemma

$$\begin{aligned}\frac{d\mathbb{P}^N_{\alpha,m}}{d\mathbb{Q}^N_m}(x) &= \exp\left(N[\alpha S_N(x,m) + C_\alpha(\pi_N(x),m)]\right)\\ \mathbb{P}^N_{1,m} &= \mathbb{P}^N_{1,m'} = \mathbb{P}^N_1\end{aligned}$$

$$\pi^N : x \in \Omega^N \to \pi^N(x) \in (M,d)$$

$$\overline{\mathbb{P}}^N_{\alpha,m} = \mathbb{P}^N_{\alpha,m} \circ \pi_N^{-1} \quad \text{and} \quad \overline{\mathbb{Q}}^N_m = \mathbb{Q}^N_m \circ \pi_N^{-1}$$

Integral Lemma

-$\overline{\mathbb{Q}}_m^N$ *LDP* H_m.
-$C_\alpha(\,\centerdot\,, m)$ *cont. at each* m, $C_\alpha(m,m) = 0$.
-*for some* $(m, n) \in M \times (1, \infty)$ (*expo. moment condition*)

$$\limsup_{N\to\infty} \frac{1}{N} \log \int \exp nN[S_N(x,m) + C_1(\pi_N(x), m)] \quad d\mathbb{Q}_m^N(x) < \infty$$

$$\Downarrow$$

$$\overline{\mathbb{P}}_1^N \text{ } \mathit{LDP} \text{ } I(m) = H_m(m).$$