SOME CONTRIBUTIONS TO NON LINEAR ESTIMATION AND INTERACTING PARTICLE SYSTEMS

Pierre DEL MORAL
Lab. Stat. & Probab.-C.N.R.S
Université Paul Sabatier, F-Toulouse.

Feynman-Kac or Boltzmann-Gibbs' formula

Physics $\longleftrightarrow$ Biology $\longleftrightarrow$ Engineering

Continuous time

$$d\mathbb{Q}_t = \frac{1}{Z_t} \quad \exp\left\{-\int_0^t V_s(\mathcal{X}_s)\ ds\right\} \quad d\mathbb{P}_t$$

Discrete time

$$\mathbb{Q}_n = \frac{1}{Z_n} \quad \exp\left\{-\sum_{p=0}^{n} V_p(\mathcal{X}_p)\right\} \ \mathbb{P}_n = \frac{1}{Z_n} \ \left\{ \ \prod_{p=0}^{n} \ G_p(\mathcal{X}_p) \ \right\} \ \mathbb{P}_n$$

Traditional Monte-Carlo

$$\mathbb{Q}_n^N = \sum_{i=1}^{N} \frac{e^{-\sum_{p=0}^{n} V_p(\mathcal{X}_p^i)}}{\sum_{j=1}^{N} e^{-\sum_{q=0}^{n} V_q(\mathcal{X}_q^j)}} \ \delta_{(\mathcal{X}_0^i,,\ldots,\mathcal{X}_n^i)} \Rightarrow \quad \text{Degenerate weighted-grids}$$

$$\Downarrow$$

Interacting particle systems

$\in \big\{$Adaptative stochastic grids, Simulation technique$\big\}$

$$\mathbb{Q}_n^N = \frac{1}{N} \sum_{i=1}^{N} \ \delta_{(\xi_{0,n}^i,,\ldots,\xi_{n,n}^i)} \longrightarrow \textbf{genealogical particle tree}$$

Topics/Collaborations

- **Feynman-Kac formulae-particle methods.**

  -**Non linear filtering**
  F. Cérou, D. Crisan, A. Guionnet, J. Jacod, M. Kouritzin, M. Ledoux,
  F. Le Gland, T.J. Lyons, L. Miclo, Ph. Protter
  -**Rare event analysis/flexible polymers**
  F. Cérou, F. Le Gland, P. Lézaud
  -**Large deviations** A. Guionnet, T. Zajic
  -**General FK/path spaces** L. Miclo

- **Genetic algorithms-global optimization.** L. Miclo

- **Contraction Markov semigroups**
  and nonlinear Feynman-Kac-Schrödinger semigroups.
  A. Guionnet, L. Miclo, M. Ledoux

- **Idempotent probability calculus and optimal control theory.** M. Doisy

Initializations

Heuristic particle methods $\rightarrow$ Radar, Sonar, G.P.S.

H. Carvalho, G. Rigal, G. Salut, J.C. Noyer, P. DM.

STCAN-DRET/DIGILOG-LAAS/CNRS

contracts (1989-1994)

$$\Downarrow$$

Genetic particle syst. $\rightarrow$ 2 types of analysis

- Fixed population size N, time $n \rightarrow \infty$.
  $[Cerf - 94] \Longrightarrow$**Generalized simulated annealing**

- Fixed time n, population size $N \rightarrow \infty$.
  [96] $\Longrightarrow$ **Measure valued and interacting processes**

$\hookrightarrow$ *Recently (pop. size, time)$\rightarrow \infty$,*

- **fixed points measure valued processes**
  ($\sim$ Annealed-Feynman-Kac models)

$$\mathbb{Q}_n^{\beta} = \frac{1}{Z_n(\beta)} \quad \exp\left\{-\sum_{p=0}^{n} \beta(p) V(\mathcal{X}_p)\right\} \quad \mathbb{P}_n$$

- **self-interacting Markov chains**
  ($\sim$ Genetic reinforced Markov chains)

- **space/time interaction processes**
  ($\sim$ Collective Ants' type global search)

Pop. size N, time $n \to \infty$. *[β inverse temp. parameter]*

$$\Downarrow$$

Genetic alg.= Markov proc. $\in E^N$,

$$\begin{aligned} G_\beta^{(1)} &= M_\beta S_\beta \\ G_\beta^{(2)} &= \alpha M_\beta + (1-\alpha)\ S_\beta, \qquad\qquad \alpha \in (0,1), \end{aligned}$$

mutation/selection

$$\begin{aligned} M_\beta(x,dy) &= \prod_{i=1}^{N} K_\beta(x^i, dy^i) \\ S_\beta(x,dy) &= \prod_{i=1}^{N} \sum_{j=1}^{N} \frac{e^{-\beta U(x^j)}}{\sum_{k=1}^{N} e^{-\beta U(x^k)}}\ \delta_{x^j}(dy^i) \end{aligned}$$

$$\Downarrow$$

GA low temp. $\in$ **Markov chains/rare transitions**

$\longrightarrow$ Markov kernels $\sim$ two parameters (β, γ)

$$Q_{\gamma,\beta}(x, dy) \simeq q(x, dy)\ e^{-\beta V_\gamma(x,y)} \quad \text{with} \quad \lim_{\gamma \to \infty} V_\gamma(x, y) = V(x, y)$$

$$\Downarrow$$

Th $(|E| < \infty)$, $\exists(\gamma_t, \beta_t) \uparrow \infty \longrightarrow$ Markov process $\xi_t \sim L_{\gamma_t,\beta_t} = Q_{\gamma_t,\beta_t} - I$

$$\lim_{t \to \infty} \mathrm{Ent}(\mathrm{Law}(\xi_t) \mid \pi_{\gamma_t,\beta_t}) = 0 \quad \text{with} \quad \pi_{\gamma,\beta} L_{\gamma,\beta} = 0$$

-Mean potential optimization $\quad \rightarrow U(x) = \int \mathcal{U}(x,u)\mu(du)$

Random & non homog. potentials

$$V_{\gamma_t}(x,y) = [U_t(y) - U_t(x)]^+ \quad \text{with} \quad U_t(x) = \frac{1}{t}\int_0^t \mathcal{U}(x, Y_s)\ ds$$

$$\downarrow$$

$$V(x,y) \;=\; [U(y) - U(x)]^+$$

$$\Downarrow$$

$$\lim_{t\rightarrow\infty} \mathbb{P}(\xi_t \in U^\star) = 1\,, \quad U^\star = \{x\,,\ U(x) = \inf_E U\}$$

-**Concentration genetic algo.** $\in E^N$

elementary mutations $\longrightarrow K_\beta(x^i, dy^i) \simeq k(x^i, dy^i)\ e^{-\beta a(x^i, y^i)}$

$$\Downarrow$$

Th

$\exists N^{(1)} > N^{(2)}$ (critical sizes) : $\forall N \geq N^{(\alpha)}, \quad \lim_{t \to \infty} \mathbb{P}(\xi_t^{(\alpha)} \in U_a^\star) = 1$

a-equiv. & U-optimal configurations

$$U_a^\star = \{x \in (U^\star)^N,\ a(x^i, x^j) = 0\}$$

sg tech.+log-Sob [96] $\hookrightarrow$ simplification+extension of Cerf's [94]

$\left(\textbf{Pop. size } N \to \infty\right) \longrightarrow \left[\textbf{new discrete generation interacting systems}\right]$

$$\Downarrow$$

Measure valued process

$$\Phi_n : \eta_{n-1} \in \mathcal{P}(E) \longrightarrow \qquad \eta_n = \Phi_n(\eta_{n-1}) \qquad (\in \mathcal{P}(E))$$

Interacting particle system (IPS)

$$\xi_n = (\xi_n^i)_{1 \leq i \leq N} \in E^N \quad \textbf{"iid"} \quad \sim \; \Phi_n\left(\frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_{n-1}^i}\right)$$

<u>**Th**</u> $\quad \Phi_n \quad$ unif. cont. $\Rightarrow \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \delta_{\xi_n^i} \xrightarrow{law} \eta_n$

$\hookrightarrow \mathbb{L}_p +$exponential rates $\sim$ Φ_n regularity

Extensions:

Markov interpretation $\exists \mathcal{K}_{n,\eta}(x,dy)$

$$\eta_n = \eta_{n-1}\mathcal{K}_{n,\eta_{n-1}} \quad \text{(not unique)}$$

McKean Measure

$$\mathbb{K}_n(d(x_0,\ldots,x_n)) = \eta_0(dx_0)\ \mathcal{K}_{1,\eta_0}(x_0,dx_1)\ \ldots\ \mathcal{K}_{n,\eta_{n-1}}(x_{n-1},dx_n)$$

$$\Downarrow$$

Interacting particle system

$$\xi_n^i \quad \sim \quad \mathcal{K}_{n,\frac{1}{N}\sum_{i=1}^N \delta_{\xi_{n-1}^i}}(\xi_{n-1}^i,\,\cdot\,)$$

Ex.: Feynman-Kac models

$(G_n, \mathcal{M}_n)$ =potential/Markov kernel

$$\eta_n(f) = \gamma_n(f)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f) = \mathbb{E}(f(\mathcal{X}_n) \prod_{p=0}^{n-1} G_p(\mathcal{X}_p))$$

One step transf.

$$\Phi_{n+1}(\eta) = \Psi_n(\eta)\mathcal{M}_{n+1}$$

Boltzmann-Gibbs' transf. $\quad \Psi_n(\eta)(dx) = \frac{1}{\eta(G_n)} \, G_n(x) \, \eta(dx)$

$$\Downarrow$$

$$\Phi_{n+1}\left(\frac{1}{N}\sum_{i=1}^{N} \delta_{x^i}\right) = \sum_{i=1}^{N} \frac{G_n(x^i)}{\sum_{j=1}^{N} G_n(x^j)} \, \mathcal{M}_{n+1}(x^i, \boldsymbol{.})$$

Genetic (selection/mutation)

$$\eta_n^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_n^i} \stackrel{law}{\longrightarrow} \eta_n$$

First proof+rates estimates [96]$\longrightarrow$Heuristics schemes

–Particle filters (Salut/Rigal/DM-[contract report 92])

–Sampling-Importance-Resampling

(Gordon/Salmon/Smith-93)

–Genetic filters (Kitagawa-96)

Related branching particle filters

–Branching particle systems (Crisan-Gaines-Lyons)(Crisan-Lyons-DM)

–Sequential Monte-Carlo, Condensation filters.../...

Feynman-Kac path measures $\longleftrightarrow$ Genealogical particle models

$$\eta_n(f_n) = \gamma_n(f_n)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f_n) = \mathbb{E}(f_n(\mathcal{X}_n) \prod_{p=0}^{n-1} G_p(\mathcal{X}_p))$$

BUT $\mathcal{X}_n$=historical process of $\mathcal{X}'$

$$\mathcal{X}_n \;=\; (\mathcal{X}'_0, \ldots, \mathcal{X}'_n) \in E_n = E'_0 \times \ldots \times E'_n \quad \text{and} \quad G_n(\mathcal{X}_n) = G'_n(\mathcal{X}'_n)$$

$$\Downarrow$$

Path particles=Ancestral lines [path-selection/ path-extensions]

$$\xi^i_n = (\xi^i_{0,n}, \ldots, \xi^i_{n,n}) \in E_n = E'_0 \times \ldots \times E'_n$$

$$\downarrow$$

Genetic path-particle $\sim (G_n, \mathcal{X}_n) =$ Genealogy particle model $\sim (G'_n, \mathcal{X}'_n)$

Un-normalized Feynman-Kac models

-Representation formula

$$\gamma_n(f) \;=\; \mathbb{E}(f(\mathcal{X}_n) \prod_{p=0}^{n-1} G_p(\mathcal{X}_p)) = \eta_n(f) \;\times\; \prod_{p=0}^{n-1} \eta_p(G_p)$$

$$\Downarrow$$

-Unbias estimate $\left(\sim\right.$-martingale interpretation/semi-groups method$\left.\right)$

$$\gamma_n^N(f) \;=\; \eta_n^N(f) \;\times\; \prod_{p=0}^{n-1} \eta_p^N(G_p)$$

$\overset{Varadhan's form.}{\Longrightarrow}$ Particle approx. Lyap. expo. F-K-Schrödinger semigroups

Research program $\longrightarrow$ Feynman-Kac formulae

- **Stochastic particle methods**

  Branching and Interacting particle systems

  (random or fixed pop. size)

  Genealogical tree based algorithms

  Discrete and continuous time models

  Self-interacting Markov chains

- **Applications**

  Signal processing

  –Filtering/smoothing/path estimation

  –Discrete time models

–Zakai & Kushner-Stratonovitch equations (Stoch. PDE)

–Semi-discrete models (+ Euler schemes)

–Interacting Kalman-Bucy filters

Rare event analysis (network overflows/exit times)

Particle polymerizations

Spectral analysis of FK-Schrödinger sg

- **Stochastic Analysis**

  Asymptotic stability of semigroup

  –Contraction of Markov kernels

  –Contraction of non linear semigroups

  Feynman-Kac flows in random media

  Annealing and concentration properties

- **Asymptotic behavior**

  $\mathbb{L}_p$, expo. estimates

  Weak convergence of empirical processes

  Uniform estimates w.r.t. time

  Increasing propagation of chaos

  Central limit theorems

  Large deviations principles

Applications

-**Filtering** $G_n(x) =$likelihood observation/signal

$$\eta_n = \text{Law}(\mathcal{X}_0, \ldots, \mathcal{X}_n | \mathcal{Y}_0, \ldots, \mathcal{Y}_{n-1})$$

-**Polymers** $G_n(x_0, \ldots, x_n) = 1_{E-\{x_0,\ldots,x_{n-1}\}}(x_n)$

$$\eta_n = \text{Law}(\mathcal{X}_0, \ldots, \mathcal{X}_n | \forall p < q < n \quad \mathcal{X}_p \neq \mathcal{X}_q)$$

-**Rare events**

rare target level $\quad B = B_{m+1} \subset B_m \subset \ldots \subset B_1 \subset B_0 \quad$ (initial level)

n-th excursion or "return to B_0"

$$\begin{array}{rcl} \mathcal{X}_n & = & (X_t \;,\; T_{n-1} \leq t \leq T_n) \\ G_n(\mathcal{X}_n) & = & 1_{B_n}(X_{T_n}) \end{array}$$

$\eta_n =$
$\text{Law}([X_0, (X_t\,, T_0 \leq t \leq T_1), \ldots, (X_t \;,\; T_{n-1} \leq t \leq T_n)] \mid T_n < \text{Return time})$

–Spectral analysis

$$\begin{aligned}
\Lambda(V) &= \log \mathsf{Lyap}(e^V \times \mathcal{M}) = \log \mathsf{Spr}(e^V \times \mathcal{M}) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}(\exp \sum_{p=0}^{n-1} V(\mathcal{X}_p)) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{p=0}^{n-1} \log \eta_p[\exp(V)] = \eta_\infty[\exp(V)] \\
&\Downarrow \\
\Lambda(V) &\sim \Lambda_n^N(V) = \frac{1}{n} \sum_{p=0}^{n-1} \log \eta_p^N[\exp(V)]
\end{aligned}$$

continuous time: Schrödinger op. $L^v = L + V \longrightarrow$ top Lyap. expo

$$\Lambda(V) \sim \Lambda_t^N(V) = \frac{1}{t} \int_0^t \eta_s^N(V)\, ds$$

Stochastic Analysis

Semigroup contractions [→generalizations of Dobrushin's work]

Dobrushin's contraction coef.

$$\|\mu\mathcal{M} - \eta\mathcal{M}\|_{tv} \;\leq\; (1-\alpha(\mathcal{M}))\;\|\mu-\eta\|_{tv}$$

Ex.:

$$(\mathcal{M})_m \qquad \mathcal{M}^m(x,\,.) \;\geq\; \epsilon\;\mathcal{M}^m(y,\,.) \Rightarrow \alpha(\mathcal{M}^m) \geq \epsilon$$

$$\Downarrow$$

No invariant measures$+(E,\mathcal{E})$ **measurable space**

I. **Contraction FKS nonlinear sg** $\quad [+(\mathrm{G}) \longrightarrow G_n(x) \;\geq\; r\; G_n(y)]$

$$\gamma_n = \gamma_p \mathcal{Q}_{p,n} \;\longrightarrow\; \eta_n(f) = \Phi_{p,n}(\eta_p)(f) = \frac{\eta_p(G_{p,n}\mathcal{M}_{p,n}(f))}{\eta_p(G_{p,n})}$$

$$with \quad G_{p,n} = \mathcal{Q}_{p,n}(1) \;\; \text{and} \;\; \mathcal{M}_{p,n}(f) = \frac{\mathcal{Q}_{p,n}(f)}{\mathcal{Q}_{p,n}(1)}$$

Th

$$\sup_{(\mu,\eta)} \|\Phi_{p,n}(\mu) - \Phi_{p,n}(\eta)\|_{tv} = 1 - \alpha(\mathcal{M}_{p,n}) \leq \left(1 - \epsilon^2\; r^{m-1}\right)^{[(n-p)/m]}$$

$$\|\Phi_{p,n}(\mu) - \Phi_{p,n}(\eta)\|_{tv} \leq \frac{1}{\epsilon^2 r^m}\; (1 - \alpha(\mathcal{M}_{p,n}))\; \|\mu - \eta\|_{tv}$$

.../...

II. **Relative ϕ-entropy**

$$\phi(x,y) \text{ homogenous convex } \longrightarrow H_\phi(\mu,\eta) = \int \phi(d\mu, d\eta)$$

ϕ-contraction coef.

$$H_\phi(\mu\mathcal{M}, \eta\mathcal{M}) \leq \ (1 - \alpha_\phi(\eta, \mathcal{M}))\ H_\phi(\mu,\eta)$$

Ex.:

$$\begin{aligned}
\phi_0(x,y) = |x-y| \quad &\Longrightarrow \quad H_\phi(\mu,\eta) = \|\mu - \eta\|_{tv} \\
\phi_p(x,y) = y|x/y - 1|^p \quad &\Longrightarrow \quad H_\phi(\mu,\eta) = \|\frac{d\mu}{d\eta} - 1\|_{p,\eta}^p \\
\phi_{ent}(x,y) = x \log(x/y) \quad &\Longrightarrow \quad H_\phi(\mu,\eta) = \mathrm{Ent}(\mu|\eta) \\
\phi_{hell}(x,y) = y\ (x/y - (x/y)^\alpha) \quad &\Longrightarrow \quad H_\phi(\mu,\eta) = 1 - \int (d\mu)^\alpha\ (d\eta)^{1-\alpha}
\end{aligned}$$

Right object = Ajoint Markov op.

$$\langle f, \mathcal{M}(g)\rangle_\eta = \langle \mathcal{M}_\eta^\star(f), g\rangle_{\eta\mathcal{M}} \longrightarrow \mathcal{M}_\eta^\star(f) = \frac{d(f\eta)\mathcal{M}}{d\eta\mathcal{M}}$$

Properties

$$\eta \quad \mathcal{M}\mathcal{M}_\eta^\star - \text{rev.} \Longrightarrow \mathcal{E}_{\eta,\mathcal{M}\mathcal{M}_\eta^\star}(f,g) = \mathcal{E}_{\eta,\mathcal{M}\mathcal{M}_\eta^\star}(g,f)$$

$$\Downarrow$$

<u>Th</u>

$$\alpha_{\phi_2}(\eta,\mathcal{M}) \;=\; Spgap(\eta,\mathcal{M}\mathcal{M}_\eta^\star) \quad \text{and} \quad \alpha_{\phi_{ent}}(\eta,\mathcal{M}) \geq LogSob(\eta,\mathcal{M}\mathcal{M}_\eta^\star)$$

$\longrightarrow$ *Universal estimate :* $\quad \alpha_\phi(\eta,\mathcal{M}) \geq \alpha(\mathcal{M}) \quad (\;\geq\; \epsilon \;\Longleftarrow (\mathcal{M})_{m=1})$ .../...

Asymptotic behavior $(N \to \infty)$ $\quad$ [(G): $G_n(x) \;\geq\; r\; G_n(y)$]

-Un-normalized meas.

$$\gamma_n^N(.) = [\prod_{p=0}^{n-1} \eta_p^N(G_p)\ ] \ \times\ \eta_n^N(.) \quad \longrightarrow \quad \gamma_n(.) = \gamma_n(1) \times \eta_n(.)$$

-Particle McKean meas.

$$\mathbb{K}_n^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{(\xi_0^i,\ldots,\xi_n^i)} \longrightarrow \mathbb{K}_n = \eta_0 \ \times \mathcal{K}_{1,\eta_0} \times \ \ \ldots \ \ \times \mathcal{K}_{n,\eta_{n-1}}$$

-Particle density profiles

$$\eta_n^N \ = \ \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_n^i} = \frac{1}{N}\sum_{i=1}^{N} \delta_{(\xi_{0,n}^i,\ldots,\xi_{n,n}^i)} \longrightarrow \eta_n = \gamma_n(.)/\gamma_n(1)$$

$\mathbb{L}_p$-estimates ($\hookrightarrow$(CLT))

McKean meas. apprx

$$\mathbb{E}(|\mathbb{K}_n^N(F) - \mathbb{K}_n(F)|^p)^{1/p} \leq c(n)/\sqrt{N}$$

Particle density profiles

$$\mathbb{E}(\sup_{f\in\mathcal{F}} |\eta_n^N(f) - \eta_n(f)|^p)^{1/p} \leq c(n)I(\mathcal{F})/N^\alpha$$

$$\begin{array}{rcll}
(G) & \Longrightarrow & \alpha = 1/2 \text{ and } & c(n) < \infty \\
(\Phi_n \text{ expo. stable}) + (G) & \Longrightarrow & \alpha > 0 \text{ and } & c = \sup_n c(n) < \infty \\
(\mathcal{M})_m + (G) & \Longrightarrow & \alpha = 1/2 \text{ and } & c = \sup_n c(n) < \infty
\end{array}$$

Expo. estimates $(\hookrightarrow$**(LDP)**$)$

$$\mathbb{P}(\sup_{f\in\mathcal{F}} |\eta_n^N(f) - \eta_n(f)| \geq \epsilon) \leq c(n)\ \mathcal{N}(\epsilon,\mathcal{F})\ e^{-N\epsilon^2/c(n)}$$

Talagrand's $\leq$ for interacting systems (Polynomial covering)

$$\forall 0 < \epsilon < C\,, \qquad \mathcal{N}(\epsilon,\mathcal{F}) \leq \left(\frac{C}{\epsilon}\right)^V$$

$$\Longrightarrow \mathbb{P}(\sqrt{N} \sup_{f\in\mathcal{F}} |\eta_n^N(f) - \eta_n(f)| > \delta\sigma_n) \leq \left(\frac{D\,\delta}{\sqrt{V}}\right)^V e^{-2\,\delta^2}$$

$\uparrow$ Propagation of chaos $\hookrightarrow$indepedence+adequation

$$\mathbb{P}_n^{(q,N)} = \mathsf{Law}((\xi_0^i,\ldots,\xi_n^i)_{1\leq i\leq q}) \quad \text{and} \quad \mathbb{P}_{[n]}^{(q,N)} = \mathsf{Law}((\xi_n^i)_{1\leq i\leq q})$$

$$(G) \Longrightarrow \|\mathbb{P}_{[n]}^{(N,q)} - \eta_n^{\otimes q}\|_{tv} \leq \frac{\exp\left(q\,c(n)\right)}{N}$$

-Uniform estimates

$$\begin{array}{ccc} c = \sup_n c(n) < \infty & & \\ \Uparrow & & \\ (\mathcal{M})_m + (G) & \Longrightarrow & \mathsf{Ent}(\mathbb{P}_n^{(N,q)} \mid \mathbb{K}_n^{\otimes q}) \leq c\,\dfrac{q\ n}{N} \end{array}$$

$q(N)n(N) = o(N) \Rightarrow\uparrow$ **prop.chaos w.r.t.**$\uparrow$ (**time horizon**/ **particle blocks**)

Fluctuations on path space $\quad [\mathcal{M}_n(x,.) << \mathcal{M}_n(y,.)]$

Regular Laplace-formulation

$$d\mathbb{P}_n^{(N,N)} = \exp H_n^N \quad d\mathbb{K}_n^{\otimes N}$$

$\hookrightarrow$ **interaction potential**

$$H_n^N((x_0^i,\ldots,x_n^i)_{1\le i\le N}) = N \sum_{p=1}^{n} \int \log \frac{d\Phi_p(m(x_{p-1}))}{d\Phi_p(\eta_{p-1})} \, dm(x_p)$$

Th [extension of Shiga-Tanaka's th $\to$ nonlinear interaction]

$$\lim_{N\to\infty} \mathbb{E}_N(\exp(iW_n^N(\varphi))) = \exp\left(-\frac{1}{2}\|(I-A_n)^{-1}\varphi\|^2_{2,\mathbb{K}_n}\right)$$

$$W_n^N = \sqrt{N}(\mathbb{K}_n^N - \mathbb{K}_n)$$

Large deviations on path space

(N=precision parameter, $\alpha \in \mathbb{R}$ "interaction degree")

$$\pi_N : x \in \Omega_N \to \pi_N(x) = m \in (M, d)$$

Radon-Nykodim's structure

$$\frac{d\mathbb{P}^N_{\alpha,m}}{d\mathbb{Q}^N_m}(x) = \exp\left(N[\alpha S_N(x,m) + \underbrace{C_\alpha(\pi_N(x), m)}_{\text{cont.}+C_\alpha(m,m)=0}\ ]\right)$$

$$\mathbb{P}^N_{1,m} = \mathbb{P}^N_1$$

<u>**Integral transfer lemma**</u> *[+expo. condition]*

$$\overline{\mathbb{Q}}^N_m = \mathbb{Q}^N_m \circ \pi_N^{-1} \quad \textbf{LDP } H_m(.)$$

$$\Downarrow$$

$$\overline{\mathbb{P}}^N_1 = \mathbb{P}^N_1 \circ \pi_N^{-1} \quad \textbf{LDP } I(m) = H_m(m)$$

LDP bips path space

$\Sigma_n = E^{n+1}$, $[\mathcal{K}_{n,\eta} \sim \mathcal{K}_{n,\mu}]$

$$\eta \in M = \mathcal{P}(\Sigma_n) \ \rightarrow \phi(\eta) = (\eta_p)_p \rightarrow \textbf{McKean meas. } \mathbb{K}_{\phi(\eta)} \in \mathcal{P}(\Sigma_n)$$

$$\begin{array}{rcl} \mathbb{P}^N_{\alpha,\eta} & = & \mathsf{Law}((\xi^i_0, \ldots, \xi^i_n)_{1 \leq i \leq N}) \longrightarrow \textbf{bips} \ \sim \left[\dfrac{d\mathcal{K}_{n,\mu}(u, \centerdot)}{d\mathcal{K}_{n,\eta_{n-1}}(u, \centerdot)}\right]^\alpha \ \mathcal{K}_{n,\eta_{n-1}}(u, dv) \\ \mathbb{Q}^N_\eta & = & \mathbb{K}^{\otimes N}_{\phi(\eta)} \end{array}$$

$$\pi_N : \omega \in \Omega_N = (\Sigma_n)^N \rightarrow \pi^N(\omega) = \frac{1}{N} \sum_{i=1}^N \delta_{\omega^i} \in (M, d)$$

$$\overline{\mathbb{Q}}^N_\eta \quad \textbf{LDP } H_\eta \overset{Sanov}{=} \mathsf{Ent}(\centerdot | \mathbb{K}_{\phi(\eta)}) \quad \Rightarrow \quad \overline{\mathbb{P}}^N_1 \quad \textbf{LDP } I(\eta) = \mathsf{Ent}(\eta | \mathbb{K}_{\phi(\eta)})$$

Research program

-Self-interacting and reinforced Markov chains

$$\mathbb{P}(\mathcal{X}_n \in dx | \mathcal{X}_0, \ldots, \mathcal{X}_{n-1}) = K_{\frac{1}{n}\sum_{p=0}^{n-1} \delta_{\mathcal{X}_p}}(\mathcal{X}_{p-1}, dx)$$

ex.: genetic interaction

$$K_{\frac{1}{n}\sum_{p=0}^{n-1} \delta_{x_p}}(x_{p-1}, \,.\,) = \Phi\left(\frac{1}{n}\sum_{p=0}^{n-1} \delta_{x_p}\right) = \sum_{p=0}^{n-1} \frac{G(x_p)}{\sum_{q=0}^{n-1} G(x_q)} \; \mathcal{M}(x_p, \,.\,)$$

<u>Th</u> $\lim_{n\to\infty} \frac{1}{n}\sum_{p=0}^{n-1} \delta_{\mathcal{X}_p} = \eta_\infty = \Phi(\eta_\infty)$ ($\mathbb{L}_p$-rates$\sim\Phi$-regularity)

-Self-interacting particle models $\xi_p \in E^N \rightarrow m(\xi_p) = \frac{1}{N}\sum_{i=1}^N \delta_{\xi_p^i} \in \mathcal{P}(E)$

$$\mathbb{P}(\xi_n \in dx | \xi_0, \ldots, \xi_{n-1}) = \prod_{i=1}^N \Phi\left(\frac{1}{n}\sum_{p=0}^{n-1} m(\xi_p)\right)(dx^i)$$

-Particle approximations signed-measure flows

$$\begin{aligned} \gamma_n = \gamma_n^+ - \gamma_n^- &= \Gamma_n(\gamma_{n-1}) \\ &= \Gamma_n^+(\gamma_{n-1}) - \Gamma_n^-(\gamma_{n-1}) \quad \longrightarrow \quad \text{(+/-)-bips} \end{aligned}$$