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A Lyapunov approach to stability of positive semigroups: an overview with illustrations

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ABSTRACT

The stability analysis of possibly time varying positive semigroups on non-necessarily compact state spaces, including Neumann and Dirichlet boundary conditions is a notoriously difficult subject. These crucial questions arise in a variety of areas of applied mathematics, including nonlinear filtering, rare event analysis, branching processes, physics and molecular chemistry. This article presents an overview of some recent Lyapunov-based approaches, focusing principally on practical and powerful tools for designing Lyapunov functions. These techniques include semigroup comparisons as well as conjugacy principles on non-necessarily bounded manifolds with locally Lipschitz boundaries. All the Lyapunov methodologies discussed in the article are illustrated in a variety of situations, ranging from conventional Markov semigroups on general state spaces to more sophisticated conditional stochastic processes possibly restricted to some non-necessarily bounded domains, including locally Lipschitz and smooth hypersurface boundaries, Langevin diffusions as well as coupled harmonic oscillators.

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1. Introduction

This review article outlines some of the main points of the stability theory of possibly time varying positive semigroups on non-necessarily compact state spaces, including Neumann and Dirichlet boundary conditions. We present an overview of some recent Lyapunov-based approaches, focusing principally on practical and powerful tools for designing Lyapunov functions.

Foster-Lyapunov criterion dates back to the 1950s with the seminal articles [1,2]. These criteria are nowadays an essential tool to analyze the stability properties of Markov semigroups on general state spaces [3–8]. There is also a vast literature on subgeometric convergence rates for Markov chains, starting with the foundational articles [9,10] based on sequences of Lyapunov-type functions defined in terms of some well-chosen subgeometrical rate, followed by the control of modulated moments of the return-time to some regular set. More practical Foster-Lyapunov conditions are

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presented in [11–15]. Subgeometric convergence rates for continuous time Markov processes are also discussed in [5, 16,17], see also the more recent article [18].

The use of Foster-Lyapunov criteria in the context of positive semigroup arising in discrete time nonlinear filtering goes back to the pioneering articles [19,20], based on coupling techniques developed in [21,22]. The extension of Foster-Lyapunov criterion to discrete or continuous time varying positive semigroups and their normalized versions on general state spaces were further developed in [23], extending Dobrushin's ergodic coefficient techniques introduced in [24,25] and further developed in [26–30] to unbounded state space models.

Recall that the Dobrushin's ergodic coefficient of a Markov semigroup is the operator norm of the Markov integral operator acting on probability measures equipped with the total variation norm (see for instance [28] and references therein). In the same vein, the V-Dobrushin's ergodic coefficient of a Markov transition introduced in [30] is defined as the operator norm of the Markov integral operator acting on probability measures equipped with the V-norm. In this framework, the contraction w.r.t. V-norms is deduced by coupling the Foster-Lyapunov criterion with a local contraction on a sufficiently large compact sub-level set of the Lyapunov function.

This operator-theoretic framework is discussed in Section 2 in the context of discrete time and homogeneous Markov semigroups. Section 2.1 is dedicated to V-norm contraction coefficients and the exponential convergence of Markov semigroups. This rather elementary operator-theoretic framework is further extended in Section 2.2 to derive in a rather simple way subexponential convergence rates, stripped of all analytical superstructure, and probabilistic irrelevancies.

The extension of this framework to more general classes of time varying Markov semigroups with possibly continuous time indices is discussed in Section 2.3 as well as in Section 2.4 in the context of diffusion semigroups. Exponential stability theorems for more general classes of positive semigroups are discussed in Section 3.1.

To take the discussion one step further and underline the role of Lyapunov conditions, we emphasize that local contraction principles (a.k.a. local minorization conditions) on the compact sub-level sets of a prescribed Lyapunov function are generally easily verifiable conditions. This property is often deduced from a Doeblin type local minorization property of integral operators on the compact sub-level sets of the Lyapunov function. For instance, this local minorization condition is satisfied as soon as the semigroup is lower bounded by an absolutely continuous integral operator (a.k.a. transition kernel operator). This class of models includes hypo-elliptic diffusion semigroups as well as some regular jump processes on non-necessarily bounded domains.

We also underline that for diffusion semigroups with smooth densities on bounded manifolds with entrance boundaries (i.e. boundary states that cannot be reached from the inside), the existence of a sufficiently strong Lyapunov function is essential to ensure the stability of the semigroup. In this context, the transition densities are null on entrance boundary states so that the local minorization condition alone applied to some exhausting sequence of compact subsets is not sufficient to ensure the stability of the process. The exhausting sequence of compact subsets needs to be equivalent to the sub-level sets of some sufficiently strong Lyapunov function near entrance boundaries. For a

more thorough discussion on this subject we refer to Section 2 and the article [23], see also the series of Riccati-type diffusions discussed in Section 6.

The general problem of constructing Lyapunov functions for positive semigroups, including for Markov semigroups often requires having some good intuition about a candidate for a Lyapunov function on some particular class of model. As for deterministic dynamical systems, the design of Lyapunov functions for sub-Markov semigroups associated with a non-absorbed stochastic process requires to use some physical insight on the stability and the behavior of the free evolution stochastic process near possible absorbing boundaries.

Constructing Lyapunov functions for general classes of positive semigroups is well known as a very hard problem in system theory as well as in applied probability literature. The main subject of this article is to find *practical ways* to design these Lyapunov functions for various classes of positive semigroups that have been discussed in the literature, including conditional diffusions on manifolds with Neumann and Dirichlet boundaries. We did our best to cover the subject as broadly as possible; we also refer to the article [23] for additional historical and reference pointers. Due to the vast literature on this subject we apologize for possible omissions of some important contributions due to the lack of knowledge.

The remainder of this article is structured as follows:

In Section 2, we begin with a brief review on the stability of Markov semigroup. The extension of these results to time varying positive semigroups is discussed in Section 3. Section 3.1 is dedicated to exponential stability theorems for normalized semigroups. In Section 3.2, we present some consequences of these results in the context of time homogenous models, including existence of ground states and quasi-invariant measures. Section 3.3 presents different tools to design Lyapunov functions for continuous time Markov semigroups and sub-Markov semigroups. We also illustrate these results through different examples of semigroups arising in physics and applied probability, including over damped Langevin diffusions, Langevin and hypo-elliptic diffusions, as well as typical examples of solvable one-dimensional sub-Markov semigroups such as the harmonic oscillator, the half-harmonic oscillator and the Dirichlet heat kernel. General comparison and conjugacy principles to construct Lyapunov functions for positive semigroups are provided in Section 4. Boundary problems are discussed in some detail in Section 5. We then turn in Section 6 to the design of Lyapunov functions for Riccati type processes, including positive definite matrix valued diffusions, logistic and multivariate birth and death processes arising respectively in Ensemble Kalman-Bucy filter theory and population dynamic analysis.

In Section 7, we illustrate the power of the Lyapunov approach in the context of multivariate conditional diffusions. Section 8 is dedicated to illustrations with explicit computations of geometrical objects for the Lyapunov functions discussed in Section 5.3 in the context of hypersurface Dirichlet boundaries.

1.1. Some basic notation

Let $\mathcal{B}(E)$ be the algebra of locally bounded measurable functions on a locally compact Polish space E. We denote by $\mathcal{B}_b(E) \subset \mathcal{B}(E)$ the sub-algebra of bounded measurable functions endowed with the supremum norm $||.||$.

For a given uniformly positive function $V \in \mathcal{B}(E)$, we let $\mathcal{B}_V(E) \subset \mathcal{B}(E)$ be the subspace of functions $f \in \mathcal{B}(E)$ equipped with the norm $||f||_V := ||f/V||$.

We also let $\mathcal{B}_\infty(E) \subset \mathcal{B}(E)$ be the subalgebra of locally bounded and uniformly positive functions V that grow at infinity; that is, $\sup_K V < \infty$ for any compact set $K \subset E$, and for any $r \geqslant V_\star := \inf_E V > 0$ the r-sub-level set $\mathcal{V}(r) := \{V \leqslant r\} \subset E$ is a non-empty compact subset. We denote by $\mathcal{B}_0(E) := \{1/V \ : \ V \in \mathcal{B}_\infty\} \subset \mathcal{B}_b(E)$ the sub-algebra of (bounded) positive functions, locally lower bounded and that vanish at infinity. For a given $V \in \mathcal{B}_\infty(E)$, consider the subspace

$$\mathcal{B}_{0,V}(E) := \big\{f \ \in \mathcal{B}(E) \ : \ |f|/V \in \mathcal{B}_0(E)\big\}.$$

We denote by $\mathcal{C}(E) \subset \mathcal{B}(E)$ the sub-algebra of continuous functions and by $\mathcal{C}_b(E) \subset \mathcal{C}(E)$ the sub-algebra of bounded continuous functions.

We also set $\mathcal{C}_V(E) := \mathcal{B}_V(E) \cap \mathcal{C}(E)$, $\mathcal{C}_0(E) := \mathcal{B}_0(E) \cap \mathcal{C}(E)$ and $\mathcal{C}_\infty(E) := \mathcal{B}_\infty(E) \cap \mathcal{C}(E)$ and $\mathcal{C}_{0,V}(E) := \mathcal{B}_{0,V}(E) \cap \mathcal{C}(E)$. Note that none of the sub-algebras $\mathcal{B}_0(E)$ and $\mathcal{B}_\infty(E)$ have a unit unless E is compact, the null function $0 \notin \mathcal{B}_0(E)$ but the unit function $1 \in \mathcal{C}_{0,V}(E)$ as soon as $V \in \mathcal{B}_\infty(E)$.

Let $\mathcal{M}_b(E)$ be the set of bounded signed measures μ on E equipped with the total variation norm $||\mu||_{tv} := |\mu|(E)/2$, where $|\mu| := \mu_+ + \mu_-$ stands for the total variation measure associated with a Hahn-Jordan decomposition $\mu = \mu_+ - \mu_-$ of the measure. Also let $\mathcal{P}(E) \subset \mathcal{M}_b(E)$ be the subset of probability measures on E. Recall that for any $\mu_1, \mu_2 \in \mathcal{P}(E)$ and $\epsilon \in]0,1]$ we have

$$||\mu_1 - \mu_2||_{tv} \leqslant 1 - \epsilon \Longleftrightarrow (\exists \nu \in \mathcal{P}(E) \ : \ \mu_1 \geqslant \epsilon\ \nu \quad \text{and} \quad \mu_2 \geqslant \epsilon\ \nu). \tag{1}$$

With a slight abuse of notation, we denote by 0 and 1 the null and unit scalars as well as the null and unit function on E.

Let $Q_{s,t}$, be a semigroup of positive integral operators on $\mathcal{B}_b(E)$ indexed by continuous time indices $s,t \in \mathcal{T} = \mathbb{R}_+ := [0,\infty[$ or by a discrete time index set $\mathcal{T} = \mathbb{N}$, with $s \leqslant t$. The action of $Q_{s,t}$ on $\mathcal{B}_b(E)$ is given for any $f \in \mathcal{B}_b(E)$ by the formulae

$$Q_{s,t}(f)(x) := \int Q_{s,t}(x,dy)\ f(y). \tag{2}$$

The left action of $Q_{s,t}$ on $\mathcal{M}_b(E)$ is given for any $\eta \in \mathcal{M}_b(E)$ by the formulae

$$(\eta\ Q_{s,t})(dy) := \int \eta(dx)\ Q_{s,t}(x,dy). \tag{3}$$

In this notation, the semigroup property takes the following form

$$Q_{s,u}Q_{u,t} = Q_{s,t} \quad \text{with} \quad Q_{s,s} = I, \quad \text{the identity operator.} \tag{4}$$

In the above display, $Q_{s,u}Q_{u,t}$ is a shorthand notation for the composition $Q_{s,u} \circ Q_{u,t}$ of the left or right-action operators. Unless otherwise stated, all the semigroups discussed in this article are indexed by conformal indices $s \leqslant t$ in the set $\mathcal{T}$. To avoid repetition, we often write $Q_{s,t}$ without specifying the order $s \leqslant t$ of the indices $s,t \in \mathcal{T}$.

We denote by $\mathcal{M}_V(E)$ be the space of measures $\mu \in \mathcal{M}_b(E)$ equipped with the operator V-norm $|||\mu|||_V := |\mu|(V)$, and by $\mathcal{P}_V(E) \subset \mathcal{M}_V(E)$ be the convex set of probability measures. Whenever $V \geqslant 1$, for any $\rho > 0$ we have the norm equivalence formulae

$$\rho|||\mu|||_V \leqslant |||\mu|||_{1+\rho V} \leqslant (1+\rho)|||\mu|||_V \tag{5}$$

We associate with a function $h \in \mathcal{B}_{0,V}(E)$ the Boltzmann-Gibbs transformation

$$\Psi_h \ : \ \mu \in \mathcal{P}_V(E) \mapsto \Psi_h(\mu) \in \mathcal{P}_{V^h}(E) \tag{6}$$

with the probability measure

$$\Psi_h(\mu)(dx) := \frac{h(x)}{\mu(h)}\ \mu(dx) \quad \text{and} \quad V^h := V/h \in \mathcal{B}_\infty(E).$$

We also denote by $|||Q|||_V$ the operator norm of a bounded linear operator $Q : f \in \mathcal{B}_V(E) \mapsto Q(f) \in \mathcal{B}_V(E)$; that is

$$|||Q|||_V := \sup\{||Q(f)||_V \ : \ f \in \mathcal{B}_V(E) \quad \text{such that} \quad ||f||_V \leqslant 1\}. \tag{7}$$

In terms of the V-conjugate semigroup

$$f \in \mathcal{B}_b(E) \mapsto Q^V(f) := Q(Vf)/V \in \mathcal{B}_b(E)$$

we have

$$|||Q|||_V = ||Q^V(1)|| = |||Q^V||| := \sup\{||Q^V(f)|| \ : \ f \in \mathcal{B}_b(E) \quad \text{such that} \quad ||f|| \leqslant 1\}.$$

For a given measurable function f and a given measurable subset, we use the shorthand notation

$$-\infty \leqslant \inf_A f := \inf_{x\in A} f(x) \leqslant \sup_A f := \sup_{x\in A} f(x) \leqslant +\infty.$$

For a given $s \in \mathcal{T}$ and $\tau \in \mathcal{T}$ with $\tau > 0$, we consider the time mesh

$$[s,\infty[_\tau := \{s + n\tau \in [s,\infty[\ : \ n \in \mathbb{N}\}.$$

Throughout, unless otherwise is stated we write c for some positive constants whose values may vary from line to line, and we write c_α, as well as $c(\beta)$ and $c_\alpha(\beta)$ when their values may depend on some parameters α, β defined on some parameter sets. We also set $a\wedge b = \min(a,b)$, $a\vee b = \max(a,b)$, and $a_+ = a\vee 0$ for $a, b \in \mathbb{R}$.

1.2. V-positive semigroups

We say that $Q_{s,t}$ is a V-positive semigroup on $\mathcal{B}_V(E)$ for some Lyapunov function $V \in \mathcal{B}_\infty(E)$ as soon as there exists some $\tau > 0$ and some function $\Theta_\tau \in \mathcal{B}_0(E)$ such that for any $0 < f \in \mathcal{B}_V(E)$ and $s < t$ we have $0 < Q_{s,t}(f) \in \mathcal{B}_{0,V}(E)$ as well as

$$Q_{s,s+\tau}(V)/V \leqslant \Theta_\tau \quad \text{and} \quad \sup_{|t-s|\leqslant\tau} \left(|||Q_{s,t}|||\vee|||Q_{s,t}|||_V\right) < \infty. \tag{8}$$

As shown in Section 4.1, the l.h.s. criterion in (8) can be seen as a uniform Foster-Lyapunov condition (a.k.a. drift condition).

The irreducibility condition $f > 0 \Rightarrow Q_{s,t}(f) > 0$ is satisfied if and only if we have $Q_{s,t}(1) > 0$. We check this claim by contradiction. Assume that $Q_{s,t}(1) > 0$ and consider a function $f > 0$ and some $x \in E$ such that $Q_{s,t}(f)(x) = 0$. In this case, for any $\epsilon > 0$ we would have

$$\epsilon\ Q_{s,t}(1_{f\geqslant\epsilon})(x) \leqslant Q_{s,t}(f)(x) = 0$$

by Fatou's lemma we would find the contradiction

$$\liminf_{\epsilon \to 0} Q_{s,t}(1_{f \geqslant \epsilon})(x) = 0 \geqslant Q_{s,t}(1)(x) \Rightarrow Q_{s,t}(1)(x) = 0.$$

Without further mention, all semigroups $Q_{s,t}$ considered in this article are assumed to be semigroups of positive integral operators $Q_{s,t}$ on $\mathcal{B}_b(E)$ satisfying the irreducibility condition $Q_{s,t}(1) > 0$ for any $s \leqslant t$. Notice that the condition

$$0 < f \in \mathcal{B}_V(E) \quad \Rightarrow \quad \forall s < t \qquad 0 < Q_{s,t}(f) \in \mathcal{B}_{0,V}(E)$$

is met as soon as $Q_{s,t}$ is a strong V-Feller semigroup (i.e. for any $s < t$ we have $Q_{s,t}(\mathcal{B}_V(E)) \subset \mathcal{C}_V(E)$ and when we have $Q_{s,t}(V)/V \in \mathcal{B}_0(E)$). To check this claim, observe that for any positive function $f \in \mathcal{B}_V(E)$ and $s < t$ the function $Q_{s,t}(f)$ is positive and continuous; and thus locally lower bounded. In this situation, whenever $||f||_V \leqslant 1$, for any $s < t$ we have the comparison property

$$Q_{s,t}(f)/V \leqslant Q_{s,t}(V)/V \in \mathcal{B}_0(E) \Rightarrow Q_{s,t}(f)/V \in \mathcal{B}_0(E) \Longleftrightarrow Q_{s,t}(f) \in \mathcal{C}_{0,V}(E).$$

In summary, a strong V-Feller semigroup $Q_{s,t}$ is V-positive on $\mathcal{B}_V(E)$ as soon as there exists some $\tau > 0$ and some function $\Theta_\tau \in \mathcal{B}_0(E)$ such that the l.h.s. condition in (8) is met and for any $s < t$ we have

$$Q_{s,t}(V)/V \in \mathcal{B}_0(E) \quad \text{and} \quad Q_{s,s+\tau}(V)/V \leqslant \Theta_\tau \in \mathcal{B}_0(E).$$

When $V \in \mathcal{C}_\infty(E)$, we say that $Q_{s,t}$ is a V-positive semigroup on $\mathcal{C}_V(E)$ as soon as $Q_{s,t}(\mathcal{C}_V(E)) \subset \mathcal{C}_{0,V}(E)$ for any $s < t$ and condition (8) is met.

A V-Feller semigroup $Q_{s,t}$ for some $V \in \mathcal{C}_\infty(E)$, in the sense that for any $s < t$ we have $Q_{s,t}(\mathcal{C}_V(E)) \subset \mathcal{C}_V(E)$, is also said to be V-positive on $\mathcal{C}_V(E)$ as soon as there exists some $\tau > 0$ and some function $\Theta_\tau \in \mathcal{B}_0(E)$ such that the l.h.s. condition in (8) is met and for any $s < t$ we have

$$Q_{s,t}(V)/V \in \mathcal{C}_0(E) \quad \text{and} \quad Q_{s,s+\tau}(V)/V \leqslant \Theta_\tau \in \mathcal{B}_0(E).$$

Last but not least, observe that positive semigroups $\mathcal{Q}_{s,t}$ with continuous time indices $s \leqslant t \in \mathbb{R}_+$ can be turned into discrete time models by setting $Q_{p,n} = \mathcal{Q}_{p\tau,n\tau}$ for any $p \leqslant n \in \mathbb{N}$ and some parameter $\tau > 0$. Up to a time rescaling, the parameter $\tau > 0$ arising in the definition of a discrete time V-positive semigroups $Q_{p,n}$ can be chosen as the unit time parameter. In this context, the r.h.s. condition in (8) is automatically satisfied.

2. A brief review on Markov semigroups

The stability analysis of positive semigroups presented in this article is mainly based on discrete time operator-type contraction techniques combining Lyapunov inequalities with local minorization conditions. This section presents a brief overview of this operator-theoretic framework. Our presentation is nearly self-contained and follows that of Section 8 in the book [30] (see also Section 2 in [23]).

2.1. V-norm contraction coefficients

In this section, we are mainly interested in the contraction properties of discrete time Markov integral operators. We only consider time homogeneous Markov semigroups

$P := P_{t,t+1}$ on $\mathcal{B}_b(E)$, so that $P_t := P_{0,t} = PP_{1,t}$. In what follows E is assumed to be a Polish space. One key mathematical object is the V-norm contraction coefficient.

We further assume that there exists a Lyapunov function $V \in \mathcal{B}_\infty(E)$ and parameters $\epsilon \in]0,1[$ and $c < \infty$ such that

$$P(V) \leqslant \epsilon \ V + c. \tag{9}$$

Note that the class of Markov semigroups considered in this section is more general than the one discussed in (8). Indeed, in our context the Lyapunov condition stated in the l.h.s. of (8) takes the form

$$P(V)/V \leqslant \Theta \in \mathcal{B}_0(E).$$

This condition ensures that *for any* $0 < \epsilon < ||\Theta||$, the set $K_\epsilon := \{\Theta \geqslant \epsilon\}$ is a non-empty compact subset and we have

$$P(V) \leqslant \epsilon \ 1_{E-K_\epsilon} V + 1_{K_\epsilon}(\Theta \ V) \leqslant \epsilon \ V + c_\epsilon \quad \text{with} \quad c_\epsilon := ||\Theta|| \sup_{K_\epsilon} V.$$

Replacing V by $2^{-1}(1 + \epsilon V/c)$ there is no loss of generality to assume that $c = 1/2$ and $V \geqslant 1/2$. Also assume there exists some $r_0 \geqslant 1$ and some function $\alpha : r \in [r_0, \infty[\ \mapsto \alpha(r) \in]0,1]$, such that for any $r \geqslant r_0$ we have

$$\sup_{(x,y)\in\mathcal{V}(r)^2} ||\delta_x P - \delta_y P||_{tv} \leqslant 1 - \alpha(r) \quad \text{with} \quad \mathcal{V}(r) := \{V \leqslant r\}. \tag{10}$$

The V-Dobrushin coefficient $\beta_V(P)$ of P is defined by the V-norm operator

$$\beta_V(P) := \sup_{\mu,\eta\in\mathcal{P}_V(E)} |||(\mu - \eta)P|||_V / |||\mu - \eta|||_V. \tag{11}$$

As show in Section 8 in [30] (see also Section 2.3 in [23]), the supremum in (11) is attained on Dirac masses $(\mu, \eta) = (\delta_x, \delta_y)$; that is, we have

$$\beta_V(P) = \sup_{(x,y)\in E^2} \frac{||\delta_x P - \delta_y P||_V}{V(x) + V(y)}.$$

The terminology V-Dobrushin coefficient comes from the fact that we recover the standard Dobrushin coefficient $\beta(P) := \beta_{1/2}(P)$ by choosing the constant function $V = 1/2$. Theorem 8.2.21 in [30] (see also Lemma 2.3 in [23]) shows that the Lyapunov inequality (9) combined with the local minorization condition (10) yield a V-norm contraction estimate for some well-chosen Lyapunov function.

Lemma 2.1 ([30]). *Assume (9) and (10). In this situation, for any* $r \geqslant r_0 \vee r_\epsilon$ *with* $r_\epsilon := 1/(1-\epsilon)$ *we have*

$$\beta_{V_{\epsilon,r}}(P) \leqslant 1 - \alpha_\epsilon(r) \tag{12}$$

with the rescaled Lyapunov function

$$V_{\epsilon,r} := \frac{1}{2}\left(1 + \frac{1}{1+\epsilon} \ \frac{\alpha(r)}{r} \ V\right) \quad \text{and} \quad \alpha_\epsilon(r) := \frac{\alpha(r)}{2} \ \frac{(1-\epsilon)}{(1+\epsilon) + \frac{\alpha(r)}{2}}\left(1 - \frac{r_\epsilon}{r}\right) > 0.$$

For the convenience of the reader a proof of the V-norm estimate (12) is provided in the Appendix, on page 75. As an aside, note that whenever (9) is met with $c = 1/2$ we have

$$P(V_{\epsilon,r}) \leqslant \epsilon \ V_{\epsilon,r} + c_{\epsilon,r} \quad \text{with} \quad c_{\epsilon,r} := \frac{1}{2}\left((1-\epsilon) + \frac{1}{1+\epsilon}\ \frac{\alpha(r)}{2r}\right)$$

The equivalence of the V-norm and the $V_{\epsilon,r}$-norm yields without further work the following contraction theorem.

Theorem 2.2. *For any $t \in \mathbb{N}$ and any $\mu, \eta \in \mathcal{P}_V(E)$ we have*

$$|||(\mu-\eta)P_t|||_V \leqslant c_{\epsilon,r}\ \beta_{V_{\epsilon,r}}(P)^t\ |||\mu-\eta|||_V \quad \text{with} \quad c_{\epsilon,r} := 1 + 2r(1+\epsilon)/\alpha(r). \tag{13}$$

The contraction estimate (13) ensures the existence of a single invariant probability measure $\mu_\infty = \mu_\infty P_t \in \mathcal{P}_V(E)$. Similar approaches are presented in the article [5], simplifying the Foster-Lyapunov methodologies and the small-sets return times estimation techniques developed in [7]. While the geometric convergence of discrete time Markov semigroups toward their invariant measure μ_∞ (a.k.a. Harris-type theorem) are well known, the V-norm contraction coefficients techniques developed in [30] provides a very direct and short proof of Theorem 2.2. In Section 2.3, this natural operator-theoretic framework is extended without difficulties to time varying and continuous time indices. For instance, for any collection of Lyapunov functions $V_t \in \mathcal{B}_\infty(E)$ and any Markov transitions M_t indexed by $t \in \mathbb{N}$ from $\mathcal{B}_{V_t}(E)$ into $\mathcal{B}_{V_{t-1}}(E)$ we have

$$P_t := P_{t-1}M_t = M_1 \ldots M_t \Rightarrow |||(\mu-\eta)P_t|||_{V_t} \leqslant \left(\prod_{1 \leqslant s \leqslant t} \beta_{V_{s-1},V_s}(M_s)\right)\ |||\mu-\eta|||_{V_0}$$

with the V-norm contraction coefficients

$$\beta_{V_{t-1},V_t}(M_t) := \sup_{\mu,\eta \in \mathcal{P}_V(E)} |||(\mu-\eta)M_t|||_{V_t} / |||\mu-\eta|||_{V_{t-1}} = \sup_{(x,y)\in E^2} \frac{||\delta_x M_t - \delta_y M_t||_{V_t}}{V_{t-1}(x) + V_{t-1}(y)}.$$

Whenever all the Markov transitions M_t satisfy (9) and (10) with the same Lyapunov function V and the same function $\alpha(r)$, choosing the function $V_t := V_{\epsilon,r}$ defined in Lemma 2.1 we have $\beta_{V_{t-1},V_t}(M_t) \leqslant (1-\alpha_\epsilon(r))$, with the parameter $\alpha_\epsilon(r)$ defined in Lemma 2.1. More general time-inhomogeneous models can probably be handle extending the analysis developed in [24, 26] as well as in Theorem 4.18 and Theorem 4.20 in [31] and in Proposition 1 in [32] using the standard Dobrushin coefficient to non-necessarily compact spaces in terms of V-norms contraction coefficients.

Remark 2.3. *Whenever $P(V)/V \leqslant \Theta$ for some $\Theta \in \mathcal{B}_0(E)$ a direct application of (13) yields for any $t \geqslant 1$ the estimate*

$$|||(\mu-\eta)P_{t+1}|||_V \leqslant c_{\epsilon,r}\ \beta_{V_{\epsilon,r}}(P)^t\ |||\mu-\eta|||_{V_\Theta} \quad \text{with} \quad V_\Theta := V\ \Theta$$

In some situations (cf. for instance Section 6), we can choose $\Theta = 1/V$. In this context, for any $\mu, \eta \in \mathcal{P}(E)$ we have the uniform total variation norm estimates

$$2\ |||(\mu-\eta)P_{t+1}|||_{tv} \leqslant |||(\mu-\eta)P_{t+1}|||_V \leqslant 2c_{\epsilon,r}\ \beta_{V_{\epsilon,r}}(P)^t\ |||\mu-\eta|||_{tv} \leqslant 2c_{\epsilon,r}\ \beta_{V_{\epsilon,r}}(P)^t$$

In this situation, besides the state space E may not be compact, for sufficiently large $t>0$ the standard Dobrushin contraction coefficient $\beta(P_t)<1$ of the Markov transition P_t yields exponential decays.

The extension to more general positive semigroups is slightly more involved and relies on the stability properties of triangular arrays of Markov operators (cf. [23]). In Section 3, we present an overview on the stability properties of this class of models.

Combining (1) with (10), for any $r \geqslant r_0$ there exists some probability measure ν_r such that for any $\mu=(\mu_+-\mu_-)\in \mathcal{M}_V(E)$ with $\mu(1)=0$ and any bounded positive function $f\geqslant 0$ we have

$$\mu_\pm(P(f)) \geqslant \mu_\pm(1_{\mathcal{V}(r)})\ \alpha(r)\ \nu_r(f).$$

This implies that

$$\frac{||\mu P||_{tv}}{||\mu||_{tv}}=\left\|\left(\frac{\mu_+}{\mu_+(1)}-\frac{\mu_-}{\mu_-(1)}\right)P\right\|_{tv}\leqslant 1-\alpha(r)\left(\frac{\mu_+(1_{\mathcal{V}(r)})}{\mu_+(1)}\wedge\frac{\mu_-(1_{\mathcal{V}(r)})}{\mu_-(1)}\right).$$

On the other hand, by Markov inequality we have

$$1-\frac{1}{r}\left(\frac{\mu_+(V)}{\mu_+(1)}\vee\frac{\mu_-(V)}{\mu_-(1)}\right)\leqslant\frac{\mu_+(1_{\mathcal{V}(r)})}{\mu_+(1)}\wedge\frac{\mu_-(1_{\mathcal{V}(r)})}{\mu_-(1)}.$$

This implies that

$$\frac{||\mu P||_{tv}}{||\mu||_{tv}}\leqslant 1-\alpha(r)\left(1-\frac{1}{r}\ \frac{|||\mu|||_V}{||\mu||_{tv}}\right).$$

We summarize the above discussion in the following lemma.

Lemma 2.4. *Assume that (10) is met. In this case for any $r\geqslant r_0$ and $\mu\in\mathcal{M}_V(E)$ with $\mu(1)=0$ we have*

$$||\mu P||_{tv}\leqslant(1-\alpha(r))\ ||\mu||_{tv}+\frac{\alpha(r)}{r}\ |||\mu|||_V. \tag{14}$$

Whenever (9) is met with $c=1/2$, for any $\mu\in\mathcal{M}_V(E)$ with $\mu(1)=0$, recalling that $||\mu||_{tv}=|||\mu|||_{1/2}=\mu_+(1)=\mu_-(1)$ we have

$$|||\mu P|||_V\leqslant\epsilon\ |||\mu|||_V+||\mu||_{tv}.$$

This yields for any $\rho>0$ the estimate

$$|||\mu P|||_{1/2+\rho V}\leqslant((1-\alpha(r))+\rho)\ ||\mu||_{tv}+\left(\frac{\alpha(r)}{\rho r}+\epsilon\right)\ \rho\ |||\mu|||_V$$

from which we readily check the following lemma.

Lemma 2.5. *Assume (9) and (10) are met with $c=1/2$. In this situation, for any (r,ρ) such that $r_\epsilon/r<\rho/\alpha(r)<1$ we have*

$$\beta_{1/2+\rho V}(P)\leqslant 1-\alpha_\epsilon^\rho(r)$$

with the parameter r_ϵ defined in Lemma 2.1 and $\alpha_\epsilon^\rho(r)$ defined by

$$\alpha_\epsilon^\rho(r) := (\alpha(r) - \rho) \wedge \left((1-\epsilon)\left(1 - \frac{\alpha(r)}{\rho}\frac{r_\epsilon}{r}\right)\right) > 0.$$

For instance, for any $\delta > 0$ we have

$$r > (1+\delta)\ r_\epsilon \quad \text{and} \quad \rho := (1+\delta)\ \alpha(r)\ \frac{r_\epsilon}{r}$$
$$\Rightarrow \alpha_\epsilon^\rho(r) \geqslant \left(\alpha(r)\ \left(1 - (1+\delta)\ \frac{r_\epsilon}{r}\right)\right) \wedge \left((1-\epsilon)\ \frac{\delta}{1+\delta}\right) > 0.$$

We end this section with a weaker version of another popular condition ensuring the non-expansive property of Markov semigroups w.r.t. V-norms (see for instance [11, 16, 33,34] and references therein). Instead of (9), we further assume there exist an increasing function $\varphi : v \in [1, \infty[\mapsto \varphi(v) \in [0, \infty[$ some constant $c \geqslant 0$ and some function $V \in \mathcal{B}_\infty(E)$ such that $V \geqslant 1$ and

$$P(V) \leqslant V - \varphi(V) + c \quad \text{and} \quad \varphi(V)/V \in \mathcal{B}_0(E). \tag{15}$$

Note that when $\varphi(V) = (1-\epsilon)V$ the r.h.s. condition in the above display is not met (unless E is compact) but the l.h.s. inequality coincides with the Lyapunov condition (9). In contrast with the Lyapunov condition introduced in [11], the function φ is not required to be concave.

Whenever (10) is satisfied, Lemma 2.4 ensures that for any $r \geqslant r_1 := \varphi(r_0)$ and $\mu \in \mathcal{M}_{\varphi(V)}(E)$ with $\mu(1) = 0$ we have

$$||\mu P||_{tv} \leqslant (1 - \alpha_1(r))\ ||\mu||_{tv} + \frac{\alpha_1(r)}{r}\ |||\mu|||_{\varphi(V)} \quad \text{with} \quad \alpha_1(r) := \alpha\big(\varphi^{-1}(r)\big). \tag{16}$$

On the other hand, by (15) $\mu \in \mathcal{M}_V(E)$ with $\mu(1) = 0$ we have

$$|||\mu P|||_V \leqslant |||\mu|||_V - |||\mu|||_{\varphi(V)} + 2c||\mu||_{tv}.$$

This yields for any $\rho > 0$ the inequality

$$|||\mu P|||_{1+\rho V} \leqslant |||\mu|||_{1+\rho V} - \left(2(\alpha_1(r) - \rho c)\ ||\mu||_{tv} + \left(\rho - \frac{2\alpha_1(r)}{r}\right)\ |||\mu|||_{\varphi(V)}\right) \tag{17}$$

from which we readily check the following uniform estimates.

Lemma 2.6. *Assume (10) and (15) are met. In this situation, (16) is met for some parameters (r_1, α_1). In addition, for any $r \geqslant r_1$ and parameter $\rho > 0$ such that $\rho c \leqslant \alpha_1(r)$ and $\rho \geqslant 2\alpha_1(r)/r$ we have*

$$\sup_{t \geqslant 0} \beta_{1+\rho V}(P_t) \leqslant 1 \quad \textit{or equivalently} \quad \sup_{t \geqslant 0} |||\mu P_t|||_{1+\rho V} \leqslant |||\mu|||_{1+\rho V} \tag{18}$$

for any $\mu \in \mathcal{M}_V(E)$ with $\mu(1) = 0$.

2.2. Subgeometric convergence

Consider the discrete time and homogeneous Markov semigroups discussed in Section 2.1. Without further mention, we shall assume that conditions (10) and (15) are met for some increasing function φ and some Lyapunov function $1 \leqslant V \in \mathcal{B}_\infty(E)$.

Consider a concave increasing differentiable function $\varphi_1 : [1,\infty[\mapsto [1,\infty[$ such that $\varphi_1(V)/V \in \mathcal{B}_0(E)$. We further assume there exists some parameters $\chi > 0$ and $\kappa_2 > 0$ such that

$$\partial\varphi_1(V)\ \varphi(V) \geqslant \varphi_2(V) \quad \text{with} \quad \varphi_2(V) := \kappa_2\ V(\varphi_1(V)/V)^{1+\chi} \in \mathcal{B}_\infty(E) \tag{19}$$

Observe that

$$\kappa_2\ \varphi_2(V)/V \leqslant \partial\varphi_1(V)\ \varphi(V)/V \leqslant \partial\varphi_1(1)\ \varphi(V)/V \in \mathcal{B}_0(E)$$

Thus, whenever the function $\varphi_1(V)/V$ is locally lower bounded and upper semi-continuous the above estimate ensures that

$$\varphi(V)/V \in \mathcal{B}_0(E) \Rightarrow \varphi_1(V)/V, \varphi_2(V)/V \in \mathcal{B}_0(E)$$

The prototype of model we have in mind is the case

$$\varphi(v) := \kappa_0\ v^\delta \quad \text{and} \quad \varphi_1(v) := \kappa_1\ v^{1-\upsilon\delta} \tag{20}$$

for some parameters $\upsilon, \delta \in]0,1[$ and $\kappa_1 \geqslant 1$. In this context, we have

$$\partial\varphi_1(V)\ \varphi(V) = \kappa_0\kappa_1(1-\upsilon\delta)\ V^{\delta(1-\upsilon)} = \varphi_2(V) := \kappa_2\ V\ (\varphi_1(V)/V)^{1+\chi} \in \mathcal{B}_\infty(E)$$

with the parameters

$$\kappa_2 := \kappa_0\kappa_1^{\frac{1-\delta}{\upsilon\delta}}(1-\upsilon\delta) \quad \text{and} \quad \chi := \frac{1-\delta}{\upsilon\delta}.$$

Applying Jensen's inequality and using (15) we prove that

$$P(\varphi_1(V)) \leqslant \varphi_1(V) - \varphi_2(V) + c_1 \quad \text{with} \quad c_1 = c\partial\varphi_1(1). \tag{21}$$

We check this estimate using the fact that $\partial\varphi_1$ is decreasing. Thus, for any $0 \leqslant u \leqslant v$ we have

$$\varphi_1(v-u) \leqslant \varphi_1(v) - \partial\varphi_1(v)\ u.$$

In the same vein, for any $0 \leqslant v \leqslant v-u$ we have

$$\forall w \in [v, v-u] \quad \partial\varphi_1(v) \geqslant \partial\varphi_1(w) \quad \text{and therefore} \quad \varphi_1(v-u) \leqslant \varphi_1(v) - \partial\varphi_1(v)\ u.$$

The Lyapunov inequality (21) applied to (20) is closely related to Lemma 3.5 in [13] We further assume there exists some $r_2 \geqslant 1$ and some function $\alpha_2 : r \in [r_2,\infty[\ \mapsto \alpha_2(r) \in\ ]0,1]$, such that for any $r \geqslant r_2$ we have

$$\sup_{(x,y)\in\{\varphi_2(V)\leqslant r\}^2} ||\delta_x P - \delta_y P||_{tv} \leqslant 1 - \alpha_2(r). \tag{22}$$

The above condition is automatically met with $r_2 = \varphi_2(r_0)$ and $\alpha_2(r) := \alpha(\varphi_2^{-1}(r))$ as soon as (10) is satisfied and φ_2 is increasing. In this situation, arguing as above, Lemma 2.4 ensures that for any $r \geqslant r_2$ and $\mu \in \mathcal{M}_{\varphi_2(V)}(E)$ with $\mu(1) = 0$ we have

$$||\mu P||_{tv} \leqslant (1-\alpha_2(r))\ ||\mu||_{tv} + \frac{\alpha_2(r)}{r}\ |||\mu|||_{\varphi_2(V)}.$$

On the other hand, by (21) we have

$$|||\mu P|||_{\varphi_1(V)} \leqslant |||\mu|||_{\varphi_1(V)} - |||\mu|||_{\varphi_2(V)} + 2c_1||\mu||_{tv}.$$

Applying Jensen's inequality for any $\mu \in \mathcal{M}_V(E)$ with $\mu(1) = 0$ we find that

$$|||\mu|||_{\varphi_2(V)} = \kappa_2 \ \frac{|\mu|(V \ (\varphi_1(V)/V)^{1+\chi})}{|\mu|(V)} \ |\mu|(V) \geqslant \kappa_2 \ |||\mu|||^{1+\chi}_{\varphi_1(V)}/|||\mu|||^{\chi}_V \tag{23}$$

Following word-for-word the proof of (17) we readily check the following lemma.

Lemma 2.7. *For any $r \geqslant r_2 \vee r_1$ and $\rho > 0$ such that*

$$\rho(c_1 \wedge c) \leqslant \alpha_1(r) \wedge \alpha_2(r) \quad \text{and} \quad \delta_\rho(r) := \kappa_2 \left(\rho - 2 \ \frac{\alpha_2(r) \vee \alpha_1(r)}{r} \right) > 0$$

and for any $\mu \in \mathcal{M}_V(E)$ with $\mu(1) = 0$ we have

$$|||\mu P|||_{1+\rho\varphi_1(V)} \leqslant |||\mu|||_{1+\rho\varphi_1(V)} - \delta_\rho(r) \ |||\mu|||^{1+\chi}_{\varphi_1(V)}/|||\mu|||^{\chi}_V.$$

To take the final step, recall that $|||\mu|||_{1+\rho\varphi_1(V)} \leqslant (1+\rho)|||\mu|||_{\varphi_1(V)}$. Combining this estimate with the uniform estimate (18) for any $t \in \mathcal{T} = \mathbb{N}$ we check that

$$|||\mu P_{t+1}|||_{1+\rho\varphi_1(V)} - |||\mu P_t|||_{1+\rho\varphi_1(V)} \leqslant -\varsigma_\mu \Big(|||\mu P_t|||_{1+\rho\varphi_1(V)} \Big)$$

with the function $\varsigma_\mu(u) := \omega_\mu(\rho) \ u^{1+\chi}$ and the parameters

$$\omega_\mu(\rho) := \omega(\rho) \ |||\mu|||^{-\chi}_{1+\rho V} \quad \text{and} \quad \omega(\rho) := \rho^\chi \ \delta_\rho(r)/(1+\rho)^{1+\chi} \tag{24}$$

Lemma 2.8 ([35]). *Consider a decreasing sequence of positive numbers u_t such that for any $t \in \mathbb{N}$ we have*

$$u_{t+1} - u_t \leqslant -\varsigma(u_t)$$

for some continuous increasing function ς from $]0, u_0]$ into $]0, \infty[$. In this situation, for any $t \geqslant 1$ we have

$$u_t \leqslant I_\varsigma^{-1}(t) \quad \text{with} \quad I_\varsigma(u) := \int_u^{u_0} \frac{dv}{\varsigma(v)}.$$

Proof. Since ς is increasing we have $v \in [u_{n+1}, u_n] \Rightarrow \varsigma(v) \leqslant \varsigma(u_n)$. On the other hand, by the mean value theorem there exists some $v \in [u_{n+1}, u_n]$ such that

$$I_\varsigma(u_{n+1}) - I_\varsigma(u_n) = -\frac{u_{n+1} - u_n}{\varsigma(v)} \geqslant \frac{\varsigma(u_n)}{\varsigma(v)} \geqslant 1.$$

We conclude that $I_\varsigma(u_n) \geqslant n + I_\varsigma(u_0)$. This ends the proof of the lemma. ∎

Observe that for any $t \geqslant 1$ we have

$$\begin{aligned} \varsigma(v) = \ v^{1+\chi} &\Rightarrow I_\varsigma(u) = \frac{1}{\chi} \ (u^{-\chi} - u_0^{-\chi}) \\ &\Rightarrow I_\varsigma^{-1}(\omega t) = \left(\frac{1}{u_0^{-\chi} + t\omega \ \chi} \right)^{1/\chi} \leqslant (\chi\omega)^{-1/\chi} \ t^{-1/\chi}. \end{aligned}$$

The above lemma readily yields the following polynomial convergence theorem.

Theorem 2.9. *Assume conditions (19) and (22) are satisfied for some function φ_1 and some parameter. In this situation, for any $t \geqslant 1$ and any $\mu \in \mathcal{M}_V(E)$ with $\mu(1) = 0$ we have the polynomial convergence estimates*

$$|||\mu P_t|||_{1+\rho\varphi_1(V)} \leqslant \; c_\rho^\chi \; t^{-1/\chi} \; |||\mu|||_{1+\rho V} \quad \text{with} \quad c_\rho^\chi := (\chi\omega(\rho))^{-1/\chi}$$

In the above display, ρ and $\omega(\rho)$ stands for the parameters defined in Lemma 2.7 and in (24).

Remark 2.10. *Without further work, we recover the sequence of polynomial rates of convergence discussed in* [13] *by choosing for any $1 < i < n$ the parameters*

$$\delta := (n-1)/n \qquad \upsilon := (i-1)/(n-1)$$

$$\Rightarrow 1 - \upsilon\delta = 1 - (i-1)/n \qquad \delta(1-\upsilon) = 1 - i/n \qquad 1/\chi = i - 1.$$

As expected, the operator-theoretic framework described above can be extended easily to situations where the function φ_2 in (19) has the following form

$$\varphi_2(V) := V \; \psi(\varphi_1(V)/V) \in \mathcal{B}_\infty(E) \tag{25}$$

for some convex increasing function $\psi \; : \; [0,\infty[\mapsto [0,\infty[$ such that $\psi(0) = 0$. In this situation, arguing as in (23) for any $\mu \in \mathcal{M}_V(E)$ with $\mu(1) = 0$ we have

$$|||\mu|||_{\varphi_2(V)} \geqslant |||\mu|||_V \; \psi\Big(|||\mu|||_{\varphi_1(V)}/|||\mu|||_V\Big)$$

Using the fact that $\psi(\lambda\nu) \leqslant \lambda\psi(\nu)$, for any $\lambda \in [0,1]$ and $\nu \geqslant 0$, for any $t \geqslant 0$ we check that

$$\begin{aligned} |||\mu P_t|||_{\varphi_2(V)} &\geqslant \sup_{s\geqslant 0} |||\mu P_s|||_V \; \frac{|||\mu P_t|||_V}{\sup_{s\geqslant 0} |||\mu P_s|||_V} \; \psi\Big(|||\mu P_t|||_{\varphi_1(V)}/|||\mu P_t|||_V\Big) \\ &\geqslant \sup_{s\geqslant 0} |||\mu P_s|||_V \; \psi\Big(|||\mu P_t|||_{\varphi_1(V)}/\sup_{s\geqslant 0}|||\mu P_s|||_V\Big) \end{aligned}$$

On the other hand, by (18) we have $\rho \sup_{s\geqslant 0}|||\mu P_s|||_V \leqslant (1+\rho)|||\mu|||_V$. This yields the rather crude estimate

$$|||\mu P_t|||_{\varphi_2(V)} \geqslant |||\mu|||_V \; \psi\left(\frac{\rho}{1+\rho}\frac{|||\mu P_t|||_{\varphi_1(V)}}{|||\mu|||_V}\right).$$

Thus, recalling the norm equivalence formulae (5) we prove that

$$|||\mu P_t|||_{\varphi_2(V)} \geqslant |||\mu|||_{1+\rho V} \; \psi_\rho\Big(|||\mu P_t|||_{1+\rho\varphi_1(V)}/|||\mu|||_{1+\rho V}\Big)$$

with the rescaled convex function

$$\psi_\rho(\nu) := \frac{1}{1+\rho} \; \psi\left(\frac{\rho^2}{(1+\rho)^2} \; \nu\right). \tag{26}$$

This shows that

$$v_0 := |||\mu|||_{1+\rho V} \quad \text{and} \quad u_t := |||\mu P_t|||_{1+\rho\varphi_1(V)} \Rightarrow u_{t+1} - u_t \leqslant -\,\delta_\rho(r)\ \varsigma_{v_0}(u_t)$$

with the same $\delta_\rho(r)$ as in Lemma 2.7 (with $\kappa_2 = 1$) and the function $\varsigma_{v_0}(u) := v_0\ \psi_\rho(u/v_0)$. In this context, Lemma 2.8 readily yields the following convergence theorem.

Theorem 2.11. *Assume there exists some function φ_1 satisfying (19), with the function φ_2 defined in (25) in terms of some convex increasing function ψ. In this situation, for any $t \geqslant 0$ and any $\mu \in \mathcal{M}_V(E)$ with $\mu(1) = 0$ we have*

$$|||\mu P_t|||_{1+\rho\varphi_1(V)} \leqslant J^{-1}_{\psi_\rho}(t)\ \ |||\mu|||_{1+\rho V}$$

with the parameter ρ defined in Lemma 2.7, the function ψ_ρ defined in (26) and

$$J_{\psi_\rho}(u) := \int_u^\iota \frac{dv}{\psi_\rho(v)} \quad \text{with} \quad \iota := 1 \vee ||\varphi_1(V)/V||.$$

Proof. Observe that

$$I_{\varsigma_{v_0}}(u) := \int_u^{u_0} \frac{dv}{\varsigma_{v_0}(v)} = \int_{u/v_0}^{u_0/v_0} \frac{dv}{\psi_\rho(v)} =: I_{\psi_\rho}(u/v_0).$$

On the other hand, we have $u_0 \leqslant \iota\ v_0$ and $I_{\psi_\rho}(u) \leqslant J_{\psi_\rho}(u)$. Since I_{ψ_ρ} and J_{ψ_ρ} are decreasing their inverse are also decreasing and choosing $u = I^{-1}_{\psi_\rho}(t)$ in the above display we have

$$t = I_{\psi_\rho}(I^{-1}_{\psi_\rho}(t)) \leqslant J_{\psi_\rho}(I^{-1}_{\psi_\rho}(t))$$

$$\Rightarrow J^{-1}_{\psi_\rho}(t) \geqslant I^{-1}_{\psi_\rho}(t) \Rightarrow I^{-1}_{\varsigma_{v_0}}(t) = v_0\ I^{-1}_{\psi_\rho}(t) \leqslant v_0\ J^{-1}_{\psi_\rho}(t).$$

The end of the proof is now a direct consequence of Lemma 2.8. ∎

2.3. Some V-norm stability theorems

Let $P_{s,t}$ be a semigroup of Markov integral operators $P_{s,t}$ on $\mathcal{B}_b(E)$ indexed by continuous time indices $s, t \in \mathcal{T} = \mathbb{R}_+ := [0, \infty[$ or by discrete time indices $s, t \in \mathcal{T} = \mathbb{N}$. We further assume that

$$P_{s,s+\tau}(V) \leqslant \epsilon_\tau\ V + c_\tau \tag{27}$$

for some parameter $\epsilon_\tau \in]0, 1[$ and some finite constant $c_\tau < \infty$. The geometric drift condition (27) ensures that the sequence $|||P_{s,s+n\tau}|||_V$ indexed by $s \geqslant 0$ and $n \geqslant 1$ is uniformly bounded. In this context, (8) applied to $Q_{s,t} = P_{s,t}$ ensures that the operator norms of $P_{s,t}$ are uniformly bounded w.r.t. any time horizon. More precisely, whenever (27) is met we have the equivalence

$$\sup_{s \geqslant 0} \sup_{t \geqslant s} |||P_{s,t}|||_V < \infty \Longleftrightarrow \sup_{|t-s| \leqslant \tau} |||P_{s,t}|||_V < \infty. \tag{28}$$

Note that the condition (28) is automatically satisfied whenever (27) is met for any $\tau > 0$ with $\sup_{\tau\in[0,1]} c_\tau < \infty$. For instance, consider the Markov transition semigroup $P_{s,t}$ of a continuous time stochastic flow $X_{s,t}(x)$ on some locally compact normed vector space $(E, ||.||)$ with generator L_t defined on some common domain $\mathcal{D}(L) \subset \mathcal{B}(E)$. In this context, for any non-negative function $V \in \mathcal{D}(L)$ and any parameters $a > 0$, $c < \infty$ and $\tau > 0$ we have

$$\begin{aligned} &\forall t \in \mathcal{T} = \mathbb{R}_+ \qquad L_t(V) \leqslant -a\ V + c \\ &\Rightarrow (27) \quad \text{and } (28) \quad \text{with} \quad \epsilon_\tau = (1+a\tau)^{-1} < 1 \quad \text{and} \quad c_\tau = c\tau. \end{aligned} \tag{29}$$

The above estimate is rather well known, a detailed proof is provided in the appendix on page 74. Further examples of Markov diffusion semigroups on $\mathbb{R}^n$ satisfying (27) are discussed in Section 2.4. We further assume there exists some $r_0 \geqslant 1$ and some function $\alpha_\tau : r \in [r_0, \infty[\ \mapsto \alpha_\tau(r) \in\ ]0,1]$, such that for any $r \geqslant r_0$ we have

$$\sup_{(x,y)\in\mathcal{V}(r)^2} \left|\left|\delta_x P_{s,s+\tau} - \delta_y P_{s,s+\tau}\right|\right|_{tv} \leqslant 1 - \alpha_\tau(r), \tag{30}$$

with the compact level sets $\mathcal{V}(r)$ introduced in (10). By Theorem 2.2, conditions (27), (28) and (30) ensure the existence of some parameter $\bar\tau > 0$ such that

$$\sup_{|t-s|\leqslant\bar\tau} \beta_V(P_{s,t}) < \infty \quad \text{and} \quad \sup_{s\geqslant 0} \beta_V(P_{s,s+\bar\tau}) < 1. \tag{31}$$

In the above display, $\beta_V(P_{s,t})$ stands for the V-Dobrushin coefficient of the Markov operator $P_{s,t}$ introduced in (11). The next exponential contraction theorem is a direct consequence of the operator norm estimates (31) and it is valid on abstract measurable spaces as well as for any function $V \geqslant 1$.

Theorem 2.12. *Let $P_{s,t}$ be a semigroup of Markov integral operators $P_{s,t}$ on some measurable state space E satisfying condition (31) for some function $V \geqslant 1$ and some parameter $\bar\tau > 0$. In this situation, there exists a parameter $b > 0$ and some finite constant $c < \infty$ such that for any $s \leqslant t$ and $\mu, \eta \in \mathcal{P}_V(E)$ we have the exponential estimate*

$$|||(\mu-\eta)P_{s,t}|||_V \leqslant\ c\ e^{-b(t-s)}\ |||\mu-\eta|||_V. \tag{32}$$

In particular, the above exponential Lipschitz estimates are met as soon as conditions (27), (28) and (30) are satisfied. The estimates (32) also hold for any $s \geqslant 0$ and $t \in [s, \infty[_\tau$ as soon as (27) and (30) are satisfied for some $\tau > 0$ and $\epsilon_\tau \in]0,1[$.

Theorem 2.12 can be seen as an extension of Harris' theorem to time varying Markov semigroup. The proof of Theorem 2.12 is based on the *discrete time* V-norm operator contraction techniques presented in Section 2.1. The r.h.s. condition in (28) is a technical condition only made for continuous time semigroups to ensure that (32) also holds for continuous time indices. Note that the strength of conditions (27) and (30) depends on the strength of the function V: when the function V is bounded, the geometric drift condition (27) and the uniform norm condition (28) are trivially met but in this case condition (30) is a uniform contraction condition on the state E. In the reverse angle, when $V \in \mathcal{B}_\infty(E)$ is a function with compact sub-level sets, the geometric drift condition (27) combined with (28) ensures that $\mu P_{s,t}$ is a tight collection of

probability measures indexed by $s \leqslant t$. For time homogenous models $P_{s,s+t} = P_t$, following Remark 2.3, uniform total variation estimates can be derived for any $t \geqslant \tau$ as soon $||P_\tau(V)|| < \infty$. Some examples satisfying this condition are discussed in Section 6.

Using (1) we readily check that the local contraction condition (30) is met if and only if for any $s \geqslant 0$ and any $(x,y) \in \mathcal{V}(r)^2$ there exists some probability measure μ on E (that may depends on the parameters (τ, r, s, x, y)) such that

$$\forall z \in \{x,y\} \qquad \delta_z P_{s,s+\tau}(dy) \geqslant \alpha_\tau(r)\ \mu(dy).$$

For instance, the above condition is met as soon as

$$P_{s,s+\tau}(x,dy) \geqslant p_{s,s+\tau}(x,y)\ \nu_\tau(dy) \tag{33}$$

for some Radon positive measure ν_τ on E and some density function $p_{s,s+\tau}$, satisfying for any $r \geqslant r_0$ the local minorization condition

$$0 < \inf_{s\in\mathcal{T}} \inf_{\mathcal{V}(r)^2} p_{s,s+\tau} \quad \text{and} \quad 0 < \nu_\tau(\mathcal{V}(r)) < \infty. \tag{34}$$

For locally compact Polish spaces condition $0 < \nu_\tau(\mathcal{V}(r)) < \infty$ is met as soon as V has compact sub-levels sets $\mathcal{V}(r)$ with non-empty interior and ν_τ is a Radon measure of full support; that is ν_τ is finite on compact sets and strictly positive on non-empty open sets. For time homogeneous models, also note that the l.h.s. minorization condition (34) is satisfied as soon as $(x,y) \in (\overset{\circ}{E})^2 \mapsto p_\tau(x,y)$ is a continuous positive function on the interior $\overset{\circ}{E}$ of the set E.

Several illustrations of Theorem 2.12 are discussed in Section 2.4 in the context of diffusion processes on Euclidean spaces as well as in Section 6 in the context of Riccati-type diffusion on positive definite matrix spaces and multivariate birth and death jump type processes on countable state spaces. The stability of Markov semigroups on manifolds with entrance boundaries can also be analyzed using the Lyapunov techniques developed in Section 5. For instance, as shown in Section 5.1, any absolutely continuous Markov semigroup $P_{s,t}$ on a bounded connected subset $E \subset \mathbb{R}^n$ with locally Lipschitz boundary ∂E satisfies the conditions of Theorem 2.12 with the (non unique) Lyapunov function $V(x) = 1/\sqrt{d(x,\partial E)}$ and the distance to the boundary defined for any $x \in E$ by

$$d(x,\partial E) := \inf\big\{||x-y|| \ :\ y \in \partial E\big\}.$$

We illustrate the above discussion with some elementary one dimensional examples.

Example 2.13. *Consider a one dimensional Brownian motion $X_t(x)$ starting at $X_0(x) = x \in E := [0,1]$ and reflected at the boundaries $\partial E = \{0,1\}$. We recall that the Markov transition of the process $t \in \mathcal{T} := \mathbb{R}_+ \mapsto X_t(x) \in E$ is symmetric and absolutely continuous; that is we have*

$$P_t(x,dy) := \mathbb{P}(X_t(x) \in dy) = p_t(x,y)\ \nu(dy) \qquad \text{with} \quad \nu(dy) := 1_{[0,1]}(y)dy$$

and the density $p_t(x,y)$ is given by the spectral decomposition

$$p_t(x,y) = 1 + 2\sum_{n\geqslant 1} e^{-(n\pi)^2 t/2}\ \cos(n\pi x)\ \cos(n\pi y)$$

In this situation, $P_t = P_{0,t}$ coincides with the Neumann heat semigroup on $[0,1]$. Since the Neumann heat kernel $p_t(x,y)$ is smooth as well as bounded and strictly positive on the compact interval $[0,1]$, the conditions of Theorem 2.12 are satisfied with the unit Lyapunov function $V(x) = 1$, as well as for any of the Lyapunov functions $V(x) = 1/\sqrt{x}$, $V(x) = 1/\sqrt{1-x}$ or $V(x) = 1/\sqrt{x} + 1/\sqrt{1-x}$. Indeed, note that the minorization condition (34) holds for any of the Lyapunov functions V discussed above. Since $\nu(V) \leqslant \infty$, we have $||P_\tau(V)|| < \infty$ for any $\tau > 0$, so that the Lyapunov condition (27) is also met.

The same reasoning applies to the one dimensional positive Riccati-type diffusions with an entrance boundary at the origin discussed in Section 6. Reflecting this class of positive diffusions at $x = 1$, the conditions of Theorem 2.12 are satisfied on $E =]0,1]$ with the Lyapunov functions $V(x) = 1/\sqrt{x}$ as well as for $V(x) = 1/\sqrt{x} + 1/\sqrt{1-x}$.

In the same vein, assume there exists some increasing differentiable concave function $\varphi \;:\; v \in [1,\infty[\mapsto \varphi(v) \in [0,\infty[$ with bounded differential $||\partial\varphi|| < \infty$ and some function $V \in \mathcal{B}_\infty(E)$ such that $V \geqslant 1$ and $V, \varphi(V) \in \mathcal{D}(L)$. In addition, there exists some finite constant $c > \varphi(1)$ such that for any $t \in \mathcal{T} = \mathbb{R}_+$ we have

$$L_t(V) \leqslant -\varphi(V) + c \quad \text{and} \quad \varphi(V)/V \in \mathcal{B}_0(E) \tag{35}$$

We set

$$V_\tau := V + \tau\ \varphi(V) \quad \text{and} \quad \varphi_\tau(v) = \varphi(v/\beta_\tau) \quad \text{with} \quad \beta_\tau := 1 + \tau||\varphi(V)/V||. \tag{36}$$

Observe that

$$\varphi_\tau(V_\tau) = \varphi(V(1+\tau\varphi(V)/V)/\beta_\tau) \leqslant \varphi(V) \quad \text{with} \quad \beta_\tau := 1 + \tau||\varphi(V)/V||$$

and

$$\varphi_\tau(V_\tau)/V_\tau \leqslant \varphi(V)/V \in \mathcal{B}_0(E)$$

This estimate ensures that $\varphi_\tau(V_\tau)/V_\tau \in \mathcal{B}_0(E)$ as soon as $\varphi_\tau(V_\tau)/V_\tau$ is locally lower bounded and upper semi-continuous.

Lemma 2.14. *Assume condition (35) is satisfied for some functions (φ, V) and some constant $c > \varphi(1)$. In this situation, for any $s \in \mathcal{T}$ and $\tau > 0$ we have*

$$\forall s \in \mathcal{T} \qquad P_{s,s+\tau}(V_\tau) \leqslant V_\tau - \tau\ \varphi_\tau(V_\tau) + c_\tau$$

with the function V_τ defined in (36) and $c_\tau := c\ \tau\ (1 + ||\partial\varphi||\ \tau/2)$.

The proof of the above lemma is provided in the Appendix, on page 74. Next theorem is the continuous time version of the polynomial convergence theorem, Theorem 2.9 presented in Section 2.2. The continuous time version of Theorem 2.11 can be obtained using the same lines of arguments, thus it is left to the reader.

Theorem 2.15. *Assume (35) is met for some function (φ, V) and the function φ_τ defined in (36) satisfies (19) for some parameters $\tau, \chi > 0$ and some functions $(\varphi_{\tau,1}, \varphi_{\tau,2})$ such that $\varphi_\tau(V_\tau)/V_\tau, \varphi_{\tau,1}(V_\tau)/V_\tau \in \mathcal{B}_0(E)$ and $\varphi_{\tau,2}(V_\tau) \in \mathcal{B}_\infty(E)$. We also assume that (30) is satisfied with the compact level sets of the function V_τ. In this situation, there exists some constant $c < \infty$ (that may depends on τ) such that for any $s \geqslant 0$ and any $t \in [s,\infty[_\tau$ and $\mu, \eta \in \mathcal{P}_V(E)$we have*

$$|||(\mu-\eta)P_{s,t}|||_{tv} \leqslant c\ (t-s)^{-1/\chi}\ |||\mu-\eta|||_V$$

The above polynomial convergence estimates also holds for any continuous time indices as soon as $\sup_{|s-t|\leqslant\tau}|||P_{s,t}|||_V<\infty$.

Proof. Applying Theorem 2.9, there exists some parameter $\rho_\tau>0$ and some finite constant c_τ such that for any $n\in\mathbb{N}$ we have the polynomial convergence estimates

$$|||\mu P_{s,s+n\tau}|||_{1+\rho_\tau\varphi_{\tau,1}(V_\tau)} \leqslant\ c_\tau\ n^{-1/\chi}\ |||\mu|||_{1+\rho_\tau V_\tau} \leqslant c_\tau\ (1+\rho_\tau)\ n^{-1/\chi}\ |||\mu|||_{V_\tau}$$

This implies that

$$|||\mu P_{s,s+n\tau}|||_{tv} \leqslant c\ n^{-1/\chi}\ |||\mu|||_V \qquad \text{with} \quad c=c_\tau\ \beta_\tau(1+\rho_\tau)$$

This ends the proof of the theorem. ■

2.4. Diffusion semigroups

This section is mainly concerned with the design of Lyapunov functions for continuous time Markov semigroups. To simplify notation, we only consider time homogeneous models. All the semigroups discussed in this section satisfy condition (28). Thus, by (33) the contraction theorem, Theorem 2.12, applies to all the Markov semigroups discussed in this section as soon as the transition semigroups have a continuous density with respect to the Lebesgue measure.

Section 2.4.1 presents some elementary principles based on spectral conditions on the drift function and a simple way to design Lyapunov functions in terms of the generator of diffusion process. These generator-type techniques are illustrated in Section 2.4.2 in the context of overdamped Langevin diffusions. The design of Lyapunov functions for hypo-elliptic diffusions and Langevin diffusions are discussed respectively in Sections 2.4.3 and 2.4.4.

2.4.1. Some general principles

Consider the Markov semigroup P_t of a diffusion flow $X_t(x)$ on $E=\mathbb{R}^n$ defined by

$$dX_t(x)=b(X_t(x))\ dt+\sigma(X_t(x))\ dB_t. \tag{37}$$

In the above display, B_t is a n_1-dimensional Brownian motion starting at the origin for some $n\geqslant 1$, b is a differentiable drift function from $\mathbb{R}^n$ into itself with gradient-matrix $\nabla b=(\partial_{x_i}b^j)_{1\leqslant i,j\leqslant n}$, and σ stands for some diffusion function from $\mathbb{R}^n$ into $\mathbb{R}^{n\times n_1}$. We set $\Sigma^2:=\sigma\sigma'$, where $\sigma'(x):=\sigma(x)'$ stands for the transposition of the matrix $\sigma(x)$, so that $\Sigma^2(x):=\sigma(x)\sigma'(x)$. The absolutely continuity of the transition semigroup $P_t(x,dy)=\mathbb{P}(X_t(x)\in dy)=p_t(x,y)\nu(dy)$ for some continuous transition densities $p_t(x,y)$ (w.r.t. the Lebesgue measure $\nu(dy)$) is ensured as soon as (b,σ) are globally Lipschitz continuous and the diffusion matrix is invertible or more generally satisfying a parabolic Hörmander condition (see for instance [36–38] and references therein). The generator L of the diffusion flow $X_t(x)$ and its carré du champ operator Γ_L are given respectively by the formula

$$L(f) := b'\nabla f + \frac{1}{2} \quad \mathrm{Tr}\big(\Sigma^2 \nabla^2 f\big) \quad \text{and} \quad \Gamma_L(f,g) := (\nabla f)' \Sigma^2 \nabla g. \tag{38}$$

The next proposition provides a rather elementary way to design a Lyapunov function.

Proposition 2.16. *Assume that $\sigma(x) = \sigma_0$ for some $\sigma_0 \in \mathbb{R}^{n \times n_1}$ and we have*

$$\nabla b + (\nabla b)' \leqslant -2\lambda \ I \quad \text{for some} \quad \lambda > 0. \tag{39}$$

Then for any $v > 0$ and $t > 0$ there exists some $\delta_t > 0$ such that

$$V(x) := \exp(v||x||) \Rightarrow P_t(V)/V \leqslant c_t / V^{\delta_t}. \tag{40}$$

The proof of the above proposition is rather technical, thus it is provided in the appendix on page 76.

The next proposition is a slight extension of Theorem 2.6 [39] on reversible semi-groups to stochastic flows in Euclidean spaces. It provides a rather simple way to design Lyapunov functions in terms of generators.

Proposition 2.17. *Assume there exists some function $W \geqslant 0$ as well as some parameters $\alpha > 0$, $\beta \in \mathbb{R}$ and $0 < \epsilon < 1$ such that*

$$\alpha \ W + \beta + L(W) \leqslant -\epsilon \ \Gamma_L(W, W). \tag{41}$$

In this situation, for any $t > 0$ we have

$$V := \exp(2\epsilon W) \Rightarrow P_t(V)/V \leqslant v_t / V^{\delta_t} \tag{42}$$

with the parameters

$$v_t = \exp\big(-2\beta\epsilon \ (1 - e^{-\alpha t})/\alpha\big) \quad \text{and} \quad \delta_t := (1 - e^{-\alpha t}).$$

The proof of the above proposition follows word-for-word the proof of Theorem 2.6 in [39], thus it is provided in the appendix on page 77.

We further assume that P_t satisfies for any $t > 0$ the sub-Gaussian estimate

$$P_t(x, dy) \leqslant c_t \ \exp\left(-\frac{1}{2\sigma_t^2} \ ||y - m_t(x)||^2\right) \ dy \tag{43}$$

for some parameters $\sigma_t > 0$ and some function m_t on $\mathbb{R}^n$ such that

$$||m_t(x)|| \leqslant c_t \ (1 + ||x||).$$

In this situation, for any $n \geqslant 1$ and $t \geqslant 0$ we have

$$V(x) := 1 + ||x||^n \Rightarrow ||P_t(V)/V|| < \infty.$$

More refined estimates can be found when the function m_t is such that

$$|m_t(x)| \leqslant \epsilon_t \ |x| \quad \text{with} \quad \epsilon_t \in]0, 1[\tag{44}$$

for some norm $|.|$ on $\mathbb{R}^n$. In this situation, observe that any $v \geqslant 0$ and any centered Gaussian random variable Y on $\mathbb{R}^n$ with identity covariance matrix I_n we have

$$e^{-v|x|} \ \mathbb{E}\big(e^{v|m_t(x) + \sigma_t Y|}\big) \leqslant c_t \ e^{-v(1-\epsilon_t)|x|}.$$

This yields the following lemma.

Lemma 2.18. *Consider a Markov semigroup P_t satisfying the sub-Gaussian estimate (43) as well as (44) for some norm $|.|$ on $\mathbb{R}^n$. Then for any $v \geqslant 0$ and $t > 0$ there also exists some finite constant $\delta_t > 0$ such that*

$$V(x) := \exp(v|x|) \Rightarrow P_t(V)/V \leqslant c_t/V^{\delta_t}.$$

2.4.2. Overdamped Langevin diffusion

Let $W(x)$ be some twice differentiable potential function from $\mathbb{R}^n$ into $\mathbb{R}$. The overdamped Langevin diffusion is defined by choosing in (37) the drift function

$$b(x) := -\gamma\ \nabla W(x) \quad \text{and} \quad (n_1, \sigma(x)) = (n, \rho\ I) \quad \text{for some} \quad \gamma, \rho > 0.$$

In this context, we have

$$(39) \Longleftrightarrow \nabla^2 W \geqslant (\lambda/\gamma)\ I \quad \text{for some} \quad \lambda > 0.$$

Also observe that

$$(41) \Longleftrightarrow \alpha\ W + \beta + \frac{\rho^2}{2} \quad \mathrm{Tr}(\nabla^2 W) \leqslant \left(\gamma - \epsilon\ \rho^2\right)\ ||\nabla W||^2.$$

The above condition is clearly met when W behaves as $||x||^m$ with $m \geqslant 1$ at infinity; that is, there exists some sufficiently large radius r such that for any $||x|| \geqslant r$ we have

$$\left|\ \mathrm{Tr}(\nabla^2 W(x))\right| \leqslant c_1\ ||x||^{(m-2)_+} \quad \text{and} \quad \|\nabla W(x)\|^2 \geqslant c_2\ ||x||^{2(m-1)}.$$

2.4.3. Hypo-elliptic diffusions

Consider the $\mathbb{R}^n$-valued diffusion (37) with $(b(x), \sigma(x)) = (Ax, \Sigma)$, for some matrices (A, Σ) with appropriate dimensions. We assume that A is stable (a.k.a. Hurwitz); that is its spectral abscissa $\varsigma(A)$ defined below is negative

$$\varsigma(A) := \sup\left\{\mathrm{Re}(\lambda(A))\ :\ \lambda(A) \in\ \mathrm{Spec}(A)\right\} < 0. \tag{45}$$

In the above display $\mathrm{Spec}(A)$ denotes the spectrum of the matrix A, and $\mathrm{Re}(\lambda(A))$ the real part of $\lambda(A)$. We also assume that $R := \Sigma\Sigma'$ is positive semi-definite and the pair of matrices $(A, R^{1/2})$ are controllable, in the sense that the $(n \times n^2)$-matrix

$$\left[R^{1/2}, AR^{1/2} \ldots, A^{r-1}R^{1/2}\right] \quad \text{has rank } n. \tag{46}$$

Whenever $\varsigma(A) < 0$ we have

$$P_t(x, dy) = \frac{1}{\sqrt{\det(2\pi C_t)}} \exp\left(-\frac{1}{2}\left(y - m_t(x)\right)' C_t^{-1}\left(y - m_t(x)\right)\right)\ dy \tag{47}$$

with the mean value function

$$x \mapsto m_t(x) := e^{tA}x \rightarrow_{t\to\infty} 0$$

and the covariance matrices C_t defined for any $t > 0$ by

$$0 < C_t := \int_0^t e^{sA} R e^{sA'} \, ds \to_{t\to\infty} C_\infty := \int_0^\infty e^{sA} R e^{sA'} \, ds.$$

Since A is stable, there exists some norm $|.|$ on $\mathbb{R}^n$ such that the corresponding operator norm satisfies $|e^{tA}| \leqslant e^{l(A)t}$ for some log-norm parameter $l(A) < 0$. This implies that

$$|m_t(x)| = |e^{tA}x| \leqslant e^{l(A)t} \, |x|. \tag{48}$$

This clearly shows that the semigroup P_t of the hypo-elliptic Ornstein-Ulhenbeck diffusion satisfies (43) and (44), and thus the conditions of Lemma 2.18 are met.

Let $\mathcal{P}_t$ be the Markov semigroup of the $\mathbb{R}^n$-valued linear diffusion

$$d\mathcal{X}_t(x) = (A\mathcal{X}_t(x) + a(\mathcal{X}_t(x)))dt + \Sigma \, dB_t \tag{49}$$

with some bounded drift function a on $\mathbb{R}^n$, an $(n \times n)$-matrix A satisfying (45), some n_1-valued Brownian motion B_t starting at the origin and some $(n \times n_1)$-matrix Σ satisfying the rank condition (46).

Using the stochastic interpolation formula (cf. Theorem 1.2 in [40]) given by

$$\mathcal{X}_t(x) - X_t(x) = \int_0^t e^{(t-s)A'} \, a(\mathcal{X}_s(x)) \, ds$$

we check the almost sure estimate

$$|X_t(x) - \mathcal{X}_t(x)| \leqslant c \quad \text{for some finite constant } c < \infty.$$

This yields the following proposition.

Proposition 2.19. *For any $v > 0$ and $t > 0$ there exists some $\delta_t > 0$ such that*

$$V(x) := \exp(v|x|) \Rightarrow \mathcal{P}_t(V)/V \leqslant c_t/V^{\delta_t}.$$

2.4.4. Langevin diffusion

Consider the Langevin diffusion flow $\mathcal{X}_t(z) = (X_t(z), Y_t(z)) \in (\mathbb{R}^r \times \mathbb{R}^r)$ starting at $z = (x, y) \in (\mathbb{R}^r \times \mathbb{R}^r)$ and given by

$$\begin{aligned} dX_t(z) &= Y_t(z)/m \, dt \\ dY_t(z) &= (b(X_t(z)) - \beta Y_t(z)/m) \, dt + \sigma \, dB_t. \end{aligned}$$

In the above display, B_t stands for an r-dimensional Brownian motion starting at the origin, $\sigma, \beta, m > 0$ some parameters and b a function of the form

$$b(x) := -\gamma \, x + a(x) \quad \text{with} \quad \gamma > 0 \quad \text{and} \quad ||a|| < \infty.$$

In statistical physics, the above diffusion represents the evolution of N particles $X_t(z) = (X_t^i(z))_{1 \leqslant i \leqslant N} \in \mathbb{R}^{3N}$ with mass $m > 0$, position $X_t(z) \in \mathbb{R}^{3N}$ and momenta $Y_t(z)$. In this context, $\gamma > 0$ stands for some friction parameter, and the diffusion parameter $\sigma > 0$ is related to the Boltzmann constant and the temperature of the system. In this context, the function $b(x) = -\nabla W(x)$ is often described by the gradient of some potential function W. For instance, for a quadratic confinement we have

$$W(x) := \gamma ||x||^2/2 + w(x) \quad \text{with} \quad ||\nabla w|| < \infty$$

$$\Rightarrow b(x) = -\nabla W(x) := -\gamma\ x + a(x) \quad \text{and} \quad a(x) = \nabla w(x).$$

Notice that $\mathcal{X}_t(z)$ can be rewritten in vector form as in (49) with $n = 2r$, $a(x,y) = \begin{pmatrix} 0 \\ a(x) \end{pmatrix}$ and the matrices

$$A = \begin{pmatrix} 0 & m^{-1}\ I_{n\times n} \\ -\gamma\ I_{n\times n} & -\beta m^{-1}\ I_{n\times n} \end{pmatrix} \quad \text{and} \quad \Sigma := \begin{pmatrix} 0 & 0 \\ 0 & \sigma I_{n\times n} \end{pmatrix}. \tag{50}$$

It is a simple exercise to check that A satisfies (45) and (46).

Consider the $\mathbb{R}^2$-valued stochastic process $X_t = (q_t, p_t)$ defined by

$$\begin{cases} dq_t &= \beta\ \dfrac{p_t}{m}\ dt \\ dp_t &= -\beta \left(\dfrac{\partial W}{\partial q}(q_t) + \dfrac{\sigma^2}{2}\ \dfrac{p_t}{m} \right)\ dt + \sigma\ dB_t \end{cases} \tag{51}$$

with some positive constants β, m, σ, a Brownian motion B_t, and a smooth positive function W on $\mathbb{R}$ such that for sufficiently large r we have

$$\forall\ |q| \geqslant r \qquad q\frac{\partial W}{\partial q}(q) \geqslant \delta\ \left(W(q) + q^2\right)$$

for some positive constant δ. This condition is clearly met when W behaves as q^{2l} for certain $l \geqslant 1$ at infinity. We let $V(q, p)$ be the function on $\mathbb{R}^2$ defined by

$$V(q,p) = 1 + \frac{1}{2m}\ p^2 + W(q) + \frac{\epsilon}{2}\ \left(\frac{\sigma^2}{2}\ q^2 + 2pq\right) \quad \text{with} \quad \epsilon < \frac{\sigma^2}{2m}.$$

In this situation, there exists some $a > 0$ and $c < \infty$ such that

$$L(V) \leqslant -\ aV\ + c. \tag{52}$$

The proof of the above estimate is rather technical, thus it is provided in the appendix on page 80.

3. Stability of *V*-positive semigroups

3.1. Normalized semigroups

For non-necessarily Markov V-positive semigroups $Q_{s,t}$ one natural idea is to normalize the semigroups. For any probability measure $\eta \in \mathcal{P}_V(E)$ we let $\Phi_{s,t}(\eta) \in \mathcal{P}_V(E)$ be the normalized distribution defined for any $f \in \mathcal{B}_V(E)$ by the formula

$$\Phi_{s,t}(\eta)(f) := \frac{\eta Q_{s,t}(f)}{\eta Q_{s,t}(1)} \quad \text{and we set} \quad \bar{Q}_{s,t}(f)(x) := \frac{Q_{s,t}(f)(x)}{Q_{s,t}(1)(x)} = \Phi_{s,t}(\delta_x)(f). \tag{53}$$

The mapping $\Phi_{s,t}$ is a well-defined semigroup on $\mathcal{P}_V(E)$. The denormalisation formula connecting these semigroups is given for any $t \in [s, +\infty[_\tau$ by

$$\mu Q_{s,t}(f) = \Phi_{s,t}(\mu)(f) \prod_{u\in[s,t[_\tau} \Phi_{s,u}(\mu)(Q_{u,u+\tau}(1)). \tag{54}$$

with

$$[s,t[_\tau := \{s + n\tau \in [s,t[\ : \ n \in \mathbb{N}\}.$$

To check this claim, observe that for any $t := s + n\tau$ we have

$$\Phi_{s,s+p\tau}(\mu)(Q_{s+p\tau,s+(p+1)\tau}(1)) = \mu Q_{s,s+(p+1)\tau}(1)/\mu Q_{s,s+p\tau}(1)$$

$$\text{and therefore} \quad \textstyle\prod_{0\leqslant p<n} \ \Phi_{s,s+p\tau}(\mu)(Q_{s+p\tau,s+(p+1)\tau}(1)) = \mu Q_{s,s+n\tau}(1).$$

The above formula coincides with the product formula relating the unnormalized operators $Q_{s,t}$ with the normalized semigroup $\Phi_{s,t}$ discussed in [[26], Section 1.3.2], see also [[29], Proposition 2.3.1] and [[41], Section 12.2.1].

Observe that for any $u \in [s,t[_\tau$ and $f \in \mathcal{B}_b(E)$ we have

$$\bar{Q}_{s,t} = R^{(t)}_{s,u}\bar{Q}_{u,t} \qquad \text{with} \quad R^{(t)}_{s,u}(f) := Q_{s,u}(f \ Q_{u,t}(1))/Q_{s,u}(Q_{u,t}(1))$$

Note that $\{R^{(t)}_{u,u+\tau}, \ u \in [s,t[_\tau\}$ forms a triangular array of time varying Markov semigroups. In terms of the standard Dobrushin coefficient discussed in Section 2.1 assume there exists some $\delta_\tau \in]0,1[$ such that for any $t \in [s,+\infty[_\tau$ and $u \in [s,t[_\tau$ we have the strong contraction condition $\beta(R^{(t)}_{u,u+\tau}) \leqslant (1-\delta_\tau)$. In this situation, for any $t \in [s,+\infty[_\tau$ we have the operator norm contraction

$$\beta(\bar{Q}_{s,t}) \leqslant \prod_{u\in[s,t[_\tau} \beta(R^{(t)}_{u,u+\tau}) \leqslant (1-\delta_\tau)^{(t-s)/\tau}$$

For a more thorough discussion on these strong contraction conditions we refer the reader to Section 4 in [23] and references therein. Unfortunately, the strong contraction condition discussed above is rarely satisfied for non-compact state spaces and we need to resort to Lyapunov conditions. In this connection, note that the Lyapunov condition in (8) is stronger than the one discussed in (27) in the context of Markov semigroups. We also strengthen (33) and assume that for any $s \geqslant 0$ and $\tau > 0$, the integral operator $Q_{t,t+\tau}$ has a density $q_{s,s+\tau}$ with respect to some Radon positive measure ν_τ on E; that is we have the formula

$$Q_{s,s+\tau}(x,dy) = q_{s,s+\tau}(x,y) \ \nu_\tau(dy). \tag{55}$$

We also assume there exists some $r_0 > 1$ such that for any $r \geqslant r_0$ we have

$$0 < \imath_r(\tau) := \inf_{s\in\mathcal{T}} \inf_{\mathcal{V}(r)^2} q_{s,s+\tau} \leqslant \sup_{s\in\mathcal{T}} \sup_{\mathcal{V}(r)^2} q_{s,s+\tau} < \infty \qquad \text{and} \quad \nu_\tau(\mathcal{V}(r)) > 0. \tag{56}$$

In this situation, for any $r \geqslant r_0$ and $\bar{r} \geqslant r$ we have the uniform estimate

$$\inf_{\mathcal{V}(r)} Q_{s,s+\tau}(1) \geqslant \jmath_{r,\bar{r}}(\tau) := \inf_{\mathcal{V}(r)} Q_{s,s+\tau}(1_{\mathcal{V}(\bar{r})}) \geqslant \imath_r(\tau) \ \nu_\tau(\mathcal{V}(r)) > 0.$$

We associate with a given $\mu \in \mathcal{P}_V(E)$ and some function $H \in \mathcal{B}_{0,V}(E)$ the finite rank (and hence compact) operator

$$f \in \mathcal{B}_V(E) \mapsto T_{s,t}^{\mu,H}(f) := \frac{Q_{s,t}(H)}{\mu_s(Q_{s,t}(1))}\ \ \mu_t(f) \in \mathcal{C}_V(E)$$

with the flow of measures $\mu_t = \Phi_{s,t}(\mu_s)$ starting at $\mu_0 = \mu$. With this notation at hand, one has the following theorem.

Theorem 3.1 ([23]). *Consider a V-positive semigroups $Q_{s,t}$ with a density (55) satisfying (56) for some parameter $\tau > 0$ and some $r_0 > 1$. In this situation, there exists a parameter $b > 0$ such that for any $\mu, \eta \in \mathcal{P}_V(E)$ and any $s \geqslant 0$ and $t \in [s, \infty[_\tau$ we have the local Lipschitz estimate*

$$|||\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)|||_V \leqslant c(\mu,\eta)\ \ e^{-b(t-s)}\ \ |||\mu - \eta|||_V. \tag{57}$$

For any $(\mu, H) \in (\mathcal{P}_V(E) \times \mathcal{B}_{0,V}(E))$ there exists some finite constant $c_H(\mu) < \infty$ such that for any $s \geqslant 0$ and $t \in [s, \infty[_\tau$ we have

$$\left|\left|\left| \frac{Q_{s,t}}{\mu_s Q_{s,t}(1)} - T_{s,t}^{\mu,H} \right|\right|\right|_V \leqslant\ c_H(\mu)\ e^{-b(t-s)}. \tag{58}$$

For continuous time semigroups, the above estimates also hold for any continuous time indices $s \leqslant t$ as soon as for any $r \geqslant r_0$ there exists some $\bar{r} \geqslant r$ such that $\inf_{\delta \in [0,\tau]} \mathrm{J}_{r,\bar{r}}(\delta) > 0$.

The proof of Theorem 3.1 is based on discrete time type V-norm operator contraction techniques combining the geometric drift condition stated in the l.h.s. of (8) with the local minorization condition stated in (56). The condition $\inf_{\delta \in [0,\tau]} \mathrm{J}_{r,\bar{r}}(\delta) > 0$ is a technical condition only made for continuous time semigroups to ensure that (57) and (58) also hold for continuous time indices.

Theses regularity conditions are rather flexible as we will now explain.

Absolutely continuous integral operators arise in a natural way in discrete time settings [19,20, 26, 29] and in the analysis of continuous time elliptic diffusion absorption models [42–45]. In connection to this, two-sided estimates for stable-like processes are provided in [46–49]. Two sided Gaussian estimates can also be obtained for some classes of degenerate diffusion processes of rank 2, that is when the Poisson brackets of the first order span the whole space [50]. This class of diffusions includes frictionless Hamiltonian kinetic models.

Diffusion density estimates can be extended to sub-Markovian semigroups using the multiplicative functional methodology developed in [27]. Whenever the trajectories of these diffusion flows, say $t \mapsto X_t(x)$, where $x \in E$ is the initial position, are absorbed on the smooth boundary ∂E of an open connected domain E, for any $\tau > 0$ the densities $q_\tau(x,y)$ of the sub-Markovian semigroup Q_τ (with respect to the trace of the Lebesgue measure on E) associated with the non-absorption event are null at the boundary. Nevertheless, whenever these densities are positive and continuous on the open set E^2 for some $\tau > 0$, they are uniformly positive and bounded on any compact subset of E; thus condition (56) is satisfied.

In this context, whenever $T(x)$ stands for first exit time from E and $T_r(x)$ the first exit time from the compact level set $\mathcal{V}(r) \subset E$ starting from $x \in \mathcal{V}(r)$, for any $\delta \in [0,\tau]$ and $r_+ > r$ we have the estimate

$$Q_\delta(1_{\mathcal{V}(r_+)})(x) := \mathbb{E}\big(1_{\mathcal{V}(r_+)}(X_\delta(x))\ 1_{T(x)>\delta}\big) \geqslant \mathbb{P}\big(T_{r_+}(x) > \delta\big) \geqslant \mathbb{P}\big(T_{r_+}(x) > \tau\big).$$

In this context, we have

$$\inf_{x\in\mathcal{V}(r)} \mathbb{P}\big(T_{r_+}(x) > \tau\big) > 0 \Rightarrow \inf_{\delta\in[0,\,\tau]} \inf_{\mathcal{V}(r)} Q_\delta(1) \geqslant \inf_{\delta\in[0,\,\tau]} \jmath_{r,\,r_+}(\delta) > 0. \tag{59}$$

Whenever the interior $E_+ := \mathcal{V}(r_+)^\circ$ is a connected domain, the l.h.s. estimate in (59) is met as soon as the sub-Markovian semigroup Q_τ^+ associated with the non-absorption event at the boundary ∂E_+ has a strictly positive continuous density $(x,y) \in E_+^2 \mapsto q_\tau^+(x,y)$. To check this claim, observe that for any $x \in \mathcal{V}(r)$ we have

$$\begin{aligned}\mathbb{P}\big(T_{r_+}(x) > \tau\big) = Q_\tau^+(1)(x) \geqslant Q_\tau^+(1_{\mathcal{V}(r)})(x) &= \int q_\tau^+(x,y)\ 1_{\mathcal{V}(r)}(y)\ \nu_\tau(dy)\\ &\geqslant \nu_\tau(\mathcal{V}(r))\ \inf_{\mathcal{V}(r)^2} q_\tau^+ > 0.\end{aligned}$$

It is out of the scope of this article to review the different classes of absolutely continuous operators and related two-sided Gaussian estimates arising in the analysis of continuous time elliptic diffusion and particle absorption models. For a more thorough discussion on this topic we refer to the series of reference pointers presented above.

Needless to say that the design of Lyapunov functions is a crucial and challenging problem in the stability analysis of positive semigroups. We have chosen to concentrate our review on presenting practical and general principles for designing Lyapunov functions.

3.2. Time homogenous models

For time homogeneous models we use the notation

$$(\Phi_t, Q_t, \bar{Q}_t) := (\Phi_{0,t}, Q_{0,t}, \bar{Q}_{0,t}).$$

As expected for time homogeneous semigroups a variety of results follow almost immediately from the estimates obtained in Theorem 3.1. Following [23] (cf. for instance Sections 4.1 and 4.3), these results include the existence of a unique leading Eigen-triple

$$(\rho, \eta_\infty, h) \in (\mathbb{R} \times \mathcal{P}_V(E) \times \mathcal{B}_{0,V}(E)) \quad \text{with} \quad \eta_\infty(h) = 1 \tag{60}$$

in the sense that for any $t \in \mathcal{T}$ we have

$$Q_t(h) = e^{\rho t}\ h \quad \text{and} \quad \eta_\infty Q_t = e^{\rho t}\ \eta_\infty \quad \text{or equivalently} \quad \Phi_t(\eta_\infty) = \eta_\infty. \tag{61}$$

The Eigen function h is sometimes called the ground state and the fixed point measure η_∞ the quasi-invariant measure. For any $x \in E$ we also have the product series formulation

$$0 < h(x) := \prod_{n\geqslant 0} \big\{1 + e^{-\rho\tau}[\Phi_{n\tau}(\delta_x)(Q_\tau(1)) - \Phi_{n\tau}(\eta_\infty)(Q_\tau(1))]\big\}.$$

In this context, choosing $(\mu, H) = (\eta_\infty, h)$ in (58), we readily check that

$$T_{s,s+t}^{\eta_\infty,h}(f) = T(f) := \frac{h}{\eta_\infty(h)}\ \eta_\infty(f) \quad \text{and} \quad |||e^{-\rho t}\ Q_t - T|||_V \leqslant c_h(\eta_\infty)\ e^{-bt}.$$

In terms of the Boltzmann-Gibbs transformation Ψ_h introduced in (6), for any $\eta \in \mathcal{P}_V(E)$ we have the conjugate formulae

$$\Psi_h(\Phi_t(\eta)) = \Psi_h(\eta)P_t^h \tag{62}$$

with the Doob h-transform of Q_t defined by the Markov semigroup

$$P_t^h \ : \ f \in \mathcal{B}_{V^h}(E) \mapsto P_t^h(f) := e^{-\rho t}\frac{1}{h}\ Q_t(hf) \in \mathcal{B}_{V^h}(E).$$

Observe that

$$\eta_\infty = \Phi_t(\eta_\infty) \Longleftrightarrow \quad \eta_\infty^h := \Psi_h(\eta_\infty) = \eta_\infty^h P_t^h.$$

The Markov semigroup P_t^h is sometimes called the transition semigroup of the h-process, a.k.a. the process evolving in the ground state.

We further assume that Q_t is a sub-Markov semigroup of *self-adjoint operators* on $\mathbb{L}_2(\nu)$ with respect to some locally finite measure ν on E. In addition, there exists an orthonormal basis $(\varphi_n)_{n\geqslant 1}$ associated with a decreasing sequence of eigenvalues $\rho_n \leqslant 0$ such that

$$Q_t(x,dy) = \sum_{n\geqslant 1}\ e^{\rho_n t}\ \varphi_n(x)\ \varphi_n(y)\ \nu(dy). \tag{63}$$

In this context, the formulae (61) are satisfied with the parameters

$$(\rho, h) = (\rho_1, \varphi_1) \quad \text{and} \quad \eta_\infty(dx) = \Psi_h(\nu)(dx) := \frac{1}{\nu(h)}\ h(x)\ \nu(dx).$$

Note that in this case h has unit norm $\nu(h^2) = 1$. The spectral resolution (63) yields for any $t \geqslant 0$ and $f \in \mathbb{L}_2(\nu)$ the following decomposition

$$e^{-\rho t}Q_t(f)(x) - \frac{h(x)}{\eta_\infty(h)}\ \eta_\infty(f) = \sum_{n\geqslant 2} e^{\rho_n^h t}\ \varphi_n(x)\ \nu(\varphi_n f) \quad \text{with} \quad \rho_n^h = \rho_n - \rho_1. \tag{64}$$

This yields the following result.

Proposition 3.2. *For any time horizon $t \geqslant 0$ and any $f \in \mathbb{L}_2(\nu)$ we have the exponential estimates*

$$\left\| e^{-\rho t}Q_t(f) - \frac{h}{\eta_\infty(h)}\ \eta_\infty(f)\right\|_{\mathbb{L}_2(\nu)} \leqslant e^{\rho_2^h t}\ \left(\nu(f^2) - \nu(hf)^2\right)^{1/2}. \tag{65}$$

Whenever Q_t is a positive semigroup of *self-adjoint operators* on $\mathbb{L}_2(\nu)$ the Doob h-transform P_t^h is a semigroup of self-adjoint operators on $\mathbb{L}_2(\eta_\infty^h)$ and we have the following spectral decomposition

Lemma 3.3. *For any $t \geqslant 0$ and $f \in \mathbb{L}_2(\eta_\infty^h)$ we have*

$$P_t^h(x,dy) = \eta_\infty^h(dy) + \sum_{n\geqslant 2}\ e^{\lambda_n t}\ h_n(x)\ h_n(y)\ \eta_\infty^h(dy) \tag{66}$$

with the $\mathbb{L}_2(\eta_\infty^h)$ orthonormal basis $(h_n)_{n\geqslant 2}$ defined for any $n \geqslant 2$ by

$$h_n := \varphi_n/h \quad \text{and} \quad \lambda_n = \rho_n - \rho_1 < 0 \quad \text{and} \quad \eta^h_\infty = \Psi_{h^2}(\nu).$$

Note that the density of the integral operator $P^h_t(x,dy)$ w.r.t. $\eta^h_\infty(dy)$ is given by

$$p^h_t(x,y) = e^{-\rho_1 t}\ \frac{q_t(x,y)}{h(x)h(y)} = 1 + \sum_{n \geqslant 2}\ e^{\lambda_n t}\ h_n(x)\ h_n(y). \tag{67}$$

We further assume that $h \in \mathcal{B}_0(E)$ and P^h_t is ultra contractive, in the sense that for any $t>0$ we have

$$|||P^h_t|||_{\mathbb{L}_2(\eta^h_\infty)\mapsto \mathbb{L}_\infty(\eta^h_\infty)} = e^{-\rho_1 t} \sup_{(x,y)\in E^2} \frac{q_t(x,y)}{h(x)h(y)} = \sup_{(x,y)\in E^2} p^h_t(x,y) < \infty. \tag{68}$$

Proposition 3.4. *Assume that $\nu(E) < \infty$ and $h \in \mathcal{B}_0(E)$. In addition, for any $t>0$ (68) holds and the mapping $x \mapsto \int\ p^h_t(x,y)\ \nu(dy)$ is upper semi-continuous and locally lower bounded. In this situation, the function $V := 1/h \in \mathcal{B}_\infty(E)$ and for any $t>0$ we have $Q_t(V)/V \in \mathcal{B}_0(E)$. In addition, for any $t>0$ we have*

$$Q_t(V)/V \leqslant c_t/V^2 \in \mathcal{B}_0(E). \tag{69}$$

3.3. Sub-Markov semigroups

Sub-Markov semigroups are prototype-based models of positive integral operators. In time homogeneous settings, these stochastic models are defined in terms of a stochastic flow $X_t(x)$ evolving on some metric Polish space $(\mathcal{E},d)$, some non-negative absorption potential function U on some non-necessarily bounded Borel subset $E \subset \mathcal{E}$. For a given $x \in E$ we denote by $T(x)$ the exit time of the flow $X_t(x)$ from E.

We associate with these objects, the sub-Markov semigroup $Q^{[U]}_t$ defined for any $f \in \mathcal{B}_b(E)$ and $x \in E$ by

$$Q^{[U]}_t(f)(x) = \mathbb{E}\left(f(X_t(x))\ 1_{T(x)>t}\ \exp\left(-\int_0^t U(X_s(x))ds\right)\right). \tag{70}$$

The above model can be interpreted as the distribution of a stochastic flow evolving in an absorbing medium with hard and soft obstacles. Before killing, the flow starts at $x \in E$ and evolves as $X_t(x)$. Then, it is killed at rate U or as soon as it exits the set E. In the case $E = \mathcal{E}$, the flow cannot exit the set E and it is only killed at rate U. This situation is sometimes referred a sub-Markov semigroup with soft obstacles represented by the absorbing potential function U on E. When the flow may exit the set $E \subset \mathcal{E}$, the complementary subset $C := \mathcal{E} - E$ is interpreted as a hard obstacle, a.k.a. an infinite energy barrier.

We illustrate the V-positive semigroup analysis developed in this article through three typical examples of solvable sub-Markov semigroups arising in physics and applied probability.

3.3.1. The harmonic oscillator

Consider the case $E = \mathcal{E} = \mathbb{R}$, and let $X_t(x) = B_t(x)$ be a Brownian motion starting at $x \in \mathbb{R}$ and let $U(x) = x^2/2$. In this situation, the semigroup $Q_t^{[U]} = Q_t$ defined in (70) coincides with the one dimensional harmonic oscillator. For any $t > 0$, the integral operator Q_t has a continuous density w.r.t. the uniform measure ν on E given by

$$q_t(x,y) = \sum_{n \geqslant 1} e^{\rho_n t} \ \varphi_n(x)\varphi_n(y) \tag{71}$$

with the $\mathbb{L}_2(\nu)$ orthonormal basis Eigen states

$$\varphi_n(x) = (2^{n-1}(n-1)!\sqrt{\pi})^{-1/2} \ e^{-x^2/2} \ \mathbb{H}_{n-1}(x)$$

associated with the eigenvalues

$$\rho_n = -(n-1/2) \quad \text{and the Hermite polynomials} \quad \mathbb{H}_n(x) = (-1)^n \ e^{x^2} \ \partial^n e^{-x^2}.$$

In this context, the Eigen state associated with the top eigenvalue $\rho = \rho_1 = -1/2$ is given by the harmonic function

$$h(x) = \varphi_1(x) = \pi^{-1/4} \ e^{-x^2/2}. \tag{72}$$

The spectral resolution of integral operator $P_t^h(x, dy)$ and its density $p_t^h(x,y)$ with respect to the invariant measure

$$\eta_\infty^h(dy) = \frac{1}{\sqrt{\pi}} \ e^{-y^2} \ dy$$

are given as in (66) and (67) with $\mathbb{L}_2(\eta_\infty^h)$ orthonormal basis defined for any $n \geqslant 2$ by

$$h_n = (2^{n-1}(n-1)!)^{-1/2} \ \mathbb{H}_{n-1} \quad \text{and} \quad \rho_n^h = \rho_n - \rho_1 = -(n-1).$$

In this context, the h-process is given by the Ornstein-Uhlenbeck diffusion

$$dX_t^h(x) = \partial \log h(X_t^h(x)) \ dt + dB_t = -X_t^h(x) \ dt + dB_t. \tag{73}$$

In the above display, $B_t = B_t(0)$ stands for the one dimensional Brownian motion starting at the origin. The conjugate formula

$$Q_t(hf)/Q_t(h) = P_t^h(f) \Longleftrightarrow Q_t(f) = e^{\rho t} h \ P_t^h(f/h) \tag{74}$$

yields the following proposition.

Proposition 3.5. *For any time horizon $t \geqslant 0$ we have*

$$Q_t(x, dy) = \frac{1}{\sqrt{\cosh(t)}} \ \exp\left(-\frac{x^2}{2} \ p_t\right) \ \frac{1}{\sqrt{2\pi p_t}} \ \exp\left(-\frac{(y - m_t(x))^2}{2p_t}\right) \ dy$$

with the mean and variance parameters $(m_t(x), p_t)$ defined by

$$m_t(x) = x/\cosh(t) \quad \text{and} \quad p_t = \tanh(t).$$

The proof of the above proposition is a direct consequence of the conjugate formula (74), thus it is provided in the appendix, on page 78.

Choosing $V(x) = 1 + |x|^n$, for some $n \geqslant 1$, we readily check that

$$V \in \mathcal{C}_\infty(E) \quad \text{and} \quad Q_t(V)/V \leqslant \nu_t \ Q_t(1) \ \in \mathcal{C}_0(E) \tag{75}$$

where ν_t is a constant depending only on t.

3.3.2. The half-harmonic oscillator

Consider the case $E =]0, \infty[\subset \mathcal{E} = \mathbb{R}$, and let $X_t(x) = B_t(x)$ be a Brownian motion starting at $x \in \mathcal{E}$ and let $U(x) = x^2/2$. In this situation, the semigroup $Q_t^{[U]} = Q_t$ defined in (70) coincides with the harmonic oscillator with an infinite barrier at the origin $\partial E = \{0\}$ (a.k.a. the half-harmonic oscillator). Using the fact that

$$e^{x^2/2} \ \frac{1}{2} \ \partial^2 \ e^{-x^2/2} = U(x) - 1/2$$

by an exponential change of probability measure (cf. for instance Section 18.3 in [30]) we have the conjugate formula

$$Q_t(f)(x) = e^{-t/2} \ e^{-x^2/2} \ \mathbb{E}\Big(f(Y_t(x)) \ e^{Y_t(x)^2/2} \ 1_{T^Y(x)>t}\Big)$$

with the Ornstein-Uhlenbeck diffusion

$$\begin{aligned} &dY_t(x) = -Y_t(x) \ dt + dB_t \\ &\text{and} \quad T^Y(x) := \inf\big\{t \geqslant 0 \ : \ Y_t(x) \in \partial E\big\} \quad \text{with} \quad \partial E = \{0\}. \end{aligned} \tag{76}$$

Note that the stochastic flow $Y_t(x)$ coincides with the h-process of the harmonic oscillator discussed in (73). Thus, by reflection arguments we have

$$\begin{aligned} Q_t^Y(f)(x) :&= \mathbb{E}\big(f(Y_t(x)) \ 1_{T^Y(x)>t}\big) = \mathbb{E}\big(f(B_{\sigma_t}(\epsilon_t x)) \ 1_{T(\epsilon_t x)>t}\big) \\ &= \int_0^\infty f(y) \ q_t^Y(x,y) \ dy \quad \text{with} \quad q_t^Y(x,y) := (r_t(x,y) - r_t(x,-y)). \end{aligned}$$

In the above display, (ϵ_t, σ_t) stands for the parameters

$$(\epsilon_t, \sigma_t) := \left(e^{-t}, \sqrt{\frac{1-\epsilon_t^2}{2}}\right) \quad \text{and} \quad r_t(x,y) = \frac{1}{\sqrt{2\pi\sigma_t^2}} \ \exp\left(-\frac{1}{2\sigma_t^2}(y - \epsilon_t x)^2\right).$$

This yields the following proposition.

Proposition 3.6. *For any $t > 0$ and $x \in E$ the normalized semigroup $\bar{Q}_t$ defined in (53) is given by*

$$\bar{Q}_t(x, dy) = \frac{\sinh(y \ m_t(x))}{\mathbb{P}\big(0 \leqslant Z \leqslant m_t(x)/\sqrt{p_t}\big)} \ \times \frac{1}{\sqrt{2\pi p_t}} \ \exp\left(-\frac{y^2 + m_t(x)^2}{2p_t}\right) \ \nu(dy).$$

In the above display, $\nu(dy) := 1_{[0,\infty[}(y) \ dy$ stands for the trace of the Lebesgue measure on the half-line, Z is a centered Gaussian variable with unit variance and $(m_t(x), p_t)$ are the mean and variance parameters defined in Proposition 3.5. In addition, the total mass function $Q_t(1)(x)$ is given by the formula

$$Q_t(1)(x) = 2\ \frac{e^{-\frac{x^2}{2}p_t}}{\sqrt{\cosh(t)}} \times \mathbb{P}\big(0 \leqslant Z \leqslant m_t(x)/\sqrt{p_t}\big) \in \mathcal{C}_0(E).$$

The proof of the above proposition follows the same lines of arguments as the proof of Proposition 3.5; it is provided in the appendix, on page 79.

Choosing $V(x) = x^n + 1/x$, for some $n \geqslant 1$, we readily check that

$$V \in \mathcal{C}_\infty(E) \quad \text{and} \quad Q_t(V)/V \leqslant c_t/V \ \in \mathcal{C}_0(E). \tag{77}$$

The proof of the above estimate follows elementary but lengthly calculations, thus it is provided in the appendix on page 81.

For any $t > 0$, the integral operator Q_t has a continuous density w.r.t. the uniform measure ν on E given by

$$q_t(x,y) = \sum_{n \geqslant 1}\ e^{\rho_n t}\ \varphi_n(x)\ \varphi_n(y)$$

with the $\mathbb{L}_2(\nu)$ orthonormal basis Eigen states

$$\varphi_n(x) = \sqrt{2}\ (2^{2n-1}(2n-1)!\sqrt{\pi})^{-1/2}\ e^{-x^2/2}\ \mathbb{H}_{2n-1}(x)$$

associated with the eigenvalues

$$\rho_n = -((2n-1) + 1/2).$$

In this context, the Eigen state associated with the top eigenvalue $\rho = \rho_1 = -3/2$ is given for any $x \in]0,\infty[$ by the harmonic function

$$h(x) = \varphi_1(x) = 2\pi^{-1/4}\ x\ e^{-x^2/2} = h_0(x)\ \mathbb{H}_1(x)$$

with the ground state h_0 of the harmonic oscillator discussed in (72). Note that h coincides with the restriction on $]0,\infty[$ of the first excited state of the harmonic-oscillator (negative on $]-\infty,0]$ and crossing the origin at $x=0$).

The spectral resolution of integral operator $P^h_t(x,dy)$ and its density $p^h_t(x,y)$ with respect to the invariant measure

$$\eta^h_\infty(dy) = \frac{4}{\sqrt{\pi}}\ y^2\ e^{-y^2}\ 1]0,\infty[(y)\ dy$$

are given for any $x,y \in]0,\infty[$ as in (66) and (67) with $\mathbb{L}_2(\eta^h_\infty)$ orthonormal basis defined for any $n \geqslant 2$ and $x \in]0,\infty[$ by the odd Hermite functions

$$h_n(x) = (2^{2n}(2n-1)!)^{-1/2}\ \mathbb{H}_{2n-1}(x)/x \quad \text{and} \quad \rho^h_n = -2(n-1).$$

In this context, the h-process is given by the diffusion

$$dX^h_t(x) = \partial \log h(X^h_t(x))\ dt + dB_t = \left(\frac{1}{X^h_t(x)} - X^h_t(x)\right)\ dt + dB_t. \tag{78}$$

3.3.3. The Dirichlet heat kernel

Let $X_t(x) = B_t(x)$ be a Brownian motion starting at $x \in E :=]0,1[\subset \mathcal{E} := \mathbb{R}$ and $T(x)$ be the first time $t \geqslant 0$ the process $B_t(x) \in \partial E := \{0,1\}$. Choosing $U = 0$ in (70), the

semigroup $Q_t^{[U]} = Q_t$ takes the following form

$$Q_t(f)(x) := \mathbb{E}(f(B_t(x))\ 1_{T(x)>t}).$$

For any $t>0$, the integral operator Q_t has a continuous density w.r.t. the uniform measure ν on E given by the Dirichlet heat kernel

$$q_t(x,y) = \sum_{n \geqslant 1} e^{\rho_n t}\ \varphi_n(x)\varphi_n(y) \tag{79}$$

with the $\mathbb{L}_2(\nu)$ orthonormal basis Eigen states

$$\varphi_n(x) = \sqrt{2}\ \sin(n\pi x) \quad \text{associated with the eigenvalues} \quad \rho_n = -(n\pi)^2/2.$$

In this context, the Eigen state $h(x) = \varphi_1(x) = \sqrt{2}\ \sin(\pi x)$ associated with the top eigenvalue $\rho = \rho_1 = -\pi^2/2$ is strictly positive except at the boundary $\{0, 1\}$. By removing the boundary, the semigroup P_t^h of the process evolving in the ground state $h(x)$ on the open interval E is a self-adjoint operators on $\mathbb{L}_2(\eta_\infty^h)$ with

$$\eta_\infty^h(dx) = h^2(x)\ \nu(dx) = 2\ \sin^2(\pi x)\ 1_E(x)\ dx.$$

In addition, we have the spectral decomposition (66) with the $\mathbb{L}_2(\eta_\infty^h)$ orthonormal basis Eigen states

$$h_n(x) := \sin(n\pi x)/\sin(\pi x)$$

associated with the eigenvalues

$$\lambda_n = -\pi^2(n^2-1)/2 < 0.$$

Our next objective is to estimate the density $p_t^h(x,y)$ of the integral operator $P_t^h(x,dy)$ w.r.t. η_∞^h defined in (67). Recalling that $|\sin(ny)| \leqslant n|\sin(y)|$, for any $n \geqslant 1$ and $y \in \mathbb{R}$, for any $x \in E$ we have the diagonal estimate

$$p_t^h(x,x) - 1 = \sum_{n \geqslant 2} e^{\rho_n^h t}\ h_n(x)^2$$

with

$$h_n(x)^2 = \left(\frac{\sin(n\pi x)}{\sin(\pi x)}\right)^2 \leqslant n^2 \quad \text{so that condition (68) is satisfied.}$$

Observe that the function

$$V \ : \ x \in E \mapsto V(x) := \sqrt{2}/h(x) \in [1, \infty[$$

is locally bounded with compact level sets given for any $0 < \epsilon \leqslant 1$ by the formulae

$$\mathcal{K}_\epsilon := \{x \in]0,1[\ : \ V(x) \leqslant 1/\epsilon\} = \{x \ : \ \sin(\pi x) \geqslant \epsilon\} \subset E.$$

In any dimension we can use the intrinsic ultracontractivity to produce a Lyapunov function V. Let E be a bounded domain of $\mathbb{R}^n$ for some $n \geq 1$ and assume that it is a $C^{1,\alpha}$ domain for some $\alpha > 0$. Denote by $q_t(x,y)$ the Dirichlet heat kernel on E. By [51] one has

$$q_t(x,y) \leq c_t\ d(x,\partial E)d(y,\partial E)$$

for some constant c_t independent on x and y. Here $d(x, \partial E)$ denotes the distance from x to the boundary of E. Set $V(x) = \frac{1}{d(x,\partial E)}$. The above intrinsic ultracontractivity implies

$$Q_t(V)(x) = \int_E q_t(x,y)V(y)dy \leq c_t|E| \ d(x,\partial E)$$

which in turn gives $Q_t(V)/V \leq c_t|E|/V^2 \in \mathcal{B}_0(E)$, where $|E|$ stands for the volume of the bounded set E.

4. Lyapunov design principles

The aim of this section is to present some general principles to construct Lyapunov functions for positive semigroups. Section 4.1 provides equivalent formulations of the Lyapunov condition in (8) encountered in the literature in terms of exhausting sequences of compact level sets. This section also presents simple ways to design Lyapunov functions for sub-Markov semigroups on normed spaces in terms of their generators. Section 4.2 presents some principles to construct Lyapunov functions for positive semigroups dominated by semigroups with known Lyapunov functions. Section 4.3 is dedicated to the design of Lyapunov functions for conjugate semigroups. All the principles discussed in this section are illustrated in Section 6 as well as in Section 7 in the context of conditional diffusions.

4.1. Foster-Lyapunov conditions

For time homogeneous models $Q_{s,s+t} := Q_t$, the l.h.s. condition in (8) takes the form $Q_\tau(V)/V \leqslant \Theta_\tau \in \mathcal{B}_0(E)$. In terms of the compact sets $K_\epsilon := \{\Theta_\tau \geqslant \epsilon\}$, the l.h.s Lyapunov condition in (8) yields for any $\tau > 0$ the estimate

$$Q_\tau(V)(x) \leqslant \epsilon \ V(x) + 1_{K_\epsilon}(x) \ c_\epsilon \tag{80}$$

for any $\epsilon > 0$ with the parameter $c_\epsilon := \sup_{K_\epsilon}(V\Theta_\tau) < \infty$. This implies that for any $n \geqslant 1$ we have

$$Q_\tau(V)(x) \leqslant \epsilon_n \ V(x) + 1_{K_{\epsilon_n}}(x) \ c_{\epsilon_n} \tag{81}$$

where $K_{\epsilon_n} \subset E$ stands for some increasing sequence of compacts sets and c_{ϵ_n} some finite constants, indexed by a decreasing sequence of parameters $\epsilon_n \in [0,1]$ such that $\epsilon_n \to 0$ as $n \to \infty$. In the reverse angle, assume that $Q_\tau(V)/V$ is locally lower bounded and lower semicontinuous. In this situation, condition (81) ensures that $Q_\tau(V)/V \in \mathcal{B}_0(E)$ for any $\tau > 0$. Indeed, for any $\delta > 0$, there exists some $n \geqslant 1$ such that $\epsilon_n < \delta$ and we have

$$\{Q_\tau(V)/V \geqslant \delta\} \subset \{Q_\tau(V)/V > \epsilon_n\} \subset K_{\epsilon_n}.$$

Since $\{Q_\tau(V)/V \geqslant \delta\}$ is a closed subset of a compact set it is also compact.

More generally, whenever (81) is met for some exhausting sequence of compact sets K_{ϵ_n}, in the sense that for any compact subset $K \subset E$ there exists some $n \geqslant 1$ such that $K \subset K_{\epsilon_n}$ we have

$$\inf_K Q_\tau(V)/V \geqslant \inf_{K_{\epsilon_n}} Q_\tau(V)/V \geqslant \epsilon_n.$$

This ensures that the function $Q_\tau(V)/V$ is necessarily locally lower bounded. In this situation, we have $Q_\tau(V)/V \in \mathcal{B}_0(E)$ as soon as $Q_\tau(V)/V$ is lower semicontinuous.

Notice that the sub-level set $\mathcal{V}(r) := \{V \leqslant r\}$ of the Lyapunov function $V \in \mathcal{B}_\infty(E)$ and the ϵ-super-level sets $K_\epsilon := \{\Theta_\tau \geqslant \epsilon\}$ of $\Theta_\tau \in \mathcal{B}_0(E)$ are equivalent compact exhausting sequences, in the sense that for any $r \geqslant 1$ we have

$$\mathcal{V}(r) \subset K_{\epsilon_r} \subset \mathcal{V}(r_\epsilon) \quad \text{with} \quad \epsilon_r := \inf_{\mathcal{V}(r)} \Theta_\tau \quad \text{and} \quad r_\epsilon := \sup_{K_{\epsilon r}} V.$$

Whenever E is a locally compact Polish space, the abstract sequence $C_n := K_{\epsilon_n}$ in (81) is automatically exhausting; that is, we have that $E = \cup_{n \geqslant 0} C_n$ with C_n is included in the interior C_{n+1}° of the compact set C_{n+1}. To check this claim, observe that for any $n \geqslant 1$ there exists some $m_n \geqslant n$ such that

$$C_n \subset \{\Theta_\tau \geqslant \inf_{C_n} \Theta\} \subset C_{m_n} \subset \{\Theta_\tau \geqslant \inf_{C_{m_n}} \Theta\}.$$

Thus, the exhausting sequence C_n is equivalent to the one defined by the super-level sets of Θ_τ.

The rather abstract condition (81) is often presented in the literature as an initial condition to check on a case-by-case basis to analyze the stability property of time homogenous sub-Markov semigroups (see for instance [52,53], as well as Section 17.5 in [30] in the context of Markov semigroups and the references therein).

We end this section with a brief discussion on condition (81) in the context of the sub-Markov semigroup discussed in (70). Note that this semigroup can be turned into a Markov semigroup by sending the killed process into a cemetery state, say Δ, at the killing time. In this interpretation, functions on E are extended to $E_\Delta = E \cup \{\Delta\}$ by setting $f(\Delta) = 0$. More interestingly, whenever E is locally compact its topology coincides with the weak topology induced by $\mathcal{C}_0(E) := \mathcal{B}_0(E) \cap \mathcal{C}_b(E)$, and inversely (cf. Proposition 2.1 in [54]). In this context a continuous function f vanishes at infinity if and only if its extension to the one point compactification (a.k.a. Alexandroff compactification) $E_\Delta := E \cup \{\Delta\}$ (obtained by setting $f(\Delta) = 0$) is continuous. For locally compact spaces, we also recall that the one point extension E_Δ is compact.

Whenever it exists, the generator L^U of these sub-Markov semigroups $Q_t^{[U]}$ are defined on domain of functions $D(L^U) \subset \mathcal{B}_0(E)$. As expected, the analysis of this class of models in terms of generators often requires developing a sophisticated analysis taking into account the topological structure of the set E. To the best of our knowledge, there is no simple sufficient condition to check (81) in terms of these generators.

The situation is greatly simplified for sub-Markov semigroups with soft obstacles. When $E = \mathcal{E}$ is a locally compact normed space $(E, ||.||)$ we let L be the generator of the flow $X_t(x)$. In this situation, the generator of the sub-Markov semigroup $Q_t^{[U]}$ is given by $L^U = L - U$. We further assume that L and L^U are defined on some common domain $\mathcal{D}(L) \subset \mathcal{B}(E)$.

Lemma 4.1 ([52]). *Let $V, V_0 \in \mathcal{D}(L)$ be a couple of functions such that $V, V_0 \geqslant 1$ and*

$$V(x) \to_{||x|| \to \infty} \infty \quad \text{and} \quad V(x)/V_0(x) \to_{||x|| \to \infty} \infty. \tag{82}$$

In this situation, condition (81) is satisfied as soon as there exists some finite constant $c_0 < \infty$ such that

$$L^U(V_0)/V_0 \leqslant c_0 \quad \text{and} \quad L^U(V)(x)/V(x) \to_{||x|| \to \infty} -\infty. \tag{83}$$

Note that in this context, the compact sets in (81) are given for some sufficiently large radii $r_\epsilon > 0$ by the closed balls:

$$\mathcal{K}_\epsilon = \bar{\mathbb{B}}(0, r_\epsilon) := \{x \in E \ : \ ||x|| \leqslant r_\epsilon\}. \tag{84}$$

4.2. Semigroup domination

For a given $p \geqslant 1$ we clearly have

$$V \in \mathcal{B}_\infty(E) \Longleftrightarrow V^p \in \mathcal{B}_\infty(E) \quad \text{and} \quad \mathcal{B}_{V^{1/p}}(E) \subset \mathcal{B}_V(E) \subset \mathcal{B}_{V^p}(E).$$

We say that a V-positive semigroup $Q_{s,t}$ is p-dominated by a collection of integral operators $\mathcal{Q}_{s,t}$ on $\mathcal{B}_{V^p}(E)$ and we write $Q \ll_p \mathcal{Q}$ as soon as for any non-negative function $f \in \mathcal{B}_V(E)$ and any $s \leqslant t$ we have

$$Q_{s,t}(f) \leqslant c_{t-s}(p) \ \mathcal{Q}_{s,t}(f^p)^{1/p}.$$

To simplify notation, when $p = 1$ we write $Q \ll \mathcal{Q}$ instead of $Q \ll_1 \mathcal{Q}$. Observe that

$$Q \ll_p \mathcal{Q} \quad \Rightarrow \quad \forall s \leqslant t \qquad (Q_{s,t}(V)/V)^p \leqslant c_{t-s}(p)^p \ \mathcal{Q}_{s,t}(V^p)/V^p.$$

This yields for any $\tau > 0$ and $\theta_\tau \in \mathcal{B}_0(E)$ the Lyapunov estimate

$$\mathcal{Q}_{s,s+\tau}(V^p)/V^p \leqslant \theta_\tau^p \Rightarrow Q_{s,s+\tau}(V)/V \leqslant c_\tau \ \theta_\tau. \tag{85}$$

We illustrate the above domination property with the Langevin diffusion flow $\mathcal{X}_t^{(a)}(z) = (X_t(z), Y_t(z)) \in (\mathbb{R}^n \times \mathbb{R}^n)$ starting at $z = (x, y) \in (\mathbb{R}^n \times \mathbb{R}^n)$ and defined by the hypo-elliptic diffusion

$$\begin{aligned} dX_t(z) &= Y_t(z)/m \ dt \\ dY_t(z) &= (a(X_t(z)) - \gamma X_t(z) - \beta Y_t(z)/m) \ dt + \sigma \ dB_t. \end{aligned} \tag{86}$$

In the above display, $\sigma, \gamma, \beta, m > 0$ stands for some parameters and a some Lipschitz function on $\mathbb{R}^n$, with $n \geqslant 1$. Notice that when $a = 0$, the flow $\mathcal{X}_t^{(0)}(z)$ resumes to an hypo-elliptic Ornstein-Ulhenbeck on $\mathbb{R}^{2n}$. Consider a bounded open connected domain $D \subset \mathbb{R}^n$ and set

$$\forall z \in E := D \times \mathbb{R}^n \qquad T^{(a)}(z) := \inf\left\{t \geqslant 0 \ : \ \mathcal{X}_t^{(a)}(z) \in \partial E\right\}.$$

We associate with these objects, the sub-Markov semigroup defined for any $f \in \mathcal{B}_b(E)$ and $z = (x, y) \in E$ by

$$\mathcal{Q}_t^{(a)}(f)(z) := \mathbb{E}\left(f(\mathcal{X}_t^{(a)}(z)) \ 1_{T^{(a)}(z) > t}\right).$$

In this situation, we have

$$\sup_D a < \infty \quad \Rightarrow \quad \forall p > 1 \qquad \mathcal{Q}^{(a)} \ll_p \mathcal{Q}^{(0)}. \tag{87}$$

The proof of the above assertion is a direct consequence of Girsanov's theorem and Hölder's inequality. For the convenience of the reader, a detailed proof is provided in the appendix on page 83.

To emphasize the role of the absorption in sub-Markov semigroups we return to the class of models discussed in (70). We let P_t be the free evolution Markov semigroup associated with the stochastic flow $X_t(x)$. Assume that $Q_t^{[U]}(1) \in \mathcal{B}_0(E)$ and

$$||Q_t^{[U]}(V)/V|| < \infty \quad \text{for some } t > 0 \text{ and } V \in \mathcal{B}_\infty(E). \tag{88}$$

Applying Hölder's inequality and choosing $V_p := V^{1/p} \in \mathcal{B}_\infty(E)$ with $p > 1$ we readily check the estimate

$$Q_t^{[U]}(V_p)/V_p \leqslant c_t(p)\ Q_t^{[U]}(1)^{1-1/p} \in \mathcal{B}_0(E). \tag{89}$$

The next lemma provides several practical conditions to check the uniform estimate (88) for sub-Markov semigroups associated with soft obstacles.

Lemma 4.2. *Consider the sub-Markov semigroup discussed in (70) when $E = \mathcal{E}$ is a locally compact normed space $(E, ||.||)$. Assume that the generators L and L^U of the flows P_t and $Q_t^{[U]}$ are defined on some common domain $\mathcal{D}(L) \subset \mathcal{B}(E)$. In this situation, for any $V \in \mathcal{B}_\infty(E) \cap \mathcal{D}(L)$ and parameter $a > 0$ we have*

$$L^U(V) \leqslant -aV + c \quad \Rightarrow \quad \forall t \geqslant 0 \qquad ||Q_t^{[U]}(V)/V|| < \infty. \tag{90}$$

Whenever $U \in \mathcal{B}_\infty(E) \cap \mathcal{D}(L)$, for any $a_0 \geqslant 0$ and $a_1 \in \mathbb{R}$ we have

$$L(U) \leqslant a_0 + a_1 U \quad \Rightarrow \quad \forall t \geqslant 0 \qquad ||Q_t^{[U]}(U)|| < \infty. \tag{91}$$

The proof of the above lemma follows essentially the same lines of arguments as the proof of Lemma 4.1; thus it is provided in the appendix, on page 82.

Whenever $E = \mathcal{E}$ and the absorption potential function U is bounded, we have $P \ll Q^{[U]} \ll P$. In this context, *there is no hope to have that $Q_t^{[U]}(1) \in \mathcal{B}_0(E)$ for some $t > 0$.* Nevertheless, for any $V \in \mathcal{B}_\infty(E)$ and any time horizon $t > 0$ we have

$$Q_t^{[U]}(V)/V \in \mathcal{B}_0(E) \Longleftrightarrow P_t(V)/V \in \mathcal{B}_0(E).$$

In this situation, the design of Lyapunov functions V satisfying (8) or equivalently Foster-Lyapunov conditions of the form (81) is equivalent to the problem of finding a Lyapunov function for the Markov semigroup P_t.

Whenever P_t is stable, in the sense that it has a Lyapunov $V \in \mathcal{B}_\infty(E)$ such that $P_t(V)/V \in \mathcal{B}_0(E)$ for some $t > 0$, then the domination property $Q^{[U]} \ll P$ yields automatically a Lyapunov function for $Q_t^{[U]}$.

Whenever P_t is not necessarily stable but we have $||P_t(V)/V|| < \infty$ for some $t > 0$ and $V \in \mathcal{B}_\infty(E)$, applying (89) the domination property $Q^{[U]} \ll P$ ensures that for any $p > 1$ we have $V_p := V^{1/p} \in \mathcal{B}_\infty(E)$ and

$$Q_t^{[U]}(1) \in \mathcal{B}_0(E) \quad \Rightarrow \quad Q_t^{[U]}(V_p)/V_p \in \mathcal{B}_0(E).$$

Last, but not least, note that the above discussion extends without difficulties to time varying models.

4.3. Some conjugacy principles

For any given $V \in \mathcal{B}_\infty(E)$, observe that for any positive function H,

$$H \in \mathcal{B}_{0,V}(E) \Longleftrightarrow V^H := V/H \in \mathcal{B}_\infty(E).$$

Thus, $\mathcal{Q}_t$ is a V-positive semigroup on $\mathcal{B}_V(E)$ if and only if the H-conjugate semigroup $\mathcal{Q}_t^H(f) := \mathcal{Q}_t(fH)/H$ is a V^H-positive semigroup on $\mathcal{B}_{V^H}(E)$. In this situation, any semigroup $Q \ll \mathcal{Q}^H$ dominated by $\mathcal{Q}^H$ yields for any $s \geqslant 0$ and $t > 0$ the Lyapunov estimate

$$Q_{s,s+t}(V^H)/V^H \leqslant c_t \;\; \mathcal{Q}_t(V)/V \in \mathcal{B}_0(E).$$

To get one step further, observe that

$$\mathcal{Q}_t(V)/V = \mathcal{Q}_t(1) \;\; \bar{\mathcal{Q}}_t(V)/V.$$

In this notation, for any $H \in \mathcal{B}_{0,V}(E)$ and any V-positive semigroup $\mathcal{Q}_t$ on $\mathcal{B}_V(E)$ such $\mathcal{Q}_t(1) \in \mathcal{B}_0(E)$ and $|||\bar{\mathcal{Q}}_t|||_V < \infty$ we have

$$Q \ll \mathcal{Q}^H \Rightarrow Q_{s,s+t}(V^H)/V^H \leqslant c_t \;\; \mathcal{Q}_t(1) \in \mathcal{B}_0(E). \tag{92}$$

We illustrate the above comparison principles with an elementary example. Let $\mathcal{E} := \mathbb{R}$ and $W \in \mathcal{B}_\infty(\mathbb{R})$ be some non-negative function. Consider the stochastic flow $X_t^W(x)$ of a one-dimensional Langevin diffusion on $\mathcal{E}$ with generator

$$L(f) = \frac{1}{2} \;\; e^{2W} \partial \big(e^{-2W} \partial f \big). \tag{93}$$

We associate with a given open connected interval $E \subset \mathcal{E}$, the sub-Markov semigroup Q_t on $\mathcal{B}_b(E)$ defined by

$$Q_t(f)(x) := \mathbb{E}\big(f(X_t^W(x)) \;\; 1_{T_{\partial E}^W(x) > t} \big) \quad \text{with} \quad T_{\partial E}^W(x) := \inf\big\{ t \geqslant 0 \;\; : \;\; X_t^W(x) \in \partial E \big\}. \tag{94}$$

Observe that

$$H := e^{-W} \Rightarrow U := H^{-1} \frac{1}{2} \;\; \partial^2 H = \frac{1}{2} \;\; \big((\partial W)^2 - \partial^2 W \big). \tag{95}$$

When $W = 0$ the flow $X_t^0(x) = B_t(x)$ coincides with the Brownian flow $B_t(x)$ starting at x. Thus, by a change of probability we check that

$$Q_t = \mathcal{Q}_t^H \quad \text{with} \quad \mathcal{Q}_t(f)(x) := \mathbb{E}\left(f(B_t(x)) \;\; 1_{T_{\partial E}^0(x) > t} \;\; e^{-\int_0^t U(B_s(x)) \;\; ds} \right). \tag{96}$$

Whenever $E =]0,1[$ the semigroup $\mathcal{Q}_t$ is dominated by the Dirichlet heat kernel on $]0,1[$ discussed in Section 3.3.3. When $E = \mathbb{R}$, respectively $E =]0,\infty[$, and $U(x) \geqslant c + \varsigma \; x^2/2$, for some $c < \infty$ and $\varsigma > 0$, the semigroup $\mathcal{Q}_t$ is dominated by the harmonic

oscillator discussed in Section 3.3.1, respectively the half-harmonic oscillator discussed in Section 3.3.2. All of these dominating semigroups are completely solvable with $\mathcal{Q}_t(1) \in \mathcal{B}_0(E)$ and known Lyapunov functions.

5. Boundary problems

Let $(\mathcal{E}, d)$ be a locally compact Polish space with a distinguished complete metric $d : (x,y) \in E^2 \mapsto d(x,y) \in \mathbb{R}_+$. We recall that these metric spaces are complete σ-compact and locally compact metric spaces, thus they have the Heine-Borel property, that is each closed and bounded subsets in $\mathcal{E}$ are compact.

We also recall that a subspace $E \subset \mathcal{E}$ is Polish if and only if it is the intersection of a countable collection of open subsets. The distance from $x \in \mathcal{E}$ to a measurable subset $\mathcal{A} \subset \mathcal{E}$ is denoted by

$$d(x, \mathcal{A}) := \inf\{d(x,y) \ : \ y \in \mathcal{A}\}.$$

We also denote by $\partial E := \bar{E} - E^{\circ}$ the boundary of some domain (open and connected) $E \subset \mathcal{E}$, where $\bar{E}$ and E° stand for the closure and the interior of a subset E.

In the further development of the article, χ stands for some decreasing positive function χ on $]0, \infty[$ such that for any $0 < \alpha < 1$ we have

$$\lim_{\alpha \to 0} \chi(\alpha) = +\infty \qquad \chi(\alpha) < 1/\alpha \quad \text{and} \quad \bar{\chi}(\alpha) := \int_0^{\alpha} \chi(u) du < \infty.$$

Definition 5.1. *We associate with χ the function $V_{\partial} \in \mathcal{C}(E)$ defined by*

$$V_{\partial} \ : \ x \in E \ \mapsto \ V_{\partial}(x) := \chi(d(x, \partial E)) \in]0, \infty[. \tag{97}$$

For instance, we can choose $\chi(u) = 1/u^{1-\epsilon}$, for some $\epsilon \in]0,1[$. For any $r > 0$ the r-sub-level sets of V_{∂} are given by the closed subsets

$$\mathcal{V}_{\partial}(r) := \{x \in E \ : \ V_{\partial}(x) \leqslant r\} = \{x \in E \ : \ d(x, \partial E) \geqslant \chi^{-1}(r)\}.$$

Note that $V_{\partial} \in \mathcal{C}_{\infty}(E)$ as soon as $\bar{E}$ is compact.

5.1. Bounded domains

Let $E \subset \mathcal{E} := \mathbb{R}^n$ be some bounded domain with locally Lipschitz boundary ∂E, for some $n \geqslant 1$. Consider a semigroup of integral operators

$$Q_t(x, dy) = q_t(x,y) \ dy \tag{98}$$

having for any $t > 0$ a bounded density $(x,y) \in E^2 \mapsto q_t(x,y) \in [0, \infty[$ w.r.t. the trace of the Lebesgue measure $\nu(dy) = dy$ on E. In this situation, we have the following lemma.

Lemma 5.2. *For any $t > 0$ we have*

$$V_{\partial} \in C_{\infty}(E) \quad \text{and} \quad ||Q_t(V_{\partial})|| \leqslant c_t \int_E \chi(d(x, \partial E)) \ dx < \infty. \tag{99}$$

The proof of the above lemma follows from an elementary change of variable formulae, thus it is provided in the appendix, on page 83.

The estimate (99) clearly applies to the class of sub-Markov semigroups $Q_t^{[U]}$ defined in (70) for any choice of the absorption potential function, as soon as the semigroup $Q^{[U]} \ll Q$ is dominated by a collection of integral operators $Q_t(x, dy)$ having a bounded density $q_t(x, y)$ on E^2 w.r.t. the Lebesgue measure on E. For instance, when the transition semigroup of the free evolution flow $X_t(x)$ in (70) has a density $p_t(x, y)$ for any non negative function f on E and any $x \in E$ we have

$$Q_t^{[U]}(f)(x) \leqslant \int q_t(x,y)\ f(y)\ dy \quad \text{with} \quad q_t(x,y) := p_t(x,y)\ 1_E(y).$$

We summarize the above discussion with the following proposition.

Proposition 5.3. *Assume that $Q^{[U]} \ll Q$ is dominated by a collection of integral operators Q_t satisfying (98). Then,*

$$Q_t^{[U]}(V_\partial)/V_\partial \leqslant c_t/V_\partial \in \mathcal{B}_0(E).$$

The choice of the Lyapunov function V is clearly not unique. For instance, when $E =]0,1[$ instead of V_∂ we can choose $V(x) := 1/\sqrt{x} + 1/\sqrt{1-x}$. For the Dirichlet heat kernel discussed in Section 3.3.3 we can also choose $V(x) = 1/\sin(\pi x)$.

We emphasize that sub-Markov integral operators on the compact interval $E = [0,1]$ with *a positive* continuous density w.r.t. the Lebesgue measure on E arise when the free evolution process is reflected at both sides of the interval. In this context the process is not conditioned by any type of non-absorption at the boundaries. In this context, the unit function $V = 1$ belongs to $\mathcal{B}_\infty(E)$. In the same vein, sub-Markov integral operators with mixed boundary conditions on the left-closed interval $E = [0,1[$, or respectively on the right-closed interval $E =]0,1]$ arise when the free evolution process is reflected at the Neumann boundary $\partial_N E := \{0\}$ and non-absorbed at the Dirichlet boundary $\partial_D E = \{1\}$, or respectively reflected at $\partial_N E := \{1\}$ and non-absorbed at $\partial_D E = \{0\}$.

More generally, consider a bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$ consisting of two disjoint connected components $\partial_D\Omega$ and $\partial_N\Omega$ closed in $\mathbb{R}^n$, and set $E := \Omega \cup \partial_N\Omega$. In this notation, the function $V_\partial(x) := \chi(d(x, \partial_D E))$ belongs to $\mathcal{C}_\infty(E)$. In addition, for any bounded density $q_t(x, y)$ on E^2 we have the uniform estimate

$$\int_E q_t(x,y)\ V_\partial(y)\ dy \leqslant c_t \int_E V_\partial(y)\ dy < \infty.$$

The above estimate also holds for the function $V_\partial(x) = \chi(d(x, \partial E))$.

5.2. Unbounded domains

When the domain E is not bounded the function $V_\partial \notin \mathcal{B}_\infty(E)$. In this context, one natural way to design a Lyapunov function $V \in \mathcal{B}_\infty(E)$ is to consider an auxiliary function $V_\mathcal{E} \in \mathcal{C}_\infty(\mathcal{E})$ with $V_\mathcal{E}(x) \geqslant 1$ for any $x \in E$. In this situation, we have

$$V := V_\partial + V_{\mathcal{E}} \in \mathcal{C}_\infty(E).$$

To check this claim, observe that the sub-level sets of V_∂ are given by the closed subsets

$$\mathcal{V}_\partial(r) := \{V_\partial \leqslant r\} = \{x \in E \ : \ d(x, \partial E) \geqslant \chi^{-1}(r)\} \subset E$$

and we have the compact inclusion

$$\mathcal{V}(r) := \{V \leqslant r\} \subset \mathcal{V}_{\mathcal{E}}(r) \cap \mathcal{V}_\partial(r) \quad \text{with} \quad \mathcal{V}_{\mathcal{E}}(r) := \{x \in \mathcal{E} \ : \ V_{\mathcal{E}}(x) \leqslant r\}.$$

This yields the following easily checked proposition.

Proposition 5.4. *For any $t > 0$ we have*

$$||Q_t(V_\partial)|| \vee ||Q_t(V_{\mathcal{E}})|| < \infty \quad \Rightarrow \quad Q_t(V)/V \leqslant c_t/V \in \mathcal{B}_0(E).$$

When $Q_t(1) \in \mathcal{B}_0(E)$ we also have

$$||\bar{Q}_t(V_\partial)|| \vee ||\bar{Q}_t(V_{\mathcal{E}})/V_{\mathcal{E}}|| < \infty \Rightarrow Q_t(V)/V \leqslant c_t \ \ Q_t(1) \in \mathcal{B}_0(E).$$

The design of a function $V_{\mathcal{E}}$ is rather flexible. For instance, assume that $\bar{Q} \ll P$ is dominated by some Markov integral operators P_t on $\mathcal{B}_b(\mathcal{E})$ such that $||P_t(V_{\mathcal{E}})/V_{\mathcal{E}}|| < \infty$ for some $V_{\mathcal{E}} \in \mathcal{B}_\infty(\mathcal{E})$. In this situation, we have $||\bar{Q}_t(V_{\mathcal{E}})/V_{\mathcal{E}}|| < \infty$ as well as

$$||V_{\mathcal{E}} \ \ Q_t(1)|| < \infty \Rightarrow ||Q_t(V_{\mathcal{E}})|| < \infty.$$

For instance, when P_t satisfies the sub-Gaussian estimates (43) on $\mathcal{E} = \mathbb{R}^n$ we can choose $V_{\mathcal{E}}(x) := 1 + ||x||^k$, for some $k \geqslant 1$, as soon as the function $Q_t(1)(x) \to_{||x|| \to \infty} 0$ faster than $||x||^{-k}$.

When the domain E and its boundary ∂E are both non-necessarily bounded, it may happens that $Q_t(1) \in \mathcal{B}_0(E)$ but $\bar{Q}_t(V_\partial) \notin \mathcal{B}_b(E)$. In this situation, we can use the following proposition.

Proposition 5.5. *Assume there exists some $V_{\mathcal{E}} \in \mathcal{C}_\infty(\mathcal{E})$ with $V_{\mathcal{E}}(x) \geqslant 1$ for any $x \in E$ and such that*

$$||\bar{Q}_t(V_\partial)/V_{\mathcal{E}}|| \vee ||\bar{Q}_t(V_{\mathcal{E}})/V_{\mathcal{E}}|| \vee ||Q_t(1)V_{\mathcal{E}}|| < \infty.$$

Then we have

$$Q_t(V)/V \leqslant c_t/V \in \mathcal{B}_0(E).$$

Proof. Using the following decompositions

$$Q_t(V_\partial) = Q_t(1)V_{\mathcal{E}} \ \bar{Q}_t(V_\partial)/V_{\mathcal{E}} \quad \text{and} \quad Q_t(V_{\mathcal{E}}) = Q_t(1)V_{\mathcal{E}} \ \bar{Q}_t(V_{\mathcal{E}})/V_{\mathcal{E}}$$

and applying Proposition 5.4 we have

$$||Q_t(V_\partial)|| \vee ||Q_t(V_{\mathcal{E}})|| < \infty \quad \text{and therefore} \quad Q_t(V)/V \leqslant c_t/V \in \mathcal{B}_0(E).$$

This ends the proof of the proposition. ■

The case $Q_t(1) \notin \mathcal{B}_0(E)$ can also be handle whenever the pair $(V_\partial, V_{\mathcal{E}})$ can be chosen so that

$$\forall \delta > 0 \qquad V_\partial \ V_{\mathcal{E}}^{\delta} \in \mathcal{C}_\infty(E). \tag{100}$$

For instance we can choose for some $\nu > 0$ and $\epsilon \in]0,1[$ the functions

$$V_{\mathcal{E}}(x) := \exp(\nu||x||) \quad \text{and} \quad \chi(u) := 1/u^{1-\epsilon}.$$

Observe that

$$d(x,\partial E) \leqslant ||x|| + d(0,\partial E) \quad \text{and} \quad V_\partial(x) \geqslant \chi(||x|| + (1 \vee d(0,\partial E)))$$

and for any $m \geqslant 0$ and $\delta > 0$ we have

$$V_{\mathcal{E}}(x) \geqslant c_\nu(m,\delta)\ (1 + ||x||)^{(m+1)/\delta}.$$

This implies that

$$V_{\mathcal{E}}(x)^\delta V_\partial(x) \geqslant c_2 \frac{(1+||x||)^{m+1}}{(||x|| + (1 \vee d(0,\partial E)))^{1-\epsilon}} \geqslant c\ (1+||x||)^{m+\epsilon}.$$

Using the fact that $V_{\mathcal{E}}(x) \geqslant 1$ for any $x \in E$, this implies that

$$\{x \in E \ : \ V_{\mathcal{E}}(x)^\delta V_\partial(x) \leqslant r\} \subset \{x \in E \ : \ c\ (1+||x||)^{m+\epsilon} \leqslant r\} \cap \{x \in E \ : \ V_\partial(x) \leqslant r\}.$$

We conclude that $V_{\mathcal{E}}^\delta V_\partial$ has compact level sets and (100) is satisfied.
In this context, we have the following proposition.

Proposition 5.6. *Consider a couple of functions $(V_\partial, V_{\mathcal{E}})$ satisfying (100). Assume there exists some parameters $t > 0$, $\delta_t > 0$ and $\epsilon \geqslant 0$ such that*

$$Q_t(V_{\mathcal{E}})/V_{\mathcal{E}} \leqslant c_t/V_{\mathcal{E}}^{\delta_t} \quad \text{and} \quad Q_t(V_\partial) \leqslant c_t\ V_{\mathcal{E}}^{\epsilon\delta_t}. \tag{101}$$

In this situation, for any $p > 1 + \epsilon$ we have

$$V := V_{\mathcal{E}}^{1-1/p}\ V_\partial^{1/p} \in \mathcal{C}_\infty(E)$$

as well as

$$Q_t(V)/V \leqslant c_t/(V_{\mathcal{E}}^{\delta_t\epsilon_p} V_\partial^{1/p}) \in \mathcal{C}_0(E) \quad \textit{with} \quad \epsilon_p := (1 - (1+\epsilon)/p) > 0.$$

Proof. Observe that for any $p > 1 + \epsilon$ we have

$$V_\partial V_{\mathcal{E}}^{p-1} \in \mathcal{C}_\infty(E) \quad \text{and therefore} \quad V := V_\partial^{1/p}\ V_{\mathcal{E}}^{1-1/p} \in \mathcal{C}_\infty(E).$$

In the same vein, for any $\epsilon \geqslant 0$ we have

$$(100) \Rightarrow V_\partial V_{\mathcal{E}}^{p\delta_t\epsilon_p} \in \mathcal{C}_\infty(E) \quad \text{and therefore} \quad V_\partial^{1/p}\ V_{\mathcal{E}}^{\delta_t\epsilon_p} \in \mathcal{C}_\infty(E).$$

On the other hand, using Hölder's inequality, we have

$$\begin{aligned} Q_t(V)/V &\leqslant (Q_t(V_{\mathcal{E}})/V_{\mathcal{E}})^{1-1/p}\ (Q_t(V_\partial)/V_\partial)^{1/p} \\ &\leqslant c_t(1)(Q_t(V_\partial)/(V_{\mathcal{E}}^{\delta_t(p-1)}V_\partial))^{1/p} \leqslant c_t(2)(1/(V_{\mathcal{E}}^{p\delta_t(1-(1+\epsilon)/p)}V_\partial))^{1/p}. \end{aligned}$$

This ends the proof of the proposition. ∎

The design of a function $V_{\mathcal{E}}$ satisfying (101) is rather flexible. For instance, (101) is automatically satisfied when $Q \ll P$ is dominated by some Markov integral operators P_t on $\mathcal{B}_b(\mathcal{E})$ such that

$$P_t(V_{\mathcal{E}})(x)/V_{\mathcal{E}}(x) \leqslant c_t(1)/V_{\mathcal{E}}(x)^{\delta_t}.$$

Section 2.4 discusses a variety of Lyapunov functions $V_{\mathcal{E}}$ satisfying the above condition for Markov diffusion semigroups. These Lyapunov functions can also be designed using the domination principles presented in Section 4.2. For instance, consider the semigroup $Q_t := \mathcal{Q}_t^{(a)}$ associated with the Langevin diffusion flow on a cylinder discussed in (87). In this situation, combining (85) with Proposition 2.18, for any $\nu \geqslant 0$ and $t > 0$ there exists some finite constant $\delta_t > 0$ such that

$$V_{\mathcal{E}}(x) := \exp(\nu|x|) \Rightarrow Q_t(V_{\mathcal{E}})/V_{\mathcal{E}} \leqslant c_t/V_{\mathcal{E}}^{\delta_t}.$$

Next, we illustrate the r.h.s. condition in (101) when q_t are sub-Gaussian densities; in the sense that for any $x, y \in E$ we have

$$q_t(x,y) \leqslant c_t \; g_t(x,y) \quad \text{with} \quad g_t(x,y) := \frac{1}{(2\pi\sigma_t^2)^{n/2}} \; \exp\left(-\frac{1}{2\sigma_t^2} \; ||y - m_t(x)||^2\right) \tag{102}$$

for some parameter $\sigma_t > 0$ and some non-necessarily bounded function m_t on E.

Proposition 5.7. *Let φ be a Lipschitz function on $\mathbb{R}^{n-1}$ with uniformly bounded gradient and set*

$$E := \{x = (x_i)_{1 \leqslant i \leqslant n} \in \mathbb{R}^n \; : \; x_n > \varphi(x_{-n})\} \quad \textit{with} \quad x_{-n} := (x_i)_{1 \leqslant i < n} \in \mathbb{R}^{n-1}.$$

Then the r.h.s. condition in (101) is met with $\epsilon = 0$ for any positive semigroup satisfying (102). The same property holds when the boundary ∂E can be decomposed as a finite union of graphs of differentiable functions on $\mathbb{R}^{n-1}$ with uniformly bounded gradients.

Proof. We choose $\alpha > 0$ sufficiently small so that for any

$$x \in D_\alpha(E) := \{x \in \bar{E} \; : \; d(x, \partial E) \leqslant \alpha\}$$

there exists a projection $\bar{x} \in \partial E$ with $d(x, \partial E) = ||x - \bar{x}||$. Let $\mathcal{C}_\varpi(\bar{x})$ be an interior cone with a given base vertex $\bar{x} = (x_{-n}, \varphi(x_{-n})) \in \partial E$ and a given half-opening angle ϖ around the axis $\mathcal{A}(\bar{x}) := \{(x_{-n}, x_n) \; : \; x_n \geqslant \varphi(x_{-n})\}$. For any $x \in \mathcal{A}(\bar{x})$ there exists a projection $\hat{x} \in \partial\mathcal{C}_\varpi(\bar{x})$ on the boundary $\partial\mathcal{C}_\varpi(\bar{x})$ with

$$d(x, \partial\mathcal{C}_\varpi(\bar{x})) = d(x, \hat{x}) = \cos\left(\frac{\pi}{2} - \varpi\right) \; (x_n - \varphi(x_{-n})) \leqslant d(x, \bar{x})$$

On the other hand, for any $y \in \partial E$ we have

$$z := (y_{-n}, \varphi(x_{-n})) \Rightarrow 0 \leqslant \frac{\pi}{2} - \varpi \leqslant \widehat{y\bar{x}z} \quad \text{and} \quad \tan(\widehat{y\bar{x}z}) = \frac{|\varphi(x_{-n}) - \varphi(y_{-n})|}{||x_{-n} - y_{-n}||}.$$

This yields the estimate

$$\cos\left(\frac{\pi}{2} - \varpi\right) \geqslant \cos\left(\widehat{y\bar{x}z}\right) = \frac{1}{\sqrt{1 + \tan^2(\widehat{y\bar{x}z})}}$$

from which we conclude that

$$0 \leqslant x_n - \varphi(x_{-n}) \leqslant \kappa \ d(x, \bar{x})$$

with

$$\kappa := \sqrt{1 + ||\nabla\varphi||^2} \quad \text{and} \quad ||\nabla\varphi|| := \sup_{y \in \mathbb{R}^{n-1}} ||\nabla\varphi(y)|| < \infty.$$

This implies that

$$\begin{aligned}
&\int_{D_\alpha(E)} \chi\big(d(y, \partial E)\big) q_t(x, y) \ dy \\
&\leqslant c_t(1) \int_{D_\alpha(E)} \chi\big(\big(y_n - \varphi(y_{-n})\big)/\kappa\big)\big) \ \exp\left(-\frac{1}{2\sigma_t^2}\left((y_n - \varphi(y_{-n})) + (\varphi(y_{-n}) - (m_t(x))_n)\right)^2\right) \\
&\quad \times \ \exp\left(-\frac{1}{2\sigma_t^2} \ ||y_{-n} - (m_t(x))_{-n}||^2\right) \ dy_n dy_{-n}.
\end{aligned}$$

Using the change of variables

$$z := \big(y_n - \varphi(y_{-n})\big)/\kappa \Rightarrow dy_n = \kappa \ dz$$

we find that

$$\begin{aligned}
&\int_{D_\alpha(E)} \chi\big(d(y, \partial E)\big) q_t(x, y) \ dy \\
&\leqslant \kappa \ c_t(1) \ \bar{\chi}(\alpha) \ \int_{\mathbb{R}^{n-1}} \ \exp\left(-\frac{1}{2\sigma_t^2} \ ||y_{-n} - (m_t(x))_{-n}||^2\right) \ dy_{-n} \leqslant c_t(2).
\end{aligned}$$

On the other hand, for any $\alpha > 0$ we have

$$\int_{E - D_\alpha(E)} \chi\big(d(y, \partial E)\big) \ q_t(x, y) \ dy \leqslant \chi(\alpha) \ ||Q_t(1)|| \leqslant c_t(3).$$

This ends the proof of the proposition. ■

5.3. Smooth boundaries

Next, we illustrate the Lyapunov conditions on V_∂ in the context of absolutely continuous sub-Markov semigroup of the form (98) with a bounded density $q_t(x, y)$ on a non-necessarily bounded domain $E \subset \mathbb{R}^n$ with smooth non-necessarily bounded $\mathcal{C}^2$-boundary with uniformly bounded interior curvature.

We assume that there exists $\alpha > 0$ sufficiently small so that every point of the α-offset of ∂E (a.k.a. α-tubular neighborhood) defined by

$$Tub_\alpha(\partial E) := \{x \in \mathbb{R}^n \ : \ d(x, \partial E) \leqslant \alpha\}$$

lies on some normal ray passing through a point on ∂E and no two normal rays passing through different points of ∂E intersect in $Tub_\alpha(\partial \mathrm{E})$. We let $N(z)$ be the unit normal vector at $z \in Tub_\alpha(\partial \mathrm{E})$ pointing inward E, and let $\mathcal{D}_r(E)$ the closed subset defined for any $r \leqslant \alpha$ by

$$\mathcal{D}_r(E) := \{x \in \bar{E} \ : \ d(x, \partial E) \leqslant r\} \quad \text{and} \quad \mathcal{D}_{-r}(E) := \{x \in \mathbb{R}^n - E \ : \ d(x, \partial E) \leqslant r\}.$$

In this notation, the inverse of the normal coordinate map

$$F \ : \ (z, r) \in \partial E \times [-\alpha, \alpha] \mapsto F(z, r) = z + r \ N(z) \in Tub_\alpha(\partial \mathrm{E}) \tag{103}$$

is given for any $x \in Tub_\alpha(\partial \mathrm{E})$ by

$$F^{-1}(x) = \big(\mathrm{proj}_{\partial E}(x), d_\alpha(x, \partial E)\big)$$

where $\mathrm{proj}_{\partial E}(x)$ stands for the projection of $x \in Tub_\alpha(\partial \mathrm{E})$ onto ∂E and $d_\alpha(x, \partial E)$ stands for the signed distance function

$$d_\alpha(x, \partial E) := d(x, \partial E) \ 1_{\mathcal{D}_\alpha(E)}(x) - d(x, \partial E) \ 1_{\mathcal{D}_{-\alpha}(E)}(x) \in [-\alpha, \alpha].$$

In addition, the inward normal $N(x)$ at any x on the $\mathcal{C}^2$ boundary ∂E is given by

$$\nabla d_\alpha(x, \partial E) = N(x).$$

The Hessian of the signed distance function on the boundary ∂E gives the Weingarten map $\mathcal{W}(x)$. With this notation at hand, we have

$$\int_{\mathcal{D}_\alpha(E)} f(d(y, \partial E)) \ q_t(x, y) \ dy = \int_0^\alpha f(r) \ q_t^\partial(x, r) \ dr$$

with the level-set density function

$$\begin{aligned} q_t^\partial(x, r) &:= \int_{\partial E_r} q_t(x, y) \ \sigma_{\partial, r}(dy) \\ &= \int_{\partial E} q_t(x, z + rN(z)) \ |\det(I - r \ \mathcal{W}(z))| \ \sigma_\partial(dz). \end{aligned} \tag{104}$$

In the above display, $\sigma_{\partial, r}(dz)$ stands for the Riemannian volume measure on the r-extended boundary

$$\partial E_r := \{x \in E \ : \ d(x, \partial E) = r\}.$$

Moreover, since E has uniformly bounded interior curvature, for any $r \leqslant \alpha$ we have

$$\kappa_\partial(\alpha) := \sup|\det(I - r \ \mathcal{W}(z))| < \infty \quad \text{and} \quad \kappa_\partial^-(\alpha) := \sup\big|\det\big(I + r \ \mathcal{W}(y)\big)\big| < \infty.$$

In the above display, the supremum is taken over all $z \in \partial E$, $y \in \partial E_r$, and $r \leqslant \alpha$. Several examples of hypersurface boundaries satisfying the above conditions are discussed in Section 8 (cf. for instance Proposition 8.4).

We denote by $\bar{q}_t^\partial \geqslant q_t^\partial$ the function defined as q_t^∂ by replacing q_t by $\bar{q}_t$. Using the fact that

$$\bar{Q}_t(V_\partial)(x) \leqslant \chi(\alpha) + \int_0^\alpha \ \chi(r) \ \bar{q}_t^\partial(x, r) \ dr$$

we readily check the following proposition.

Proposition 5.8. *For any $t > 0$ we have*

$$\begin{aligned} \sup_{0 \leqslant r \leqslant \alpha} \sup_{x \in E} q_t^\partial(x, r) < \infty &\Rightarrow Q_t(V_\partial) \leqslant \chi(\alpha) \ Q_t(1) + c_t(\alpha) \ \bar{\chi}(\alpha) \\ \sup_{0 \leqslant r \leqslant \alpha} \sup_{x \in E} \bar{q}_t^\partial(x, r) < \infty &\Rightarrow \bar{Q}_t(V_\partial) \leqslant \chi(\alpha) + c_t(\alpha) \ \bar{\chi}(\alpha). \end{aligned} \tag{105}$$

When the boundary ∂E is bounded, for any $t>0$ we have the estimate

$$||\bar{Q}_t(V_\partial)|| \leqslant c_t(\alpha) \left(\chi(\alpha) + \bar{\chi}(\alpha) \sup_{0\leqslant r\leqslant \alpha} \sigma_{\partial,r}(\partial E_r)\right). \tag{106}$$

We end this section with some practical tools to estimate the level-set density functions discussed in Section 5.3. Most of our estimates are based on the following technical lemma.

Lemma 5.9. *Consider a couple of non-negative functions f, g on $\mathbb{R}^n$ and some parameter $\alpha>0$ such that*

$$\sup_{||u||\leqslant\alpha} f(z+u) \leqslant \imath(\alpha)\ g(z) \quad \text{for some } \imath(\alpha)<\infty.$$

In this situation, we have the uniform estimate

$$\sup_{0\leqslant r\leqslant\alpha} \int_{\partial E_r} f(z)\ \sigma_{\partial,r}(dz) \leqslant \imath(\alpha)\ \kappa_\partial(\alpha) \int_{\partial E} g(z)\ \sigma_\partial(dz)$$

as well as the co-area estimate

$$\int_{\partial E} f(z)\ \sigma_\partial(dz) \leqslant \frac{1}{\alpha}\ \imath(\alpha)\ \kappa_\partial^-(\alpha) \int_{\mathcal{D}_\alpha(E)} g(z)\ dz.$$

The proof of the above lemma is provided in the appendix, on page 85.

Note that the level-set density function defined in (104) can be estimated for any $0\leqslant r\leqslant\alpha$ by the formula

$$q_t^\partial(x,r) \leqslant \kappa_\partial(\alpha) \int_{\partial E} q_t(x, z+rN(z))\quad \sigma_\partial(dz).$$

Proposition 5.10. *Assume that $q_t(x,y)\leqslant \varpi_t\ g_t(x,y)$ is dominated by some probability density $y\mapsto g_t(x,y)$ on $\mathbb{R}^n$ for some $t>0$ and some parameter $\varpi_t<\infty$. In addition, we have*

$$\sup_{||u||\leqslant\alpha} g_t(x,y+u) \leqslant \imath_t(\alpha)\ g_{\alpha,t}(x,y) \tag{107}$$

for some probability density $y\mapsto g_{\alpha,t}(x,y)$ and some $\imath_t(\alpha)<\infty$. In this situation, we have the uniform density estimates

$$\sup_{0\leqslant r\leqslant\alpha}\ \sup_{x\in E} q_t^\partial(x,r) \leqslant \varpi_t\ \imath_t(\alpha)\kappa_\partial^-(\alpha)\kappa_\partial(\alpha)/\alpha. \tag{108}$$

Proof. By (107) for any $0\leqslant r\leqslant\alpha$ we have

$$q_t^\partial(x,r) \leqslant \varpi_t \quad \kappa_\partial(\alpha) \int_{\partial E} g_t(x, z+rN(z))\quad \sigma_\partial(dz).$$

On the other hand, we have

$$\int_{\mathcal{D}_\alpha(E)} g_{t,\alpha}(x,y)\ dy \leqslant 1.$$

The estimate (108) is now a direct consequence of the co-area estimate stated in Lemma 5.9. This ends the proof of the proposition. ■

We illustrate the above condition when q_t are the sub-Gaussian densities discussed in (102). In this situation, using the fact that $2a'b \leqslant \frac{1}{\epsilon}||a||^2 + \epsilon||b||^2$ for any $0 < \epsilon < 1$ and $||u|| \leqslant \alpha$ we check that

$$-\frac{1}{2\sigma_t^2}\ ||(y+u) - m_t(x)||^2 \leqslant -\frac{(1-\epsilon)}{2\sigma_t^2}\ ||y - m_t(x)||^2 + \frac{1}{2\sigma_t^2}\left(\frac{1}{\epsilon} - 1\right)\ \alpha^2.$$

In this context, condition (107) is met with the Gaussian density

$$g_{\alpha,t}(x,y) := \frac{e^{-\frac{1}{2\sigma_t(\epsilon)^2}\ ||y-m_t(x)||^2}}{(2\pi\sigma_t(\epsilon)^2)^{n/2}}$$

with

$$\iota_t(\alpha) := c_t\ e^{\frac{\alpha^2/\epsilon}{2\sigma_t(\epsilon)^2}} \quad \text{and} \quad \sigma_t(\epsilon)^2 := \sigma_t^2/(1-\epsilon).$$

6. Riccati type processes

6.1. Positive diffusions

Consider the Riccati type diffusion on $E =]0, +\infty[$ defined for any $x \in E$ by

$$dX_t(x) = \left(a_0 + a_1\ X_t(x) - b\ X_t(x)^2\right)\ dt + \sigma_1(X_t(x))\ dB_t^1 + \sigma_2(X_t(x))\ dB_t^2,\ X_0(x) = x$$

for some Brownian motion (B_t^1, B_t^2) on $\mathbb{R}^2$, the diffusion functions

$$\sigma_1(x) := \varsigma_1\ \sqrt{x} \qquad \sigma_2(x) := \varsigma_2\ x$$

and the parameters

$$a_1 \in \mathbb{R} \qquad a_0 > \varsigma_1^2 \qquad b > 0 \quad \text{and} \quad \varsigma_1, \varsigma_2 \geqslant 0.$$

Applying Itô's formula, we readily check that

$$\partial_t \mathbb{E}(X_t(x)) \leqslant\ \text{Ricc}(\mathbb{E}(X_t(x))) \quad \text{and} \quad \partial_t \mathbb{E}(1/X_t(x)) \leqslant\ \text{Ricc}^-(\mathbb{E}(1/X_t(x)))$$

with the Riccati drift functions defined by

$$\text{Ricc}(z) := a_0 + a_1 z - bz^2 \quad \text{and} \quad \text{Ricc}^-(z) := a_0^- + a_1^- z - b^- z^2 \tag{109}$$

with the parameters

$$a_0^- := b \quad a_1^- := (\varsigma_2^2 - a_1) \quad \text{and} \quad b^- := a_0 - \varsigma_1^2.$$

Consider the Lyapunov function $V \in \mathcal{B}_\infty(E)$ defined by $V(x) := x + 1/x$. By well-known properties of Riccati flows, for any $t > 0$ we have $||P_t(V)|| < \infty$. For a more thorough discussion on this class of one-dimensional Riccati diffusions, we refer to the article [55].

6.2. Matrix valued diffusions

Let $\bar{E}$ and E be the space of $(n \times n)$-positive semi-definite and definite matrices respectively. Also let $\lambda_1(x) \geqslant ... \geqslant \lambda_n(x)$ denote the ordered eigenvalues of $x \in \bar{E}$. Let $\mathcal{W}_t$ denotes an $(n \times n)$-matrix with independent Brownian entries. Also let A be an $(n \times n)$-matrix with real entries and let $R, S \in E$. We associate with these objects the E-valued diffusion

$$dX_t \;=\; (AX_t + X_tA' + R - X_tSX_t)\; dt \;+\; \frac{\epsilon}{2}\Big[X_t^{1/2} d\mathcal{W}_t\; R^{1/2} + R^{1/2}\; d\mathcal{W}_t'\; X_t^{1/2}\Big].$$

Whenever $\epsilon \leqslant 2/\sqrt{n+1}$, the diffusion X_t has a unique strong solution that never hits the boundary $\partial E = \bar{E} - E$. In addition, the transition semigroup P_t of X_t is strongly Feller and admits a smooth density w.r.t. the Lebesgue measure on E, thus it is irreducible. Furthermore, when $\epsilon^2(1+n)/2 \leqslant \lambda_n(R)/\lambda_1(R)$ then the function $V(x) = \mathrm{Tr}(x) + \mathrm{Tr}(x^{-1})$ is a Lyapunov function with compact level subsets. For a detailed proof of the above assertion for more general classes of Riccati matrix valued diffusions we refer to [56] (see for instance the stability Theorem 2.4 and Section 5.4 in [56]).

6.3. Logistic birth and death process

Let $X_t(x)$ be the stochastic flow on $E := \mathbb{N} - \{0\}$ with generator L defined for any $f \in \mathcal{B}_b(E)$ and $x \geqslant 2$ by

$$L(f)(x) = J(x, x-1)\;\; (f(x-1) - f(x)) + J(x, x+1)\;\; (f(x+1) - f(x))$$

and for $x = 1$ by

$$L(f)(1) = J(1,2)(f(2) - f(1)).$$

In the above display, the birth and death rates are given by

$$J(x, x+1) := \lambda_b\;\, x + \upsilon_b \quad \text{and} \quad J(x, x-1) := \lambda_d\;\, x + \lambda_l\;\, x(x-1) + \upsilon_d \tag{110}$$

for some non-negative parameters $\lambda_d, \lambda_b, \upsilon_b, \upsilon_d \geqslant 0$ and $\lambda_l > 0$. Consider the identity function $V : x \in E \mapsto V(x) = x$. For any $x \geqslant 2$ we have

$$L(V)(x) = J(x, x+1) - J(x, x-1) = (F^{\circ} V)(x)$$

with the concave function

$$z \in \mathbb{R}_+ \mapsto F(z) := (\upsilon_b - \upsilon_d) + (\lambda_b + \lambda_l - \lambda_d)\;\, z - \lambda_l\;\, z^2 \in \mathbb{R}. \tag{111}$$

Observe that

$$L(V)(1) - F(V(1)) = J(1,2) - F(1) = J(1,0) = \upsilon_d + \lambda_d.$$

This yields the estimate

$$\begin{aligned} P_t(L(V))(x) &= P_t(1_{[2,\infty[}\;\, L(V))(x) + P_t(1_{\{1\}}\;\, L(V))(x) \\ &= P_t((F^{\circ} V))(x) + P_t(1_{\{1\}})(x)\;\; (L(V)(1) - F(V(1))) \\ &\leqslant F(P_t(V))(x) + J(1,0) \end{aligned}$$

from which we check that

$$\partial_t P_t(V)(x) \leqslant \ \mathrm{Ricc}(P_t(V)(x))$$

with the Riccati drift function defined in (109) with the parameters

$$a_0 := \upsilon_b + \lambda_d \qquad a_1 := \lambda_b + \lambda_l - \lambda_d \quad \text{and} \quad b := \lambda_l > 0.$$

By well-known properties of Riccati flows, for any $t > 0$ we conclude that $||P_t(V)|| < \infty$.

6.4. Multivariate birth and death processes

We denote by $e := \{e_i,\ 1 \leqslant i \leqslant n\}$ the collection of column vector e_i on $\{0,1\}^n$ with entries $e_i(j) = 1_{i=j}$ and with a slight abuse of notation we denote by 0 the null state in $\mathbb{N}^n$. Let $X_t(x)$ be a stochastic flow on $E = \mathbb{N}^n - \{0\}$ with generator L defined by

$$L(f)(x) := \sum_{y \in E} J(x,y)\ (f(y) - f(x)). \tag{112}$$

Let $\lambda, \mu, \upsilon, \varsigma$ be some column vectors and let C, D some $(d \times d)$-matrices with real entries such that for any $1 \leqslant i \leqslant d$ and any $x \in E$ we have

$$J(x, x + e_i) := \upsilon_i + x_i\ (\lambda_i + (Cx)_i) \geqslant 0 \quad \text{and} \quad J(x, x - e_i) := \varsigma_i + x_i\ (\mu_i + (Dx)_i) \geqslant 0.$$

We also set

$$J(x,y) = 0 \quad \text{as soon as} \quad |x - y| \geqslant 2.$$

We further assume that

$$|\upsilon| \geqslant |\varsigma| \qquad B := (D - C) \geqslant b\ I > 0 \quad \text{for some } b\ >\ 0.$$

and we set

$$a_0 := |\upsilon| - |\varsigma| \geqslant 0 \qquad a_1 := \vee_{1 \leqslant i \leqslant n}(\lambda_i - \mu_i)$$

and for any $x \in \mathbb{N}^n$

$$||x|| := \left(\sum_{1 \leqslant i \leqslant n} x_i^2 \right)^{1/2} \geqslant |x| := \sum_{1 \leqslant i \leqslant n} x_i.$$

Consider the Lyapunov function

$$x \in E \mapsto V(x) = |x| \in \mathbb{N}_+.$$

Note that V is locally bounded with finite level sets and for any $x \in E - e$ we have

$$L(V)(x) = \sum_{1 \leqslant i \leqslant n} \left(\left(\upsilon_i + x_i\ (\lambda_i + (Cx)_i) \right) - \left(\varsigma_i + x_i\ (\mu_i + (Dx)_i) \right) \right).$$

In this situation, we have the formula

$$L(V)(x) = a_0 + (\lambda - \mu)'x - x'Bx \leqslant a_0 + a_1|x| - b||x||^2. \tag{113}$$

On the other hand, for any $y = e_j$ we have

$$L(V)(y) = \sum_{1 \leqslant i \leqslant n} J(y, y + e_i).$$

This implies that

$$\begin{aligned} P_t(L(V)) &= P_t(1_{E-e}\ L(V)) + P_t(1_e\ L(V)) \\ &= a_0 + a_1\ P_t(V) - b\ P_t(V^2) \\ &\quad + \sum_{1 \leqslant j \leqslant n} P_t(1_{e_j})\ \left(L(V)(e_j) - \left(a_0 + (\lambda - \mu)' e_j - e_j' B e_j\right)\right) \end{aligned}$$

from which we readily check that

$$\partial_t \mathbb{E}(V(X_t(x))) \leqslant a_0^+ + a_1 \mathbb{E}(V(X_t(x))) - b\ \left(\mathbb{E}(V(X_t(x)))\right)^2$$

with

$$\begin{aligned} a_0^+ &:= a_0 + \sum_{1 \leqslant j \leqslant d} \left(\sum_{1 \leqslant i \leqslant d} J(e_j, e_j + e_i) - \left(|\upsilon| - |\varsigma| + (\lambda - \mu)' e_j - e_j'(D - C) e_j\right)\right) \\ &= a_0 + \sum_{1 \leqslant j \leqslant d} \left(|\varsigma| + \mu' e_j + e_j' D e_j\right). \end{aligned}$$

We conclude that $||P_t(V)|| < \infty$, for any $t > 0$. The semigroup analysis discussed above can be extended without difficulties to more general process on countable spaces models satisfying condition (113). The extension to time varying models can also be handle using a more refined analysis on time varying Riccati equations.

We also mention that the case $|\upsilon| = 0 = |\varsigma|$ coincides with the competitive and multivariate Lotka-Volterra birth and death process discussed in Theorem 1.1 in [57].

7. Some conditional diffusions

7.1. Coupled harmonic oscillators

Consider the $\mathbb{R}^n$-valued diffusion (37) with $(b(x), \sigma(x)) = (Ax, \Sigma)$, for some *non-necessarily stable* drift matrix A and some diffusion matrix Σ with appropriate dimensions. We associate with a given semi-definite positive $(n \times n)$ matrix $S \geqslant 0$ the potential function

$$U(x) := \frac{1}{2}\ x' S x \quad \text{and we set} \quad R = \Sigma \Sigma'. \tag{114}$$

We assume that the pairs $(A, R^{1/2})$ and $(A', S^{1/2})$ are both controllable. Let $Q_t = Q_t^{[U]}$ be the sub-Markov semigroup defined in (70) on the Euclidean space $\mathcal{E} = E = \mathbb{R}^n$. As shown in [58], the leading-triple (ρ, h, η_∞) discussed in (61) is given by

$$\begin{aligned} \rho &= -\ \mathrm{Tr}(R q_\infty)/2 = -\ \mathrm{Tr}(p_\infty S)/2 \\ h(x) &= \exp\left(-x' q_\infty x/2\right) \quad \text{and} \quad \eta_\infty(dx) = \frac{\exp\left(-x' p_\infty^{-1} x/2\right)}{\sqrt{\det(2\pi p_\infty)}}\ dx, \end{aligned} \tag{115}$$

with the positive fixed points p_∞ and q_∞ of the dual algebraic Riccati matrix equation

$$A p_\infty + p_\infty A' + R - p_\infty S p_\infty = 0 \quad \text{and} \quad A' q_\infty + q_\infty A + S - q_\infty R q_\infty = 0.$$

In this context, the h-process, denoted $(X_t^h(x))_{t \geq 0}$ and defined by the stochastic differential equation

$$dX_t^h(x) = A^h X_t^h(x)\ dt + \Sigma\ dB_t \quad \text{with} \quad A^h := A - R\ q_\infty. \tag{116}$$

Our controllability conditions ensure that A^h is a stable matrix. Note that $X_t^h(x)$ is an $\mathbb{R}^n$-valued Gaussian random variable with mean $m_t^h(x)$ and covariance matrix $p_t^h \in \mathbb{R}^{n\times n}$ given for any $t>0$ by

$$m_t^h(x) = \exp(A^h t)\ x \quad \text{and} \quad p_t^h = \int_0^t \exp(A^h s)\ R \exp\left((A^h)' s\right)\ ds > 0.$$

This yields the explicit formula

$$P_t^h(x,dy) = \frac{1}{\sqrt{\det(2\pi p_t^h)}}\ \exp\left(-\frac{1}{2}(y-m_t^h(x))'(p_t^h)^{-1}(y-m_t^h(x))\right)\ dy.$$

Moreover the invariant measure $\eta_\infty^h = \eta_\infty^h \mathcal{P}_t^h$ is unique and given by

$$\eta_\infty^h(dx) = \frac{1}{\sqrt{\det(2\pi p_\infty^h)}}\ \exp\left(-\frac{1}{2}y'(p_\infty^h)^{-1}y\right)\ dy$$

with the limiting covariance matrix

$$p_\infty^h := \int_0^\infty \exp(A^h s)\Sigma^2 \exp\left((A^h)' s\right)\ ds = (p_\infty^{-1} + q_\infty)^{-1} > 0.$$

For any time horizon $t \geqslant 0$ and any measurable function F on the set $\mathcal{C}([0,t],\mathbb{R}^n)$ of continuous paths from $[0,t]$ into $\mathbb{R}^n$ we have the path space exponential change of measure Feynman-Kac formula

$$\mathbb{E}\left(F(\mathbb{X}_t(x))\exp\left(\int_0^t \mathbb{U}_s(\mathbb{X}_s(x))\ ds\right)\right) = e^{\rho t}\ h(x)\ \mathbb{E}\Big(F(\mathbb{X}_t^h(x))/h(X_t^h(x))\Big)$$

with the historical processes

$$\mathbb{X}_t(x) := (X_s(x))_{0\leqslant s\leqslant t}, \qquad \mathbb{X}_t^h(x) := (X_s^h(x))_{0\leqslant s\leqslant t} \quad \text{and} \quad \mathbb{U}_s(\mathbb{X}_s(x)) := U_s(X_s(x)).$$

This yields the conjugate formulae

$$Q_t(f) = e^{\rho t}\ h\ P_t^h(f/h).$$

We denote by $(m_t(x), p_t) \in (\mathbb{R}^n \times \mathbb{R}^{n\times n})$ the mean and covariance parameters satisfying the linear evolution and the Riccati matrix differential equations

$$\begin{cases} \partial_t m_t(x) &= (A - p_t S)\ m_t(x) \\ \partial_t p_t &= A p_t + p_t A' + \Sigma^2 - p_t S p_t \quad \text{with} \quad (m_0(x), p_0) = (x, 0). \end{cases} \tag{117}$$

The next proposition provides an explicit description of these semigroups.

Proposition 7.1 ([58]). *For any time horizon $t>0$ we have $p_t > 0$ and the normalized semigroup $\bar{Q}_t$ defined in (53) is given by*

$$\bar{Q}_t(x,dy) = \frac{1}{\sqrt{\det(2\pi p_t)}}\ \exp\left(-\frac{1}{2}(y-m_t(x))' p_t^{-1}(y-m_t(x))\right)\ dy \tag{118}$$

as well as

$$-2\log Q_t(1)(x) = x'\left(\int_0^t F_s' S F_s \; ds\right)x + \int_0^t \mathrm{Tr}(S p_s) \; ds$$

with the fundamental matrix semigroup F_t starting at $F_0 = I$ given by

$$\partial_t F_t = (A - p_t S) \; F_t.$$

Observe that the normalized Markov operator $\bar{Q}_t$ satisfies (43) and (44) with the parameters

$$c_t = \frac{1}{\sqrt{\det(2\pi p_t)}}, \;\; \sigma_t^2 = \lambda_{\max}(p_t) \quad \text{and} \quad \epsilon_\tau = |e^{\tau(A - p_\infty S)}| \to 0 \quad \text{as} \quad \tau \to \infty \tag{119}$$

for some matrix norm $|.|$. The r.h.s. assertion is a direct consequence of the Floquet representation theorem presented in [59] (cf. (1.3) and Theorem 1.1) and the fact that $(A - p_\infty S)$ is a stable matrix. Applying Lemma 2.18 for any $\nu \geqslant 0$ and $t > 0$ there also exists some finite constant $\delta_t > 0$ such that

$$V(x) := \exp(\nu|x|) \Rightarrow \bar{Q}_t(V)/V \leqslant c_t/V^{\delta_t}.$$

Using Proposition 7.1, for any $k \geqslant 0$ and $t \geqslant 0$ it is also readily checked that

$$V(x) := (1 + ||x||)^k \Rightarrow ||\bar{Q}_t(V)/V|| < \infty \quad \text{and} \quad ||Q_t(V)|| < \infty.$$

7.2. Half-harmonic linear diffusions

For one dimensional models, the coupled harmonic oscillator discussed in Section 7.1 resumes to one dimensional linear diffusion

$$dX_t(x) = a \; X_t(x) \; dt + dB_t \quad \text{and the potential} \quad U(x) = \varsigma x^2/2 \tag{120}$$

for some parameters $\varsigma > 0$ and $a \in \mathbb{R}$. We set $\beta := a + \sqrt{a^2 + \varsigma}$. In this notation, the leading pair $(\rho, h) = (\rho_1, \varphi_1)$ is given by

$$\rho = -\beta/2 \quad \text{and} \quad h(x) = ((\beta - a)/\pi)^{1/4} \exp\left(-\beta x^2/2\right). \tag{121}$$

The quasi-invariant measure is therefore given by

$$\eta_\infty(dx) = \sqrt{\frac{\varsigma}{2\pi\beta}} \; \exp\left(-\varsigma x^2/(2\beta)\right) \; dx.$$

Therefore, the h-process resumes to the Ornstein-Uhlenbeck diffusion

$$dX_t^h(x) = -b \; X_t^h(x) \; dt + dB_t \tag{122}$$

with the invariant measure

$$\eta_\infty^h(dx) := \sqrt{\frac{b}{\pi}} \; \exp(-b \; x^2) \; dx \quad \text{with} \quad b := (\beta - a) = \sqrt{a^2 + \varsigma} > 0.$$

Note that any Ornstein-Uhlenbeck process can be seen as the h-process associated with a non absorbed (possibly transient) linear diffusion evolving in some quadratic potential well.

In this context, Proposition 7.1 is also satisfied with the mean and variance parameters

$$\begin{cases} \partial_t m_t(x) &= (a - p_t\varsigma)\ m_t(x) \\ \partial_t p_t &= 2ap_t + 1 - \varsigma p_t^2 \quad \text{with} \quad (m_0(x), p_0) = (x, 0). \end{cases} \tag{123}$$

We also have

$$-\frac{2}{\varsigma}\log \mathcal{Q}_t(1)(x) = \bar{p}_t + \chi_t\ x^2 \tag{124}$$

with

$$\chi_t := \int_0^t \exp\left(-2\int_0^s (a - p_u\varsigma)du\right)\ ds \quad \text{and} \quad \bar{p}_t := \int_0^t p_s ds.$$

The half-harmonic semigroup associated with the flow $X_t(x)$ is defined for any $x \in E :=]0, \infty[$ and $f \in \mathcal{B}_b(E)$ by the formulae

$$Q_t(f)(x) := \mathbb{E}\left(f(X_t(x))\ 1_{T(x)>t}\ \exp\left\{-\int_0^t U(X_s(x))\ ds\right\}\right). \tag{125}$$

In the above display, $T(x)$ stands for the hitting time of the origin. In terms of the h-process of the flow in the harmonic potential (122) we also have the conjugate formula

$$Q_t(f)(x) = e^{t\rho}\ e^{-\beta x^2/2}\ \mathbb{E}\Big(f(Y_t(x))\ e^{\beta Y_t(x)^2/2}\ 1_{T^Y(x)>t}\Big) \tag{126}$$

with the parameters (ρ, β) defined in (121) and the Ornstein-Uhlenbeck diffusion flow defined by

$$dY_t(x) = -b\ Y_t(x)\ dt + dB_t \quad \text{with} \quad b := (\beta - a) > 0.$$

In the above display, $T^Y(x)$ stands for the hitting time of the origin by the flow $Y_t(x)$ starting at $x > 0$. Arguing as in Section 3.3.2 we check that

$$\begin{aligned} &Q_t(x, dy) \\ &= \sinh(y\ m_t(x))\ \exp\left(-\frac{\varsigma}{2}\left(\chi_t\ x^2 + \bar{p}_t\right)\right)\ \times \sqrt{\frac{2}{\pi p_t}}\ \exp\left(-\frac{y^2 + m_t(x)^2}{2p_t}\right)\ 1]0,\infty[(y)\ dy \end{aligned}$$

with the parameters $(m_t(x), p_t)$ and $(\chi_t, \bar{p}_t)$ defined in (123) and (124).

Arguing as in (77), choosing the Lyapunov function $V(x) = x^n + 1/x$, for some $n \geqslant 1$, we readily check that

$$V \in \mathcal{C}_\infty(E) \quad \text{and} \quad Q_t(V)/V \leqslant c_t/V\ \ \in \mathcal{C}_0(E). \tag{127}$$

7.3. Linear diffusions in some domains

Consider the one-dimensional stochastic flow $Y_t(x)$ of an Ornstein-Uhlenbeck

$$dY_t(x) = -b\ Y_t(x)\ dt + dB_t \quad \text{for some} \quad b > 0.$$

In the above display, B_t is a one-dimensional Brownian motion starting at the origin. For a given $x \in E :=]0, \infty[$, we let $T^Y(x)$ be the hitting time of the origin by the flow $Y_t(x)$ starting at $x > 0$. Consider the semigroup

$$Q_t^Y(f)(x) := \mathbb{E}(f(Y_t(x))\ 1_{T^Y(x)>t}).$$

Choosing $(a, \varsigma, \beta, \rho) = (0, b^2, b, -b/2)$ in (121), formula (126) takes the form

$$Q_t^Y(f)(x) = e^{-\rho t}\ H(x)^{-1} Q_t(fH)(x) \quad \text{with} \quad H(x) = \exp\left(-bx^2/2\right)$$

with the semigroup Q_t defined in (125) with $U(x) = b^2x/2$.

For any given $n \geqslant 1$ we have

$$V(x) := x^n + 1/x \Rightarrow V \in \mathcal{C}_\infty(E) \quad \text{and} \quad V^H := V/H \in \mathcal{C}_\infty(E).$$

Using (127) we conclude that

$$V^H \in \mathcal{C}_\infty(E) \quad \text{and} \quad Q_t^Y(V^H)/V^H = e^{-\rho t}\ Q_t(V)/V \leqslant c_t/V \in \mathcal{C}_0(E).$$

The long time behavior of the positive semigroup Q_t^Y is also studied in [60], and more recently in [61] in terms of the tangent of the h-process.

More generally, consider the $\mathbb{R}^n$-valued diffusion flow $X_t(x)$ defined in (37) with $(b(x), \sigma(x) = (Ax, \Sigma)$, for some matrices (A, Σ) with appropriate dimensions. Assume that $R := \Sigma\Sigma'$ is positive semi-definite and the pair of matrices $(A, R^{1/2})$ are controllable. In this situation, the Markov semigroup P_t of the stochastic flow $X_t(x)$ satisfies the sub-Gaussian estimate (102) for some parameters $(\sigma_t, m_t(x))$.

Consider a domain $E \subset \mathbb{R}^n$ with $\mathcal{C}^2$-boundary with uniformly bounded interior curvature. For any given $x \in E$, let Q_t be the sub-Markov semigroup

$$Q_t(f)(x) := \mathbb{E}(f(X_t(x))\ 1_{T(x)>t}) \quad \text{with} \quad T(x) := \inf\left\{t \geqslant 0\ :\ X_t(x) \in \partial E\right\}. \tag{128}$$

We clearly have $Q_t(V_\partial) \leqslant P_t(V_\partial)$, with the function V_∂ defined in (97). When E is none necessarily bounded but its boundary ∂E is bounded we known from (106) that $||Q_t(V_\partial)|| < \infty$. For non-necessarily bounded boundaries the sub-Gaussian property (102) ensures that $||Q_t(V_\partial)|| < \infty$.

When E is bounded, applying Lemma 5.2 (see also Proposition 5.3) we have

$$V_\partial \in \mathcal{C}_\infty(E) \quad \text{and} \quad Q_t(V_\partial)/V_\partial \leqslant c_t/V_\partial \in \mathcal{C}_0(E).$$

For unbounded domains we need to ensure that A is stable so that (48) is satisfied for some norm $|.|$ on $\mathbb{R}^n$. In this situation, applying Proposition 2.19 for any $t > 0$ there exists some $\delta_t > 0$ such that

$$V_\mathcal{E}(x) := \exp(\nu|x|) \Rightarrow Q_t(V_\mathcal{E})/V_\mathcal{E} \leqslant c_t/V_\mathcal{E}^{\delta_t}.$$

Applying Proposition 5.6 with $\epsilon = 0$, for any $p > 1$ we conclude that

$$V_p := V_{\mathcal{E}}^{1-1/p}\ V_{\partial}^{1/p} \in \mathcal{C}_\infty(E) \quad \text{and} \quad Q_t(V_p)/V_p \leqslant c_t\ \Theta_{p,t} \tag{129}$$

with the function

$$\Theta_{p,t} := 1/(V_{\mathcal{E}}^{\delta_t(1-1/p)} V_{\partial}^{1/p}) \in \mathcal{C}_0(E).$$

7.4. Langevin diffusions in some domains

Consider the semigroup Q_t of the one-dimensional Langevin diffusion defined in (94) with $E =]0, \infty[$ and a quadratic confinement potential

$$W(x) = x^2/2 \Rightarrow H(x) := e^{-W(x)} = e^{-x^2/2} \quad \text{and} \quad U := \frac{1}{2}\ (x^2 + 1).$$

In this case, the semigroup $\mathcal{Q}_t$ defined in (96) coincides with the semigroup of the half-harmonic oscillator discussed in Section 3.3.2. By (77) for any $n \geqslant 1$ we have

$$V(x) := x^n + 1/x \Rightarrow \mathcal{Q}_t(V)/V \leqslant c_t/V \ \in \mathcal{C}_0(E).$$

Notice that

$$V^H(x) := V(x)/H(x) = x^n\ e^{x^2/2} + \frac{e^{x^2/2}}{x}. \tag{130}$$

Using (96) we conclude that

$$V^H \in \mathcal{C}_\infty(E) \quad \text{and} \quad Q_t(V^H)/V^H = \mathcal{Q}_t(V)/V \leqslant c_t/V \in \mathcal{C}_0(E).$$

More generally, consider the case $E =]0, \infty[$ with at least a quadratic confinement potential U, in the sense that

$$U(x) = \frac{1}{2}\ \left((\partial W)^2 - \partial^2 W\right)(x) \geqslant\ U_2(x) := c + \varsigma\ x^2/2 \quad \text{for some } \varsigma > 0.$$

In this situation, $\mathcal{Q}_t \ll \mathcal{Q}^{[U_2]}$ is dominated by the semigroup $\mathcal{Q}^{[U_2]}$ of the half-harmonic oscillator discussed in Section 3.3.2. Arguing as in (130) we have

$$H := e^{-W} \qquad V^H := V/H \in \mathcal{C}_\infty(E) \quad \text{and} \quad Q_t(V^H)/V^H \leqslant c_t/V \in \mathcal{C}_0(E).$$

For instance, whenever the confinement potential W is chosen so that

$$W(x) \geqslant \epsilon_0 \log x + W_1(x) \quad \text{for some } 0 \leqslant \epsilon_0 < 1$$

and some function $W_1 \geqslant 1$ such that $W_1(x) \to_{x \to \infty} \infty$ we have

$$H = e^{-W} \Rightarrow V^H(x) := V(x)/H(x) = x^n\ e^{W(x)} + \frac{e^{W(x)}}{x} \geqslant x^n\ e^{W(x)} + \frac{e^{W_1(x)}}{x^{1-\epsilon_0}}.$$

Using (96) we conclude that

$$V^H \in \mathcal{C}_\infty(E) \quad \text{and} \quad Q_t(V^H)/V^H = \mathcal{Q}_t(V)/V \leqslant c_t/V \in \mathcal{C}_0(E).$$

We illustrate the above result, with the logistic diffusion discussed in [62]. Consider the generalized Feller diffusion

$$dY_t(x) := \left(2a\ Y_t(x) - (8b/\sigma^2)\ Y_t(x)^2\right)\ dt + \sigma\ \sqrt{Y_t(x)}\ dB_t$$

starting at $x \in E :=]0, \infty[$. In the above display, B_t is a one dimensional Brownian motion starting at the origin and $a, b, \sigma > 0$ some parameters. Observe that

$$X_t(x) := (2/\sigma)\sqrt{Y_t(x)} \Rightarrow dX_t(x) = -\partial W(X_t(x))\ dt + dB_t$$

with the potential function

$$\partial W(x) = \frac{1}{2x} - a\ x + b\ x^3 \qquad \text{with} \quad a, b > 0.$$

Thus, choosing

$$W(x) = \frac{1}{2}\ \log x + b\frac{x^4}{4} - a\frac{x^2}{2}$$

we readily check that

$$V^H(x) := e^{W(x)}(x^n + 1/x) = \left(x^{n-1/2} + 1/\sqrt{x}\right)\ e^{b\frac{x^4}{4} - a\frac{x^2}{2}} \Rightarrow V^H \in \mathcal{C}_\infty(E).$$

More generally, consider the Langevin diffusion flow

$$\mathcal{X}_t(z) = (X_t(z), Y_t(z)) \in (\mathbb{R}^n \times \mathbb{R}^n)$$

starting at $z = (x, y) \in (\mathbb{R}^n \times \mathbb{R}^n)$ and defined by the hypo-elliptic diffusion (86). We further assume that $\sup_D a < \infty$ for some bounded open connected domain $D \subset \mathbb{R}^n$ with $\mathcal{C}^2$-boundary, and for any $z \in E := D \times \mathbb{R}^n$ and $f \in \mathcal{B}_b(E)$ we set

$$\mathcal{Q}_t(f)(z) := \mathbb{E}\left(f(\mathcal{X}_t(z))\ 1_{T(z)>t}\right) \qquad \text{with} \quad T(z) := \inf\left\{t \geqslant 0\ :\ X_t(z) \in \partial D\right\}.$$

We know from (87) that for any $q > 1$ we have $\mathcal{Q} \ll_q Q$ is q-dominated by the sub-Markov semigroup Q_t associated with the Ornstein-Uhlenbeck diffusion on E defined in (128), with the matrices (A, Σ) defined in (50). In terms of the functions $(V_p, \Theta_{p,t})$ defined in (129), combining (85) with (129) for any $p, q > 1$ we conclude that

$$\mathcal{Q}_t(V_{p,q})/V_{p,q} \leqslant c_t(p, q) \quad \Theta_{p,q,t}$$

with the collection of Lyapunov functions

$$V_{p,q} := V_p^{1/q} \in \mathcal{C}_\infty(E) \qquad \text{and the function} \quad \Theta_{p,q,t} := \Theta_{p,t}^{1/q} \in \mathcal{C}_0(E).$$

7.5. Coupled oscillators in some domains

Consider the $\mathbb{R}^n$-valued diffusion $X_t(x)$ and the quadratic potential function U discussed in Section 7.1, for some $n \geqslant 2$ and set $E :=]0, \infty[\times \mathbb{R}^{n-1}$. Let Q_t be the semigroup defined for any $f \in \mathcal{B}_b(E)$ and $x \in E$ by the formulae

$$Q_t(f)(x) := \mathbb{E}\left(f(X_t(x))\ 1_{T(x)>t}\ \exp\left(-\int_0^t U(X_s(x))ds\right)\right) \tag{131}$$

with the quadratic function U in (114) and the exit time $T(x)$ given by

$$T(x) := \inf\{t \geqslant 0 \ : \ X_t(x) \in \partial E\} \quad \text{with} \quad \partial E = \{0\} \times \mathbb{R}^{n-1}.$$

In terms of the h-process $Y_t(x) := X_t^h(x)$ associated with the leading pair (ρ, h) defined in (116) we also have the conjugate formula

$$Q_t(f) = e^{\rho t} \ h \ Q_t^Y(f/h) \quad \text{with} \quad Q_t^Y(f)(x) := \mathbb{E}\big(f(Y_t(x)) \ 1_{T^Y(x)>t}\big).$$

In the above display, $T^Y(x)$ stands for the boundary hitting time

$$T^Y(x) := \inf\{t \geqslant 0 \ : \ Y_t(x) \in \partial E\}.$$

When $n=2$, the linear diffusion $X_t(x)$ associated to the matrices $A_{1,2} = A_{2,1} = A_{2,2} = 0$ and $A_{1,2} = 1$ and $\Sigma_{1,1} = \Sigma_{1,2} = \Sigma_{2,1} = 0$ and $\Sigma_{2,2} = 1$ coincides with the integrated Wiener process model discussed in [63–65]. In a seminal article [65], McKean obtained the joint distribution of the pair $(T(x), X^2_{T(x)})$ in the absence of soft absorption, that is when $U=0$. To the best of our knowledge, an explicit description of the distribution of this pair and the corresponding sub-Markov semigroup is unknown in more general situations.

Observe that for any $x \in E$ and any non-negative function $f \in \mathcal{B}_b(\mathbb{R}^n)$ we have

$$Q_t(f)(x) \leqslant \mathcal{Q}_t(f)(x) := e^{\rho t} \ h(x) \ \mathbb{E}\big(f(Y_t(x))/h(Y_t(x))\big).$$

The semigroup $\mathcal{Q}_t$ defined above coincides with the semigroup of the coupled harmonic oscillator discussed in Section 7.1. We know from (119) that $\bar{\mathcal{Q}}_t$ satisfies the sub-Gaussian estimates (43) with

$$c_t = \frac{1}{\sqrt{\det(2\pi p_t)}} \quad \text{and} \quad \sigma_t^2 = \lambda_{\max}(p_t)$$

with the solution p_t of the Riccati-matrix equation (117). Using Proposition 7.1 for any $k \geqslant 1$ we have

$$V_{\mathcal{E}}(x) := 1 + ||x||^k \Rightarrow ||\bar{\mathcal{Q}}_t(V_{\mathcal{E}})/V_{\mathcal{E}}|| < \infty \Rightarrow Q_t(V_{\mathcal{E}}) \leqslant c_t \ \mathcal{Q}_t(1) V_{\mathcal{E}}.$$

Recalling that $\mathcal{Q}_t(1)(x)$ tends to 0 exponentially fast as $||x|| \to \infty$, this implies that

$$\forall t > 0 \qquad ||Q_t(V_{\mathcal{E}})|| < \infty.$$

On the other hand for any $y = (y_1, y_{-1}) \in E = (]0, \infty[\times \mathbb{R}^{n-1})$, the distance to the boundary is given by $d(y, \partial E) = y_1$. In terms of the function V_∂ defined in (97) his implies that

$$Q_t(V_\partial)(x) \leqslant \int \ \mathcal{Q}_t(x, dy) \ 1]0,1[(y_1) \ \chi(y_1) + \chi(1) \ \mathcal{Q}_t(1)(x)$$

from which we check that $||Q_t(V_\partial)|| < \infty$. Applying Proposition 5.4, we conclude that

$$V := V_\partial + V_{\mathcal{E}} \in \mathcal{C}_\infty(E) \quad \text{and} \quad Q_t(V)/V \leqslant c_t/V \in \mathcal{C}_0(E).$$

The same analysis applies by replacing the half line E_1 by the unit interval $E_1 :=]0,1[$. In this context, the boundary is given by the two infinite potential walls

$$\partial E = (\{0\} \times \mathbb{R}^{n-1}) \cup (\{1\} \times \mathbb{R}^{n-1}) \quad \text{and} \quad d(x, \partial E) = x_1 \wedge (1 - x_1).$$

More generally, consider a domain $E \subset \mathbb{R}^n$ with $\mathcal{C}^2$-boundary with uniformly bounded interior curvature. In this situation, the sub-Gaussian property (102) ensures that $||\bar{\mathcal{Q}}_t(V_\partial)|| < \infty$ and therefore

$$||Q_t(V_\partial)|| \leqslant ||\mathcal{Q}_t(V_\partial)|| = ||\mathcal{Q}_t(1)\bar{\mathcal{Q}}_t(V_\partial)|| < \infty.$$

Applying Proposition 5.4, we conclude that

$$V := V_\partial + V_\mathcal{E} \in \mathcal{C}_\infty(E) \quad \text{and} \quad Q_t(V)/V \leqslant c_t/V \in \mathcal{C}_0(E).$$

8. Some hypersurface boundaries

8.1. Defining functions and charts

Consider a smooth function $y \in \mathbb{R}^{n-1} \mapsto \varphi(y) \in \mathbb{R}$ with non-empty and connected level set, for some $n \geqslant 2$. Consider a domain E in $\mathbb{R}^n$ with a smooth boundary $\partial E = \bar{\varphi}^{-1}(\{0\})$ defined as the null level set of the function

$$x = (x_i)_{1 \leqslant i \leqslant n} \in \mathbb{R}^n \mapsto \bar{\varphi}(x) := \varphi(x_{-n}) - x_n \quad \text{with} \quad x_{-n} := (x_i)_{1 \leqslant i < n} \in \mathbb{R}^{n-1}.$$

Consider the column vectors $\nabla\varphi(x_{-n}) := (\partial_{x_i}\varphi(x_{-n}))_{1 \leqslant i < n}$. In this notation, the unit normal vector $N(x)$ at $x \in \partial E$ is given by the column vectors

$$N(x) = \frac{\nabla\bar{\varphi}(x)}{||\nabla\bar{\varphi}(x)||} = \frac{1}{\sqrt{1 + ||\nabla\varphi(x_{-n})||^2}} \begin{pmatrix} \nabla\varphi(x_{-n}) \\ -1 \end{pmatrix}.$$

Observe that the vector $N(x)$ is the outward-pointing normal direction to E as soon as $E = \bar{\varphi}^{-1}(]-\infty, 0[)$ and the inward-pointing normal direction to E when $E = \bar{\varphi}^{-1}(]0, +\infty[)$.

Consider the column vectors $e_i := (1_i(j))_{1 \leqslant i < n}$, with $1 \leqslant i < n$. In this notation, the $(n-1)$ tangential column vectors $T_i(x)$ at $x \in \partial E$ are given for any $1 \leqslant i < n$ by the column vectors

$$T_i(x) := \begin{pmatrix} e_i \\ \partial_{x_i}\varphi(x_{-n}) \end{pmatrix}.$$

The inner product $g(x)$ on the tangent space $\mathbb{T}_x(\partial E)$ (a.k.a. the first fundamental form on ∂E) is given by the Gramian matrix

$$g(x) = \Big(T_i(x)'T_j(x)\Big)_{1 \leqslant i,j < n} = T(x)T(x)' \quad \text{with} \quad T(x)' := (T_1(x), ..., T_{n-1}(x)).$$

This yields the matrix formula

$$g(x) = \big(I, \nabla\varphi(x_{-n})\big) \begin{pmatrix} I \\ \nabla\varphi(x_{-n})' \end{pmatrix} = I + \nabla\varphi(x_{-n})\nabla\varphi(x_{-n})'.$$

In this notation, the projection $\mathrm{proj}_{\mathbb{T}_x(\partial E)}$ on the tangent space $\mathbb{T}_x(\partial E)$ is defined for any column vector $V = (V^i)_{1 \leqslant i \leqslant n} \in \mathbb{R}^n$ by

$$\mathrm{proj}_{\mathbb{T}_x(\partial E)}(V) := (T_1(x), ..., T_{n-1}(x))g(x)^{-1}\begin{pmatrix} T_1(x)' \\ \vdots \\ T_{n-1}(x)' \end{pmatrix}\begin{pmatrix} V^1 \\ \vdots \\ V^n \end{pmatrix}.$$

In matrix notation, the projection of m column vectors $V_i \in \mathbb{R}^n$, with $i \in \{1, ..., m\}$ and any $m \geqslant 1$ takes the synthetic form

$$\begin{aligned}\mathrm{proj}_{\mathbb{T}_x(\partial E)}(V_1, ..., V_m) &= \big(T(x)'g(x)^{-1}T(x)\big)(V_1, ..., V_m) \\ &= \Big(\ \mathrm{proj}_{\mathbb{T}_x(\partial E)}(V_1), ...,\ \mathrm{proj}_{\mathbb{T}_x(\partial E)}(V_m)\Big).\end{aligned}$$

Equivalently, if $g(x)^{i,j}$ denotes the (i, j)-entry of the inverse matrix $g(x)^{-1}$, the projection of a column vector $V \in \mathbb{R}^n$ onto $\mathbb{T}_x(\partial E)$ is defined by

$$\mathrm{proj}_{\mathbb{T}_x(\partial E)}(V) = \sum_{1 \leqslant i,j<n} g(x)^{i,j}\ \Big(T_j(x)'V\Big)\ T_i(x).$$

8.2. The shape matrix

Consider the Monge parametrization

$$\psi\ :\ \theta = (\theta_i)_{1 \leqslant i<n} \in \mathcal{S} := \mathbb{R}^{n-1} \mapsto \psi(\theta) = \begin{pmatrix} \theta \\ \varphi(\theta) \end{pmatrix} \in \partial E \subset \mathbb{R}^n. \tag{132}$$

In this chart, the tangent vectors and the normal unit vector at $x = \psi(\theta)$ are given for any $1 \leqslant i < n$ by

$$T_i^{\psi}(\theta) := \partial_{\theta_i}\psi(\theta) = T_i(\psi(\theta)) \in \mathbb{T}_x(\partial E) \quad \text{and} \quad N^{\psi}(\theta) := N(\psi(\theta)) \in \mathbb{T}_x^{\perp}(\partial E).$$

For any $1 \leqslant i, j < n$ we have

$$\big(\partial_{\theta_i}\psi(\theta)\big)'N^{\psi}(\theta) = 0$$

$$\Rightarrow \Omega(\psi(\theta))_{i,j} := \big(\partial_{\theta_i,\theta_j}\psi(\theta)\big)'N^{\psi}(\theta) = -\big(\partial_{\theta_i}\psi(\theta)\big)'\partial_{\theta_j}N^{\psi}(\theta).$$

Observe that for $x = \psi(\theta)$,

$$\partial_{\theta_i}N^{\psi}(\theta) = \sum_{1 \leqslant k \leqslant n} (\partial_{x_k}N)(x)\ \partial_{\theta_i}\psi^k(\theta) = (\nabla N(x))'\ \partial_{\theta_i}\psi(\theta)$$

from which we check that for any $1 \leqslant i, j < n$ the coefficients of the second fundamental form can be computed as follows:

$$\Omega(x)_{i,j} = -\big(\partial_{\theta_i}\psi(\theta)\big)'(\nabla N(x))'\partial_{\theta_i}\psi(\theta).$$

We set

$$\big(\partial N^{\psi}(\theta)\big)' := \big(\partial_{\theta_1}N^{\psi}(\theta), ..., \partial_{\theta_{n-1}}N^{\psi}(\theta)\big) \in \big(\mathbb{T}_{\psi(\theta)}(\partial E)\big)^{n-1}.$$

In this notation, for any $x = \psi(\theta)$ we have the matrix formulation

$$\Omega(x) := -\partial\psi(\theta)\ \left(\partial N^{\psi}(\theta)\right)' = \left(\left(\partial_{\theta_i,\theta_j}\psi(\theta)\right)' N(x)\right)_{1\leqslant i,j<n}$$
$$= -\frac{\nabla^2\varphi(\theta)}{\sqrt{1+||\nabla\varphi(\theta)||^2}} \quad \text{with} \quad \nabla^2\varphi(\theta) := \left(\partial_{\theta_i,\theta_j}\varphi(\theta)\right)_{1\leqslant i,j<n}.$$

We also readily check the matrix formulation of the Weingarten's equations

$$\left(\partial N^{\psi}(\theta)\right)' = \left((\partial\psi(\theta))' g(\psi(\theta))^{-1}\right)(\partial\psi(\theta))\left(\partial N^{\psi}(\theta)\right)' = -(\partial\psi(\theta))'\mathcal{W}(x).$$

In the above display, $\mathcal{W}(x)$ stands for the shape matrix (a.k.a. the Weingarten map or the mixed second fundamental form) defined by

$$\mathcal{W}(x) := g(x)^{-1}\Omega(x)$$
$$= -\frac{1}{\sqrt{1+||\nabla\varphi(x_{-n})||^2}}\ \left(I + \nabla\varphi(x_{-n})\nabla\varphi(x_{-n})'\right)^{-1}\nabla^2\varphi(x_{-n}).$$

We summarize the above discussion in the following proposition.

Proposition 8.1. *For any $1\leqslant i<n$ we have the Weingarten's equations*

$$\partial_{\theta_i}N^{\psi}(\theta) = -\sum_{1\leqslant k<n}\mathcal{W}(\psi(\theta))_{k,i}\ \partial_{\theta_k}\psi(\theta) \in \mathbb{T}_{\psi(\theta)}(\partial E).$$

Example 8.2. *For $n=2$ we have $x\in\mathbb{R}\mapsto\bar{\varphi}(x)=\varphi(x)-x$, so that the boundary $\partial E = \bar{\varphi}^{-1}(\{0\})$ coincides with the graph of the function φ. In this context, the metric and Weingarten map at $x\in\partial E = \{x=(x_1,x_2)\in\mathbb{R}^2\ :\ x_2=\varphi(x_1)\}$ take the form*

$$g(x) = 1+||\partial\varphi(x_1)||^2 \quad \text{and} \quad \mathcal{W}(x) = -\frac{1}{\left(1+||\partial\varphi(x_1)||^2\right)^{3/2}}\ \partial^2\varphi(x_1).$$

Example 8.3. *For $n=3$, the boundary ∂E is given by the surface in $\mathbb{R}^3$ defined*

$$\partial E := \{x=(x_i)_{1\leqslant i\leqslant 3}\in\mathbb{R}^3\ :\ x_3=\varphi(x_1,x_2)\}.$$

The Monge parametrization is given by

$$\psi\ :\ \theta=(\theta_1,\theta_2)\in\mathbb{R}^2\mapsto\psi(\theta)=\begin{pmatrix}\theta_1\\ \theta_2\\ \varphi(\theta_1,\theta_2)\end{pmatrix}\in\partial E\subset\mathbb{R}^3.$$

In this situation, the tangent vectors at $x\in\partial E$ are given by

$$T_1(x)=\begin{pmatrix}1\\0\\ \partial_{x_1}\varphi(x)\end{pmatrix} \quad \text{and} \quad T_2(x)=\begin{pmatrix}0\\1\\ \partial_{x_2}\varphi(x)\end{pmatrix}.$$

In the same vein, whenever $E=\{x\in\mathbb{R}^3\ :\ \varphi(x_1,x_2)\leqslant x_3\}$ the outward pointing unit normal at $x\in\partial E$ is given by

$$N(x) = \frac{1}{\sqrt{1 + (\partial_{x_1}\varphi(x))^2 + (\partial_{x_2}\varphi(x))^2}} \begin{pmatrix} \partial_{x_1}\varphi(x) \\ \partial_{x_2}\varphi(x) \\ -1 \end{pmatrix}.$$

The inner product g(x) is easily computed and given by

$$g(x) = \begin{pmatrix} 1 + (\partial_{x_1}\varphi(x))^2 & (\partial_{x_1}\varphi(x))(\partial_{x_2}\varphi(x)) \\ (\partial_{x_1}\varphi(x))(\partial_{x_2}\varphi(x)) & 1 + (\partial_{x_2}\varphi(x))^2 \end{pmatrix}.$$

The inverse metric is given by

$$g(x)^{-1} = \frac{1}{\det(g(x))} \begin{pmatrix} 1 + (\partial_{x_2}\varphi(x))^2 & -(\partial_{x_1}\varphi(x))(\partial_{x_2}\varphi(x)) \\ -(\partial_{x_1}\varphi(x))(\partial_{x_2}\varphi(x)) & 1 + (\partial_{x_1}\varphi(x))^2 \end{pmatrix}$$

with

$$\det(g(x)) = 1 + (\partial_{x_1}\varphi(x))^2 + (\partial_{x_2}\varphi(x))^2 = 1 + ||\nabla\varphi(x)||^2.$$

The second fundamental form is also given by

$$\Omega(x) = -\frac{1}{\sqrt{1 + ||\nabla\varphi(x)||^2}} \begin{pmatrix} \partial^2_{x_1}\varphi(x) & \partial_{x_1,x_2}\varphi(x) \\ \partial_{x_1,x_2}\varphi(x) & \partial^2_{x_2}\varphi(x) \end{pmatrix}$$

and the Weingarten map is defined by

$$\mathcal{W}(x) = -\frac{1}{(1 + ||\nabla\varphi(x)||^2)^{3/2}}$$
$$\times \begin{pmatrix} (1 + (\partial_{x_2}\varphi(x))^2)\partial^2_{x_1}\varphi(x) - (\partial_{x_1}\varphi(x))(\partial_{x_2}\varphi(x))\partial_{x_1,x_2}\varphi(x) & (1 + (\partial_{x_2}\varphi(x))^2)\partial_{x_1,x_2}\varphi(x) - -(\partial_{x_1}\varphi(x))(\partial_{x_2}\varphi(x))\partial^2_{x_2}\varphi(x) \\ -(\partial_{x_1}\varphi(x))(\partial_{x_2}\varphi(x))\partial^2_{x_1}\varphi(x) + (1 + (\partial_{x_1}\varphi(x))^2)\partial_{x_1,x_2}\varphi(x) & -(\partial_{x_1}\varphi(x))(\partial_{x_2}\varphi(x))\partial_{x_1,x_2}\varphi(x) + (1 + (\partial_{x_1}\varphi(x))^2)\partial^2_{x_2}\varphi(x) \end{pmatrix}.$$

8.3. Surface and volume forms

The surface form σ_∂ on the boundary ∂E expressed in the chart ψ introduced in (132) is given by

$$\left(\sigma_\partial{}^\circ\psi^{-1}\right)(d\theta) = \sqrt{\det\left(g(\psi(\theta))\right)}\ d\theta$$

with the Gramian of the coordinate chart

$$\begin{aligned} g(\psi(\theta)) &:= \ \mathrm{Gram}\left(\partial_{\theta_1}\psi(\theta), ..., \partial_{\theta_{n-1}}\psi(\theta)\right) \\ &:= (\partial\psi(\theta))(\partial\psi(\theta))' = I + \nabla\varphi(\theta)(\nabla\varphi(\theta))' \end{aligned}$$

with the coordinates tangent vectors $\partial\psi(\theta) = T^\psi(\theta) := T(\psi(\theta))$. To check this claim recall that the surface area spaced by the column vectors

$$\partial\psi(\theta)' := \left(\partial_{\theta_1}\psi(\theta), ..., \partial_{\theta_{n-1}}\psi(\theta)\right)$$

is equal to the volume of the parallelepided generated by the column vectors

$$\left(\partial\psi(\theta)', N(\psi(\theta))\right) := \left(\partial_{\theta_1}\psi(\theta), ..., \partial_{\theta_{n-1}}\psi(\theta), N(\psi(\theta))\right)$$

which is given by the determinant of the column vectors, so that

$$\big(\sigma_\partial \circ \psi^{-1}\big)(d\theta) = |\ \det\big(\partial\psi(\theta)', N(\psi(\theta))\big)|\ d\theta.$$

On the other hand, we have

$$\begin{pmatrix} \partial\psi(\theta) \\ N(\psi(\theta))' \end{pmatrix} \big(\, \partial\psi(\theta)', N(\psi(\theta))\,\big)$$
$$= \begin{pmatrix} (\partial_{\theta_1}\psi(\theta))' \\ \vdots \\ (\partial_{\theta_1}\psi(\theta))' \\ N(\psi(\theta))' \end{pmatrix} \big(\, \partial_{\theta_1}\psi(\theta), ..., \partial_{\theta_1}\psi(\theta), N(\psi(\theta))\,\big) = \begin{pmatrix} (\partial\psi(\theta))(\partial\psi(\theta))' & 0_{n-1,1} \\ 0_{1,n-1} & 1 \end{pmatrix}.$$

This implies that

$$\begin{aligned} |\ \det(\partial\psi(\theta)', N(\psi(\theta)))| &= \sqrt{|\ \det((\partial\psi(\theta)', N(\psi(\theta)))'(\partial\psi(\theta)', N(\psi(\theta))))|} \\ &= \sqrt{\ \det((\partial\psi(\theta))(\partial\psi(\theta))')}. \end{aligned}$$

Using the determinant perturbation formula w.r.t. rank-one matrices $\det(I + uv') = 1 + v'u$ which is valid for any column vectors $u, v \in \mathbb{R}^n$ we check that

$$\det\big(I + \nabla\varphi(\theta)\nabla\varphi(\theta)'\big) = 1 + ||\nabla\varphi(\theta)||^2.$$

This yields the formula

$$\big(\sigma_\partial \circ \psi^{-1}\big)(d\theta) = \sqrt{1 + ||\nabla\varphi(\theta)||^2}\ d\theta.$$

The mapping F defined in (103) can also be rewritten as a chart $\bar{\psi}$ on $\mathcal{D}_r(E)$ defined for any $(\theta, u) \in (\mathcal{S} \times [0, r])$ defined by

$$\bar{\psi}(\theta, u) := F(\psi(\theta), u) = \psi(\theta) + u\ N(\psi(\theta)) \in \mathcal{D}_r(E).$$

The Jacobian matrix of $\bar{\psi}$ is given by

$$\mathrm{Jac}(\bar{\psi})(\theta, u) = \big(\partial_{\theta_1}\bar{\psi}(\theta, u), ..., \partial_{\theta_{n-1}}\bar{\psi}(\theta, u), N(\psi(\theta))\big).$$

By Proposition 8.1, we have

$$\begin{aligned} \partial_{\theta_i}\bar{\psi}(\theta, u) &= \partial_{\theta_i}\psi(\theta) + u\ \partial_{\theta_i}N^{\psi}(\theta) \\ &= \partial_{\theta_i}\psi(\theta) - u \sum_{1 \leqslant k < n} \partial_{\theta_k}\psi(\theta)\ \mathcal{W}(\psi(\theta))_{k,i}. \end{aligned}$$

This yields the formula

$$\big(\partial_{\theta_1}\bar{\psi}(\theta, u), ..., \partial_{\theta_{n-1}}\bar{\psi}(\theta, u)\big) = \big(\partial_{\theta_1}\psi(\theta, u), ..., \partial_{\theta_{n-1}}\psi(\theta)\big)\ (I - u\ \mathcal{W}(\psi(\theta)))$$

from which we check that

$$\Big|\det\big(\ \mathrm{Jac}(\bar{\psi})(\theta, u)\big)\Big| = \sqrt{\ \det(g(\psi(\theta))}\ |\det(I - u\ \mathcal{W}(\psi(\theta)))|.$$

Note that $\bar{\psi}(\theta, 0) = \psi(\theta)$, and for any given $u < r$, the mapping $\theta \mapsto \bar{\psi}(\theta, u)$ is a chart on ∂E_u. This yields the following proposition. For the convenience of the reader, a more detailed proof of the next proposition is provided in the appendix on page 85.

Proposition 8.4. *For any $u \leqslant r$, the surface form $\sigma_{\partial,u}$ on the boundary ∂E_u expressed in the chart $\theta \in \mathcal{S} \mapsto \bar{\psi}(\theta,u) := F(\psi(\theta),u)$ is given by the formula*

$$\left(\sigma_{\partial,u}{}^{\circ}\bar{\psi}(.,u)^{-1}\right)(d\theta) = |\det(I - u\ \mathcal{W}(\psi(\theta)))|\ \left(\sigma_{\partial}{}^{\circ}\psi^{-1}\right)(d\theta)$$

with

$$|\det(I - u\ \mathcal{W}(\psi(\theta)))|$$
$$= |\det\left(I + \frac{u}{\sqrt{1 + ||\nabla\varphi(\theta)||^2}}\ \left(I + \nabla\varphi(\theta)\nabla\varphi(\theta)'\right)^{-1}\nabla^2\varphi(\theta)\right)|.$$

In addition, the volume form $\sigma_{\mathcal{D}_r(E)}$ on $\mathcal{D}_r(E)$ expressed in the chart $\bar{\psi}$ is given by

$$\left(\sigma_{\mathcal{D}_r(E)}{}^{\circ}\bar{\psi}^{-1}\right)(d(\theta,u)) = |\det(I - u\ \mathcal{W}(\psi(\theta)))|\ \left(\sigma_{\partial}{}^{\circ}\psi^{-1}\right)(d\theta)\ du.$$

Using Jacobi's formula for the derivative of determinants, we also have

$$\partial_u \log\ \det(I - u\ \mathcal{W}(x)) = -\ \mathrm{Tr}\left((I - u\ \mathcal{W}(x))^{-1}\mathcal{W}(x)\right).$$

The level-set density function defined in (104) expressed in the chart ψ is given by the formula

$$q_t^{\partial}(x,r)$$
$$= \int_{\mathcal{S}} q_t(x,\psi(\theta) + r\ N(\psi(\theta)))\ |\det(I - r\ \mathcal{W}(\psi(\theta)))|\ \sqrt{\det(g(\psi(\theta)))}\ d\theta.$$

8.4. Boundary decompositions

For some given coordinate index $k \in \{1,...,n\}$ and $x = (x_i)_{1 \leqslant i \leqslant n} \in \mathbb{R}^n$ we set

$$x_{-k} := (x_i)_{i \in \mathcal{I}} \quad \text{with} \quad \mathcal{I} := \{1,...,n\} - \{k\}$$

We further assume that

$$\partial E = \left\{x \in \mathbb{R}^n\ :\ x_{-k} \in \mathcal{S} \quad \text{and} \quad \varphi(x_{-k}) = x_k\right\} = \bar{\varphi}^{-1}(\{0\})$$

is defined as the null level set of some global defining function of the form

$$\bar{\varphi}\ :\ x \in \{(x_i)_{1 \leqslant i \leqslant n} \in \mathbb{R}^n\ :\ x_{-k} \in \mathcal{S}\} \mapsto \bar{\varphi}(x) := \varphi(x_{-k}) - x_k \in \mathbb{R}$$

for some open domain $\mathcal{S} \subset \mathbb{R}^{n-1}$.

Example 8.5 (Cylindrical boundaries). *Let $1 \leqslant k \leqslant n_1$ and $n = n_1 + n_2$ for some $n_1 > 1$ and $n_2 \geqslant 1$. Consider a domain $\mathcal{S}$ of the form $\mathcal{S} = (\hat{\mathcal{S}} \times \mathbb{R}^{n_2})$ with $\hat{\mathcal{S}} \subset \mathbb{R}^{n_1-1}$ and assume that*

$$\forall y \in \mathbb{R}^{n_1}\ s.t.\ y_{-k} \in \hat{\mathcal{S}} \quad \text{and} \quad \forall z \in \mathbb{R}^{n_2} \quad \text{we have} \quad \varphi(y_{-k},z) := \hat{\varphi}(y_{-k}).$$

In this situation, the set ∂E is a cylindrical boundary given by the formula

$$\partial E = \partial\hat{E} \times \mathbb{R}^{n_2} \quad \text{with} \quad \partial\hat{E} := \left\{y \in \mathbb{R}^{n_1}\ :\ y_{-k} \in \hat{\mathcal{S}} \quad \text{and} \quad \hat{\varphi}(y_{-k}) = y_k\right\}.$$

In this context, the coordinates of the outward normal by

$$N^j(x) = \frac{\epsilon}{\sqrt{1 + ||\nabla\varphi(x_{-k})||^2}} \ \big(1_{\mathcal{I}}(j) \ \partial_{x_j}\varphi(x_{-k}) + 1_k(j) \ (-1)\big)$$

with the orientation parameter $\epsilon = 1$ when $E = \bar{\varphi}^{-1}(]-\infty, 0[)$; and $\epsilon = -1$ when $E = \bar{\varphi}^{-1}(]0, +\infty[)$. In the same vein, the entries $T_i^j(x)$ of the tangent vectors $T_i(x)$ indexed by $i \in \mathcal{I}$ are given for any $1 \leqslant j \leqslant n$ by

$$T_i^j(x) = 1_i(j) \ + 1_k(j) \ \partial_{x_i}\varphi(x_{-k}).$$

Consider the $(n \times (n-1))$-matrix

$$T(x)' := (T_1(x), ..., T_{k-1}(x), T_{k+1}(x), ..., T_n(x)).$$

In this notation, the inner product $g(x)$ on the tangent space $\mathbb{T}_x(\partial E)$ is given by the $((n-1) \times (n-1))$-square Gramian matrix

$$g(x) = T(x)T(x)' = I + \nabla\varphi(x_{-k})\nabla\varphi(x_{-k})'$$

with the gradient column vector

$$\nabla\varphi(x_{-k}) := \big(\partial_{x_i}\varphi(x_{-k})\big)_{i\in\mathcal{I}} = \begin{pmatrix} \partial_{x_1}\varphi(x_{-k}) \\ \vdots \\ \partial_{x_{k-1}}\varphi(x_{-k}) \\ \partial_{x_{k+1}}\varphi(x_{-k}) \\ \vdots \\ \partial_{x_n}\varphi(x_{-k}) \end{pmatrix} \in \mathbb{R}^{n-1}.$$

We check this claim using the fact that for any $i_1, i_2 \in \mathcal{I}$ we have

$$\begin{aligned} T_{i_1}(x)'T_{i_2}(x) &= \sum_{1\leqslant j\leqslant n} \big(1_{i_1}(j) \ + 1_k(j) \ \partial_{x_{i_1}}\varphi(x_{-k})\big)\big(1_{i_2}(j) \ + 1_k(j) \ \partial_{x_{i_2}}\varphi(x_{-k})\big) \\ &= 1_{i_1=i_2} + \partial_{x_{i_1}}\varphi(x_{-k}) \ \partial_{x_{i_2}}\varphi(x_{-k}). \end{aligned}$$

The parametrization of the hyper surface ∂E is now given by the chart function

$$\psi \ : \ \theta = (\theta_i)_{i\in\mathcal{I}} \in \mathcal{S} \mapsto \psi(\theta) \in \partial E$$

with

$$\forall 1 \leqslant j \leqslant n \qquad \psi^j(\theta) := 1_{\mathcal{I}}(j) \ \theta_j \ + 1_k(j) \ \varphi(\theta).$$

For any $1 \leqslant j \leqslant n$ and $i_1, i_2 \in \mathcal{I}$ observe that

$$\partial_{\theta_{i_1}}\psi(\theta) = T_{i_1}^{\psi}(\theta) := T_{i_1}(\psi(\theta)) \quad \text{ and } \quad \partial_{\theta_{i_1},\theta_{i_2}}\psi^j(\theta) = 1_{k_\iota}(j) \ \partial_{\theta_{i_1},\theta_{i_2}}\varphi(\theta).$$

This implies that

$$\Big(\partial_{\theta_{i_1},\theta_{i_2}}\psi(\theta)\Big)'N(\psi(\theta)) = -\epsilon \ \frac{\big(\nabla^2\varphi(\theta)\big)_{i_1,i_2}}{\sqrt{1 + ||\nabla\varphi(x_{-k})||^2}}$$

$$\text{with} \quad \nabla^2\varphi(\theta) := \big(\partial_{\theta_{i_1},\theta_{i_2}}\varphi(\theta)\big)_{(i_1,i_2)\in\mathcal{I}^2}.$$

We set $(\partial\psi(\theta))' := T(\psi(\theta))'$ and $N^\psi(\theta) := N(\psi(\theta))$. In this notation, we also have

$$\begin{aligned}&\big(\partial N^\psi(\theta)\big)'\\ &:= \Big(\partial_{\theta_1} N^\psi(\theta), ..., \partial_{\theta_{k_\iota-1}} N^\psi(\theta), \partial_{\theta_{k_\iota+1}} N^\psi(\theta), ..., \partial_{\theta_n} N^\psi(\theta)\Big)\\ &= -\mathcal{W}(\psi(\theta)) := -g(\psi(\theta))^{-1}\Omega(\psi(\theta)) \quad \text{with} \quad \Omega(\psi(\theta)) = -\epsilon\ \frac{\nabla^2\varphi(\theta)}{\sqrt{1+||\nabla\varphi(\theta)||^2}}.\end{aligned}$$

Example 8.6. *For the cylindrical boundary discussed in Example 8.5, the inner product and the Weingarten map on the boundary $\partial\hat{E}$ are given for any $y \in \partial\hat{E}$ by the matrices*

$$\hat{g}(y) = I_{(n_1-1,\,n_1-1)} + \nabla\hat{\varphi}(y_{-k})\nabla\hat{\varphi}(y_{-k})' \quad \text{and} \quad \hat{\mathcal{W}}(y) := \epsilon\ \hat{g}(y)^{-1}\frac{\nabla^2\hat{\varphi}(y_{-k})}{\sqrt{1+||\nabla\hat{\varphi}(y_{-k})||^2}}$$

with the gradient column vector and the Hessian matrix given by

$$\begin{aligned}\nabla\hat{\varphi}(y_{-k}) &:= \big(\partial_{y_i}\hat{\varphi}(y_{-k})\big)_{i\in\hat{\mathcal{I}}}\\ \nabla^2\hat{\varphi}(y_{-k}) &:= \big(\partial_{y_{i_1},y_{i-2}}\hat{\varphi}(y_{-k})\big)_{i_1,\,i_2\in\hat{\mathcal{I}}} \quad \textit{with} \quad \hat{\mathcal{I}} := \{1,...,n_1\} - \{k\}.\end{aligned}$$

Observe that

$$\det(\hat{g}(y)) = 1 + ||\nabla\hat{\varphi}(y_{-k})||^2 \quad \text{and} \quad \nabla^2\varphi(y_{-k},z) = \begin{pmatrix}\nabla^2\hat{\varphi}(y_{-k}) & 0_{(n_1-1,\,n_2)}\\ 0_{(n_2,\,n_1-1)} & 0_{(n_2,\,n_2)}\end{pmatrix}.$$

In this case, the inner product and the Weingarten map on the boundary ∂E are given for any point $x = (y,z) \in (\partial\hat{E}\times\mathbb{R}^{n_2})$ by the matrices

$$g(x) = \begin{pmatrix}\hat{g}(y) & 0_{(n_1-1,\,n_2)}\\ 0_{(n_2,\,n_1-1)} & I_{(n_2,\,n_2)}\end{pmatrix} \quad \text{and} \quad \mathcal{W}(x) = \begin{pmatrix}\hat{\mathcal{W}}(y) & 0_{(n_1-1,\,n_2)}\\ 0_{(n_2,\,n_1-1)} & 0_{(n_2,\,n_2)}\end{pmatrix}.$$

Observe that the above matrices are bounded (w.r.t. any matrix norm) as soon as $\partial\hat{E}$ is bounded.

More generally, assume that the boundary $\partial E \subset \cup_{\iota\in\mathcal{J}}\mathcal{O}(\iota) \subset \mathbb{R}^n$ admits a finite covering by open connected subsets $\mathcal{O}(\iota)\subset\mathbb{R}^n$ indexed by some finite set $\mathcal{J}$. In addition, there exists some local defining smooth functions $\bar{\varphi}_\iota$ with non-vanishing gradients on $\mathcal{O}(\iota)$ such that

$$\partial E(\iota) := \partial E \cap \mathcal{O}(\iota) = \bar{\varphi}_\iota^{-1}(\{0\}) \quad \text{and} \quad E(\iota) := E\cap\mathcal{O}(\iota) = \bar{\varphi}_\iota^{-1}(]0,\infty[).$$

Up to shrinking the set $\mathcal{O}(\iota)$, by the implicit function theorem there is no loss of generality to assume that the defining functions are given by

$$\bar{\varphi}_\iota \ : \ x = (x_i)_{1\leqslant i\leqslant n} \in \mathcal{O}(\iota) \mapsto \bar{\varphi}_\iota(x) = \varphi_\iota(x_{-k_\iota}) - x_{k_\iota}$$

for some parameter $1\leqslant k_\iota\leqslant n$ and some smooth function φ_ι on some ball $\mathcal{S}(\iota)\subset\mathbb{R}^{n-1}$. We set $\mathcal{I}_\iota := \{1,...,n\} - \{k_\iota\}$. In this notation, the parametrization of the hyper surface $\partial E(\iota)$ is now given by the smooth homeomorphism

$$\psi_\iota \ : \ \theta = (\theta_i)_{i\in\mathcal{I}_\iota} \in \mathcal{S}(\iota) \mapsto \psi_\iota(\theta) \in \partial E(\iota) \quad \text{with} \quad \psi_\iota^j(\theta) := 1_{\mathcal{I}_\iota}(j)\ \theta_j \ + 1_{k_\iota}(j)\ \varphi_\iota(\theta). \tag{133}$$

The first and second fundamental forms on $\mathbb{T}_x(\partial E(\iota))$ as well as the Weingarten map at $x \in \partial E(\iota)$ are given by

$$g_\iota(x) = I + \nabla\varphi_\iota(x_{-k_\iota})\nabla\varphi(x_{-k_\iota})'$$
$$\Omega_\iota(x) = -\frac{\nabla^2\varphi_\iota(x_{-k_\iota})}{\sqrt{1+||\nabla\varphi_\iota(x_{-k_\iota})||^2}} \quad \text{and} \quad \mathcal{W}_\iota(x) := g_\iota(x)^{-1}\Omega_\iota(x).$$

The atlas $\mathcal{A} = (\psi_\iota, \mathcal{S}_\iota)_{\iota\in\mathcal{J}}$ represents a collection of local coordinate systems of the boundary $\partial E = \cup_{\iota\in\mathcal{J}}\partial E(\iota)$. In this situation, the surface form on ∂E and the volume form $\sigma_{\mathcal{D}_r(E)}$ on $\mathcal{D}_r(E)$ expressed in the atlas $\mathcal{A}$ are defined by the formulae

$$\sigma_\partial^{\mathcal{A}}(d\theta) := \sum_{\iota\in\mathcal{J}} \ \pi_\iota(\psi_\iota(\theta))\ 1_{\mathcal{S}(\iota)}(\theta)\ \sqrt{1+||\nabla\varphi_\iota(\theta)||^2}\ d\theta$$
$$\sigma_{\mathcal{D}_r(E)}^{\mathcal{A}}(d(\theta,u)) := \sum_{\iota\in\mathcal{J}} \ \pi_\iota(\psi_\iota(\theta))\ 1_{\mathcal{S}(\iota)}(\theta)|\det(I - u\ \mathcal{W}_\iota(\psi_\iota(\theta)))|\ \sqrt{1+||\nabla\varphi_\iota(\theta)||^2}\ du.$$

In the above display, $\pi_\iota \ : \ \partial E \mapsto [0,1]$ stands for some partition of unity subordinate to the open cover of the boundary induced by the atlas.

Example 8.7. *Observe that the metric in the graph model discussed in Example 8.2 is not necessarily bounded. In this context, we can also use for any $a < a_+ < b_- < b$ a covering of the form*

$$\mathcal{O}(0) =]a,b[\times\mathbb{R} \qquad \mathcal{O}(-1) =]b_-, +\infty[\times\mathbb{R} \quad \text{and} \quad \mathcal{O}(1) =]-\infty, a_+[\times\mathbb{R}.$$

For instance when $\varphi(z) = z^2$ and $(a, a_+, b_-, b) = (-2, -1, 1, 2)$ we have

$$\partial E(0) = \{(x_1,x_2) \in]-2,2[\times]4,\infty[\ : \ x_2 = \varphi_0(x_1)\}$$
$$\partial E(1) = \{(x_1,x_2) \in]-\infty,-1[\times]1,+\infty[\ : \ x_1 = \varphi_1(x_2)\}$$
$$\partial E(-1) = \{(x_1,x_2) \in]1,\infty[\times]1,+\infty[\ : \ x_1 = \varphi_{-1}(x_2)\}$$

with the functions

$$\varphi_0(z) = z^2 \quad \text{and} \quad \forall\epsilon \in \{-1,1\} \qquad \varphi_\epsilon(z) = -\epsilon\sqrt{z}.$$

Whenever E is the sub-graph of φ, the parameter $\epsilon \in \{-1,1\}$ plays the role of the orientation and the outward pointing unit normal vector at $x \in \partial E(0)$ and $y \in \partial E(\epsilon)$ are given by

$$N_0(x) = \frac{1}{\sqrt{1+4x_1^2}}\begin{pmatrix} 2x_1 \\ -1 \end{pmatrix} \quad \text{and} \quad N_\epsilon(y) = \frac{\epsilon}{\sqrt{1+1/(4y_2)}}\begin{pmatrix} -1 \\ -\epsilon/\sqrt{4y_2} \end{pmatrix}.$$

The tangent vectors at $x \in \partial E(0)$ and at $y \in \partial E(\epsilon)$ are defined by

$$T_0(x) = \begin{pmatrix} 1 \\ 2x_1 \end{pmatrix} \quad \text{and} \quad T_\epsilon(y) = \begin{pmatrix} -\epsilon/\left(\sqrt{4y_2}\right) \\ 1 \end{pmatrix}.$$

The above sub-graphs can be described with 3 charts $\{\psi_0, \psi_{+1}, \psi_{-1}\}$ defined for any $\epsilon \in \{-1,1\}$ by

$$\psi_0 \ : \ \theta \in]-2,2\Big[\mapsto \psi_0(\theta) = \begin{pmatrix} \theta \\ \theta^2 \end{pmatrix} \quad \text{and} \quad \psi_\epsilon \ : \ \theta \in \Big]1,\infty\Big[\mapsto \psi_\epsilon(\theta) := \begin{pmatrix} -\epsilon\sqrt{\theta} \\ \theta \end{pmatrix}.$$

In this situation, the tangent vectors are given by

$$\partial_\theta \psi_0(\theta) = T_0(\psi_0(\theta)) = \begin{pmatrix} 1 \\ 2\theta \end{pmatrix} \quad \textit{and} \quad \partial_\theta \psi_\epsilon(\theta) = T_\epsilon\big(\psi_\epsilon(\theta)\big) = \begin{pmatrix} -\epsilon/\big(\sqrt{4\theta}\big) \\ 1 \end{pmatrix}.$$

In this context, for any $\theta \in]-2,2[$ we have

$$g(\psi_0(\theta)) = 1 + 4\theta^2 \quad \textit{and} \quad \mathcal{W}(\psi_0(\theta)) = -2\ \big(1 + 4\theta^2\big)^{-3/2}.$$

In addition, for any $\theta \in]1,\infty[$ we have

$$g(\psi_\epsilon(\theta)) = 1 + 1/(4\theta) \quad \textit{and} \quad \mathcal{W}(\psi_\epsilon(\theta)) = -2\epsilon\ \big(1 + 4\theta\big)^{-3/2}.$$

Observe that the metric expressed in the chart $\{\psi_0, \psi_{+1}, \psi_{-1}\}$ is defined in terms of bounded functions.

Example 8.8. *Consider the hyperbolic paraboloid boundary*

$$\begin{aligned} \partial E &= \{(y_1,y_2,y_3) \in \mathbb{R}^3 \ : \ y_3 = y_1^2 + y_2^2\} \\ &= \partial E(0) \cup \partial E(1,1) \cup \partial E(1,-1) \cup \partial E(2,1) \cup \partial E(2,-1). \end{aligned}$$

In the above display, $\partial E(0)$ and $\partial E(i,\epsilon)$ with $i \in \{1,2\}$ and $\epsilon \in \{-1,1\}$ stands for the partition defined for any $\epsilon \in \{-1,1\}$ by

$$\begin{aligned} \partial E(0) &:= \{y \in \mathbb{R}^3 \ : \ (y_1,y_2) \in \mathcal{S}_0 \quad y_3 = \varphi_0(y_1,y_2) := y_1^2 + y_2^2\} \\ \partial E(1,\epsilon) &:= \{y \in \mathbb{R}^3 \ : \ (y_1,y_3) \in \mathcal{S} \quad y_2 = \varphi_{1,\epsilon}(y_1,y_3) := \epsilon\sqrt{y_3 - y_1^2}\} \\ \partial E(2,\epsilon) &:= \{y \in \mathbb{R}^3 \ : \ (y_2,y_3) \in \mathcal{S} \quad y_1 = \varphi_{2,\epsilon}(y_1,y_2) := \epsilon\sqrt{y_3 - y_2^2}\} \end{aligned}$$

with the sets

$$\begin{aligned} \mathcal{S}_0 &:= \{(y_1,y_2) \in \mathbb{R}^2 \ : \ y_1^2 + y_2^2 < 2\} \\ \mathcal{S} &:= \{(y_2,y_3) \in \mathbb{R}^2 \ : \ y_3 > 1 \qquad |y_2| < \sqrt{3y_3/4}\}. \end{aligned}$$

On the truncated boundary $\partial E(0)$ we use a single chart defined by

$$\psi_0 \ : \ \theta = (\theta_1,\theta_2) \in \mathcal{S}_0 \mapsto \psi_0(\theta) = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_1^2 + \theta_2^2 \end{pmatrix} \in \partial E(0).$$

On $\partial E(1,\epsilon)$ we use the chart defined by

$$\psi_{1,\epsilon} \ : \ \theta = (\theta_1,\theta_3) \in \mathcal{S} \mapsto \psi_{1,\epsilon}(\theta) = \begin{pmatrix} \theta_1 \\ \epsilon\sqrt{\theta_3 - \theta_1^2} \\ \theta_3 \end{pmatrix} \in \partial E(1,\epsilon).$$

Finally, on $\partial E(2)$ we use the chart defined by

$$\psi_{2,\epsilon} \ : \ \theta = (\theta_2, \theta_3) \in \mathcal{S} \mapsto \psi_{2,\epsilon}(\theta) = \begin{pmatrix} \epsilon\sqrt{\theta_3 - \theta_2^2} \\ \theta_2 \\ \theta_3 \end{pmatrix} \in \partial E(2, \epsilon).$$

For any $\theta = (\theta_1, \theta_2) \in \mathcal{S}_0$ we have

$$\partial_{\theta_1}\psi_0(\theta) = \begin{pmatrix} 1 \\ 0 \\ 2\theta_1 \end{pmatrix} \quad \textit{and} \quad \partial_{\theta_2}\psi_0(\theta) = \begin{pmatrix} 0 \\ 1 \\ 2\theta_2 \end{pmatrix}.$$

In this chart, the metric is given by

$$g(\psi_0(\theta)) = \begin{pmatrix} 1 + 4\theta_1^2 & 4\theta_1\theta_2 \\ 4\theta_1\theta_2 & 1 + 4\theta_2^2 \end{pmatrix} \quad \textit{and} \quad g(\psi_0(\theta))^{-1} = \frac{1}{1 + 4(\theta_1^2 + \theta_2^2)} \begin{pmatrix} 1 + 4\theta_2^2 & -4\theta_1\theta_2 \\ -4\theta_1\theta_2 & 1 + 4\theta_1^2 \end{pmatrix}.$$

In addition, the outward pointing unit normal at $\psi_0(\theta) \in \partial E(0)$ is given by

$$N_0\big(\psi_0(\theta)\big) = \frac{1}{\sqrt{1 + 4(\theta_1^2 + \theta_2^2)}} \begin{pmatrix} 2\theta_1 \\ 2\theta_2 \\ -1 \end{pmatrix} \quad \textit{and} \quad \Omega_0\big(\psi_0(\theta)\big) = \frac{1}{\sqrt{1 + 4(\theta_1^2 + \theta_2^2)}} \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

For any $\theta = (\theta_1, \theta_3) \in \mathcal{S}$ we have

$$\partial_{\theta_1}\psi_{1,\epsilon}(\theta) = \begin{pmatrix} 1 \\ \dfrac{-\epsilon\theta_1}{\sqrt{\theta_3 - \theta_1^2}} \\ 0 \end{pmatrix} \quad \textit{and} \quad \partial_{\theta_3}\psi_{1,\epsilon}(\theta) = \begin{pmatrix} 0 \\ \dfrac{\epsilon}{2\sqrt{\theta_3 - \theta_1^2}} \\ 1 \end{pmatrix}.$$

In this chart, the metric is given by

$$g(\psi_{1,\epsilon}(\theta)) = \begin{pmatrix} 1 + \dfrac{\theta_1^2}{\theta_3 - \theta_1^2} & -\dfrac{\theta_1}{2(\theta_3 - \theta_1^2)} \\ -\dfrac{\theta_1}{2(\theta_3 - \theta_1^2)} & 1 + \dfrac{1}{4(\theta_3 - \theta_1^2)} \end{pmatrix} \quad \textit{and} \quad g(\psi_{1,\epsilon}(\theta))^{-1} = \frac{1}{1 + \frac{\theta_1^2}{\theta_3 - \theta_1^2} + \frac{1}{4(\theta_3 - \theta_1^2)}} \begin{pmatrix} 1 + \dfrac{1}{4(\theta_3 - \theta_1^2)} & \dfrac{\theta_1}{2(\theta_3 - \theta_1^2)} \\ \dfrac{\theta_1}{2(\theta_3 - \theta_1^2)} & 1 + \dfrac{\theta_1^2}{\theta_3 - \theta_1^2} \end{pmatrix}.$$

In addition, the outward pointing unit normal at $\psi_{1,\epsilon}(\theta) \in \partial E(1, \epsilon)$ is given by

$$N_{1,\epsilon}\big(\psi_{1,\epsilon}(\theta)\big) = \frac{-\epsilon}{\sqrt{1 + \frac{\theta_1^2}{\theta_3 - \theta_1^2} + \frac{1}{4(\theta_3 - \theta_1^2)}}} \begin{pmatrix} \dfrac{-\epsilon\theta_1}{\sqrt{\theta_3 - \theta_1^2}} \\ -1 \\ \dfrac{\epsilon}{2\sqrt{\theta_3 - \theta_1^2}} \end{pmatrix} \quad \textit{and} \quad \Omega_{1,\epsilon}\big(\psi_{1,\epsilon}(\theta)\big) = \frac{-\epsilon}{\sqrt{1 + \frac{\theta_1^2}{\theta_3 - \theta_1^2} + \frac{1}{4(\theta_3 - \theta_1^2)}}} \begin{pmatrix} \dfrac{\epsilon\theta_3}{(\theta_3 - \theta_1^2)^{3/2}} & \dfrac{-\epsilon\theta_1}{(\theta_3 - \theta_1^2)^{3/2}} \\ \dfrac{-\epsilon\theta_1}{(\theta_3 - \theta_1^2)^{3/2}} & -\dfrac{\epsilon}{4(\theta_3 - \theta_1^2)^{3/2}} \end{pmatrix}.$$

Finally, for any $\theta = (\theta_2, \theta_3) \in \mathcal{S}$ we have

$$\partial_{\theta_2}\psi_{2,\epsilon}(\theta) = \begin{pmatrix} \dfrac{-\epsilon\theta_2}{\sqrt{\theta_3 - \theta_2^2}} \\ 1 \\ 0 \end{pmatrix} \quad \textit{and} \quad \partial_{\theta_3}\psi_{2,\epsilon}(\theta) = \begin{pmatrix} \dfrac{\epsilon}{2\sqrt{\theta_3 - \theta_2^2}} \\ 0 \\ 1 \end{pmatrix}.$$

In this chart, the metric and the outward pointing unit normal at $\psi_{2,\epsilon}(\theta) \in \partial E(2,\epsilon)$ *are given by*

$$g(\psi_{2,\epsilon}(\theta)) = \begin{pmatrix} 1+\dfrac{\theta_2^2}{\theta_3-\theta_2^2} & -\dfrac{\theta_2}{2(\theta_3-\theta_2^2)} \\ -\dfrac{\theta_2}{2(\theta_3-\theta_2^2)} & 1+\dfrac{1}{4(\theta_3-\theta_2^2)} \end{pmatrix} \quad \textit{and} \quad N_{2,\epsilon}(\psi_{2,\epsilon}(\theta)) = -\frac{\epsilon}{\sqrt{1+\frac{\theta_2^2}{\theta_3-\theta_2^2}+\frac{1}{4(\theta_3-\theta_2^2)}}} \begin{pmatrix} \dfrac{-1}{\sqrt{\theta_3-\theta_2^2}} \\ \dfrac{\epsilon}{2\sqrt{\theta_3-\theta_2^2}} \end{pmatrix}.$$

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Appendix

Proof of (29)
We have

$$P_{s,s+t}(V) \leqslant V + \int_s^{s+t} (-aP_{s,u}(V) + c)\ du = V + ct - a\int_0^t P_{s,s+u}(V)\ du$$

and

$$\int_0^t P_{s,s+u}(V)\ du = tP_{s,s+t}(V) - \int_0^t uP_{s,s+u}(L_{s+u}(V))\ du \geqslant tP_{s,s+t}(V) - ct^2/2.$$

Combining the above estimates, we readily check that

$$P_{s,s+t}(V) \leqslant (1+at)^{-1}V + ct\ \frac{1+at/2}{1+at} \leqslant (1+at)^{-1}V + ct.$$

This ends the proof of (29). ■

Proof of Lemma 2.14 We have

$$1 \leqslant P_{s,s+t}(V) \leqslant V - \int_0^t (P_{s,s+u}(\varphi(V)) - c)\ du \tag{134}$$

By Jensen's inequality

$$\begin{aligned}(P_{s,s+u}(\varphi(V)) - \varphi(V))/u &\leqslant u^{-1}(\varphi(P_{s,s+u}(V)) - \varphi(V)) \\ &\leqslant u^{-1}\left(\varphi\left(V + c\ u - \int_0^u (P_{s,s+v}(\varphi(V))\ dv\right) - \varphi(V)\right)\end{aligned}$$

Letting $u \to 0$ we conclude that

$$L_s(\varphi(V)) \leqslant (\partial\varphi)(V)(c - \varphi(V)) \leqslant c\ (\partial\varphi)(V)$$

For any $s \leqslant u \leqslant t$, this implies that

$$P_{s,t}(\varphi(V)) \leqslant P_{s,u}(\varphi(V)) + c\int_u^t P_{s,v}((\partial\varphi)(V))\ dv$$

Integrating $u \in [s,t]$ and using (134) we conclude that

$$\begin{aligned}(t-s)P_{s,t}(\varphi(V)) &\leqslant \int_s^t P_{s,u}(\varphi(V))du + c\ ||\partial\varphi||\ \frac{(t-s)^2}{2} \\ &\leqslant V - P_{s,t}(V) + \left(c\ (t-s) + c\ ||(\partial\varphi)||\ \frac{(t-s)^2}{2}\right)\end{aligned}$$

We conclude that

$$P_{s,s+\tau}(V_\tau) = P_{s,s+\tau}(V + \tau\varphi(V)) \leqslant V + c_\tau = V_\tau - \tau\varphi(V) + c_\tau$$

with the parameter c_τ defined in (36).

This ends the proof of Lemma 2.14. ■

Proof of Lemma 2.1 Set $V_\rho := 1/2 + \rho V$ with $\rho \in]0,1[$. We associate with these parameters the function

$$\begin{aligned}\Delta_\rho(x,y) &:= \frac{||(\delta_x - \delta_y)P||_{V_\rho}}{||\delta_x - \delta_y||_{V_\rho}} \\ &\leqslant \frac{||(\delta_x - \delta_y)M||_{tv}}{1 + \rho(V(x) + V(y))} + \frac{\rho(P(V)(x) + P(V)(y))}{1 + \rho(V(x) + V(y))}.\end{aligned}$$

By (9) (with $c = 1/2$), whenever $V(x) + V(y) \geqslant r > r_0$ we have

$$\begin{aligned}\Delta_\rho(x,y) &\leqslant \frac{1}{1 + \rho(V(x) + V(y))} + \frac{\rho(V(x) + V(y))}{1 + \rho(V(x) + V(y))} \left(\epsilon + \frac{1}{r}\right) \\ &= 1 - \left(1 - \frac{1}{1 + \rho(V(x) + V(y))}\right)\left(1 - \left(\epsilon + \frac{1}{r}\right)\right).\end{aligned}$$

This yields for any $r > r_0 \vee r_\epsilon$ the estimate

$$\Delta_\rho(x,y) \leqslant 1 - d^1_\rho(r)$$

with

$$d^1_\rho(r) := (1 - \epsilon)\left(1 - \frac{1}{1 + \rho r}\right)\left(1 - \frac{r_\epsilon}{r}\right).$$

Recalling that $V \geqslant 1/2$ we readily check that $P(V)/V \leqslant (1 + \epsilon)$ and

$$\begin{aligned}&V(x) + V(y) \leqslant r \\ &\Rightarrow \frac{\rho V(x)(P(V)(x)/V(x)) + \rho V(y)\ (P(V)(y)/V(y))}{1 + \rho(V(x) + V(y))} \leqslant (1 + \epsilon)\ \frac{\rho r}{1 + \rho r},\end{aligned}$$

This yields for any (x, y) s.t. $V(x) + V(y) \leqslant r$ the estimate

$$\Delta_\rho(x,y) \leqslant 1 - d^2_\rho(r) := \frac{1 - \alpha(r)}{1 + \rho} + (1 + \epsilon)\ \frac{\rho r}{1 + \rho r}.$$

Choosing

$$\rho = \rho(r) := \frac{\alpha(r)}{2(1 + \epsilon)r} \Rightarrow \frac{\rho r}{1 + \rho r} = \frac{\alpha(r)}{2}\ \frac{1}{(1 + \epsilon) + \frac{\alpha(r)}{2}}$$

we have

$$d^1_\rho(r) = \frac{\alpha(r)}{2}\ \frac{(1 - \epsilon)}{(1 + \epsilon) + \frac{\alpha(r)}{2}}\left(1 - \frac{r_\epsilon}{r}\right) < 1,$$

and

$$1 - d^2_\rho(r) = \frac{1 - \alpha(r)/2}{1 + \rho(r)} \leqslant 1 - \alpha(r)/2.$$

We conclude that

$$\beta_{V_{\epsilon,r}}(M) = \sup_{x,y\in E} \Delta_\rho(x,y) \leqslant (1 - d^1_\rho(r)) \vee (1 - d^2_\rho(r)) \leqslant 1 - \alpha_\epsilon(r).$$

This ends the proof of Lemma 2.1. ■

Proof of Proposition 2.16 We have the following almost sure estimate $||\nabla X_\tau(x)||_2 \leqslant e^{-\lambda\tau}$, where $||A||_2$ stands for the spectral norm of a matrix A. This yields for any $x, y \in \mathbb{R}^n$ the almost sure estimate

$$||X_\tau(x) - X_\tau(y)|| \leqslant e^{-\lambda\tau} \ ||x - y||. \tag{135}$$

Applying the above to $y = 0$ we find that

$$P_\tau(V)(x) \leqslant P_\tau(V)(0) \ V(x)^{1-\delta} \quad \text{with} \quad \delta = 1 - e^{-\lambda\tau}.$$

Next, we check that $P^X_\tau(V)(0) < \infty$. We have

$$X_u(0) = \int_0^u (b(0) \ ds + \sigma \ dB_s) + \int_0^u \int_0^1 \nabla b(\epsilon X_s(0))' \ X_s(0) \ d\epsilon \ ds.$$

This implies that

$$||X_u(0)|| \leqslant \beta \ (u + ||B_u||) + \beta \int_0^u ||X_s(0)|| \ ds$$

with $\beta := \sigma \vee ||b(0)|| \vee ||\nabla b||$. Applying Grönwall lemma we check that

$$||X_u(0)|| \overset{law}{\leqslant} \beta \ (u + ||B_u||) + \beta^2 \int_0^u (s + ||B_s||) e^{\beta(u-s)} \ ds.$$

On the other hand, we have

$$\int_0^u ||B_s|| \ ds = u \int_0^1 ||B_{us}|| \ ds \overset{law}{=} u^{3/2} \int_0^1 ||B_s|| \ ds.$$

This yields the rather crude estimate

$$||X_u(0)|| \overset{law}{\leqslant} \beta \ \left(u + \ u^{1/2} \ ||B_1||\right) + \beta^2 \ u^2/2 + \beta^2 e^{\beta u} \ u^{3/2} \int_0^1 ||B_s|| \ ds.$$

For any $a \geqslant 0$ by Jensen's inequality

$$\mathbb{E}\left(e^{a\int_0^1 ||B_s|| \ ds)} \leqslant \int_0^1 \mathbb{E}(e^{a||B_s||}) \ ds \leqslant e^{a^2 r/2}.\right.$$

It is now an elementary exercise to check that $\mathbb{E}(e^{\nu||X_\tau(0)||}) < \infty$. This ends the proof of the proposition. ■

Proof of Proposition 2.17 Consider the function

$$f_t(x) := \exp\left(2\epsilon \ \left(e^{-\alpha t} \ W(x) - \beta \ \frac{1 - e^{-\alpha t}}{\alpha}\right)\right)$$

$$\Rightarrow -\partial_t \log f_t(x) = 2\epsilon \ e^{-\alpha t} \ (\alpha W(x) + \beta).$$

In the same vein, we check that

$$\partial_{x_i} f_t(x)/f_t(x) = 2\epsilon\ e^{-\alpha t}\ \partial_{x_i} W$$
$$\partial_{x_i,x_j} f_t(x)/f_t(x) = 2\epsilon\ e^{-\alpha t}\ \Big(2\epsilon\ e^{-\alpha t}\ \partial_{x_i} W \partial_{x_j} W + \partial_{x_i,x_j} W\Big).$$

This implies that

$$\big(L(f_t) - \partial_t f_t\big)/f_t$$
$$= 2\epsilon\ e^{-\alpha t}\ \big((\alpha W + \beta) + L(W) + \epsilon\ e^{-\alpha t}\ \Gamma_L(W,W)\big).$$

Combining the above with (41) we find that

$$\big(L(f_t)(x) - \partial_t f_t(x)\big) \leqslant -2\ \epsilon^2\ e^{-\alpha t}\ (1 - e^{-\alpha t})\Gamma_L(W,W)(x)\ f_t(x) \leqslant 0.$$

This yields the interpolation formula

$$\mathbb{E}\big(f_0(X_t(x))\big) - f_t(x) = \int_0^t \mathbb{E}\big(\partial_s(f_{t-s}(X_s(x)))\big)\ ds \leqslant 0.$$

We check (42) after some elementary manipulations, thus there are skipped. This ends the proof of the proposition. ■

Proof of Proposition 3.5 Notice that

$$X_t^h(x) \overset{\text{law}}{=} \epsilon_t\ x + \sigma_t\ Z = B_{\sigma_t}(\epsilon_t\ x) \quad \text{with} \quad \epsilon_t := e^{-t} \quad \text{and} \quad \sigma_t := \sqrt{\frac{1-\epsilon_t^2}{2}}$$

and some centered Gaussian random variable Z with unit variance. The conjugate formula (74) yields the integral operator equation

$$Q_t(x,dy) = e^{-t/2}\ e^{-x^2/2}\ \frac{1}{\sqrt{2\pi\sigma_t^2}}\ \exp\left(-\frac{(y-\epsilon_t x)^2}{2\sigma_t^2} + \frac{y^2}{2}\right)\ dy.$$

Observe that

$$-\frac{(y-\epsilon_t x)^2}{\sigma_t^2} + y^2 = -\frac{1}{p_t}\ \left(y - \frac{\epsilon_t}{1-\sigma_t^2}\ x\right)^2 + x^2\ \frac{\epsilon_t^2}{1-\sigma_t^2}$$

with

$$p_t := \frac{1-\epsilon_t^2}{1+\epsilon_t^2} = \tanh(t) \quad \Longleftrightarrow \quad \partial_t p_t = 1 - p_t^2 \quad \text{with} \quad p_0 = 0. \tag{136}$$

We check this claim using the fact that

$$\frac{1}{\sigma_t^2} = \frac{2}{1-\epsilon_t^2} = 1 + \frac{1+\epsilon_t^2}{1-\epsilon_t^2} = 1 + \frac{1}{p_t}.$$

On the other hand, we have

$$\frac{1-\sigma_t^2}{\epsilon_t} = \cosh(t) \quad \text{and} \quad \partial_t \log\cosh(t) = p_t = \tanh(t).$$

This implies that

$$\int_0^t p_s\ ds = \log\cosh(t) \quad \text{and} \quad \frac{\epsilon_t}{1-\sigma_t^2} = \frac{1}{\cosh(t)} = \exp\left(-\int_0^t p_s ds\right).$$

We also have

$$1-\sigma_t^2 = 1-\frac{1-\epsilon_t^2}{2} = \frac{1+\epsilon_t^2}{2} \Rightarrow 1-\frac{\epsilon_t^2}{1-\sigma_t^2} = \frac{1-\epsilon_t^2}{1+\epsilon_t^2} = p_t.$$

This implies that

$$Q_t(1)(x) = e^{-t/2}\ \frac{\sqrt{p_t}}{\sigma_t}\ \exp\left(-\frac{x^2}{2}\ p_t\right) = e^{-t/2}\ h(x)\ P_t^h(1/h)(x).$$

Notice that

$$e^{-t/2}\ \frac{\sqrt{p_t}}{\sigma_t} = \sqrt{\epsilon_t\ \frac{1-\epsilon_t^2}{1+\epsilon_t^2}\frac{2}{1-\epsilon_t^2}} = \sqrt{\frac{2}{1/\epsilon_t+\epsilon_t}} = \frac{1}{\sqrt{\cosh(t)}}$$

and

$$\partial_t m_t(x) = -p_t\ m_t(x) \quad \text{and} \quad \partial_t p_t = 1-p_t^2 \quad \text{with} \quad (m_0(x), p_0) = (x, 0).$$

This ends the proof of the proposition. ■

Proof of Proposition 3.6 Notice that

$$\begin{aligned} &e^{t/2}\ e^{x^2/2}\ Q_t(x,dy) \\ &= \frac{1}{\sqrt{2\pi\sigma_t^2}}\ \left(\exp\left(-\frac{(y-\epsilon_t x)^2}{2\sigma_t^2}+\frac{y^2}{2}\right) - \exp\left(-\frac{(y+\epsilon_t x)^2}{2\sigma_t^2}+\frac{y^2}{2}\right)\right)\ 1_{[0,\infty[}(y)\ dy. \end{aligned}$$

This implies that

$$\begin{aligned} Q_t(1)(x) \quad &= \frac{e^{-\frac{x^2}{2}\ \tanh(t)}}{\sqrt{\cosh(t)}} \\ &\times \int_0^\infty \frac{1}{\sqrt{2\pi p_t}}\ \left(\exp\left(-\frac{(y-m_t(x))^2}{2p_t}\right) - \exp\left(-\frac{(y+m_t(x))^2}{2p_t}\right)\right)\ dy. \end{aligned}$$

We conclude that

$$\begin{aligned} Q_t(1)(x) \quad &= \frac{e^{-\frac{x^2}{2}\ \tanh(t)}}{\sqrt{\cosh(t)}} \times \mathbb{P}\left(-m_t(x)/\sqrt{p_t} \leqslant Z \leqslant m_t(x)/\sqrt{p_t}\right) \\ &= 2\ \frac{e^{-\frac{x^2}{2}\ \tanh(t)}}{\sqrt{\cosh(t)}} \times \mathbb{P}\left(0 \leqslant Z \leqslant \frac{x}{\sqrt{\sinh(t)\cosh(t)}}\right) \\ &\to 0 \quad \text{as } x \to \infty \text{ or } x \to 0 \text{ or as } t \to \infty. \end{aligned}$$

In the above display, Z stands for some centered Gaussian random variable with unit variance. Note that we have used the fact that

$$m_t(x)/\sqrt{p_t} = \frac{x}{\cosh(t)\ \sqrt{\tanh(t)}} = \frac{x}{\sqrt{\sinh(t)\cosh(t)}}.$$

In addition, we have

$$\begin{aligned}\bar{Q}_t(x,dy) &= \frac{1}{\mathbb{P}\Big(0 \leqslant Z \leqslant x/\sqrt{\sinh(t)\cosh(t)}\Big)} \\ &\times \frac{1}{2\sqrt{2\pi p_t}} \left(\exp\left(-\frac{(y-m_t(x))^2}{2p_t}\right) - \exp\left(-\frac{(y+m_t(x))^2}{2p_t}\right)\right) 1_{[0,\infty[}(y)\ dy.\end{aligned}$$

This ends the proof of the proposition. ■

Proof of (52)

The generator of the process (51) is defined by

$$L(f)(q,p) = \beta\ \frac{p}{m}\ \frac{\partial f}{\partial q}\ - \beta\ \left(\frac{\partial W}{\partial q} + \frac{\sigma^2}{2}\ \frac{p}{m}\right)\frac{\partial f}{\partial p} + \frac{\sigma^2}{2}\ \frac{\partial^2 f}{\partial p^2}.$$

Recalling that $2pq \leqslant p^2 + q^2$, we prove that

$$\begin{aligned}V(q,p) &\leqslant \frac{1}{2}\left(\frac{1}{m}+\epsilon\right)\ p^2 + \frac{\epsilon}{2}\ \left(\frac{\sigma^2}{2}+1\right)\ q^2 + W(q) \\ &\leqslant C^\star(\epsilon)\ \left(1 + p^2 + q^2 + W(q)\right)\end{aligned}$$

with

$$C^\star(\epsilon) := \max\left\{\frac{1}{2}\left(\frac{1}{m}+\epsilon\right), \frac{\epsilon}{2}\ \left(\frac{\sigma^2}{2}+1\right), 1\right\}.$$

On the other hand, we have

$$\begin{aligned}L(V) &= \beta\frac{p}{m}\left(\frac{\partial W}{\partial q} + \epsilon\ \frac{\sigma^2}{2}\ q + \epsilon\ p\right) \\ &\quad - \beta\ \left(\frac{\partial W}{\partial q} + \frac{\sigma^2}{2}\ \frac{p}{m}\right)\ \left(\frac{p}{m} + \epsilon\ q\right) + \frac{\sigma^2}{2m} \\ &= -\beta\ \left[\frac{1}{m}\ \left(\frac{\sigma^2}{2m} - \epsilon\ \right)\ p^2 + \epsilon\ q\ \frac{\partial W}{\partial q}\right] + \frac{\sigma^2}{2m}.\end{aligned}$$

Under our assumptions, this implies that for any $|q| \geqslant r$ we have

$$\begin{aligned}L(V) &\leqslant -\beta\ \left[\frac{1}{m}\ \left(\frac{\sigma^2}{2m} - \epsilon\ \right)\ p^2 + \epsilon\ \delta\ \left(W(q) + q^2\right)\right] + \frac{\sigma^2}{2m} \\ &\leqslant -C_\star(\epsilon,\delta)\ \left(1 + p^2 + q^2 + W(q)\right) + c_m(\epsilon,\delta)\end{aligned}$$

with

$$C_\star(\epsilon,\delta) := \beta\ \min\left\{\left(\frac{1}{m}\ \left(\frac{\sigma^2}{2m} - \epsilon\ \right), \epsilon\ \delta\right)\right\} \quad \text{and} \quad c_m(\epsilon,\delta) := C_\star(\epsilon,\delta) + \frac{\sigma^2}{2m}.$$

We conclude that for any $|q| > r$,

$$\begin{aligned}(V^{-1}L(V))(q,p) &\leqslant -\frac{C_\star(\epsilon,\delta)\ \left(1+p^2+q^2+W(q)\right)-c_m(\epsilon,\delta)}{V(q,p)}\\ &\leqslant -\frac{C_\star(\epsilon,\delta)\ \left(1+p^2+q^2+W(q)\right)-c_m(\epsilon,\delta)}{C^\star(\epsilon)\ \left(1+p^2+q^2+W(q)\right)}\\ &= -\frac{C_\star(\epsilon,\delta)}{C^\star(\epsilon)}+\frac{c_m(\epsilon,\delta)}{C^\star(\epsilon)}\ \frac{1}{1+p^2+q^2+W(q)}\\ &\leqslant -\left[\frac{C_\star(\epsilon,\delta)}{C^\star(\epsilon)}-\frac{c_m(\epsilon,\delta)}{C^\star(\epsilon)}\ \frac{1}{1+p^2+q^2}\right].\end{aligned}$$

We choose r sufficiently large to satisfy

$$\begin{aligned}&|p| > r \quad \text{or} \quad |q| > r\\ &\Rightarrow \frac{C_\star(\epsilon,\delta)}{C^\star(\epsilon)}-\frac{c_m(\epsilon,\delta)}{C^\star(\epsilon)}\ \frac{1}{p^2+q^2} \geqslant \frac{C_\star(\epsilon,\delta)}{C^\star(\epsilon)}-\frac{c_m(\epsilon,\delta)}{C^\star(\epsilon)}\ \frac{1}{r^2} \geqslant a := \frac{C_\star(\epsilon,\delta)}{2C^\star(\epsilon)} > 0,\end{aligned}$$

and we set

$$K_r := \{(q,p) \in \mathbb{R}^2 \ : \ |p| \vee |q| \leqslant r\}.$$

In this notation, we have

$$L(V) \leqslant -aV\ 1_{E-K_r} + \sup_{K_r} L(V) \leqslant -aV\ + c \quad \text{with} \quad c := \sup_{K_r} L(V) + a \sup_{K_r} V.$$

Proof of (77)

Observe that for any $0 < y \leqslant 1$ and $z \in E =]0,\infty[$ we have

$$\sinh(yz) \leqslant y\ \sinh(z) \quad \text{and} \quad \sinh(z) \leqslant \frac{1}{2}\ e^z.$$

This implies that

$$\int_0^\infty \mathcal{Q}_t(x,dy)\ \frac{1}{y}\ \leqslant \sinh(m_t(x))\ \frac{e^{-\frac{x^2}{2}(p_t+e^{-2t}/p_t)}}{\sqrt{\cosh(t)}}\ \times \int_0^\infty \sqrt{\frac{2}{\pi p_t}}\ \exp\left(-\frac{y^2}{2p_t}\right)\ dy.$$

from which we check that

$$\int_0^\infty \mathcal{Q}_t(x,dy)\ \frac{1}{y}\ 1]0,1](y) \leqslant \ \frac{\exp\left(-\left(\frac{x^2}{2}\left(p_t+\frac{e^{-2t}}{p_t}\right)-e^{-t}x\right)\right)}{2\sqrt{\cosh(t)}}.$$

On the other hand, for any $n \geqslant 1$ we have

$$\begin{aligned}&\textstyle\int_0^\infty \mathcal{Q}_t(x,dy)\ y^n\\ &\leqslant \frac{1}{2}\ \frac{1}{\sqrt{\cosh(t)}}\ \sqrt{\frac{2}{\pi p_t}}\ \exp\left(-\frac{\epsilon_t^2 x^2}{2p_t}-\frac{x^2}{2}\ p_t\right)\int_0^\infty y^n\ \exp\left(y\epsilon_t x-\frac{y^2}{2p_t}\right)\ dy.\end{aligned}$$

Notice that

$$y\epsilon_t x-\frac{y^2}{2p_t} = -\frac{1}{2p_t}(y-\epsilon_t x p_t)^2+\frac{x^2}{2}\ \epsilon_t^2\ p_t$$

so that

$$\int_0^\infty \mathcal{Q}_t(x,dy)\ y^n \leqslant \frac{1}{2}\ \frac{1}{\sqrt{\cosh(t)}}\ \sqrt{\frac{2}{\pi p_t}} \times \exp\left(-\frac{x^2}{2}\ \left((1-\epsilon_t^2)\ p_t+\frac{\epsilon_t^2}{p_t}\right)\right)\int_0^\infty y^n\ \exp\left(-\frac{1}{2p_t}(y-\epsilon_t x p_t)^2\right)\ dy.$$

For any $n \geqslant 1$, we conclude that

$$V(x) := x^n + 1/x \Rightarrow V \in \mathcal{C}_\infty(E) \quad \text{and} \quad ||Q_t(V)|| < \infty.$$

This ends the proof of (77). ■

Proof of Lemma 4.2 To simplify notation, we write Q_t instead of $Q_t^{[U]}$. For any $V \in \mathcal{B}_\infty(E) \cap \mathcal{D}(L)$ we have

$$Q_t(V) = V + \int_0^t Q_s(L(V) - UV)\ ds \leqslant V + \int_0^t [-a\ Q_s(V) + c\ Q_s(1)]\ ds = V + c\int_0^t Q_s(1)\ ds - a\int_0^t Q_s(V)ds.$$

On the other hand, through integration by parts we have

$$\int_0^t Q_s(V)ds = [s\ Q_s(V)]_0^t - \int_0^t s\ \frac{d}{ds}Q_s(W)\ ds = t\ Q_t(V) - \int_0^t s\ Q_s(\underbrace{L(V)-UV}_{\leqslant c})\ ds \geqslant t\ Q_t(V) - c\int_0^t s\ Q_s(1)ds.$$

This implies that

$$Q_t(V) \leqslant V + c\int_0^t Q_s(1)\ ds - a\left(t\ Q_t(V) - c\int_0^t s\ Q_s(1)ds\right)$$

from which we conclude that

$$Q_t(V) \leqslant \frac{V}{1+at} + c\int_0^t Q_s(1)ds \Rightarrow Q_t(V) \leqslant \frac{V}{1+at} + ct.$$

This ends the proof of (90). Now, we come to the proof of (91). We have the forward evolution equation given for any $f \in \mathcal{D}(L)$ by

$$\partial_t Q_t(f) = Q_t(L^U(f)).$$

Applying the above to $f = U$ we readily check that

$$\partial_t Q_t(U) \leqslant a_0 + a_1\ Q_t(U) - Q_t(U^2) \leqslant a_0 + a_1\ Q_t(U) - (Q_t(U))^2/Q_t(1)$$

from which we find the Riccati estimates

$$\partial_t Q_t(U) \leqslant a_0 + a_1\ Q_t(U) - (Q_t(U))^2 \Rightarrow \forall t > 0 \quad ||Q_t(U)|| < \infty.$$

This ends the proof of the lemma. ■

Proof of (87)

By Girsanov theorem we have

$$\mathcal{Q}_t^{(a)}(f)(z) = \mathbb{E}\Big(f(\mathcal{X}_t^0(z))\ Z_t(z)\ 1_{T^0(z)>t}\Big)$$

with the exponential martingale

$$Z_t(z) = \exp\left(\frac{1}{\sigma}\int_0^t a(X_s(z))'\ dB_s - \frac{1}{2\sigma^2}\ \int_0^t ||a(X_s(z))||^2\ ds\right).$$

By Hölder's inequality, for any non-negative function f on E, any $z \in E$ and any conjugate parameters $p, q > 1$ with $1/p + 1/q = 1$ we have

$$\mathcal{Q}_t^{(a)}(f)(z) \leqslant \mathbb{E}\big(Z_t(z)^q\ 1_{T^0(z)>t}\big)^{1/q}\ \mathcal{Q}_t^{(a)}(f^p)(z)^{1/p}.$$

On the other hand, we have

$$\mathbb{E}\big(Z_t(z)^q\ 1_{T^0(z)>t}\big) = \mathbb{E}\left(\bar{Z}_t(z)\ \exp\left(\frac{q(q-1)}{2\sigma^2}\ \int_0^t ||a(X_s(z))||^2\ ds\right)\ 1_{T^0(z)>t}\right) \leqslant c_t(p) := \exp\left(\frac{pt}{2((p-1)\sigma)^2}\ \sup_D a\right)$$

with the exponential martingale

$$\bar{Z}_t(z) = \exp\left(\frac{q}{\sigma}\ \int_0^t a(X_s(z))'\ dB_s - \frac{q^2}{2\sigma^2}\ \int_0^t ||a(X_s(z))||^2\ ds\right).$$

This ends the proof of roof of (87). ■

Proof of Lemma 5.2 For any $z \in \partial E$ there exists some open ball $\mathbb{B}(z,r) \subset \mathbb{R}^n$ with $r > 0$ and some $\mathcal{C}^1$-mapping g from $\mathbb{R}^{n-1}$ into $\mathbb{R}$ such that

$$\begin{aligned} &E \cap \mathbb{B}(z,r) = \{x \in \mathbb{B}(z,r)\ :\ x_n < g(x_{-n})\} \\ &\partial E \cap \mathbb{B}(z,r) = \{x \in \mathbb{B}(z,r)\ :\ x_n = g(x_{-n})\} \quad \text{with} \quad x_{-n} := (x_1, ..., x_{n-1}). \end{aligned}$$

We make the change of variables

$$\mathcal{E}(z,r) := E \cap \mathbb{B}(z,r)$$

$$\mapsto \varsigma(x) := (x_{-n}, x_n - g(x_{-n})) \in \mathcal{O}(z,r) := \varsigma(\mathcal{E}(z,r)) \subset (\mathbb{R}^{n-1} \times \mathbb{R}_+)$$

with Jacobian

$$\nabla\varsigma(x) = \begin{pmatrix} I_{(n-1)\times(n-1)} & -\nabla g(x_{-n}) \\ 0 & 1 \end{pmatrix}.$$

Observe that

$$\varsigma\ :\ x \in \mathcal{E}_0(z,r) := (\partial E \cap \mathbb{B}(z,r))$$

$$\Rightarrow \varsigma(x) = (x_{-n}, 0) \in \mathcal{O}_0(z,r) := \varsigma(\mathcal{E}_0(z,r)) \subset (\mathbb{R}^{n-1} \times \{0\}).$$

The inverse is given by

$$y \in \mathcal{O}(z,r) \mapsto \varsigma^{-1}(y) = (y_{-n}, y_n + g(y_{-n})) \in \mathcal{E}(z,r)$$

$$\Rightarrow \nabla\varsigma^{-1}(y) = \begin{pmatrix} I_{(n-1)\times(n-1)} & \nabla g(y_{-n}) \\ 0 & 1 \end{pmatrix}.$$

On the other hand we have

$$\begin{aligned} ||\varsigma(x) - \varsigma(\bar{x})|| &= \left(||x_{-n} - \bar{x}_{-n}||^2 + \big(|x_n - \bar{x}_n| + |g(x_{-n}) - g(\bar{x}_{-n})|\big)^2\right)^{1/2} \\ &\leqslant \left(||x_{-n} - \bar{x}_{-n}||^2 + 2|x_n - \bar{x}_n|^2 + 2||\nabla g||^2 ||x_{-n} - \bar{x}_{-n}||^2\right)^{1/2} \\ &\leqslant c(g)\ ||x - \bar{x}|| \quad \text{with} \quad c(g) := \sqrt{2 \vee (1 + 2||\nabla g||^2)} \geqslant 1. \end{aligned}$$

In the same vein, we have

$$||\varsigma^{-1}(y) - \varsigma^{-1}(\bar{y})|| \leqslant c(g)\ ||y - \bar{y}|| \quad \text{so that} \quad \frac{1}{c(g)}\ ||y - \bar{y}|| \leqslant ||\varsigma^{-1}(y) - \varsigma^{-1}(\bar{y})||.$$

For any $x \in \mathcal{E}(z,r)$ and $\bar{x} \in \mathcal{E}_0(z,r)$ we have $\varsigma(\bar{x}) \in \mathcal{O}_0(z,r)$ and

$$||x - \bar{x}|| = ||\varsigma^{-1}(\varsigma(x)) - \varsigma^{-1}(\varsigma(\bar{x}))|| \geqslant \frac{1}{c(g)}\ ||\varsigma(x) - \varsigma(\bar{x})|| \geqslant \frac{1}{c(g)}\ |\varsigma(x)_n|.$$

Taking the infimum of all $\bar{x} \in \mathcal{E}_0(z,r)$ this implies that

$$d(x, \mathcal{E}_0(z,r)) \geqslant \frac{1}{c(g)}\ |\varsigma(x)_n| \quad \text{and} \quad d(\varsigma^{-1}(y), \mathcal{E}_0(z,r)) \geqslant \frac{1}{c(g)}\ |y_n|$$

for any $x \in \mathcal{E}(z,r)$ and $y \in \mathcal{O}(z,r)$. We conclude that

$$\int_{\mathcal{E}(z,r)} \chi(d(x, \mathcal{E}_0(z,r)))\ dx$$

$$= \int_{\mathcal{O}(z,r)} \chi\big(d(\varsigma^{-1}(y), \mathcal{E}_0(z,r))\big)\ |\det\big(\varsigma^{-1}(y)\big)|\ dy \leqslant \frac{1}{c(g)} \sup_{y \in \mathcal{O}(z,r)} \big|\ \det\big(\varsigma^{-1}(y)\big)\big| \int_{\mathcal{O}(z,r)} \chi(y_n)\ dy < \infty.$$

We end the proof of the lemma by covering ∂E by finitely many boundary coordinates patches $(\mathcal{E}(z_i, r_i), g_i)_{1 \leqslant i \leqslant n}$, for some $z_i \in \partial E$, $r_i > 0$ and some local defining functions g_i. ■

Proof of Lemma 5.9 Using the change of variable formulae

$$\int_{\partial E_r} f(z)\ \sigma_{\partial,r}(dz) = \int_{\partial E} f(z + rN(z))\ |\ \det(I - r\ \mathcal{W}(z))|\ \sigma_\partial(dz)$$

and

$$\int_{\partial E} f(z)\ \sigma_\partial(dz) = \int_{\partial E_r} f(z - rN(z))\ |\ \det(I + r\ \mathcal{W}(z))|\ \sigma_{\partial,r}(dz)$$

we check that

$$\int_{\partial E_r} f(z)\ \sigma_{\partial,r}(dz) \leqslant \kappa_\partial(\alpha) \int_{\partial E} f(z + rN(z)))\ \sigma_\partial(dz)$$

and

$$\int_{\partial E} f(z)\ \sigma_\partial(dz) \leqslant \kappa_\partial^-(\alpha) \int_{\partial E_r} f(z - rN(z))\quad \sigma_{\partial,r}(dz).$$

This yields the estimate

$$\int_{\partial E_r} f(z)\ \sigma_{\partial,r}(dz) \leqslant\ \imath(\alpha)\ \kappa_\partial(\alpha) \int_{\partial E} g(z)\ \sigma_\partial(dz).$$

In the same vein, we have

$$\int_{\partial E} f(z)\ \sigma_\partial(dz) \leqslant \imath(\alpha)\kappa_\partial^-(\alpha) \int_{\partial E_r} g(z)\quad \sigma_{\partial,r}(dz).$$

Integrating w.r.t. the parameter $r \in [0, \alpha]$ we check the co-area estimate

$$\alpha \int_{\partial E} f(z)\ \sigma_\partial(dz) \leqslant \imath(\alpha)\kappa_\partial^-(\alpha) \int_0^\alpha dr \int_{\partial E_r} g(z)\ \sigma_{\partial,r}(dz)$$

$$= \imath(\alpha)\ \kappa_\partial^-(\alpha) \int_{\mathcal{D}_\alpha(E)} g(z)\ dz.$$

This ends the proof of the lemma. ■

Proof of Proposition 8.4 For any given $\bar{\theta} := (\bar{\theta}_1, ..., \bar{\theta}_n) \in (\mathbb{R}^{n-1} \times [0, r])$ we set $\bar{\theta}_{-n} := (\theta_1, ..., \theta_{n-1})$. In this notation, we have

$$\begin{aligned}\bar{\psi} \ : \ \bar{\theta} \in \left(\mathbb{R}^{n-1} \times [0, r]\right) \mapsto \bar{\psi}(\bar{\theta}) \ &:= \ F(\psi(\bar{\theta}_{-n}), \bar{\theta}_n) \\ &= \ \psi(\bar{\theta}_{-n}) + \bar{\theta}_n \ N(\psi(\bar{\theta}_{-n})) \in \mathcal{D}_r(E).\end{aligned}$$

The volume form $\sigma_{\mathcal{D}_r(E)}$ on $\mathcal{D}_r(E)$ expressed in the chart $\bar{\psi}$ is given by

$$\left(\sigma_{\mathcal{D}_r(E)} \circ \bar{\psi}^{-1}\right)(d\bar{\theta}) = |\det(\mathrm{Jac}(\bar{\psi})(\theta))| = \sqrt{\det\left(\left(\partial\bar{\psi}(\bar{\theta})\right)\left(\partial\bar{\psi}(\bar{\theta})\right)'\right)}\ d\bar{\theta}.$$

Arguing as above, we have

$$\left(\partial\bar{\psi}(\bar{\theta})\right)' = \left(\left(\partial_{\bar{\theta}_{-n}}\bar{\psi}(\bar{\theta})\right)', \partial_{\bar{\theta}_n}\bar{\psi}(\bar{\theta})\right) \in \left(\mathbb{T}_{\bar{\psi}(\bar{\theta})}(\mathcal{D}_r(E))\right)^n$$

with the tangent vectors

$$\left(\partial_{\bar{\theta}_{-n}}\bar{\psi}(\bar{\theta})\right)' := \left(\partial_{\bar{\theta}_1}\bar{\psi}(\bar{\theta}), ..., \partial_{\bar{\theta}_{n-1}}\bar{\psi}(\bar{\theta})\right) \quad \text{and} \quad \partial_{\bar{\theta}_n}\bar{\psi}(\bar{\theta}) = N(\psi(\bar{\theta}_{-n})).$$

In addition, we have

$$\left(\partial_{\bar{\theta}_{-n}}\bar{\psi}(\bar{\theta})\right)' = \left(\partial\psi(\bar{\theta}_{-n})\right)' + \bar{\theta}_n \ \left(\partial(N(\psi(\bar{\theta}_{-n})))\right)' = \left(\partial\psi(\bar{\theta}_{-n})\right)'\left(I - \bar{\theta}_n \ \mathcal{W}(\psi(\bar{\theta}_{-n}))\right).$$

This yields the formula

$$\left(\partial_{\bar{\theta}_{-n}}\bar{\psi}(\bar{\theta})\right)\left(\partial_{\bar{\theta}_{-n}}\bar{\psi}(\bar{\theta})\right)' = g(\psi(\bar{\theta}_{-n}))\ \left(I - \bar{\theta}_n \ \mathcal{W}(\psi(\bar{\theta}_{-n}))\right)^2$$

from which we check that

$$\left(\partial\bar{\psi}(\bar{\theta})\right)\left(\partial\bar{\psi}(\bar{\theta})\right)' = \begin{pmatrix} g(\psi(\bar{\theta}_{-n})\left(I - \bar{\theta}_n \ \mathcal{W}(\psi(\bar{\theta}_{-n}))\right)^2 & 0_{n-1} \\ 0 & 1 \end{pmatrix}.$$

We conclude that

$$\sqrt{\det\left(\left(\partial\bar{\psi}(\bar{\theta})\right)\left(\partial\bar{\psi}(\bar{\theta})\right)'\right)} = \sqrt{\det(g(\psi(\bar{\theta}_{-n}))}\ \left|\det\left(I - \bar{\theta}_n \ \mathcal{W}(\psi(\bar{\theta}_{-n}))\right)\right|$$

and therefore

$$\left(\sigma_{\mathcal{D}_r(E)} \circ \bar{\psi}^{-1}\right)(d\bar{\theta}) = \left|\det\left(I - \bar{\theta}_n \ \mathcal{W}(\psi(\bar{\theta}_{-n}))\right)\right| \ d\bar{\theta}_n \ \left(\sigma_{\partial,0} \circ \psi^{-1}\right)(d\bar{\theta}_{-n}).$$

For any given $\bar{\theta}_n = u \in [0, r]$, the volume form $\sigma_{\partial,u}$ on the boundary ∂E_u expressed in the boundary chart

$$\bar{\psi}(., u) \ : \ \theta \in \mathbb{R}^{n-1} \mapsto \bar{\psi}(\theta, u) = F(\psi(\theta), u) \in \partial E_u$$

is given by

$$\left(\sigma_{\partial,u} \circ \bar{\psi}(., u)^{-1}\right)(d\theta) = |\det(I - u \ \mathcal{W}(\psi(\theta)))| \ \left(\sigma_{\partial,0} \circ \psi^{-1}\right)(d\theta).$$

This ends the proof of the proposition. ■