On the Stability of Non Linear Feynman-Kac Semigroups

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Abstract

The stability properties of a class of non linear Feynman-Kac semigroups in distribution space is discussed. This study is based on the use of semigroup techniques and Dobrushin's ergodic coefficient in the spirit of previous articles by A. Guionnet and one of the authors. The second part of this paper is devoted to the applications of these results. First we give conditions underwhich a killed Markov particle conditioned by non extinction forgets exponentially fast its initial condition. We also analyze the stability properties of a class of interacting processes in which the interaction goes through jumps. Finally we investigate the asymptotic stability properties of the non linear filtering equation associated to a general Markov signal with continuous paths and we examine the limiting behavior of its robust version.

Keywords : Feynman-Kac formula, asymptotic stability, non linear filtering, Dobrushin's ergodic coefficient, killed particles, Moran particle systems.

1 Introduction

Let $X = \{X_t \; ; \; t \in I\}$ be a Markov process with time space $I = \mathbb{R}_+ =_{def.} [0, \infty[$ or $I = \mathbb{N}$ and taking values in a measurable space $(E, \mathcal{E})$. Let $Z = \{Z_{s,t} \; ; \; 0 \leq s \leq t, \, , s, t \in I\}$ be a collection of multiplicative bounded positive functions such that for any $s \leq t$, $Z_{s,t}$ is a $\sigma(X_r \; ; \; s \leq r < t)$-measurable random variable. Starting from the pair (X, Z) one associate a non linear semigroup $\Phi = \{\Phi_{s,t} \; ; \; 0 \leq s \leq t\}$ on the set $\mathbf{M}_1(E)$ of all probability measures on E by setting for any distribution $\mu \in \mathbf{M}_1(E)$ and for any bounded and $\mathcal{E}$-measurable function $f : E \to \mathbb{R}$

$$\Phi_{s,t}(\mu)(f) =_{def.} \frac{\mathbb{E}_{s,\mu}\left(\, f(X_t) \; Z_{s,t} \, \right)}{\mathbb{E}_{s,\mu}\left(\, Z_{s,t} \, \right)} \tag{1}$$

where $((X_t)_{t \geq s}, \mathbb{P}_{s,\mu})$ denotes the Markov process X on $[s, +\infty)$ starting with initial distribution μ. The paper concerns the large time behavior and the stability properties of non linear Feynman-Kac semigroups Φ. A concise statement of one of our main results is the following.

Theorem 1.1 *If the transition semigroup of X is sufficiently mixing and Z is sufficiently regular then there exists some $\gamma > 0$ such that for any $t \in I$*

$$\sup_{\mu,\nu} \|\Phi_{0,t}(\mu) - \Phi_{0,t}(\nu)\|_{tv} \leq e^{-\gamma t} \tag{2}$$

where the supremum is taken over all pair distributions and $\|\,.\|_{tv}$ is the total variation norm.

This result will be reformulated in more details later in the paper where in particular we shall specify several mixing and regularity conditions underwhich the semigroup Φ is asymptotically stable. Our expressions for the exponential rates (2) will also be constructive.

To motivate our work and to get a flavor of our results let us present already and in more details a particular situation which can be handled in our framework. Let $I = \mathbb{R}_+$ and Z be given by

$$Z_{u,v} = \exp \int_u^v V_t(X_t)\, dt$$

where X is a time-homogeneous Markov process with semigroup $\{P_t \ ; \ t \in I\}$ and $\{V_t \ ; \ t \in I\}$ is a collection of measurable functions with bounded oscillations, that is for any $t \in I$

$$\mathrm{osc}(V_t) = \sup \{V_t(y) - V_t(x) \ ; \ (x,y) \in E^2\} < \infty$$

Corollary 1.2 *1. If the oscillations are integrable in the sense that*

$$\mathrm{osc}(V)^\star =_{def.} \int_0^\infty \mathrm{osc}(V_t)\, dt < \infty \tag{3}$$

then for any $p \geq 1$, $u > 0$ *and* $t \geq p.u$ *the Lyapunov exponent* γ *given in (2) satisfies*

$$\gamma \geq \frac{1}{q\ .u}\ (1 - \beta(P_u))\ \exp - (2\mathrm{osc}(V)^\star)$$

with

$$\beta(P_u) = \sup_{x,y} \|P_u(x, \boldsymbol{.}) - P_u(y, \boldsymbol{.})\|_{tv} \qquad \text{and} \quad 1/p + 1/q = 1$$

2. If the oscillations are uniformly bounded in the sense that

$$\mathrm{osc}(V) =_{def.} \sup_t \mathrm{osc}(V_t)$$

and if the semigroup $\{P_t \ ; \ t \in I\}$ *satisfies the following mixing type condition*

$$\exists (u, \mu) \in]0, \infty[\times \mathbf{M}_1(E) \quad : \quad \forall x, z \in E \qquad a\ e^{-\frac{b}{u}} \leq \frac{dP_u(x, \boldsymbol{.})}{d\mu}(z) \leq \frac{1}{a}\ e^{\frac{b}{u}} \tag{4}$$

for some constants $0 < a, b < \infty$, *then for any* $p \geq 1$, $u > 0$ *and* $t \geq p.u$ *the Lyapunov exponent* γ *given in (2) satisfies*

$$\gamma \geq \frac{a}{q\ .u}\ \exp - \left(u\ .\mathrm{osc}(V) + \frac{b}{u}\right) \qquad \text{with} \quad 1/p + 1/q = 1$$

As announced and strictly speaking the latter is not a corollary of Theorem 1.1 but rather a consequence of our constructive approach. The precise description of Theorem 1.1 will be given in section 3.

The discrete time version of Theorem 1.1 and Corollary 1.2 was first obtained by A. Guionnet and one of the authors in [6]. Our objective is to extend this study to continuous time semigroups. For a precise discussion on the origins of this problem the reader is referred to the introduction of [6].

To motivate our work let us present what is new here comparative with the previous chain of published papers.

In the first place and to our knowledge the unified treatment of continuous and discrete time space with general multiplicative functions Z presented here has never been covered in the literature. In contrast to previously referenced papers our approach is context free, its simply relies on semigroup techniques and Dobrushin's ergodic coefficient and it is applicable to a large class of Feynman-Kac type semigroups. In addition the new integrability condition (3) on the oscillations of V allows us to weaken the mixing type condition (4) usually made in the literature.

In the second part of this paper we discuss the applications of the above results. First we show how our framework can be used to study the limiting behavior of the distribution of a Markov killed particle. We also indicate how the previous stability properties can be used to study the asymptotic behavior of a class of genetic type interacting processes with continuous time space. In this connection our approach complements the study of the stability properties of McKean-Vlasov diffusions of Tamura [18] to a class of interacting processes in which the interaction goes through jumps. In addition, as noticed in [8], the semigroup Φ can be associated to a simple generalized and spatially homogeneous Boltzmann equation. Therefore our results also complements the convergence analysis for Maxwellian molecules of Carlen [2] and Carlen and al. [3] to a class equations with Feynman-Kac representations. Finally we investigate the stability properties of the filtering equation for general Markov signals with continuous paths. We also provide explicit calculations and conditions underwhich the robust version of the filtering equation is asymptotically stable. To our knowledge the asymptotic stability properties of the robust filtering equation have not been covered by the literature on the subject.

The structure of the paper is as follows: In a preliminary section 2 we give precise definitions of the main objects used in this work. We also discuss some structural properties of the semigroup Φ in distribution space and we characterize the uniform stability of Φ in terms of the Dobrushin's ergodic coefficient associated to a suitably chosen semigroup on E. The precise description of Theorem 1.1 is stated and proved in section 3 . We explain how these results relate to the discrete time space situation treated in [6]. In section 4 we derive several easily verifiable conditions on the pair (Z, X) underwhich the semigroup Φ is asymptotically stable. We also give explicit and useful estimates of the Liapunov exponent for continuous and discrete time space semigroups. Section 5 is devoted to the applications of the general results to the analysis of the stability of killed Markov particles, interacting processes and non linear filtering equations.

2 Description of the models and statement of some results

Let $\mathbf{M}(E)$ be the space of all finite and signed measurable measures on $(E, \mathcal{E})$ with the total variation norm

$$\|\mu\|_{tv} =_{def.} \frac{1}{2}\left(\sup_{A\in\mathcal{E}} \mu(A) - \inf_{A\in\mathcal{E}} \mu(A)\right)$$

where the supremum and the infimum is taken over all subsets $A \in \mathcal{E}$. We also recall that any transition function $T(x, dz)$ on E generates two integral operations. The first one acting on the set $\mathcal{B}_b(E)$ of bounded $\mathcal{E}$-measurable functions $f : E \to \mathbb{R}$ and the second one on the set $\mathbf{M}(E)$ of finite measures μ on $\mathcal{E}$

$$Tf(x) = \int_E T(x,dz)\, f(z) \qquad \mu T(A) = \int_E \mu(dx)\, T(x,A)$$

If we write $\mathbf{M}_0(E)$ the subspace of $\mathbf{M}(E)$ of measures μ such that $\mu(E) = 0$ then any Markov transition $T(x,dz)$ on E can be regarded as an operator $T : \mathbf{M}_0(E) \to \mathbf{M}_0(E)$ and its norm is given by

$$\beta(T) =_{def.} \sup_{\mu \in \mathbf{M}_0(E)} \frac{\|\mu T\|_{tv}}{\|\mu\|_{tv}} = \sup_{\mu,\nu \in \mathbf{M}_1(E)} \frac{\|\mu T - \nu T\|_{tv}}{\|\mu - \nu\|_{tv}} \tag{5}$$

The quantity $\beta(T)$ is a measure of contraction of the total variation distance of probability measures induced by T. It can also be defined as

$$\beta(T) = \sup_{x,y \in E} \|T(x,.) - T(y,.)\|_{TV} = 1 - \alpha(T)$$

where the quantity $\alpha(T)$ is usually called the Dobrushin's ergodic coefficient of T and it is defined by

$$\alpha(T) = \inf \sum_{i=1}^{m} \min\left(T(x,A_i), T(z,A_i)\right) \tag{6}$$

where the infimum is taken over all $x, z \in E$ and all resolutions of E into pairs of non intersecting subsets $\{A_i\ ;\ 1 \leq i \leq m\}$ and $m \geq 1$ (see for instance [9]).

Let $X = \{X_t\ ;\ t \in T\}$ be a non necessarily temporally homogeneous Markov process taking values in a measurable space $(E,\mathcal{E})$ and associated to a time inhomogeneous transition semigroup $P = \{P_{s,t}\ ;\ s \leq t\}$. To describe precisely our model it is convenient to realize the process X in function space. For any $s \in I$ we write Ω_s the set of all càdlàg paths from $[s,\infty)$ to E. The value of $\omega \in \Omega_s$ at time $t (\geq s)$ is denoted by $X_t(\omega)$ and $F_{s,t}$ is the smallest σ-algebra of Ω_s for which all $X_r(\omega)$, $s \leq r \leq t$ are measurable. For any $(s,x) \in I \times E$ we denote by $\mathbb{P}_{s,x}$ the probability measure on $(\Omega_s, F_{s,\infty})$ defined by

$$\mathbb{P}_{s,x}\left(X_{s_1} \in A_1, \ldots, X_{s_n} \in A_n\right) = \int_{A_1 \times \ldots \times A_n} P_{s,s_1}(x,dz_1) \ldots P_{s_{n-1},s_n}(z_{n-1},dz_n)$$

where $s < s_1 < \ldots < s_n$ and $A_1, \ldots, A_n \in \mathcal{E}$. If μ is a probability distribution over E we define for any $s \in E$

$$\mathbb{P}_{s,\mu} = \int_E \mu(dx)\, \mathbb{P}_{s,x}$$

From the above construction $(\Omega_s, (X_t)_{t \geq s}, (F_{s,t})_{t \geq s}, \mathbb{P}_{s,\mu})$ is the desired Markov process on $[s,\infty)$ with initial distribution μ and the corresponding transition semigroup $\{P_{s,t}\ ;\ s \leq t\}$.

Now let $Z = \{Z_{s,t}\ 0 \leq s \leq t\}$ be a collection of stochastic multiplicative functions satisfying the following set of conditions. For any $0 \leq s \leq r \leq t$,

- $Z_{s,t}$ is a $F_{s,t-}$-measurable positive and bounded random variable.
- $Z_{s,s} = 1$ and $Z_{s,t} = Z_{s,r}\ Z_{r,t}$

To see that the mappings Φ defined in (1) form a semigroup we first notice that

$$\mathbb{E}_{s,\mu}\left(\, f(X_t)\ Z_{s,t}\,\right) =_{def.} \mu(H_{s,t}(f))$$

where

$$\mu(H_{s,t}(f)) =_{def.} \int_E \mu(dx)\ H_{s,t}(f)(x) \quad \text{and} \quad H_{s,t}(f)(x) = \mathbb{E}_{s,x}\left(\, f(X_t)\ Z_{s,t}\,\right)$$

Then we use the multiplicative properties of Z and the Markov property of X to check that $H = \{H_{s,t}\ ;\ s \leq t\}$ is a well defined semigroup on $\mathcal{B}_b(E)$. From these observations one concludes that Φ is a well defined semigroup on $\mathbf{M}_1(E)$. To see this last claim, by the semigroup property of H, we notice that for any $s \leq r \leq t$

$$\Phi_{s,t}(\mu)(f) = \frac{\mu H_{s,t}(f)}{\mu H_{s,t}(1)} = \frac{\mu H_{s,r}(H_{r,t}(f))/\mu H_{s,r}(1)}{\mu H_{s,r}(H_{r,t}(1))/\mu H_{s,r}(1)}$$

By definition of $\Phi_{s,r}$ and $\Phi_{r,t}$ we obtain

$$\Phi_{s,t}(\mu)(f) = \frac{\Phi_{s,r}(\mu) H_{r,t}(f)}{\Phi_{s,r}(\mu) H_{r,t}(1)} = \Phi_{r,t}\left(\Phi_{s,r}(\mu)\right)(f)$$

from which we conclude that Φ is a semigroup. Before presenting some structural properties of the Feynman-Kac distributions (1) let us fix some of the terminology used in the sequel. The semigroup Φ on $\mathbf{M}_1(E)$ is said to be asymptotically stable if it satisfies the following property

$$\lim_{t\to\infty} \sup_{\mu,\nu} \|\Phi_{0,t}(\mu) - \Phi_{0,t}(\nu)\|_{tv} = 0$$

When the rate of convergence is exponential in the sense that there exists some $s \in I$ and $\gamma > 0$ such that for any $t \geq s$

$$\sup_{\mu,\nu} \|\Phi_{s,t}(\mu) - \Phi_{s,t}(\nu)\|_{tv} \leq e^{-\gamma(t-s)}$$

Φ is said to be exponentially asymptotically stable.

Our analysis will be based on the following lemma. It says that the non linear mapping $\Phi_{s,t}$ is the composite mapping of a non linear transformation and a linear semigroup in distribution space.

Lemma 2.1 *For any $s \leq t$ and $\mu \in \mathbf{M}_1(E)$ we have the following decomposition*

$$\Phi_{s,t}(\mu) = \Psi_{s,t}(\mu) K^{(t)}_{s,t} \tag{7}$$

where the mapping $\Psi_{s,t} : \mathbf{M}_1(E) \to \mathbf{M}_1(E)$ is defined by

$$\Psi_{s,t}(\mu)(f) = \frac{\mu(\ g_{s,t}\ f\ )}{\mu(\ g_{s,t}\ )} \qquad \text{where} \qquad g_{s,t} =_{def.} H_{s,t} 1$$

and $K^{(t)} = \{K^{(t)}_{s,r}\ ;\ s \leq r \leq t\}$, $t \in I$, is a collection of linear semigroups defined for any $f \in \mathcal{B}_b(E)$ and $\mu \in \mathbf{M}_1(E)$ and $s \leq r \leq t$ by

$$\mu(K^{(t)}_{s,r}(f)) = \int_E \mu(dx)\ K^{(t)}_{s,r}(f)(x) \qquad \text{and} \qquad K^{(t)}_{s,r}(f) = \frac{H_{s,r}(g_{r,t}\ f)}{H_{s,r}(g_{r,t})}$$

Proof:
Since $g_{t,t} = H_{t,t}(1) = 1$ decomposition (7) is trivial. Let us check that for any fixed time parameter $t \in I$, $K^{(t)}$ is a linear semigroup. For any $0 \leq s \leq u \leq r \leq t$ and $f \in \mathcal{B}_b(E)$ we clearly have that

$$K^{(t)}_{s,r} f = \frac{H_{s,u} H_{u,r}(g_{r,t}\ f)}{H_{s,u} H_{u,r}(g_{r,t})} = \frac{H_{s,u}(H_{u,r}(g_{r,t})\ (K^{(t)}_{u,r} f))}{H_{s,u}(H_{u,r} g_{r,t})}$$

Since $H_{u,r}(g_{r,t}) = H_{u,r} H_{r,t}(1) = H_{u,t}(1) = g_{u,t}$ one concludes that

$$K^{(t)}_{s,r} = K^{(t)}_{s,u} K^{(t)}_{u,r}$$

The end of the proof is now straightforward ■

Remark 2.2:

By construction and using the Markov property of X it is easy to see that the transition kernels $K^{(t)}_{s,r}(x,dz)$, $s \leq r \leq t$ may likewise be defined by setting

$$K^{(t)}_{s,r}(f)(x) = \frac{\mathbb{E}_{s,x}\left(f(X_r)\, Z_{s,t}\right)}{\mathbb{E}_{s,x}\left(Z_{s,t}\right)}$$

The asymptotic stability properties of Φ can be characterized in terms of the Dobrushin's ergodic coefficient of the linear semigroups $\{K^{(t)}\ ;\ t \in I\}$

Proposition 2.3 *For any $s \leq t$ and $\mu, \nu \in \mathbf{M}_1(E)$ we have that*

$$\|\Phi_{s,t}(\mu) - \Phi_{s,t}(\nu)\|_{tv} \leq \beta\left(K^{(t)}_{s,t}\right)\ \ \|\Psi_{s,t}(\mu) - \Psi_{s,t}(\nu)\|_{tv} \tag{8}$$

and

$$\sup_{\mu,\nu} \|\Phi_{s,t}(\mu) - \Phi_{s,t}(\nu)\|_{tv} = \beta\left(K^{(t)}_{s,t}\right) \tag{9}$$

where the supremum is taken over all distributions $\mu, \nu \in \mathbf{M}_1(E)$.

Proof:

The inequality (8) is a simple consequence of (5) and the decomposition (7) given in Lemma 2.1. Let us prove (9). Since for any $x \in E$ and $s \leq t$

$$\Phi_{s,t}(\delta_x) = \delta_x K^{(t)}_{s,t}$$

it follows that

$$\beta\left(K^{(t)}_{s,t}\right) = \sup_{x,y\in E} \left\|\delta_x K^{(t)}_{s,t} - \delta_y K^{(t)}_{s,t}\right\|_{tv} \leq \sup_{\mu,\nu} \|\Phi_{s,t}(\mu) - \Phi_{s,t}(\nu)\|_{tv}$$

The reverse inequality is a consequence of (8) and the proof of (9) is now completed. ■

A crucial practical advantage of (9) is that it gives a first connection between the stability properties of the non linear semigroup Φ and the ergodic coefficients associated to a collection of linear semigroup $\{K^{(t)}\ ;\ t \in I\}$.

We end this preliminary section by noting that the semigroup Φ may have completely different kinds of long time behavior. For instance, if the multiplicative functions Z are trivial in the sense that $Z_{s,t} = 1$, for any $s \leq t$ then

$$K^{(t)}_{s,t} = P_{s,t} \qquad \forall s \leq t$$

In this case the asymptotic stability properties of Φ are reduced to that of P and

$$\sup_{\mu,\nu} \|\Phi_{s,t}(\mu) - \Phi_{s,t}(\nu)\|_{tv} = \sup_{\mu,\nu} \|\mu P_{s,t} - \nu P_{s,t}\|_{tv} = \beta\left(P_{s,t}\right)$$

At the opposite if the semigroup P is trivial in the sense that $P_{s,t} = Id$, for any $s \leq t$ then

$$K^{(t)}_{s,t} = Id \qquad \forall s \leq t$$

In this situation $\beta(K^{(t)}_{s,t}) = 1$ and one cannot expect to obtain uniform stability properties. Indeed if $I = \mathbb{R}_+$ and the functions Z are given for any $s \leq t$ by

$$\log Z_{s,t} = -\int_s^t U(X_r)\,dr \qquad U : E \to \mathbb{R}_+$$

then $\Phi_{s,t}(\mu)$ can be rewritten as follows

$$\Phi_{s,t}(\mu)(f) = \Psi_{s,t}(\mu)(f) = \frac{\mu\left(e^{-(t-s)U}\ f\right)}{\mu(\ e^{-(t-s)U}\ )} \qquad \forall f \in \mathcal{B}_b(E)$$

It is then easily seen that $\Phi_{s,t}(\mu)$ tends as $t \to \infty$ and in narrow sense to the restriction of μ to the subset

$$U^\star = \{x \in E\ ;\ U(x) = \inf \mathrm{ess}_{y \in E} U(y)\}$$

where the essential infimum is understood over μ.

3 An Asymptotic Stability Theorem

The results developed in this section are a more complete form of those in [6]. Next condition on the multiplicative functions Z is pivotal.

$(\mathcal{Z})$ There exists some $t_0 \in I$ and a positive function $z : (t,u) \in I^2 \mapsto z_t(u) \in (0,1]$ such that for any $t \geq t_0$ and $x, y, y' \in E$ and $u \in I$

$$z_t(u) \leq \frac{\mathbb{E}_{t,x}\left(Z_{t,t+u}\,|X_{t+u} = y\right)}{\mathbb{E}_{t,x}\left(Z_{t,t+u}\,|X_{t+u} = y'\right)} \leq z_t^{-1}(u) \tag{10}$$

with the convention $z_t(0) = 1$.

The above hypothesis allows to extend the methodology developed in [6] to analyze the contraction properties of the semigroups $K^{(t)}$ with the collection of probability transitions $S^{(t)} = \{S^{(t)}_{s,r}\ ;\ s \leq r\}$ defined by

$$S^{(t)}_{s,r} f = \frac{P_{s,r}(g_{r,t}\ f)}{P_{s,r}(g_{r,t})} \tag{11}$$

Next we denote by $\{\,I_u(s,r)\ ;\ s \leq t\,,\ u \in I\}$ the subsets of I defined by

$$I_u(s,r) = \{s + pu\ ;\ p \in \mathbb{N}\ ,\ 0 \leq p < [(r-s)/u]\}$$

with $[a]$ the integer part of $a \in \mathbb{R}$.

Condition $(\mathcal{Z})$ is not really restrictive and it is also easily verifiable. For instance let us suppose that $I = \mathbb{R}_+$ and Z is given by the following exponentials

$$Z_{s,t} = \exp \int_s^t V_r(X_r)\ dr \qquad \forall s \leq t$$

for some positive and bounded function $V : (r,x) \in I \times E \to V_r(x) \in \mathbb{R}_+$. In this situation one gets the bounds (10) with

$$z_t(u) = \exp - \int_t^{t+u} \mathrm{osc}(V_r)\ dr \tag{12}$$

where for any $r \in I$, $\mathrm{osc}(V_r)$ denotes the oscillation of the function V_r

$$\mathrm{osc}(V_r) = \sup\{V_r(y) - V_r(x)\ ;\ (x,y) \in E^2\}$$

We also notice that in discrete time settings (10) is always satisfied for $u = 1$. More precisely assume that $I = \mathbb{N}$ and $F_{t,(t+1)-} = \sigma(X_t)$ for any $t \in \mathbb{N}$. In this settings for any $t \in \mathbb{N}$ the random variable $\{Z_{t,t+1}\ ;\ t \in \mathbb{N}\}$ have necessarily the form

$$Z_{t,t+1} = g_t(X_t) \tag{13}$$

for some positive function g_t on E. Since for any $t \in \mathbb{N}$ and $x, y \in E$ we have that

$$\mathbb{E}_{t,x}\left(Z_{t,t+1} \,|X_{t+1} = y\right) = g_t(x)$$

then (10) with $z_t(1) = 1$.

Proposition 3.1 *If $(\mathcal{Z})$ is satisfied for some $t_0 \in I$ or more generally if (10) is satisfied for some $u \in I - \{0\}$ and any $t \geq t_0$ then we have that*

$$\forall t_0 \leq s \leq r \leq t \qquad \beta\left(K_{s,r}^{(t)}\right) \leq \prod_{m \in I_u(s,r)} \left(1 - z_m(u)\, \alpha\left(S_{m,m+u}^{(t)}\right)\right) \tag{14}$$

Proof:
On the basis of the definition of H given at the beginning of section 2 it is easy to establish that for any $f \in \mathcal{B}_b(E)$, $x \in E$, $s \leq r \leq t$

$$K_{s,r}^{(t)} f(x) = \frac{\mathbb{E}_{s,x}\left(\, f(X_r)\; g_{r,t}(X_r)\; \mathbb{E}_{s,x}\left(Z_{s,r} \,|X_r\,\right)\right)}{\mathbb{E}_{s,x}\left(\, g_{r,t}(X_r)\; \mathbb{E}_{s,x}\left(Z_{s,r} \,|X_r\,\right)\right)} \tag{15}$$

Under our assumptions this clearly implies that for any non negative function $f \in \mathcal{B}_b(E)$ and $t_0 \leq s \leq s+u \leq t$

$$z_s(u)\;\; S_{s,s+u}^{(t)}(f) \leq K_{s,s+u}^{(t)}(f) \leq z_s^{-1}(u)\;\; S_{s,s+u}^{(t)}(f)$$

If we combine the above inequality with (6) we see that the ergodic properties of $K^{(t)}$ can be related to that of the transitions $S^{(t)}$. More precisely for any $t_0 \leq s \leq s+u \leq t$

$$z_s(u)\;\; \alpha\left(S_{s,s+u}^{(t)}\right) \leq \alpha\left(K_{s,s+u}^{(t)}\right) \leq z_s^{-1}(u)\;\; \alpha\left(S_{s,s+u}^{(t)}\right)$$

This implies that for any $t_0 \leq s \leq s+u \leq r \leq t$

$$\beta\left(K_{s,r}^{(t)}\right) \leq \left(1 - z_s(u)\;\; \alpha\left(S_{s,s+u}^{(t)}\right)\right)\;\; \beta\left(K_{s+u,r}^{(t)}\right) \tag{16}$$

Taking into account that

$$\beta\left(K_{s,r}^{(t)}\right) \leq \left[\prod_{m \in I_u(s,r)} \beta\left(K_{m,m+u}^{(t)}\right)\right]\; \beta\left(K_{[(r-s)/u]u,r}^{(t)}\right)$$

it is easily seen that (14) is a consequence of (16). ■

In view of the previous proposition we see that the set of Markov transitions $\{S^{(t)}\,;\; t \in I\}$ plays a pivotal role in the study of the stability properties of Φ. In order to obtain some useful estimate we will use the following assumption.

$(\mathcal{S})$ There exists some $t_1 \in I$ and $v \in I - \{0\}$ such that for any $t_1 \leq s \leq s+v \leq t$

$$\alpha\left(S_{s,s+v}^{(t)}\right) \geq \epsilon_s(v) \tag{17}$$

for some non negative constants $\epsilon_s(v) \in [0,1]$ which do not depend on the parameter t.

Theorem 3.2 *Assume that condition $(\mathcal{S})$ holds for some constants $\epsilon_s(v)$. If $(\mathcal{Z})$ is satisfied for some $t_0 \leq t_1$ (or more generally if (10) is satisfied for $u = v$ and any $t \geq t_1$) then we have for any $t_1 \leq s \leq r \leq t$*

$$\beta\left(K_{s,r}^{(t)}\right) \leq \prod_{m \in I_v(s,r)} (1 - \delta_m(v)) \qquad \text{where} \qquad \delta_m(v) = z_m(v)\, \epsilon_m(v) \tag{18}$$

Therefore the following set of implications holds.

$$\lim_{t\to\infty} \sum_{m \in I_v(t_1,t)} \delta_m(v) = \infty \implies \lim_{t\to\infty} \beta\left(K_{0,t}^{(t)}\right) = 0 \tag{19}$$

$$\lim_{t\to\infty} \frac{1}{t} \sum_{m \in I_v(t_1,t)} \delta_m(v) =_{def.} \lambda(v) \implies \limsup_{t\to\infty} \frac{1}{t} \log \beta\left(K_{0,t}^{(t)}\right) \leq -\lambda(v) \tag{20}$$

$$\inf_{m \in I_v(t_1,t)} \delta_m(v) =_{def.} \delta(v) \implies \forall t \geq p(t_1 + v) \qquad \frac{1}{t} \log \beta\left(K_{0,t}^{(t)}\right) \leq -\frac{\delta(v)}{q\, v} \tag{21}$$

for any $p, q \geq 1$ such that $1/p + 1/q = 1$.

Proof:
By Proposition 3.1, (17) implies (18). On the other hand, if we use the inequalities

$$\forall t \geq t_1 \qquad \beta\left(K_{0,t}^{(t)}\right) \leq \beta\left(K_{t_1,t}^{(t)}\right)$$

and

$$\beta\left(K_{t_1,t}^{(t)}\right) \leq \prod_{m \in I_v(t_1,t)} (1 - \delta_m(v)) \leq \exp - \sum_{m \in I_v(t_1,t)} \delta_m(v)$$

then (19) and (20) are easily checked. To prove (21) we simply notice that

$$\lim_{t\to\infty} \frac{1}{t} \text{Card}\, I_v(t_1,t) = 1/v \qquad \text{and} \qquad \text{Card}\, I_v(t_1,t) = \left[\frac{t - t_1}{v}\right] \geq \frac{t}{q\, v}$$

as soon as $t \geq p(t_1 + v)$ with $1/p + 1/q = 1$. This completes the proof of the theorem.

■

Next we return to the discrete time space. Suppose we have $I = \mathbb{N}$ and $F_{t,(t+1)-} = \sigma(X_t)$ for any $t \in \mathbb{N}$. In this situation (15) and (13) imply that for any $s < t$

$$K_{s,s+1}^{(t)} = S_{s,s+1}^{(t)}$$

Since $K^{(t)}$ is a linear semigroup this yields that for any $s \leq r \leq t$

$$K_{s,r}^{(t)} = S_{s,s+1}^{(t)} S_{s+1,s+2}^{(t)} \cdots S_{r-1,r}^{(t)}$$

From (13) we can also check that the functions $\{g_{s,t}\ ;\ 0 \leq s \leq t\}$ satisfy the backward recursions

$$\forall 0 \leq s < t \qquad g_{s,t} = g_s\ P_{s,s+1}(g_{s+1,t}) \qquad \text{and} \qquad g_{t,t} \equiv 1$$

In the discrete time case we find that for any $s \leq r \leq t$

$$\beta\left(K_{s,r}^{(t)}\right) \leq \prod_{s \leq m < (r-s)} \beta\left(S_{m,m+1}^{(t)}\right) = \prod_{s \leq m < (r-1)} \left(1 - \alpha\left(S_{m,m+1}^{(t)}\right)\right) \tag{22}$$

Therefore if $(\mathcal{S})$ holds $v = 1$ then we see that (18), (19), (20) and (21) hold true for $v = 1$ if we replace $\delta_m(1)$ by $\epsilon_m(1)$.

4 Lower Bounds for Dobrushin's Coefficient

The assumption $(\mathcal{S})$ holds for instance if the semigroup P is sufficiently mixing and/or if Z is a collection of sufficiently regular functions. In this section we present a series of conditions on the pair (P,Z) for which explicit and useful lower bounds of type (17) may be obtained. These estimating techniques will be based essentially on the properties of Dobrushin's ergodic coefficient. We will also formulate several corollaries of Theorem 3.2

Proposition 4.1 *Assume that the multiplicative function Z and the semigroup P satisfy the following condition*

$(\mathcal{ZP})$ *There exists some $t_1 \in I$ such that*

$$\forall t \geq t_1 \qquad \gamma_t =_{def.} \inf_{u\geq 0} \gamma_t(u) > 0 \qquad \text{where} \qquad \gamma_t(u) =_{def.} \inf_{x,y\in E} \frac{\mathbb{E}_{t,x}(Z_{t,t+u})}{\mathbb{E}_{t,y}(Z_{t,t+u})}$$

Then condition $(\mathcal{S})$ holds with for any $t_1 \leq s \leq s+v \leq t$

$$\gamma_{s+v} \;\; \alpha\left(P_{s,s+v}\right) \leq \alpha\left(S^{(t)}_{s,s+v}\right) \leq \gamma^{-1}_{s+v} \;\; \alpha\left(P_{s,s+v}\right) \tag{23}$$

In addition, if $\inf_{t\in I} \gamma_t =_{def.} \gamma > 0$ *then*

$$(\mathcal{S}) \Longleftrightarrow \exists (t_1,u) \in I \times (I - \{0\}) \;\; : \;\; \forall t_1 \leq s \leq s+u \qquad \alpha(P_{s,s+u}) > 0$$

Proof:
By a direct computation we have for any non negative test function $f : E \to \mathbb{R}_+$ and for any $t_1 \leq s \leq s+v \leq t$

$$\gamma_{s+v}(t-(s+v)) \; P_{s,s+v}(f) \leq S^{(t)}_{s,s+v} f \leq \gamma^{-1}_{s+v}(t-(s+v)) \; P_{s,s+v}(f)$$

thus

$$\gamma_{s+v} \; P_{s,s+v}(f) \leq S^{(t)}_{s,s+v} f \leq \gamma^{-1}_{s+v} \; P_{s,s+v}(f)$$

Therefore the desired bounds are a consequence of (6). ∎

The following special case is worth recording. Let $I = \mathbb{R}_+$ and Z be given by

$$\forall s \leq t \qquad Z_{s,t} = \exp \int_s^t V_r(X_r)\, dr \quad \text{where} \quad V_r(x) = v_r \; V(x)$$

and $V : x \in E \mapsto V(x) \in \mathbb{R}_+$ and $v : r \in I \mapsto \mathbb{R}_+$ are given. In this situation one can check that

$$v =_{def.} \int_0^\infty v_t \; dt < \infty \Longrightarrow \;\; (\mathcal{ZP}) \quad \text{with} \quad \forall t \in I \quad \gamma_t \geq e^{-v\,\mathrm{osc(V)}}$$

This also yields the bounds

$$\forall 0 \leq s \leq r \leq t \qquad e^{-v\,\mathrm{osc(V)}}\; \alpha\left(P_{s,r}\right) \leq \alpha\left(S^{(t)}_{s,r}\right) \leq e^{v\,\mathrm{osc(V)}}\; \alpha\left(P_{s,r}\right)$$

and

$$\forall t,u \in \mathbb{R}_+ \qquad z_t(u) \geq \; \exp -(v\, osc(V))$$

from which one concludes that

$$\exists v \in I - \{0\} \;\; : \;\; \forall t \in T \qquad \alpha\left(P_{t,t+v}\right) > 0 \Longrightarrow \Phi \text{ is asymptotically stable}$$

In addition, if X is time-homogeneous then (21) and (23) yields that

$$\forall t \geq 2u \qquad \frac{1}{t} \log \beta\left(K^{(t)}_{0,t}\right) \leq -\frac{\alpha(P_{0,u})}{2u} \;\; \exp -(2v\, osc(V))$$

This simple approach works in more general situations. A simple corollary of Proposition 4.1 and Theorem 3.2 is the following

Corollary 4.2 *Suppose that $I = \mathbb{R}_+$ and the multiplicative function Z is given by*

$$\forall s \leq t \qquad Z_{s,t} = \exp \int_s^t V_r(X_r)\, dr$$

where $V : (t,x) \in \mathbb{R}_+ \times E \mapsto V_t(x)$ is a measurable function such that

$$\mathrm{osc}(V)^\star =_{def.} \int_0^\infty \mathrm{osc}(V_t)\; dt < \infty$$

Then the following implication hold for any $u > 0$

$$\sum_{p\geq 1} \alpha\left(P_{(p-1)u,pu}\right) = \infty \quad \Longrightarrow \quad \lim_{t\to\infty} \beta\left(K^{(t)}_{0,t}\right) = 0$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{p=1}^{n} \alpha\left(P_{(p-1)u,pu}\right) =_{def.} \overline{\alpha}(u) \quad \Longrightarrow \quad \limsup_{t\to\infty} \frac{1}{t}\log\beta\left(K^{(t)}_{0,t}\right) \leq -\frac{\overline{\alpha}(u)}{u}\; \exp -(2\mathrm{osc}(V)^\star)$$

In addition if we have

$$\inf_{p\geq 1} \alpha\left(P_{(p-1)u,pu}\right) =_{def.} \overline{\alpha}(u)$$

then for any $t \geq p\,.\,u$ and $p, q \geq 1$ such that $1/p + 1/q = 1$

$$\frac{1}{t}\log\beta\left(K^{(t)}_{0,t}\right) \leq -\frac{\overline{\alpha}(u)}{q\,.\,u}\; \exp -(2\mathrm{osc}(V)^\star)$$

Furthermore if $\inf_{|t-s|=u} =_{def.} \overline{\alpha}(u) > 0$ *for some $u > 0$ then for any $p \geq 1$ and $T \geq pv$ and $1/p + 1/q = 1$ we have that*

$$\sup_{t\geq 0}\; \sup_{\mu,\nu\in \mathbf{M}_1(E)} \frac{1}{T}\, \|\Phi_{t,t+T}(\mu) - \Phi_{t,t+T}(\nu)\|_{tv} \leq -\frac{\overline{\alpha}(u)}{q\,.\,u}\; \exp -(2\mathrm{osc}(V)^\star)$$

Proof:
Under our assumptions, the inequality (14) implies that for any $s \leq r \leq t$

$$\beta\left(K^{(t)}_{s,r}\right) \leq \prod_{m\in I_u(s,r)} \left(1 - \alpha\left(S^{(t)}_{m,m+u}\right)\; e^{-\mathrm{osc}(V)^\star}\right)$$

Using this inequality the three implications are straightforward. The last assertion is a clear consequence of (9). ■

This special case apart, Proposition 4.1 only describes some consequence of the lower bound condition $(\mathcal{ZP})$ but does not indicate when this property holds. In the further development we give separate conditions on Z and P which suffice to check $(\mathcal{ZP})$. Before we proceed we next examine an additional sufficient condition for $(\mathcal{S})$ in terms of the mixing properties of P.

Proposition 4.3 *Assume that the semigroup P satisfies the following condition.*

$(\mathcal{P})$ *There exists some t_1 and $v \in I$ such that for any $t \geq t_1$*

$$\epsilon_t^{1/2}(v) \leq \frac{dP_{t,t+v}(x,\,.)}{d\mu_{t,v}} \leq \epsilon_t^{-1/2}(v) \qquad \forall x \in E$$

for some positive constant $\epsilon_t(v) > 0$ and some reference probability measure $\mu_{t,v} \in \mathbf{M}_1(E)$. Then condition $(\mathcal{S})$ holds with

$$\forall t_1 \leq s \leq s+v \leq t \qquad \alpha\left(S^{(t)}_{s,s+v}\right) \geq \epsilon_s(v)$$

Proof:
Under $(\mathcal{P})$ and for any non negative test function f we clearly have for any $t_1 \leq s \leq s+v \leq t$

$$\epsilon_s(v)\ \Psi_{s+v,t}(\mu)(f) \leq S^{(t)}_{s,s+v}(f) \leq \epsilon_s^{-1}(v)\ \Psi_{s+v,t}(\mu)(f)$$

Again using (6) one concludes that

$$\epsilon_s(v) \leq \alpha\left(S^{(t)}_{s,s+v}\right) \leq \epsilon_s^{-1}(v)$$

and the proof is completed. ∎

Using $(\mathcal{ZP})$ or $(\mathcal{P})$ one can obtain lower bounds for the ergodic coefficient of the transition probability functions $S^{(t)}$, $t \in I$. To see the connections between these two conditions it is convenient to strengthen condition $(\mathcal{Z})$.

$(\mathcal{Z})'$ For any $t, u \in I$ there exists a constant $z_t(u) \in (0,1]$ such that

$$z_t(u)^{1/2} \leq Z_{t,t+u} \leq z_t(u)^{-1/2} \qquad \text{P-a.s.} \tag{24}$$

As we shall this in the foregoing development this condition is met in many interesting applications.
We start by noting that $(\mathcal{Z})' \Longrightarrow (\mathcal{Z})$ and (24) $\Longrightarrow$ (10). In the same way one can also check that $\gamma_t(u) \geq z_t(u)$ for any $(u,t) \in I^2$ but the function $z_t : u \in I \mapsto z_t(u) \in (0,1]$ usually fails to be lower bounded. More precisely for any $t \in I$ we usually have that $\lim_{u\to\infty} z_t(u) = 0$ so that $(\mathcal{Z})' \not\Rightarrow (\mathcal{ZP})$. Next proposition shows that $(\mathcal{Z})' + (\mathcal{P}) \Rightarrow (\mathcal{ZP})$.

Proposition 4.4 *Assume that condition $(\mathcal{Z})'$ is satisfied for some function z. Then the following assertions hold*

$$\inf_{u\in I} z_t(u) =_{def.} z_t > 0 \quad \Longrightarrow \quad (\mathcal{ZP}) \ \text{with} \quad \gamma_t \geq z_t \tag{25}$$

$$(\mathcal{P}) \quad \Longrightarrow \quad (\mathcal{ZP}) \ \text{with} \quad \gamma_t \geq \epsilon_t(v) \inf_{u\leq v} z_t(u) \tag{26}$$

Proof:
The proof of (25) is a clear consequence of the definition of $\{\gamma_t \ ; \ t \in I\}$. To prove (26) we assume that $(\mathcal{P})$ is satisfied for some $v \in I$ and $t_1 \in I$. Since for any $u \geq v$

$$\begin{aligned} g_{t,t+u}(x) &= \mathbb{E}_{t,x}(Z_{t,t+u}) = \mathbb{E}_{t,x}(Z_{t,t+v}\, Z_{t+v,t+u}) \\ &= \mathbb{E}_{t,x}\left(\mathbb{E}_{t,x}(Z_{t,t+v}\,|X_{t+v})\ \mathbb{E}_{t,x}(Z_{t+v,t+u}\,|X_{t+v})\right) \end{aligned}$$

one obtain the lower bound

$$\forall u \geq v \qquad \gamma_t(u) \geq \epsilon_t(v)\ z_t(v)$$

from which one concludes that

$$\gamma_t \geq \min\left\{z_t(v)\epsilon_t(v), \inf_{u\leq v} z_t(u)\right\} \geq \epsilon_t(v) \inf_{u\leq v} z_t(u)$$

This ends the proof of (26). ∎

Let us now investigate another consequence of the later results. Assume that $I = \mathbb{R}_+$, and the multiplicative functions Z are given by

$$\forall s \leq t \qquad Z_{s,t} = \exp \int_s^t V(X_r)\, dr \tag{27}$$

where $V : E \to \mathbb{R}_+$ is a measurable function on E with bounded oscillation $\mathrm{osc}(V) < \infty$. As we have already noticed in (12) condition $(\mathcal{Z})$ and the lower bound (10) hold with

$$z_t(u) = z(u) = \exp -(u\, .\mathrm{osc}(V))$$

If X is a sufficiently regular diffusion on a compact manifold E then the mixing type condition $(\mathcal{P})$ holds with

$$\epsilon_t(u) = \epsilon(u) = A\ \exp -(B/u) \tag{28}$$

for some constants $0 < A, B < \infty$ and for the uniform Riemannian measure on the manifold. In this specific situation, using (21) Theorem 3.2, one concludes that for any $u \in I$, $p \geq 1$ and $t \geq p.u$

$$\frac{1}{t} \log \beta \left(K_{0,t}^{(t)}\right) \leq -\frac{A}{q\ .u}\ \exp -\left(u\ .\mathrm{osc}(V) + \frac{B}{u}\right) \qquad \text{with} \quad 1/p + 1/q = 1$$

Summarizing it can finally be seen that

Corollary 4.5 *Assume that $I = \mathbb{R}_+$ and Z is given by (27) and the semigroup of X satisfies condition $(\mathcal{P})$ with $\epsilon_t(u) = \epsilon(u)$ given by (28). Then for any $p \geq 1$ and $u > 0$ and $T \geq p\,.\,u$ we have that*

$$\sup_{\mu,\nu} \|\Phi_{t,t+T}(\mu) - \Phi_{t,t+T}(\nu)\|_{tv} \leq \exp -(\gamma\,.\,T)$$

with

$$\gamma \geq \frac{A}{q\ \ u}\ \exp -\left(\frac{B}{u} + u\,.\,\mathrm{osc}(V)\right) \qquad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1$$

The best bound in term of the constants A, B and $osc(V)$ is obtained for

$$u = u^\star =_{def.} \frac{2B}{1 + \sqrt{1 + 4B\ .\mathrm{osc}(V)}}$$

The last Feynman-Kac model which we are going to discuss will be the discrete time case. Next proposition is a useful reformulation of the above results in these settings.

Proposition 4.6 *Assume that $I = \mathbb{N}$ and $F_{t,(t+1)-} = \sigma(X_t)$ for any $t \in \mathbb{N}$ so that the random variables $\{Z_{t,t+1}\ ;\ t \in I\}$ can be defined by (13) for some measurable positive functions $g =_{def.} \{g_t\ ;\ t \in I\}$. In this situation $(\mathcal{Z})'$ holds if, and only if, g satisfies the following condition*

$$\forall t \in I, x, y \in E \qquad a_t \leq \frac{g_t(x)}{g_t(y)} \leq a_t^{-1} \tag{29}$$

for some non negative constants $\{a_t\ ;\ t \in I\}$. In addition we have that

- *Condition $(\mathcal{Z})$ and the corresponding lower bounds (10) hold with*

$$\forall (t, u) \in I^2 \qquad z_t(u) = \prod_{p=1}^{u-1} a_{t+p} \tag{30}$$

- *If $(\mathcal{P})$ holds for $v = 1$, $t_1 = 0$ and for some constant $\epsilon_t(1) =_{def.} \epsilon_t > 0$ then we have*

$$\forall s \leq t \qquad \alpha\left(S_{s,s+1}^{(t)}\right) \geq \epsilon_s$$

and therefore

$$\lim_{t\to\infty}\sum_{t\in I}\epsilon_t=\infty \quad\Longrightarrow\quad \lim_{t\to\infty}\beta\left(K^{(t)}_{0,t}\right)=0$$

$$\lim_{t\to\infty}\frac{1}{t}\sum_{s=0}^{t-1}\epsilon_s=_{def.}\epsilon \quad\Longrightarrow\quad \limsup_{t\to\infty}\frac{1}{t}\log\beta\left(K^{(t)}_{0,t}\right)\leq-\epsilon$$

$$\inf_{t\in I}\epsilon_t=_{def.}\epsilon \quad\Longrightarrow\quad \forall 0\leq s\leq r\leq t \qquad \beta\left(K^{(t)}_{s,r}\right)\leq e^{-\epsilon(r-s)}$$

- *If* $(\mathcal{P})$ *is satisfied for some* $(v,t_1)\in I^2$ *and some positive function* $\epsilon:(s,v)\in I\times E\mapsto \epsilon_s(v)\in\mathbb{R}_+$ *then* $(\mathcal{ZP})$ *holds with*

$$\forall t\geq t_1 \qquad \gamma_t\geq\epsilon_t(v)\prod_{p=0}^{v-1}a_{t+p} \tag{31}$$

and $(\mathcal{S})$ *is also satisfied with*

$$\forall t_1\leq s\leq s+v\leq t \qquad \alpha\left(S^{(t)}_{s,s+v}\right)\geq\epsilon_s(v) \tag{32}$$

If $(\mathcal{P})$ *is satisfied for some* $v\in I$ *and* $t_1=0$ *then we have that*

$$\beta\left(K^{(t)}_{0,t}\right)\leq\prod_{p=0}^{[t/v]-1}\left(1-\epsilon_{pv}(v)\prod_{q=1}^{v-1}a_{pv+q}\right) \tag{33}$$

and also

$$\beta\left(K^{(t)}_{0,t}\right)\leq\prod_{0\leq m<t-v}\left(1-\epsilon_{m+1}(v)\,\alpha\left(P_{m,m+1}\right)\prod_{q=1}^{v}a_{m+q}\right) \tag{34}$$

- *Condition* $(\mathcal{ZP})$ *holds if the series* $\sum_t\log a_t$ *converge, that is*

$$\sum_{t\geq 0}\log a_t=_{def.}\log a<\infty\Longrightarrow(\mathcal{ZP})\ \text{ with } \ \inf_{t\geq 0}\gamma_t\geq a>0 \tag{35}$$

- *If* $\inf_t a_t=_{def.}a>0$ *and* $(\mathcal{P})$ *holds for some* $v\in I$ *and* $t_1=0$ *and* $\inf_t\epsilon_t(v)=_{def.}\epsilon(v)>0$ *then for any* $p\geq 1$ *and* $t\geq p.v$

$$\frac{1}{t}\log\beta\left(K^{(t)}_{0,t}\right)\leq-\frac{1}{v.q}\;.\;a^v\;.\;\epsilon(v)\;.\;\max\left(\alpha\;.\;v\;;\;\frac{1}{a}\right) \tag{36}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $\alpha=_{def.}\inf_{n\geq 0}\alpha\left(P_{n,n+1}\right)$.

Proof:
The equivalence (29) $\Longleftrightarrow$ $(\mathcal{Z})'$ is clear. To prove that $(\mathcal{Z})$ and the bounds (10) hold with (30) it suffices to note that

$$\mathbb{E}_{t,x}\left(Z_{t,t+u}\,|X_{t+u}\right)=g_t(x)\;\mathbb{E}_{t,x}\left(\prod_{p=1}^{u-1}g_{t+p}(X_{t+p})\,|X_{t+u}\right)\qquad \text{P-a.s.}$$

The three implications are a consequence of (22). The proof of (34) is another consequence of (22) and the fact that for any $0\leq m+v<t$ and $x,y\in E$

$$\epsilon_{m+1}(v)\prod_{p=1}^{v}a_{m+p}\;\leq\frac{g_{m+1,t}(x)}{g_{m+1,t}(y)}\leq\epsilon^{-1}_{m+1}(v)\prod_{p=1}^{v}a^{-1}_{m+p} \tag{37}$$

Indeed, (37) implies that for any non negative test function f

$$\epsilon_{m+1}(v)\ \left(\prod_{p=1}^{v} a_{m+p}\right)\ P_{m,m+1}(f) \leq S^{(t)}_{m,m+1}(f) \leq \epsilon^{-1}_{m+1}(v)\ \left(\prod_{p=1}^{v} a^{-1}_{m+p}\right)\ P_{m,m+1}(f)$$

and therefore

$$\epsilon_{m+1}(v)\ \left(\prod_{p=1}^{v} a_{m+p}\right)\ \alpha\left(P_{m,m+1}\right) \leq \alpha\left(S^{(t)}_{m,m+1}\right) \leq \epsilon^{-1}_{m+1}(v)\ \left(\prod_{p=1}^{v} a^{-1}_{m+p}\right)\ \alpha\left(P_{m,m+1}\right)$$

from which (34) is a clear consequence of (22). If we combine (18) and (30) one obtain (33), that is

$$\beta\left(K^{(t)}_{0,t}\right) \leq \prod_{s\in I_v(0,t)} \left(1 - z_s(v)\ \epsilon_s(v)\right) = \prod_{s\in I_v(0,t)} \left(1 - \left(\prod_{p=1}^{v-1} a_{s+p}\right)\ \epsilon_s(v)\right)$$

Under our assumptions this implies that

$$\beta\left(K^{(t)}_{0,t}\right) \leq \left(1 - a^{v-1}\epsilon\right)^{[t/v]} \quad \text{and} \quad \beta\left(K^{(t)}_{0,t}\right) \leq \left(1 - \alpha\epsilon a^{v}\right)^{t-v}$$

from which the end of proof of (36) is straightforward. ∎

5 Applications

As we said in the introduction the analysis of Feynman-Kac semigroups as those studied in this work has motivations coming from non linear estimation such as non linear filtering and numerical function optimization but also from physics and biology. Next we present several generic examples of multiplicative functions Z and semigroup P together with some comments concerning their derivations.

5.1 Markov killed particle

Let us start from a remark that the distributions of a random particle killed at a given rate and conditioned by non extinction can be described by a Feynman-Kac non linear semigroup. More precisely let us suppose that $I = \mathbb{R}_+$ and X and V are temporally homogeneous and $V : E \to \mathbb{R}_-$ is a given negative measurable function. Let us write $\{P_t\ ;\ t \geq 0\}$ the semigroup of X. By the multiplicative property, the transitions $\{P^{(v)}_t(x, dz)\ ;\ t \geq 0\}$ defined for any measurable test function f by setting

$$P^{(v)}_t(f)(x) =_{def.} \mathbb{E}_x\left(f(X_t)\ Z_{0,t}\right) \qquad \text{and} \qquad Z_{0,t} =_{def.} \exp\int_0^t V(X_s)\,ds$$

form a semigroup which is sub-Markovian in the sense that $P^{(v)}_t(x, E) < 1$ for some x's and t's. The semigroups $P^{(v)}$ and P are related one another by the relations

$$P^{(v)}_t(x, dy) = \mathbb{E}_x\left(Z_{0,t}\,|X_t = y\right)\ P_t(x, dy)$$

To turn the sub-Markovian semigroup $\{P^{(v)}_t\ ;\ t \geq 0\}$ into the Markovian case we adjoin classically to the state space E a cemetery point denoted by Δ and we define a Markovian semigroup $\{\tilde{P}^{(v)}_t\ ;\ t \geq 0\}$ by setting for any measurable subset $A \in \mathcal{E}$

$$\tilde{P}^{(v)}_t(x, A) = P^{(v)}_t(x, A) \qquad \tilde{P}^{(v)}_t(x, \{\Delta\}) = 1 - P^{(v)}_t(x, E) \quad \text{and} \quad \tilde{P}^{(v)}_t(\Delta, \{\Delta\}) = 1$$

If $\{\tilde{X}_t^{(v)}\ ;\ t \geq 0\}$ denotes the corresponding Markov process on $E \cup \{\Delta\}$ then we have for any measurable subset $A \in \mathcal{E}$

$$\Phi_{0,t}(\delta_x)(A) = K_{0,t}^{(t)}(x,A) = \frac{\tilde{P}_t^{(v)}(x,A)}{\tilde{P}_t^{(v)}(x,E-\{\Delta\})} = \mathbb{P}_x\left(\tilde{X}_t^{(v)} \in A \,\middle|\, \tilde{T}^{(v)} > t\right)$$

where $\tilde{T}^{(v)} = \inf\{t > 0\ \tilde{X}_t^{(v)} = \Delta\}$ is the life-time of $\tilde{X}^{(v)}$. The asymptotic stability results developed in previous sections give several conditions underwhich a killed particle as defined previously and conditioned by non extension forgets exponentially fast its initial position.

Corollary 5.1 *If the non linear semigroup Φ is asymptotically stable and (2) holds for some $\gamma > 0$ then for any $t > 0$ we have that*

$$\sup_{x,y\in E}\left\|\mathbb{P}_x\left(\tilde{X}_t^{(v)} \in\ .\ \middle|\tilde{T}^{(v)} > t\right) - \mathbb{P}_y\left(\tilde{X}_t^{(v)} \in\ .\ \middle|\tilde{T}^{(v)} > t\right)\right\|_{tv} \leq \exp-(\gamma\ .\ t)$$

5.2 Stability of interacting processes

In measure valued process and genetic algorithm theory the Feynman-Kac semigroup (1) describes the evolution in time of the limiting process of Moran-type interacting particle systems (see [8]). More precisely let us assume that $I = \mathbb{R}_+$ and

- the Markov process X is associated to a collection of generators $\{L_t\ ;\ t \in I\}$ with domain $\mathcal{D} \subset \mathcal{C}_b(E)$.
- the multiplicative functions Z are defined by

$$Z_{s,t} = \exp\int_s^t V_r(X_r)\,dr \tag{38}$$

  where $V : (r,x) \in I \times E \mapsto V_r(x) \in \mathbb{R}_+$ is a bounded measurable function

In this specific example the non linear semigroup Φ represents the evolution in time of the solution of the non linear and $\mathbf{M}_1(E)$-valued process defined by

$$\forall f \in \mathcal{D} \qquad \frac{d}{dt}\eta_t(f) = \eta_t(\mathcal{L}_{t,\eta_t}(f)) \tag{39}$$

where $\{\mathcal{L}_{t,\eta}\ ;\ t \in I\ \ \eta \in \mathbf{M}_1(E)\}$ is a pregenerator on E defined on a suitable domain by

$$\mathcal{L}_{t,\eta}(f)(x) = L_t(f)(x) + \int_E (f(y) - f(x))\ V_t(y)\ \eta(dy) \tag{40}$$

The above measure valued evolution equations can be regarded as the limiting process associated to a sequence of interacting particle systems. More precisely, starting from the family of pre-generators $\{\mathcal{L}_{t,\eta}\ ;\ t \geq 0,\ \ \eta \in \mathbf{M}_1(E)\}$ we associate an N-particles system

$$(\xi_t)_{t\geq 0} = ((\xi_t^1,\ldots,\xi_t^N))_{t\geq 0} \tag{41}$$

which is a time-inhomogeneous Markov process on the product space E^N, $N \geq 1$, whose generator acts on functions ϕ belonging to a good domain by

$$\mathcal{L}_t^{(N)}(\phi)(x_1,\ldots,x_N) \ = \ \sum_{i=1}^N \mathcal{L}_{t,m(x)}^{(i)}(\phi)(x_1,\ldots,x_N) \qquad m(x) = \frac{1}{N}\sum_{i=1}^N \delta_{x^i} \tag{42}$$

where the notation $\mathcal{L}_{t,\eta}^{(i)}$ have been used instead of $\mathcal{L}_{t,\eta}$ when it acts on the i-th variable of $\phi(x_1,\ldots,x_N)$ and δ_x is the Dirac measure on $x \in E$. From (40) we notice that in these schemes the interaction between particles is expressed through jumps.

To our knowledge the earliest work on the subject of the long time behavior of non linear semigroup associated to interacting processes was that of Tamura [18]. In the latter the author studied the convergence of distributions of McKean-Vlasov type (in which the interaction goes through drifts) but he didn't look to the situation as here where interaction goes through jumps. Related genetic-type schemes for the numerical solving of Feynman-Kac formula in discrete time settings can be found in and [6, 7, 8].
In physics (1) can also be regarded as a simple generalized and spatially homogeneous Boltzmann equation (cf. [14]). The long time behavior of the Boltzmann equation for Maxwellian molecules is studied in several papers under specific assumptions on the collision kernel and/or on the initial data (see for instance [2, 3] and references therein).

Our Feynman-Kac model does not fit into these particular Maxwellian or Mc-Kean Vlasov settings. The method developed in this paper complements in some sense the work of Tamura and the papers on the convergence to equilibrium of Boltzmann's type equations. Although our approach strongly depends on the Feynman-Kac representation of (39) it exhibits and underlines some precise links between the mixing properties of the first explorating generators L_t and the asymptotic stability of (39).

As a guide to their usage next we examine how the previous stability properties can be used to obtain uniform estimates for the particle approximating models (41). To do this we need to recall some estimates presented in [8]. Let $\{\eta_t^N\ ;\ t\geq 0\}$ be the particle density profiles associated to the N-particle system $(\xi_t)_{t\geq 0}=((\xi_t^1,\ldots,\xi_t^N))_{t\geq 0}$ and defined by

$$\eta_t^N =_{def.} \frac{1}{N}\sum_{i=1}^N \delta_{\xi_t^i}$$

By Proposition 3.2.5 p.100 and the calculations given p.103 in [8] if the function V is uniformly bounded in the sense that

$$\|V\| =_{def.} \sup_t \|V_t\| < \infty$$

then we have for any $f\in\mathcal{B}_b(E)$, $\|f\|\leq 1$, $t,T\geq 0$ and $N\geq 1$

$$\mathbb{E}\left(|\eta_{t+T}^N f-\Phi_{t,t+T}(\eta_t^N)f|\right)\leq \frac{\exp{(\gamma'\,.\,T)}}{\sqrt{N}} \quad \text{and} \quad \sup_{t\leq T}\mathbb{E}\left(|\eta_t^N f-\eta_t f|\right)\leq \frac{\exp{(\gamma'\,.\,T)}}{\sqrt{N}} \tag{43}$$

for some finite constant $\gamma'>0$ which only depends on $\|V\|$. Now suppose the Feynman-Kac semigroup Φ is exponentially asymptotically stable in the sense that for some finite constants $T_0\geq 0$, $\gamma>0$ and for any $T\geq T_0$

$$\sup_{t\geq 0}\sup_{\mu,\nu}\|\Phi_{t,t+T}(\mu)-\Phi_{t,t+T}(\nu)\|_{tv}\leq \exp{-(\gamma\,.\,T)} \tag{44}$$

Then using the decomposition

$$\eta_{t+T}^N-\eta_{t+T}=\left[\eta_{t+T}^N-\Phi_{t,t+T}(\eta_t^N)\right]+\left[\Phi_{t,t+T}(\eta_t^N)-\Phi_{t,t+T}(\eta_t)\right]$$

and from (43) and (44) we find that for any $t\geq 0$, $T\geq T_0$ and $N\geq 1$

$$\mathbb{E}\left(|\eta_{t+T}^N f-\eta_{t+T}f|\right)\leq \frac{\exp{(\gamma'\,.\,T)}}{\sqrt{N}}+\exp{-(\gamma\,.\,T)}$$

It follows that for any $T\geq T_0$

$$\sup_t \mathbb{E}\left(|\eta_t^N f-\eta_t f|\right)\leq \frac{\exp{(\gamma'\,.\,T)}}{\sqrt{N}}+\exp{-(\gamma\,.\,T)}$$

This implies that for any $N \geq 1$ such that

$$T(N) =_{def.} \frac{\log N}{2(\gamma+\gamma')} \geq T_0$$

we have the uniform estimate with respect to the time parameter

$$\sup_{t\geq 0} E\left(\left|\eta_t^N f - \eta_t f\right|\right) \leq \frac{2}{N^{\alpha/2}} \quad \text{where} \quad \alpha =_{def.} \frac{\gamma}{\gamma+\gamma'}$$

5.3 Non Linear Filtering

In non linear filtering settings, the semigroup Φ represents the evolution in time of conditional distributions of a signal process with respect to its noisy observations. In these settings and if $I = \mathbb{R}_+$ the Feynman-kac formula (1) is a weak solution of the so called Kushner-Stratonovitch equation (see for instance [13]). If $I = \mathbb{N}$ the Feynman-Kac semigroup (1) can also be used to model continuous time filtering problems with discrete time observations or more classically discrete time non linear filtering problems (see for instance [6] and [7]).

5.3.1 Description of the filtering model

Let the signal $S = \{S_t \; ; \; t \in \mathbb{R}_+\}$ be an E-valued Markov process with continuous paths. We suppose S is seen through an $\mathbb{R}^d$-valued process $Y = \{Y_t \; ; \; t \in \mathbb{R}_+\}$ satisfying the following

$$dY_t = h(S_t)\, dt + \sigma\, dV_t \qquad Y_0 = 0 \tag{45}$$

where $h : E \to \mathbb{R}^d$ is a sufficiently regular function and $V = \{V_t \; ; \; t \in \mathbb{R}_+\}$ is a d-dimensional Wiener process, independent of S, and σ is an invertible $d \times d$-matrix. We further assume that the transition semigroup $Q = \{Q_t \; ; \; t \geq 0\}$ of S is associated to a pregenerator $L : \mathcal{A} \to \mathcal{C}_b(E)$ where $\mathcal{A}$ is a suitably chosen algebra $\mathcal{A} \in \mathcal{C}_b(E)$. We also make the assumption that for any $x \in E$ there exists a unique probability measure $\tilde{\mathbb{P}}_x^S$ on $\Omega_1 = \mathcal{C}(\mathbb{R}_+, E)$ such that $S_0 \circ \tilde{\mathbb{P}}_x^S = \delta_x$ and for all $f \in \mathcal{A}$ the process

$$t \in \mathbb{R}_+ \longrightarrow f(S_t) - f(x) - \int_0^t Lf(S_s)\, ds$$

is a $\tilde{\mathbb{P}}_x^S$-martingale. We will also write for any $\eta_0 \in \mathbf{M}_1(E)$

$$\tilde{\mathbb{P}}_{\eta_0}^S = \int_E \eta_0(dx)\; \tilde{\mathbb{P}}_x^S$$

To avoid technical difficulties we finally assume that $h = \left(h^1, \ldots, h^d\right) \in \mathcal{A}^d$ and to clarify notations we set $\sigma = \mathrm{Id}$. Next we denote $\Omega_2 = \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ and $Y = (Y_t)_{t\geq 0}$ is the coordinate process on Ω_2. For each $\eta_0 \in \mathbf{M}_1(E)$ we introduce on $\Omega = \Omega_1 \times \Omega_2$ a probability measure $\tilde{\mathbb{P}}_{\eta_0}$ on its usual σ-field such that its marginal on Ω_1 is $\tilde{\mathbb{P}}_{\eta_0}^S$ and such that

$$V = \{V_t \; ; \; t \geq 0\} = \left\{Y_t - \int_0^t h(S_s)\, ds \; ; \; t \geq 0\right\}$$

is a d-vector standard Brownian motion.

5.3.2 Feynman-Kac's description

In practice, this probability $\tilde{\mathbb{P}}_{\eta_0}$ is usually constructed via Girsanov's Theorem from an other reference probability measure $\widehat{\mathbb{P}}_{\eta_0}$ on Ω, under which S and Y are independent, S has law $\tilde{\mathbb{P}}^S_{\eta_0}$ and Y is a d-vector standard Brownian motion. For $t \geq 0$, let $F_t = \sigma((S_s, Y_s)\ ;\ 0 \leq s \leq t)$ be the σ-algebra of events up to time t. The probabilities $\widehat{\mathbb{P}}_{\eta_0}$ and $\tilde{\mathbb{P}}_{\eta_0}$ are in fact equivalent on F_t, and their density is given by

$$\frac{d\tilde{\mathbb{P}}_{\eta_0}}{d\widehat{\mathbb{P}}_{\eta_0}}|F_t = \tilde{Z}_{0,t}(S,Y)$$

where for any $s \leq t$

$$\log \tilde{Z}_{s,t}(S,Y) =_{def.} \int_s^t h^\star(S_s) dY_s - \frac{1}{2}\int_0^t h^\star(S_s) h(S_s) ds$$

and where $(a)^\star$ denotes the transpose of a vector $a \in \mathbb{R}^d$. If $\eta_0 \in \mathbf{M}_1(E)$ is the initial law of the signal then a version of the conditional distribution of S_t given the observations up to time t is given for any bounded measurable function f by the so-called Kallianpur-Striebel formula, namely

$$\pi_t(f) = \frac{\widehat{\mathbb{E}}_{\eta_0}\left(f(S_t)\tilde{Z}_t(S,Y)\,|\mathcal{Y}_0^t\right)}{\widehat{\mathbb{E}}_{\eta_0}\left(\tilde{Z}_t(S,Y)\,|\mathcal{Y}_0^t\right)} = \frac{\int_{\Omega_1} f(\theta_t)\ \tilde{Z}_t(\theta,Y)\ \tilde{\mathbb{P}}^S_{\eta_0}(d\theta)}{\int_{\Omega_1} \tilde{Z}_t(\theta,Y)\ \tilde{\mathbb{P}}^S_{\eta_0}(d\theta)} \qquad \tilde{\mathbb{P}}_{\eta_0} - \text{a.e.} \qquad (46)$$

Using Itô's integration by part formula, in the differential sense we have that

$$h^\star(S_s) dY_s = d(h^\star(S_s)Y_s) - Y_s^\star\, Lh(S_s) ds - Y_s^\star\, dM_s^{(h)}$$

where $L(h) = (L(h_i))_{1\leq i \leq d}\ :\ E \to \mathbb{R}^d$ and where $M^{(h)} = (M^{(h_i)})_{1\leq i\leq d}$ is a d-vector square integrable continuous martingale (relative to the natural filtration $(F_t)_{t\geq 0}$) with cross-variation processes given by

$$\forall\ 1 \leq i,j \leq d, \qquad < M^{(h_i)}, M^{(h_j)} >_t = \int_0^t \Gamma(h_i,h_j)(S_s)\, ds$$

and Γ is the "carré du champ" associated to the generator L.
For $x \in E$, we will denote by $\Gamma(h,h)(x)$ the matrix $(\Gamma(h_i,h_j)(x))_{1\leq i,j\leq d}$. This yields the decomposition

$$\ln \tilde{Z}_t(S,Y) = h^\star(S_t)Y_t - \int_0^t Y_s^\star\, Lh(S_s)\, ds - \int_0^t Y_s^\star\, dM_s^h - \frac{1}{2}\int_0^t h^\star(S_s)h(S_s) ds$$

and therefore

$$\ln \tilde{Z}_t(S,Y) = h^\star(S_t)Y_t + \int_0^t V(S_s,Y_s)\, ds + \ln \breve{Z}_t(S,Y)$$

where

$$\begin{aligned} V(x,y) &= -y^\star\, Lh(x) + \frac{1}{2}\, y^\star\, \Gamma(h,h)(x) y - \frac{1}{2} h^\star(x)h(x) \\ \ln \breve{Z}_t(S,Y) &= -\int_0^t Y_s^\star\, dM_s^{(h)} - \frac{1}{2}\int_0^t Y_s^\star \Gamma(h,h)(S_s) Y_s\, ds \end{aligned}$$

Together with (46) this decomposition implies that

$$\pi_t(f) = \frac{\displaystyle\int_{\Omega_1} f(\theta_t)\ e^{h^\star(\theta_t)Y_t + \int_0^t\ V(\theta_s,Y_s)\ ds}\ \mathbb{P}^{[Y]}_{\eta_0}(d\theta)}{\displaystyle\int_{\Omega_1} e^{h^\star(\theta_t)Y_t + \int_0^t\ V(\theta_s,Y_s)\ ds}\ \mathbb{P}^{[Y]}_{\eta_0}(d\theta)}$$

where, for any $y \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ (such that $y_0 = 0$), $\mathbb{P}^{[y]}_{\eta_0}$ is the probability measure on Ω_1 defined by its restrictions to $\mathcal{F}^{(1)}_t = \sigma(S_s,\ 0 \leq s \leq t)$:

$$\frac{d\mathbb{P}^{[y]}_{\eta_0}}{d\widehat{\mathbb{P}}_{\eta_0}}|F^{(1)}_t = \breve{Z}_t(S, y) \tag{47}$$

Using standard continuous stochastic calculus (cf. [16], particularly the Theorem 1.4 p. 313 and Novikov's criterion, Corollary 1.16 p. 319, or [10]), it is easy to realize that, as $y \in C(\mathbb{R}_+, \mathbb{R}^d)$ (s.t. $y_0 = 0$) is fixed, $\mathbb{P}^{[y]}_{\eta_0}$ is the unique solution to the martingales problem on Ω_1 associated to the initial distribution η_0 and to the time-inhomogeneous family of pregenerators $\{L^y_t\ ;\ t \geq 0\}$ defined for any $t \geq 0$ and $f \in \mathcal{A}$ by

$$L^y_t(f) = L(f) - \sum_{1 \leq i \leq d} y_{i,t}\, \Gamma(f, h_i)$$

The above formulation of the optimal filter is classically interpretated as a pathwise filter defined for any observation path parameter $y \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ (s.t. $y_0 = 0$) by

$$\pi_{t,y}(f) = \frac{\displaystyle\int_E f(x)\ e^{h^\star(x) y_t}\ \eta_{t,y}(dx)}{\displaystyle\int_E\ e^{h^\star(x') y_t}\ \eta_{t,y}(dx')} \quad \text{and} \quad \eta_{t,y}(f) = \frac{\mathbb{E}^y_{\eta_0}\left(f(X^y_t)\ Z_{0,t}(X^y, y)\right)}{\mathbb{E}^y_{\eta_0}\left(\ \ Z_{0,t}(X^y, y)\right)} \tag{48}$$

where

- For any observation path $y \in \mathcal{C}(\mathbb{R}_1, \mathbb{R}^d)$ (s.t. $y_0 = 0$) the multiplicative functions $Z(X^y, y) = \{Z_{s,t}(X^y, y)\ ;\ s \leq t\}$ are defined for any $s \leq t$ by

$$\log Z_{s,t}(X^y, y) = \int_s^t\ V(X^y_s, y_s)\ ds$$

- $\left(\Omega_1, \{F_t\ ;\ t \geq 0\}, \{X^y_t\ ;\ t \geq 0\}, \mathbb{P}^y_{\eta_0}\right)$ is a continuous time and E-valued Markov process associated to the time-inhomogeneous family of pregenerators $\{L^y_t\ ;\ t \geq 0\}$ and with initial distribution η_0.

The reader who wishes to have more details on the theory of pathwise non linear filtering is recommended to consult the pioneering papers [4, 11, 13] and [15, 17].

When studying the pathwise filter, the path observation y will always be fixed. To clarify the notations it will be dropped from our notations so that when there is no possible confusions we will write π_t, η_t, L_t, X_t and $\mathbb{P}_{\eta_0}$ instead of $\pi_{t,y}$, $\eta_{t,y}$, L^y_t, X^y_t and $\mathbb{P}^y_{\eta_0}$.

In contrast to Kallianpur-Striebel formulation (46) the above formulation does not involves stochastic integrations and it is well defined for all observation paths $y \in \mathcal{C}([0,t], \mathbb{R}^d)$. The above robust version of the optimal filter in terms of the distribution flow $\eta = \{\eta_t\ ;\ t \geq 0\}$ given in (48) allows one to construct a Moran type particle approximating model. In section 5.2 we have already presented one way to connect the long time behavior of the particle approximating model with the stability properties of the non linear semigroup $\Phi = \{\Phi_{s,t}\ ;\ 0 \leq s \leq t\}$ associated to the flow η.

By the Markov property of X it is easily seen that η satisfies the semigroup relation

$$\forall s \leq t \qquad \eta_t = \Phi_{s,t}(\eta_s)$$

where $\Phi_{s,t} : \mathbf{M}_1(E) \rightarrow \mathbf{M}_1(E)$ is given for any bounded measurable function and for any $\mu \in \mathbf{M}_1(E)$ by

$$\Phi_{s,t}(\mu) = \frac{\mathbb{E}_{s,\mu}\left(f(X_t)\ Z_{s,t}(X,y)\right)}{\mathbb{E}_{s,\mu}\left(\quad Z_{s,t}(X,y)\right)}$$

where $\mathbb{E}_{s,\mu}$ is the expectation with respect to the law of path process $X_{[s,\infty]} =_{def.} \{X_t\ ;\ t \geq s\}$ such that μ is the law of the initial value X_s. To see that the multiplicative functions $Z(X,y)$ falls into the set-up of section 3 we notice that $Z(X,y)$ satisfies condition $(\mathcal{Z})$ with

$$-\log z_t(u) = \int_t^{t+u} \operatorname{osc}\left(V(\,.\,,y_r)\right)\ dr$$

If the transition semigroup $P = \{P_{s,t}\ ;\ s \leq t\}$ of X satisfies the mixing condition $(\mathcal{P})$ then using Proposition 4.3 and Theorem 3.2 we may derive several sufficient conditions for Φ to be asymptotically stable. In section 5.3.4 we will connect the mixing properties of the signal process S with the ones of the time in-homogeneous process X. Our approach will also permit explicit lower bounds for the Liapunov exponents associated to the semigroup Φ.

The asymptotic stability of the evolution semigroup induced by $\pi = \{\pi_t\ ;\ t \geq 0\}$ also plays a prominent role in filtering literature. To see that the Kallianpur-Striebel formulation (46) falls into the set-up of this work we start by noting that after integrating the signal S in (46) we get for any bounded measurable function f

$$\pi_t(f) = \frac{\tilde{\mathbb{E}}^S_{\eta_0}\left(f(S_t)\tilde{Z}_t(S,Y)\right)}{\tilde{\mathbb{E}}^S_{\eta_0}\left(\tilde{Z}_t(S,Y)\right)} = \frac{\int_{\Omega_1}\ f(\theta_t)\ \tilde{Z}_t(\theta,Y)\ \tilde{\mathbb{P}}^S_{\eta_0}(d\theta)}{\int_{\Omega_1}\ \tilde{Z}_t(\theta,Y)\ \tilde{\mathbb{P}}^S_{\eta_0}(d\theta)} \qquad \tilde{\mathbb{P}}_{\eta_0} - \text{a.e.}$$

where $\tilde{\mathbb{E}}^S_{\eta_0}$ denotes the expectation with respect to the probability measure $\tilde{\mathbb{P}}^S_{\eta_0}$ on Ω_1. One advantage of the above realization of the optimal filter is that there is no more conditional expectation inside. The Markov property of the signal S and the multiplicative property of $\tilde{Z}(S,Y)$ imply that π again satisfies a semigroup relation, namely

$$\pi_t(f) = \tilde{\Phi}_{s,t}(\pi_s)f =_{def.} \frac{\int_{\Omega_{1,s}}\ f(\theta_t)\ \tilde{Z}_{s,t}(\theta,Y)\ \tilde{\mathbb{P}}^S_{s,\pi_s}(d\theta)}{\int_{\Omega_{1,s}}\ \tilde{Z}_{s,t}(\theta,Y)\ \tilde{\mathbb{P}}^S_{s,\pi_s}(d\theta)} \qquad \tilde{\mathbb{P}}_{\eta_0} - \text{a.e.}$$

where for any measurable subset $A \subset \Omega_{1,s} =_{def.} \mathcal{C}([s,\infty[,E)$ and $\pi \in \mathbf{M}_1(E)$

$$\tilde{\mathbb{P}}^S_{s,\pi}(A) = \int_E \pi(dx)\ \tilde{\mathbb{P}}^S_{s,x} \qquad \text{and} \qquad \tilde{\mathbb{P}}^S_{s,x} = T_s \circ \tilde{\mathbb{P}}^S_x$$

where $T_s : \Omega_1 \rightarrow \Omega_{1,s}$ is the standard shift operator given by $(T_s(\theta))_t = \theta_{t+s}$ for any $t \geq 0$. To simplify notations we will also write $\tilde{\mathbb{E}}^S_{s,\pi_s}$ the expectation with respect to the probability measure $\tilde{\mathbb{P}}^S_{s,\pi_s}$ on $\Omega_{1,s}$ so that for any $\mu \in \mathbf{M}_1(E)$ and $s \leq t$

$$\tilde{\Phi}_{s,t}(\mu)f = \frac{\tilde{\mathbb{E}}^S_{s,\mu}\left(f(S_t)\tilde{Z}_{s,t}(S,Y)\right)}{\tilde{\mathbb{E}}^S_{s,\mu}\left(\tilde{Z}_{s,t}(S,Y)\right)} \qquad \tilde{\mathbb{P}}_{\eta_0} - \text{a.e.}$$

This formulation shows that the non linear semigroup $\tilde{\Phi} = \{\tilde{\Phi}_{s,t}\ ;\ s \leq t\}$ has indeed the same form as the one given in (1) but it is random on the observation process Y. Another remark is that the definition of the multiplicative function $\tilde{Z}(S,Y)$ involves stochastic integrations and it is not immediate to check whether or not condition $(\mathcal{Z})$ holds for some random function $\tilde{z}_t(u)$. We will see in section 5.3.3 that if the signal S is sufficiently mixing then one can obtain an explicit lower bound for $\tilde{z}_t(u)$ in terms of the time parameter u, the norms

$$\|h\| = \sum_{i=1}^{d} \sup_{x\in E} |h^i(x)| \qquad \|Lh\| = \sum_{i=1}^{d} \sup_{x\in E} |Lh^i(x)| \qquad \|\Gamma(h,h)\| = \sup_{1\leq i,j\leq d} \sup_{x\in E} |\Gamma(h^i,h^j)(x)|$$

and on $\|V\|^\star_{t,u}$ where for any $v \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ and for any $s \leq t$ we denote

$$\|v\|^\star_{s,t} =_{def.} \sup_{r\in[s,s+t]} \|v_r\|$$

This lower bound will be essential to our purpose since Birkhoff ergodic theorem combined with Proposition 4.3 and Theorem 3.2 will allow us to conclude that $\tilde{\Phi}$ is almost surely asymptotically stable.

5.3.3 Stability of the NLF equation

Next we assume that the semigroup of the signal process satisfies the following mixing type condition

$(\mathcal{Q})$ There exists some reference probability measure μ on E and $t > 0$ such that for any $x \in E$ and $0 < u \leq t$, $Q_u(x,.) \sim \mu$ and

$$\epsilon^{1/2}(u) \leq \frac{dQ_u(x,.)}{d\mu}(z) \leq \frac{C}{u^{\alpha/2}} \tag{49}$$

where $C > 0$, $\epsilon(u) > 0$ and $\alpha \geq 1$ (several examples of semigroups satisfying $(\mathcal{Q})$ are given in [1] and [5]).

Our immediate goal is now to check that the multiplicative function $\tilde{Z}(S,Y)$ satisfies condition $(\mathcal{Z})$ for some $\tilde{z}_t(u,V)$ which depends on the parameter u and on $\|V\|^\star_{t,u}$.

For any $s,t \in \mathbb{R}_+$ and $x,z \in E$ we denote by $\tilde{\mathbb{P}}^S_{s,x;t,z}$ the $\tilde{\mathbb{P}}^S_{s,x}$-conditional distribution of the path signal $\{X_r\ ;\ r \geq s\}$ starting at the point $X_s = x$ and given $S_t = z$.

By definition of Y one easily check that for any $t \in \mathbb{R}_+$, $u > 0$ and $x,z,z' \in E$

$$\tilde{z}_t(u,V) \leq \frac{\int_{\Omega_{1,t}} \tilde{Z}_{t,t+u}(\theta,Y)\ \tilde{\mathbb{P}}^S_{t,x;t+u,z}(d\theta)}{\int_{\Omega_{1,t}} \tilde{Z}_{t,t+u}(\theta,Y)\ \tilde{\mathbb{P}}^S_{t,x;t+u,z'}(d\theta)} \leq \tilde{z}_t^{-1}(u,V)$$

with

$$\tilde{z}_t(u,V) \geq e^{-3\|h\|^2\cdot u} \inf_{x,z,z'} \frac{\tilde{\mathbb{E}}^S_{t,x}\left(\exp\int_t^{t+u} h^\star(S_s)\,dV_s\,|S_{t+u}=z\right)}{\tilde{\mathbb{E}}^S_{t,x}\left(\exp\int_t^{t+u} h^\star(S_s)\,dV_s\,|S_{t+u}=z'\right)}$$

where as usually $\tilde{\mathbb{E}}^S_{t,x}(.\,|S_{t+u}=z)$ denotes the expectation with respect to the probability measure $\tilde{\mathbb{P}}^S_{s,x;t+u,z}$. Using Ito integration by part formula, one obtain

$$\begin{aligned}
&\int_t^{t+u} h^\star(S_s)\,dV_s \\
&= h^\star(S_{t+u})V_{t+u} - h^\star(S_t)V_t - \int_t^{t+u} V_s^\star\,(Lh)(S_s)\,ds + \frac{1}{2}\int_t^{t+u} V_s^\star\,\Gamma(h,h)(S_s)V_s\,ds \\
&\qquad - \int_t^{t+u} V_s^\star\,dM_s^{(h)} - \frac{1}{2}\int_t^{t+u} V_s^\star\,\Gamma(h,h)(S_s)V_s\,ds
\end{aligned}$$

After some easy manipulations one concludes that for any given path $v \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$

$$\tilde{z}_t(u,v) \geq e^{-a(u,h)(\|v\|^{\star 2}_{t,u}+1)} \inf_{x,z,z'} \frac{\tilde{\mathbb{E}}^S_{0,x}\left(e^{\tilde{M}^{(v,t)}_u - \frac{1}{2}<\tilde{M}^{(v,t)}>_u} | S_u = z\right)}{\tilde{\mathbb{E}}^S_{0,x}\left(e^{\tilde{M}^{(v,t)}_u - \frac{1}{2}<\tilde{M}^{(v,t)}>_u} | S_u = z'\right)}$$

where $\{\tilde{M}^{(v,t)}_s \,;\, 0 \leq s \leq u\}$ is the $\tilde{\mathbb{P}}^S_{0,x}$-martingale given by

$$\tilde{M}^{(v,t)}_s = -\int_0^u v^\star_{t+s}\, dM^{(h)}_s \qquad \text{and} \qquad <\tilde{M}^{(v,t)}>_s = \int_0^s v^\star_{t+s}\, \Gamma(h,h)(S_s) v_{t+s}\, ds$$

and $a(u,h)$ is a finite constant such that

$$a(u,h) \leq 6\,(1+u)\ \max\left(\|h\|^2, \|h\|, \|Lh\|, \|\Gamma(h,h)\|\right)$$

Lemma 5.2 *If the semigroup of the signal process S satisfies $(\mathcal{Q})$ for some $\alpha \geq 1$, $C > 0$ and $\epsilon(u) > 0$ then for any $x, z, z' \in E$ and for any path $v \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ we have that*

$$\left|\log \frac{\tilde{\mathbb{E}}^v_{0,x}\left(e^{\tilde{M}^{(v,t)}_u - \frac{1}{2}<\tilde{M}^{(v,t)}>_u} | S_u = z\right)}{\tilde{\mathbb{E}}^v_{0,x}\left(e^{\tilde{M}^{(v,t)}_u - \frac{1}{2}<\tilde{M}^{(v,t)}>_u} | S_u = z'\right)}\right| \leq a'_\alpha(u,h)\,.\,\|v\|^{\star\,2}_{t,u} + b'_\alpha(u,h)\,.\,\|v\|^\star_{t,u}$$

where $b'_\alpha(u,h)$ and $a'_\alpha(u,h)$ are given by

$$a'_\alpha(u,h) = \alpha\,.\,u\,.\,\|\Gamma(h,h)\| \quad \text{and} \quad b'_\alpha(u,h) = 2\,A_\alpha\,\left(C\,\sqrt{\frac{u}{\epsilon(u)}}\right)^{\frac{1}{\alpha+1}}\,.\,\sqrt{\|\Gamma(h,h)\|}$$

and A_α is a universal constant which only depends on the parameter α.

The proof of Lemma 5.2 is rather technical and it will be given in the appendix. Lemma 5.2 yields that for any given path $V = v \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$

$$-\log \tilde{z}_t(u,v) \quad \leq \quad \|v\|^{\star 2}_{t,u}\ \left[a(u,h) + a'_\alpha(u,h)\right]\ + \|v\|^\star_{t,u}\, b'_\alpha(u,h) + a(u,h)$$

We may now state the main result of this section.

Theorem 5.3 *Assume that the semigroup Q of the signal satisfies $(\mathcal{Q})$ for some $\mu \in \mathbf{M}_1(E)$, $\alpha \geq 1$, $C > 0$, $0 < u \leq t$ and $\epsilon(u) > 0$. Then there exists some constant $\lambda > 0$ such that*

$$\limsup_{t\to\infty} \frac{1}{t} \sup_{\mu,\nu} \log \|\tilde{\Phi}_{0,t}(\mu) - \tilde{\Phi}_{0,t}(\nu)\|_{tv} \leq -\lambda \qquad \tilde{\mathbb{P}}_{\eta_0}\text{a.e.} \tag{50}$$

Proof:
Under our assumptions we have that for any $x \in E$ and $0 < u \leq t$, $Q_u(x,.) \sim \mu$ and

$$\tilde{\epsilon}^{1/2}(u) \leq \frac{dQ_u(x,.)}{d\mu}(z) \leq \tilde{\epsilon}^{-1/2}(u) \qquad \text{with} \qquad \tilde{\epsilon}(u) = \min\left(\epsilon(u), C^{-2}\,u^\alpha\right)$$

Assertion (50) is a consequence of Birkhoff ergodic theorem, Theorem 3.2 and Proposition 4.3. Furthermore applying Jensen's inequality to the exponential (that is $E(\exp X) \geq \exp E(X)$ for any random variable X) one concludes that

$$\lambda(u) \geq \frac{\tilde{\epsilon}(u)}{u}\ \tilde{\mathbb{E}}_{\eta_0}\left(\tilde{z}_0(u,V)\right)$$

and

$$-\log \tilde{\mathbb{E}}_{\eta_0}\left(\tilde{z}_0(u,V)\right)$$
$$\leq \tilde{\mathbb{E}}_{\eta_0}(\|V\|^{\star 2}_{0,u})\ \ (a(u,h) + a'_\alpha(u,h))\ + \tilde{\mathbb{E}}_{\eta_0}(\|V\|^\star_{0,u})\, b'_\alpha(u,h) + a(u,h)$$

Using Burkholder-Davis-Gundy inequality one concludes that $\lambda(u) > 0$ and the proof is completed. ∎

5.3.4 Stability of the robust NLF equation

Using the change of probability measures (47) and Girsanov's Theorem one can check that under $\mathbb{P}^{[y]}_{\eta_0}$, the Markov process S is a temporally in-homogeneous Markov process $X = \{X_t\ ;\ t \geq 0\}$ with initial law η_0 and transition probability measures $P_{s,s+t}(x, dz)$, $s \leq s+t$, given by

$$P_{s,s+t}(x,dz) = \tilde{\mathbb{E}}^S_{s,x}\left(\breve{Z}_{s,t}(S,y)\,|S_t = z\right)\ .\ Q_t(x,dz) \tag{51}$$

where

$$\log \breve{Z}_{s,t}(S,y) =_{def.} -\int_s^t y^\star_s\ dM^{(h)}_s - \frac{1}{2}\int_0^t y^\star_s \Gamma(h,h)(S_s) y_s\ ds$$

and where we have used the notation $\tilde{\mathbb{E}}^S_{s,x}$ to denote the expectations with respect to $\tilde{\mathbb{P}}^S_{s,\delta_x}$. Note that if the path parameter $y \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ is constant, that is $y_s = y_0$ for any $s \in \mathbb{R}_+$, then for each $s \leq s+t$

$$\log \breve{Z}_{s,t}(S,y) = y^\star_0 h(S_s) - y^\star_0 h(S_{s+t}) + \int_s^{s+t} y^\star_0\ Lh(S_\tau)\ d\tau - \frac{1}{2}\int_s^{s+t} y^\star_0 \Gamma\ (h,h)\ (S_\tau) y_0\ d\tau$$

from which one concludes that for any $s \leq s+t$ and $x \in E$ the measures $P_{s,s+t}(x,dz)$ and $Q_t(x,dz)$ are absolutely continuous and

$$\sup_{x,z\in E}\left|\log \frac{dP_{s,s+t}(x,\,.)}{dQ_t(x,\,.)}(z)\right| \leq \operatorname{osc}(y^\star_0 h) + \left(\|y^\star_0\ Lh\| + \frac{1}{2}\|y^\star_0 \Gamma\ (h,h)\ y_0\|\right)\ .\ t$$

Next proposition extends in some sense the above result to any path parameter $y \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ under a mixing condition on the semigroup Q. This result also shows that if the semigroup Q satisfies condition $(\mathcal{P})$ given page 11 then the temporally inhomogeneous semigroup P again satisfies condition $(\mathcal{P})$. The proof of Proposition 5.4 will be given in the appendix.

Proposition 5.4 *Assume that the semigroup Q of the signal process S satisfies $(\mathcal{Q})$ for some $\alpha \geq 1$, $C > 0$ and $\epsilon(u) > 0$. Then for each $0 < s \leq s+t$ and $x \in E$ the probability measures $P_{s,s+t}(x,dz)$ and $Q_s(x,dz)$ are equivalent and the Radon-Nykodim derivatives given by (51) satisfy*

$$\sup_{x,z\in E}\left|\log \frac{dP_{s,s+t}(x,\,.)}{dQ_t(x,\,.)}(z)\right| \leq \frac{\alpha}{2}\,.\,\lambda^2_{s,t}(y)\ .\ t + A_\alpha\ .\ C(t)^{\frac{1}{\alpha+1}}\ .\ \lambda_{s,t}(y)$$

where A_α is a universal constant which only depends on the parameter α,

$$C(t) = C\sqrt{\frac{t}{\epsilon(t)}} \qquad \lambda^2_{s,t}(y) = \|\Gamma(h,h)\|\ .\ \|y\|^2_{s,t} \quad \text{and} \quad \|y\|_{s,t} = \sup_{\tau\in[s,s+t[}\|y_\tau\|$$

In particular this yields that for any $0 \leq s \leq s+t$ and $x \in E$, $P_{s,s+t}(x,\,.) \sim \mu$ and

$$\sqrt{\epsilon(t)}\ \ e^{-R_\alpha(\|y\|_{s,t},t)} \leq \frac{dP_{s,s+t}(x,\,.)}{d\mu}(z) \leq \frac{C}{t^{\alpha/2}}\ \ e^{-R_\alpha(\|y\|_{s,t},t)} \tag{52}$$

with

$$R_\alpha(\|y\|_{s,t},t) = \frac{\alpha}{2}\,.\,\lambda^2_{s,t}(y)\ .\ t + A_\alpha\ .\ C(t)^{\frac{1}{\alpha+1}}\ .\ \lambda_{s,t}(y)$$

Remark 5.5:
When the observation path is the null path, that is $y = 0$, then $P_{s,s+t} = Q_t$ and one can also check that $R_\alpha(0,t) = 0$ so that (49) and (52) are equivalent.

Theorem 5.6 *We denote by $\Phi = \{\Phi_{s,t}\ ;\ s \leq t\}$ the semigroup associated to the distribution flow $\eta = \{\eta_t\ ;\ t \geq 0\}$ defined by (48) for some given path observation $y \in \mathcal{C}_b(\mathbb{R}_+, \mathbb{R}^d)$. Suppose that the semigroup Q associated to the transition probability of S satisfies the mixing condition $(\mathcal{Q})$ for some $\alpha \geq 1$, $C > 0$ and $\epsilon(u) > 0$. Then for any $p \geq 1$, $u > 0$, and $T \geq p\,.\,u$ we have that*

$$\sup_{t\geq 0}\sup_{\mu,\nu} \|\Phi_{t,t+T}(\mu) - \Phi_{t,t+T}(\nu)\|_{tv} \leq \exp -(\gamma(u,y)\,.\,T)$$

with the lower bound

$$\gamma(u,y) \geq \frac{1}{q\,.\,u}\ \min\left(C^{-2}\, u^{\alpha}, \epsilon(u)\right)\ \exp -\Lambda_\alpha(y,u)$$

where

$$\Lambda_\alpha(y,u) = u\ \left(\mathrm{osc}(V,y) + \alpha\ \lambda^2(y)\right) + B_\alpha\ C(u)^{\frac{1}{\alpha+1}}\ \lambda(y) \tag{53}$$

and B_α is a universal finite constant which only depends on the parameter α and

$$\frac{1}{p}+\frac{1}{q} = 1 \qquad \mathrm{osc}(V,y) =_{def.} \sup_{t\geq 0} \mathrm{osc}(V(\,\bullet\,,y_t)) \qquad \lambda(y) =_{def.} \sup_{s,t\geq 0} \lambda_{s,t}(y)$$

Proof:
Our assumptions imply that condition $(\mathcal{Z})$ is satisfied with

$$-\log z_t(u) \leq\ u\,.\,\mathrm{osc}(V,y)$$

On the other hand Proposition 5.4 tells us that the semigroup $\{P_{s,t}\ ;\ s \leq t\}$ satisfies the mixing condition

$$\epsilon_t^{1/2}(u) \leq \frac{dP_{t,t+u}(x,\,.\,)}{d\mu}(z) \leq \epsilon_t^{-1/2}(u)$$

with

$$\epsilon_t(u) = \min\left(\epsilon(u), u^{\alpha} C^{-2}\right)\ \exp -\left(2\,R_\alpha(\|y\|^{\star}_{t,u}, u)\right)$$

As a result the lower bound we are looking for is obtained by using Proposition 4.3 and the same arguments as in the proof of Theorem 3.2.

∎

Using the above theorem and the description (48) one can easily connect the asymptotic stability of the semigroup Φ with the asymptotic stability of the robust version of the optimal filter (48).
For any $y \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ (s.t. $y_0 = 0$) and $\mu \in \mathbf{M}_1(E)$ we write $\pi^\mu = \{\pi_t^\mu\ ;\ t \geq 0\}$ the distribution flow defined for any bounded measurable function f by

$$\pi_t^\mu(f) = \frac{\displaystyle\int_E f(x)\ e^{h^\star(x) y_t}\ \Phi_{0,t}(\mu)(dx)}{\displaystyle\int_E e^{h^\star(x') y_t}\ \Phi_{0,t}(\mu)(dx')}$$

Corollary 5.7 *Suppose that the semigroup Q associated to the transition probability of S satisfies the mixing condition $(\mathcal{Q})$ for some $\alpha \geq 1$, $C > 0$ and $\epsilon(u) > 0$. If $y \in \mathcal{C}_b(\mathbb{R}_+, \mathbb{R}^d)$ then*

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{\mu,\nu} \|\pi_t^\mu - \pi_t^\nu\|_{tv} \leq -\frac{1}{u}\ \min\left(C^{-2}\, u^{\alpha}, \epsilon(u)\right)\ \exp -\Lambda_\alpha(y,u)$$

where $\Lambda_\alpha(y,u)$ is defined in (53).

6 Appendix

The aim of this final section is to prove Proposition 5.4 and Lemma 5.2. To prove Proposition 5.4 we begin by noting that for any $0 \leq s \leq s+t$ and $x, z \in E$ we have that

$$\begin{aligned}
&\tilde{\mathbb{E}}^{S}_{s,x}\left(\breve{Z}_{s,s+t}(S,y)\,|S_{t+s}=z\right) \\
&= \tilde{\mathbb{E}}^{S}_{0,x}\left(\exp\left(-\int_0^t y^{\star}_{s+\tau}\,dM^{(h)}_{\tau} - \frac{1}{2}\int_0^t y^{\star}_{s+\tau}\Gamma\left(h,h\right)(S_\tau)y_{s+\tau}\,d\tau\right)|S_t=z\right)
\end{aligned}$$

it is clearly enough to prove (52) for $s=0$. In what follows the points $x \in E$ and $z \in E$ and the parameters t and $y \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ will be fixed and so will be dropped from our notations when there is no possible confusions. Using the above abusive notations and to clarify presentation we also write $\mathbf{Q}$ instead of $\tilde{\mathbb{P}}^{S}_{0,x}$ and we denote $\mathbf{Q}'$ the probability measure on $(\Omega, F_t, (F_s)_{0\leq s\leq t})$ defined by

$$\forall A \in F_t \qquad \mathbf{Q}'(A) = \tilde{\mathbb{E}}^{S}_{0,x}\left(1_A\ |X_t = z\right)$$

Under the assumption of the proposition for any $s \in [0,t)$ we have that

$$\frac{d\mathbf{Q}'}{d\mathbf{Q}}|F_s = \frac{dQ_{t-s}(X_s,\,.)}{dQ_t(x,\,.)}(z) =_{def.} Z'_s$$

In addition the density process $\{Z'_s\ ;\ s \in [0,t)\}$ satisfies

$$\forall s \in [0,t) \qquad Z'_s \leq \frac{C}{\epsilon(t)^{1/2}} \cdot \frac{1}{|t-s|^{\alpha/2}} \tag{54}$$

On the other hand the increasing process $\{< \widehat{M} >_s\ ;\ s \in [0,t)\}$ of the $\mathbf{Q}$-martingale

$$\widehat{M}_s =_{def.} -\int_0^s y^{\star}_r\,dM^{(h)}_r$$

induces a measure $d < \widehat{M} >_s$ which is absolutely continuous with respect to the Lebesgue measure ds on $[0,t]$ and satisfies

$$\sup_{s\in[0,t]}\left|\frac{d<\widehat{M}>_s}{ds}\right| \leq \widehat{\lambda}^2 =_{def.} \|\Gamma(h,h)\|\ .\ \sup_{s\in[0,t]}\|y_s\|^2 \tag{55}$$

The proof of (52) is a consequence of the following proposition which has its own importance and whose proof is postponed to the end of this subsection.

Proposition 6.1 *Let $t>0$ be a fixed parameter and let $(\Omega, F_t, (F_s)_{s\in[0,t]})$ be a continuous time and right filtered space endowed with two probability measures $\mathbf{Q}$ and $\mathbf{Q}'$. We assume that for any $s \in [0,t)$ the restriction $\mathbf{Q}'_s$ of $\mathbf{Q}'$ to F_s is absolutely continuous with respect to the restriction $\mathbf{Q}_s$ of $\mathbf{Q}$ to F_s and we write $Z'_s = \frac{d\mathbf{Q}'_s}{d\mathbf{Q}_s}$ the corresponding Radon-Nykodim derivatives. Let $\{M_s\ ;\ s\in[0,t]\}$ be a continuous $\mathbf{Q}$-martingale with increasing process $\{< M >_s\ ;\ s \in [0,t]\}$. We assume that the measure $d < M >_s$ is absolutely continuous with respect to the Lebesgue measure ds on $[0,t]$ and satisfies*

$$\sup_{s\in[0,t]}\left|\frac{d<M>_s}{ds}\right| \leq 1 \qquad \mathbf{Q}-\text{a.s.}$$

We further assume that the density process Z' enjoys the following property

$$\forall s \in [0,t) \qquad Z'_s \leq \frac{C(t)}{\sqrt{t}} \cdot \frac{1}{|t-s|^{\alpha/2}} \qquad \mathbf{Q}-\text{a.s.}$$

for some finite constant $C(t)$ and some parameter $\alpha \geq 1$. Then the following properties hold

1. *For any square integrable process* $\{H_s\ ;\ s \in [0,t]\}$ *we have that*

$$\sup_{s\in[0,t]} \mathbb{E}\left(|H_s|^{\alpha+1}\right) \leq 1 \Longrightarrow \left|\mathbb{E}'\left(\int_0^t H_s\, dM_s\right)\right| \leq A_\alpha \,.\, C(t)^{\frac{1}{\alpha+1}} \tag{56}$$

   where A_α *is a universal constant which only depends on the parameter* α *and* $\mathbb{E}$ *(resp.* $\mathbb{E}'$*) denote the expectation with respect to* $\mathbf{Q}$ *(resp.* $\mathbf{Q}'$*).*

2. *For any* $\lambda \geq 0$ *we have that*

$$\left|\mathbb{E}'\left(e^{\lambda M_t - \frac{\lambda^2}{2} <M>_t}\right) - 1\right| \leq A_\alpha \,.\, C(t)^{\frac{1}{\alpha+1}} \,.\, \lambda \,.\, \exp\left(\frac{\alpha}{2} \,.\, \lambda^2 \,.\, t\right) \tag{57}$$

   and also

$$\log \mathbb{E}'\left(e^{\lambda M_t - \frac{\lambda^2}{2} <M>_t}\right) \geq -A_\alpha \,.\, C(t)^{\frac{1}{\alpha+1}} \,.\, \lambda - \frac{\lambda^2}{2} \,.\, t \tag{58}$$

Thanks to the bounds (54) and (55) Proposition 6.1 applies with

$$\forall s \in [0,t] \qquad M_s = \widehat{\lambda}^{-1}\, \widehat{M}_s \quad \text{and} \quad C(t) = C\sqrt{\frac{t}{\epsilon(t)}} \quad \text{and} \quad \lambda = \widehat{\lambda}$$

More precisely using the above notations by (57) one gets easily the upper bound

$$\mathbb{E}^{\mathbf{Q}'}\left(e^{\widehat{M}_t - \frac{1}{2} <\widehat{M}>_t}\right) \leq 1 + A_\alpha \,.\, C(t)^{\frac{1}{\alpha+1}} \,.\, \widehat{\lambda} \,.\, \exp\left(\frac{\alpha}{2} \,.\, \widehat{\lambda}^2 \,.\, t\right)$$

for some universal constant A_α which only depends on the parameter α. Since for any $x, y \geq 0$

$$1 + x\, e^y \leq e^{x+y}$$

one obtain the upper bound

$$\log \mathbb{E}^{\mathbf{Q}'}\left(e^{\widehat{M}_t - \frac{1}{2} <\widehat{M}>_t}\right) \leq \frac{\alpha}{2} \,.\, \widehat{\lambda}^2 \,.\, t + A_\alpha \,.\, C(t)^{\frac{1}{\alpha+1}} \,.\, \widehat{\lambda}$$

Two lower bounds are available. Using Jensen's inequality to the exponential (that is $E(e^{-X}) \geq e^{-E(X)}$) and (58) one can also check that

$$\log \mathbb{E}^{\mathbf{Q}'}\left(e^{\widehat{M}_t - \frac{1}{2} <\widehat{M}>_t}\right) \geq -A_\alpha \,.\, C(t)^{\frac{1}{\alpha+1}} \,.\, \widehat{\lambda} - \frac{1}{2}\widehat{\lambda}^2 \,.\, t$$

On the other hand using Cauchy-Schwartz's inequality (that is $E(e^{-X})E(e^X) \geq 1$) by the same token as before one obtain the lower bound

$$-\log \mathbb{E}^{\mathbf{Q}'}\left(e^{\widehat{M}_t - \frac{1}{2} <\widehat{M}>_t}\right) \leq \frac{\alpha}{2} \,.\, \widehat{\lambda}^2 \,.\, t + A_\alpha \,.\, C(t)^{\frac{1}{\alpha+1}} \,.\, \widehat{\lambda}$$

■

The proof of Lemma 5.2 can be done using the same arguments. To prove Proposition 6.1 we need the following technical lemma.

Lemma 6.2 *Let* $\left(\Omega, \mathcal{F}, F = (F_n)_{n\in\mathbb{N}}\right)$ *be a discrete time filtered space endowed with two probability measures* $\mathbf{Q}$ *and* $\mathbf{Q}'$*. We assume that for each* $n \in \mathbb{N}$ *the restrictions* $\mathbf{Q}'_n$ *of* $\mathbf{Q}'$ *to* F_n *is absolutely continuous with respect to the restriction* $\mathbf{Q}_n$ *of* $\mathbf{Q}$ *to* F_n*. We write* $Z'_n = \frac{d\mathbf{Q}'_n}{d\mathbf{Q}_n}$ *the corresponding Radon-Nykodim derivatives and we denote by* $\mathbb{E}$ *and* $\mathbb{E}'$ *the expectations* *with respect to the measures* $\mathbf{Q}'_n$ *of* $\mathbf{Q}'$. *Let* $\mathcal{M}$ *be a martingale on* $\left(\Omega, \mathcal{F}, F=(F_n)_{n\in\mathbb{N}}, \mathbf{Q}\right)$ *such that* $\mathcal{M}_0 = 0$. *Suppose that there exists a pair of integers* $p, q > 1$ *such that* $\frac{1}{p}+\frac{1}{q}=1$ *and for any* $n \in \mathbb{N}$

$$\mathbb{E}\left(|\Delta\mathcal{M}_n|^p\right)^{\frac{1}{p}} \leq a_p \,.\, \exp\left(\lambda_p \,.\, n\right) \qquad \mathbb{E}\left(|Z'_n|^q\right)^{\frac{1}{q}} \leq a_q \,.\, \exp\left(\lambda_q \,.\, n\right) \tag{59}$$

for some constants $a_p, a_q \geq 0$ *and* $\lambda_p, \lambda_q \in \mathbb{R}$, *where* $\Delta\mathcal{M}_n =_{def.} \mathcal{M}_n - \mathcal{M}_{n-1}$ *and the convention* $\Delta\mathcal{M}_0 = 0$. *Then one has the following implication*

$$\lambda_p + \lambda_q < 0 \Longrightarrow \sup_{n\in\mathbb{N}} \left|\mathbb{E}'\left(\mathcal{M}_n\right)\right| \leq \frac{a_p \,.\, a_q}{|\lambda_p+\lambda_q|}$$

Proof:
By construction we have $\mathcal{M}_0 = 0$ and by definition of Z the process

$$\mathcal{M}'_n =_{def.} \mathcal{M}_n - \sum_{m=1}^{n} (Z'_{m-1})^{-1}\, \mathbb{E}\left(Z'_m\ \Delta\mathcal{M}_m\, |F_{m-1}\right)$$

is a $\mathbf{Q}'$-martingale such that $\mathcal{M}'_0 = 0$ (see for instance Theorem 3.46 p. 165 in [12]). This implies that

$$\forall n \in \mathbb{N} \qquad \mathbb{E}'\left(\mathcal{M}_n\right) = \sum_{m=1}^{n} \mathbb{E}\left(Z'_m\ \Delta\mathcal{M}_m\right)$$

and under the assumptions of the lemma one concludes that

$$\forall n \in \mathbb{N} \qquad \left|\mathbb{E}'\left(\mathcal{M}_n\right)\right| \leq a_p \,.\, a_q \sum_{m=1}^{n} \exp-\left(|\lambda_p+\lambda_q| n\right) \leq \frac{a_p \,.\, a_q}{e^{|\lambda_p+\lambda_q|}-1} \leq \frac{a_p \,.\, a_q}{|\lambda_p+\lambda_q|}$$

and the proof of the lemma is now completed. ∎

Proof of Proposition 6.1:
To apply Lemma 6.2 in the settings of Proposition 6.1 it is convenient to introduce the following sequence of meshes

$$\forall n \in \mathbb{N} \qquad t_n =_{def.} t \,.\, \left(1 - e^{-2(\alpha+1)n}\right)$$

It is clear that the processes $\{\mathcal{M}_n \,;\ n \in \mathbb{N}\}$ and $\{\mathcal{Z}'_n \,;\ n \in \mathbb{N}\}$ given by

$$\forall n \in \mathbb{N} \qquad \mathcal{M}_n =_{def.} \int_0^{t_n} H_s\, dM_s \quad \text{and} \quad \mathcal{Z}'_n =_{def.} Z'_{t_n}$$

are $\mathbf{Q}$-martingales with respect to the filtration $\mathcal{F}_n =_{def.} F_{t_n}$.

By Fatou's Lemma and Lemma 6.2 it clearly suffices to check that the bounds (59) hold for $p = \alpha+1$ and $q = 1+\frac{1}{\alpha}$ with

$$\lambda_p + \lambda_q = -1, \qquad \lambda_q = \alpha, \qquad a_p = a'_p\, t^{1/2}, \qquad \text{and} \qquad a_q = C(t)^{\frac{1}{1+\alpha}}\, t^{-1/2}$$

where a'_p is a universal constant which only depends on the parameter p. Under our assumptions we first observe that

$$\begin{aligned}\mathbb{E}\left((\mathcal{Z}'_n)^q\right)^{\frac{1}{q}} &= \mathbb{E}\left((\mathcal{Z}'_n)^{\frac{q}{p}} \,.\, \mathcal{Z}'_n\right)^{\frac{1}{q}} \leq C(t)^{\frac{1}{1+\alpha}}\, t^{-\frac{1}{2(1+\alpha)}}\ \left(t\ e^{-2(1+\alpha)n}\right)^{-\frac{\alpha}{2(1+\alpha)}} \\ &\leq C(t)^{\frac{1}{1+\alpha}}\, t^{-\frac{1}{2(1+\alpha)}}\, t^{-\frac{\alpha}{2(1+\alpha)}}\, e^{-\alpha n} = C(t)^{\frac{1}{1+\alpha}}\, t^{-\frac{1}{2}}\, e^{-\alpha n}\end{aligned}$$

and the desired bound for $q = 1 + \frac{1}{\alpha}$ is now proved with the desired constants λ_q and a_q. To check that the second one we use Burkholder-Davis-Gundy's inequality to check that

$$\mathbb{E}\left(|\Delta \mathcal{M}_n|^p\right) = \mathbb{E}\left(\left|\int_{t_{n-1}}^{t_n} H_s\, dM_s\right|^p\right) \leq b_p\, \mathbb{E}\left(\left|\int_{t_{n-1}}^{t_n} H_s^2\, d<M>_s\right|^{p/2}\right)$$

for some universal constant b_p which only depends on the parameter p. Under our assumptions this implies that

$$\mathbb{E}\left(|\Delta \mathcal{M}_n|^p\right)^{\frac{1}{p}} \leq b_p^{\frac{1}{p}}\, \mathbb{E}\left(\left|\int_{t_{n-1}}^{t_n} H_s^2\, ds\right|^{p/2}\right)^{\frac{1}{p}} \leq b_p^{\frac{1}{p}}\, |t_n - t_{n-1}|^{1/2}$$

Finally, since

$$|t_n - t_{n-1}|^{1/2} = t^{1/2}\, e^{-(1+\alpha)n}\, \left(e^{2(1+\alpha)} - 1\right)^{1/2}$$

the second bound for the parameter p is proved with the desired constants and $a'_p = \left(e^{2p} - 1\right)^{1/2}\, b_p^{\frac{1}{p}}$.

To prove the second part of the proposition we observe that the process

$$\forall s \in [0,t] \qquad H_s^{(\lambda)} =_{def.} e^{\lambda M_s - \frac{\lambda^2}{2} <M>_s}$$

satisfies

$$H_t^{(\lambda)} = 1 + \lambda \int_0^t H_s^{(\lambda)}\, dM_s$$

By straightforward calculations, under our assumptions one also can check that

$$\sup_{s \in [0,t]} \mathbb{E}\left(\left|H_s^{(\lambda)}\right|^{\alpha+1}\right)^{\frac{1}{\alpha+1}} \leq \exp\left(\frac{1}{2}\alpha\, \lambda^2\, t\right)$$

from which one concludes that (57) is a consequence of (56). To prove (58) we combine Jensen's inequality to the exponential (that is $E(e^X) \geq e^{E(X)}$ for any random variable X) with (57). ∎

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