Approximate Nonlinear Filtering and its Application in Navigation

B. Azimi-Sadjadi[†‡] and P.S. Krishnaprasad[‡]

† Rensselaer Polytechnic Institute
Troy, New York, 12180

‡Institute for Systems Research
University of Maryland at College Park
College Park, MD 20742

March, 2002

Abstract

In this paper we introduce for the first time particle filtering for an exponential family of densities. We prove that under certain conditions the approximated conditional density converges to the true conditional density. In the realistic setting where the conditional density does not lie in an exponential family but stays close to it, we show that under certain assumptions the error of the estimate given by an approximate nonlinear filter (which we call *projection particle filter*), is bounded. We use projection particle filtering in state estimation for a combination of inertial navigation system (INS) and global positioning system (GPS), referred to as integrated INS/GPS. We illustrate via numerical experiments that projection particle filtering outperforms regular particle filtering in navigation performance, and extended Kalman filter as well when satellite loss-of-lock occurs.

1 Introduction

Filtering problems consist of "estimating" a process $\{\mathbf{x}_t\}$ (or something about it) given a related process, $\{\mathbf{y}_t\}$, which can be observed [1]. The observation is available on an interval, i.e., $\{\mathbf{y}_s, 0 \leq s < t\}$ and the function of the state is estimated at time t. Except for the linear Gaussian system and very special cases in nonlinear settings, estimating the state given the observations results in an infinite dimensional filter [2]. Therefore, approximation methods of finite dimension are very appealing.

The most widely used approximate filtering method is the Extended Kalman Filter (EKF), which is a heuristic method based on the linearization of the state dynamics and the observation near a nominal path [2].

This research was supported in part by Army Research Office under ODDR&E MURI97 Program Grant No. DAAG55-97-1-0114 to the Center for Dynamics and Control of Smart Structures (through Harvard University), and under ODDR&E MURI01 Program Grant No. DAAD19-01-1-0465 to the Center for Communicating Networked Control Systems (through Boston University), and by the National Science Foundation Learning and Intelligent Systems Initiative Grant CMS9720334.

EKF is computationally simple but, the convergence of the estimated conditional density to the actual conditional density is not guaranteed.

Projection Filtering (PrF) is another approximation method [3][4][5][6]. In PrF it is assumed that the conditional density of the state of the system can be approximated by a member of parametric family of densities. In this case, estimating the conditional density is equivalent to estimating the parameter of the family. In [3][4][5] the exponential family of densities is chosen as the parametric family. In [6] the approach is different; there a Galerkin approximation is used to solve the Fokker-Planck equation [2].

An entirely different approach to approximate the conditional density is simulation based filtering. Grid-less simulation based filtering, now known by many different names such as Particle Filtering (PaF) [7][8], the Condensation Algorithm [9], the Sequential Monte Carlo (SMC) Method [10], and Bayesian Bootstrap Filtering [11] was first introduced in [11] and then it was rediscovered independently in [9] and [12]. Henceforth we refer to this filtering method as Particle Filtering. The results in [11] are an extension of the results in [13] and [14] to the dynamic case and is based on a method called Sampling/Importance Resampling (SIR). SIR is a key element of the grid-less simulation based filtering methods which allows these methods to have automatically high resolution grids in the areas where the conditional density is significant and low resolution in the areas where the conditional density is small.

PaF is based on the Monte Carlo method; in this method, the particles at time t_i are i.i.d. random vectors that are distributed according to the empirical conditional distribution of the state, given the observations up to time t_i. These particle/state vectors are used in the state equation to find the values of particles at time t_{i+1}. Then at time t_{i+1}, the empirical distribution is evaluated according to the values of the particles. The new observation at time t_{i+1} is taken into account through Bayes' Rule to calculate the conditional empirical distribution, this process is then repeated. In [7] it is proved that using a large enough number of particles, one can get an approximate conditional distribution that is arbitrarily close to the true conditional distribution.

In the cases where we have some prior information about the distribution, we should expect to achieve higher performance if we take this information into account. By higher performance, we mean a reduction in the computational cost and an increase in the convergence rate. Here we assume that the conditional distribution has a density in an exponential family of densities, or at least stays close to it in a sense that we will define. Using this assumption, we replace the empirical distribution in [7] with the Maximum Likelihood Estimate (MLE) of the parameters of an exponential density. We call this new method *projection particle filtering*. In Theorem 3 we show that if the conditional density of the state given the observations lies in an exponential family of densities then the estimated conditional density converges to the true conditional density in a sense that will be defined. In Theorem 4 for the case where the true conditional density stays close to an exponential family of densities we show that the error of the estimate given by projection particle filtering is bounded. Approximating the conditional density of the state given the observation by an exponential family of densities is addressed in [4]. Unlike our approach, in [4] no estimates on the closeness of the true conditional density to the approximate conditional density is reported.

One of the applications of the new particle filtering method introduced in this paper is position estimation in an integrated INS/GPS system. In [15] it was shown that when the number of GPS satellites visible to a receiver drops below a critical number, there three, EKF could no longer provide a reasonable estimate for the position, in fact the estimate given by EKF would diverge. Here we show for these critical cases, projection PaF for an exponential family of densities can provide an estimate of position that is accurate as well as smooth. We show that, on average, this filter can perform better than regular PaF.

In this paper, Section 2 states the nonlinear filtering problem. In Section 3 we review the results in [3][4][5] on projection filtering. In Section 4 we discuss particle filtering and we state the results in [7][8]. In Sections 5 and 6 we introduce a new PaF algorithm and we state the main results of this paper. In Section 7 we apply the PaF for an exponential family of densities to estimate the position in an integrated INS/GPS.

2 Nonlinear Filtering, Problem Setup

We assume that all stochastic processes are defined on a fixed probability space (Ω, F, P), and a finite time interval, $[0, T]$, on which there is defined an increasing family of σ-fields, $\{\mathcal{F}_t, 0 \leq t \leq T\}$. It is assumed that each process, $\{\mathbf{x}_t\}$, is adapted to $\mathcal{F}_t$, i.e., $\{\mathbf{x}_t\}$ is $\mathcal{F}_t$-measurable for all t. We assume that $\{\mathbf{x}_t\}$ is a vector diffusion process of the form

$$\mathbf{x}_t = \mathbf{x}_0 + \int_0^t \mathbf{f}_s(\mathbf{x}_s)ds + \int_0^t G_s(\mathbf{x}_s)d\mathbf{w}_s, \tag{1}$$

where $\mathbf{x}_t \in \mathcal{R}^n$, and $\mathbf{w}_t \in \mathcal{R}^q$ is a vector from an independent Brownian motion process; the second integral is in the Ito sense [16], and the function $\mathbf{f}_t(\cdot)$ and the matrix $G_t(\cdot)$ have the proper dimensions. The observation, $\mathbf{y}_t$, is a discrete time process given as follows:

$$\mathbf{y}_{n\tau} = \mathbf{h}_n(\mathbf{x}_{n\tau}) + \mathbf{v}_n, \tag{2}$$

where $\mathbf{y}_{n\tau} \in \mathcal{R}^d$, and $\mathbf{v}_n \in \mathcal{R}^d$ is a discrete time white Gaussian noise process with zero mean and known covariance matrix. The state dynamics and observation equations can be rewritten formally as follows:

$$\begin{array}{rcl} d\mathbf{x}_t = & \mathbf{f}_t(\mathbf{x}_t)dt + G_t(\mathbf{x}_t)d\mathbf{w}_t, & \text{given the distribution of } \mathbf{x}_0 \\ \mathbf{y}_{n\tau} = & \mathbf{h}_n(\mathbf{x}_{n\tau}) + \mathbf{v}_n & \end{array} \tag{3}$$

The noise processes $\{\mathbf{w}_t,\ t \geq 0\}$, and $\{\mathbf{v}_n,\ n = 0, 1, \cdots\}$, and the initial condition $\mathbf{x}_0$ are assumed independent. We use Q_t and R_n for the covariance matrices of the processes $\mathbf{w}_t$ and $\mathbf{v}_n$, respectively. We assume that R_n is invertible for all n's. We have the following additional assumptions [17]:

A 1 [local Lipschitz continuity] $\forall\ \mathbf{x},\ \mathbf{x}' \in B_r$ *and* $t \in [0, T]$*, where* B_r *is a ball of radius* r*, we have*

$$\begin{array}{rcll} \|\mathbf{f}_t(\mathbf{x}) - \mathbf{f}_t(\mathbf{x}')\| & \leq & k_r\|\mathbf{x} - \mathbf{x}'\|, & \textit{and} \\ \|G_t(\mathbf{x})Q_tG_t^T(\mathbf{x}) - G_t(\mathbf{x}')Q_tG_t^T(\mathbf{x}')\| & \leq & k_r\|\mathbf{x} - \mathbf{x}'\|. & \end{array} \tag{4}$$

A 2 [Non-Explosion] *There exists* $k > 0$ *such that*

$$\begin{array}{rcll} \mathbf{x}^T\mathbf{f}_t(\mathbf{x}) & \leq & k(1 + \|\mathbf{x}\|^2), & \textit{and} \\ trace(G_t(\mathbf{x})Q_tG_t^T(\mathbf{x})) & \leq & k(1 + \|\mathbf{x}\|^2). & \end{array} \tag{5}$$

$\forall\ t \in [0, T]$ *and* $\forall\ \mathbf{x} \in \mathcal{R}^n$*.*

Under assumptions (A1) and (A2), there exists a unique solution $\{\mathbf{x}_t,\ t \in [0, T]\}$ to the state equation, and $\mathbf{x}_t$ has finite moment of any order [17].

In addition to these, we assume that the probability distribution of the state $\mathbf{x}_t$, given the observation up to time t, $\pi_t(d\mathbf{x}) = P(\mathbf{x}_t \in d\mathbf{x}|\mathbf{y}^t)$, where $\mathbf{y}^t = \{\mathbf{y}_n,\ i = 1, \cdots, n,\ n\tau < t\}$, has a density p_t with respect to the Lebesgue measure on $\mathcal{R}^n$. Then $\{p_t,\ t > 0\}$ satisfies the following PDE and updating equations [4]:

$$\begin{array}{rll} \frac{\partial}{\partial t} p_t = & \mathcal{L}_t^* p_t & n\tau \le t < (n+1)\tau, \quad \text{and} \\ p_{n\tau} = & c_n \Psi_n p_{n\tau^-} & \end{array} \tag{6}$$

where

$$\begin{array}{rl} \mathcal{L}_t^*(\Phi) & = -\sum_{i=1}^n \frac{\partial}{\partial \mathbf{x}_i}[f_t^i \Phi] + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_j}[a_t^{ij}\Phi], \\ [a_t^{ij}] & = G_t Q_t G_t^T, \\ \Psi_n(\mathbf{x}) & \stackrel{\triangle}{=} \exp\left(-\frac{1}{2}(\mathbf{y}_{n\tau} - \mathbf{h}_n(\mathbf{x}))^T R_n^{-1}(\mathbf{y}_{n\tau} - \mathbf{h}_n(\mathbf{x}))\right), \end{array}$$

and c_n is a normalizing factor.

Except for the linear Gaussian case, and some very special nonlinear cases, solving System (6) constitutes an infinite dimensional filter [2]. Therefore, for practical problems it is necessary to approximate the conditional density in (6). In the next section, we discuss one of these approximation methods.

3 Projection Filtering on Exponential Families of Densities

This section is mainly a review of the results we use from [5]. We start this section with the definition of the exponential family of densities.

Definition 1 *Let $\{c_1, \cdots, c_p\}$ be affinely independent [1] scalar functions defined on $\mathcal{R}^n$, and assume that the convex set*

$$\Theta_0 = \left\{\theta \in \mathcal{R}^p :\ \Upsilon(\theta) = log \int exp\left(\theta^T \mathbf{c}(\mathbf{x})\right) d\mathbf{x} < \infty\right\},$$

has nonempty interior. Then,

$$\begin{array}{c} \mathcal{S} = \{p(\cdot, \theta),\ \theta \in \Theta\} \\ p(\mathbf{x}, \theta) := exp\left[\theta^T \mathbf{c}(\mathbf{x}) - \Upsilon(\theta)\right], \end{array}$$

where $\Theta \subseteq \Theta_0$ is open, is called an exponential family of probability densities.

We denote by $\mathcal{S}^{\frac{1}{2}}$ the space of square roots of the densities in $\mathcal{S}$, i.e., $\mathcal{S}^{\frac{1}{2}} = \{\sqrt{p(\cdot,\theta)};\ \theta \in \Theta\}$. If $p(\cdot, \theta) \in \mathcal{S}$, then $\sqrt{p(\cdot,\theta)} \in L_2$. The functions $\frac{1}{2\sqrt{p(\cdot,\theta)}} \frac{\partial p(\cdot,\theta)}{\partial \theta_i},\ i = 1, \cdots, p$ form a basis for the tangent vector space at $\sqrt{p(\cdot,\theta)}$ to the space $\mathcal{S}^{\frac{1}{2}}$, i.e., the tangent space at $\sqrt{p(\cdot,\theta)}$ is given by [19]:

$$L_{\sqrt{p(\cdot,\theta)}}\mathcal{S}^{\frac{1}{2}} = span\left\{\frac{1}{2\sqrt{p(\cdot,\theta)}}\frac{\partial p(\cdot,\theta)}{\partial \theta_1}, \cdots,\ \frac{1}{2\sqrt{p(\cdot,\theta)}}\frac{\partial p(\cdot,\theta)}{\partial \theta_p}\right\}. \tag{7}$$

[1] $\{c_1, \cdots, c_p\}$ are affinely independent if for distinct points $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{p+1}$, $\sum_{i=1}^{p+1} \lambda_i \mathbf{c}(\mathbf{x}_i) = 0$ and $\sum_{i=1}^{p+1} \lambda_i = 0$ implies $\lambda_1 = \lambda_2 = \cdots = \lambda_{p+1} = 0$ [18].

The inner product of any two basis elements is defined as follows

$$\begin{aligned}\left\langle \frac{1}{2\sqrt{p(\cdot,\theta)}}\frac{\partial p(\cdot,\theta)}{\partial\theta_i}, \frac{1}{2\sqrt{p(\cdot,\theta)}}\frac{\partial p(\cdot,\theta)}{\partial\theta_j}\right\rangle &= \tfrac{1}{4}\int \frac{1}{p(\mathbf{x},\theta)}\frac{\partial p(\mathbf{x},\theta)}{\partial\theta_i}\frac{\partial p(\mathbf{x},\theta)}{\partial\theta_j}d\mathbf{x} \\ &= \tfrac{1}{4}g_{ij}(\theta)\end{aligned} \tag{8}$$

It can be easily seen that $g(\theta) = (g_{ij}(\theta)) = (E[c_ic_j] - E[c_i]E[c_j])$ is the Fisher information matrix of $p(\cdot,\theta)$, and this construction of a Riemanian metric in Θ goes back to an early paper of Rao [20].

Any member of L_2 can be projected to the tangent space $L_{\sqrt{p(\cdot,\theta)}}\mathcal{S}^{\frac{1}{2}}$ according to the following projection formula

$$\begin{aligned}\Pi_\theta:\ L_2 \supseteq V &\rightarrow L_{\sqrt{p(\cdot,\theta)}}\mathcal{S}^{\frac{1}{2}} \\ v &\rightarrow \sum_{i=1}^{p}\sum_{j=1}^{p} 4g^{ij}(\theta)\left\langle v, \frac{1}{2\sqrt{p(\cdot,\theta)}}\frac{\partial p(\cdot,\theta)}{\partial\theta_j}\right\rangle \frac{1}{2\sqrt{p(\cdot,\theta)}}\frac{\partial p(\cdot,\theta)}{\partial\theta_i}.\end{aligned} \tag{9}$$

Projection filtering seeks a solution for p_t for (6) that lies in $\mathcal{S}$. The exponential density should be chosen so that the approximation error is small (in L_2 sense).

If we consider the square root of the density in (6), we get

$$\frac{\partial\sqrt{p_t}}{\partial t} = \frac{1}{2\sqrt{p_t}}\frac{\partial p_t}{\partial t} = \frac{1}{2\sqrt{p_t}}\mathcal{L}_t^* p_t \quad . \tag{10}$$

Define $\alpha_{t,\theta} = \frac{\mathcal{L}_t^* p_t(\cdot,\theta)}{p_t(\cdot,\theta)}$. We assume that for all $\theta\in\Theta$ and all $t\geq 0$, $E_{p(\cdot,\theta)}\{|\alpha_{t,\theta}|^2\} < \infty$, which implies that $\frac{\mathcal{L}_t^* p_t(\cdot,\theta)}{\sqrt{p_t(\cdot,\theta)}}$ is a vector of L_2 for all $\theta\in\Theta$ and all $t\geq 0$ [4].

Now assume that in equation (10), for $\{\sqrt{p_t},\ t\geq t_0\}$, starting at time $n\tau$ from the initial condition, $\sqrt{p_{n\tau}} = \sqrt{p(\cdot,\theta_{n\tau})} \in \mathcal{S}^{\frac{1}{2}}$ for some $\theta_{n\tau}\in\Theta$. Under these assumptions, the right hand side of (10) is in L_2, which can be projected into the finite dimensional tangent vector space $L_{\sqrt{p(\cdot,\theta_{n\tau})}}\mathcal{S}^{\frac{1}{2}}$. The projection filter for the exponential family, $\mathcal{S}$, in the interval $[n\tau,(n+1)\tau)$, is defined as the solution of the following differential equation in the same interval:

$$\frac{\partial\sqrt{p_t(\cdot,\theta_t)}}{\partial t} = \Pi_{\theta_t}\frac{\mathcal{L}_t^* p_t(\cdot,\theta_t)}{2\sqrt{p_t(\cdot,\theta_t)}} \quad . \tag{11}$$

We also assume that $\mathbf{h}_n(\mathbf{x})$ in equation (2) is time invariant, i.e., $\mathbf{h}_n(\mathbf{x}) = \mathbf{h}(\mathbf{x})$, and the components of $\mathbf{h}(\mathbf{x})$, $h^i(\mathbf{x})$, and $\|\mathbf{h}(\mathbf{x})\|^2_{R^{-1}}$ are linear combinations of $c_i(\mathbf{x})$, $i=1,\cdots,\ p$:

$$\frac{1}{2}\|\mathbf{h}(\mathbf{x})\|^2_{R^{-1}} = \sum_{i=1}^{p}\lambda_i^0 c_i(\mathbf{x}) \quad \text{and} \quad h^k(\mathbf{x}) = \sum_{i=1}^{p}\lambda_i^k c_i(\mathbf{x}),\quad k=1,\cdots,\ d \tag{12}$$

where $\|\mathbf{x}\|_A = \sqrt{\mathbf{x}^T A\mathbf{x}}$. Then, if $\mathbf{v}_n$ is stationary with the covariance matrix $R_n = R$, the likelihood function $\Psi_n(n)$ can be written as follows:

$$\begin{aligned}\Psi_n(\mathbf{x}) &= \exp(-\tfrac{1}{2}(\mathbf{y}_{n\tau}^T R^{-1}\mathbf{y}_{n\tau}))\exp(-\tfrac{1}{2}(\mathbf{h}^T(\mathbf{x})R^{-1}\mathbf{h}(\mathbf{x})) + (\mathbf{y}_{n\tau}^T R^{-1}\mathbf{h}(\mathbf{x}))) \\ &= A_n\ \exp\left(-\sum_{i=1}^{d}\lambda_i^0 c_i(\mathbf{x}) + \sum_{k=1}^{p}(\sum_{i=1}^{p}\lambda_i^k z_{n\tau}^k)c_i(\mathbf{x})\right),\end{aligned} \tag{13}$$

where $\mathbf{z}_{n\tau} = \mathbf{y}_{n\tau}^T R^{-1}$, and A_n is a constant depending on $\mathbf{y}_{n\tau}$. Therefore, the coefficient $\Psi_n(\mathbf{x})$ is a member of exponential family of densities. This family is closed under multiplication. Using all of these facts, we can present the following theorem [4]:

Theorem 1 [Brigo 1996] *For system (3), where* $\mathbf{w}_t$ *is a Brownian motion process with covariance* Q_t *and* $\mathbf{v}_i$ *is a white Gaussian noise with covariance* R*, we assume (A1) and (A2) to be true. We also assume that* $\frac{1}{2}\|\mathbf{h}(\mathbf{x})\|^2_{R^{-1}} = \sum_{i=1}^{p} \lambda_i^0 c_i(\mathbf{x})$*,* $h^k(\mathbf{x}) = \sum_{i=1}^{p} \lambda_i^k c_i(\mathbf{x})$*, for* $k = 1, \cdots, d$*, and* $E_{p(\cdot,\theta)}\|\frac{\mathcal{L}_t^* p(\cdot,\theta)}{p(\cdot,\theta)}\|^2 < \infty, \ \ \forall\theta \in \Theta, \ \ \forall t \geq 0$*. Then for all* $\theta \in \Theta$*, and all* $t \geq 0$*,* $\Pi_\theta \frac{\mathcal{L}_t^* p(\cdot,\theta)}{\sqrt{p(\cdot,\theta)}}$ *is a vector on the exponential manifold* $\mathcal{S}^{\frac{1}{2}}$*. The projection filter density,* $p_t^\Pi = p_t(\cdot, \theta_t)$ *is described by*

$$\begin{aligned} \frac{\partial\sqrt{p_t(\cdot,\theta_t)}}{\partial t} &= \Pi_{\theta_t} \frac{\mathcal{L}_t^* p_t(\cdot,\theta t)}{2\sqrt{p(\cdot,\theta_t)}} \ , \qquad\qquad n\tau \leq t < (n+1)\tau \\ p_{n\tau}(\cdot,\theta_{n\tau}) &= c_n \Psi_n(\mathbf{y}_{n\tau}) p_{n\tau^-}(\cdot,\theta_{n\tau^-}) \ , \end{aligned}$$

and the projection filter parameter satisfies the following combined deterministic differential equation and stochastic update map:

$$\begin{aligned} g(\theta_t)d\theta_t &= E_{\theta_t}\{\mathcal{L}_t \mathbf{c}\}dt, \qquad\qquad n\tau \leq t < (n+1)\tau, \\ \theta_{n\tau} &= \theta_{n\tau^-} - \lambda_0^0 + \textstyle\sum_{k=1}^d \lambda_0^k z_n^k, \end{aligned}$$

where

$$\mathcal{L}_t = \sum_{i=1}^{n} f_t^i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} a_t^{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

and $\lambda_0^i = [\lambda_1^i, \cdots, \lambda_p^i]^T$*,* $i = 0, \cdots, d$*, and* z_n^k *is the* k*th component of* $\mathbf{z}_{n\tau}^T = R^{-1}\mathbf{y}_{n\tau}$*.*

Henceforth, we shall use E_θ and $E_{p(\cdot,\theta)}$, $\theta_{n\tau}$ and θ_n, and $p_{n\tau}$ and p_n, interchangeably.

As can be seen from the statement of the theorem, the calculation of the conditional probability density is reduced to the calculation of the parameter of an exponential family. But still, solving the differential equation in the theorem is not an easy task. At each moment $g(\theta_t)$ and $E_{\theta_t}\{\mathcal{L}_t\mathbf{c}\}$ need to be calculated. This requires a heavy computational load. In Section 5 of this paper, we introduce a Monte Carlo method to calculate the parameter of the exponential family with a more affordable computational load.

Although PrF gives a better solution than EKF, there is no known error bound with which we can compare the distance between the real density and the density given by the PrF. In the next section we review Particle Filtering as an alternative to optimal nonlinear filtering.

Remark : The assumption on $\mathbf{h}_n(\cdot)$ and R_n in this are made only to make sure that $\Psi_n(\cdot)$ is in the family of exponential densities. These assumptions can be relaxed if $\Psi_n(\cdot)$ is otherwise guaranteed to stay in the family.

4 Particle Filtering

Consider either the continuous dynamics and discrete observation in (3) or the discrete case,

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{f}_n(\mathbf{x}_n) + G_n(\mathbf{x}_n)\mathbf{w}_n, \ \text{ given the distribution of } \ \mathbf{x}_0 \\ \mathbf{y}_n &= \mathbf{h}_n(\mathbf{x}_n) + \mathbf{v}_n. \end{aligned} \tag{14}$$

We assume that in both cases, the initial distribution for $\mathbf{x}_0$ is given. The propagation of the conditional density, at least conceptually, can be expressed as follows [2]:

- Step 1 . Initialization:

$$p_0(\mathbf{x}_0|\mathbf{y}_0) = p(\mathbf{x}_0).$$

- Step 2 . Diffusion:

$$p_{(n+1)^-}(\mathbf{x}_{n+1}|\mathcal{Y}_n) = \int p(\mathbf{x}_{n+1}|\mathbf{x}_n) p_n(\mathbf{x}_n|\mathcal{Y}_n) d\mathbf{x}_n,$$

  where $\mathcal{Y}_n = \{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$.

- Step 3 . Bayes' rule update:

$$p_{(n+1)}(\mathbf{x}_{n+1}|\mathcal{Y}_{n+1}) = \frac{p(\mathbf{y}_{n+1}|\mathbf{x}_{n+1}) p_{(n+1)^-}(\mathbf{x}_{n+1}|\mathcal{Y}_n)}{\int p(\mathbf{y}_{n+1}|\mathbf{x}_{n+1}) p_{(n+1)^-}(\mathbf{x}_{n+1}|\mathcal{Y}_n) d\mathbf{x}_{n+1}} \quad ,$$

- Step 4 . $n \leftarrow n+1$; go to Step (2).

The conditional density given by the above steps is exact, but in general it can be viewed as an infinite dimensional filter, thus, not implementable. PaF, in brief, is an approximation method that mimics the above calculations with a finite number of operations using the Monte Carlo method. The procedure for PaF is as follows [11][7]:

Algorithm 1 *Particle Filtering*

- *Step 1 . Initialization*

  - ◇ *Sample* $\mathbf{x}_0^1, \cdots, \mathbf{x}_0^N$, *N i.i.d. random vectors with the distribution* $P_0(\mathbf{x})$.

- *Step 2 . Diffusion*

  - ◇ *Find* $\hat{\mathbf{x}}_{n+1}^1, \cdots, \hat{\mathbf{x}}_{n+1}^N$ *from the given* $\mathbf{x}_n^1, \cdots, \mathbf{x}_n^N$, *using the dynamic rules:*

$$\begin{aligned} d\mathbf{x}_t = &\quad \mathbf{f}_t(\mathbf{x}_t)dt + G_t(\mathbf{x}_t)d\mathbf{w}_t, \quad n\tau \le t < (n+1)\tau \\ \textit{or} & \\ \mathbf{x}_{n+1} = &\quad \mathbf{f}_n(\mathbf{x}_n) + G_n(\mathbf{x}_n)\mathbf{v}_n. \end{aligned}$$

- *Step 3 . Find the empirical distribution*

$$P_{(n+1)^-}^N(\mathbf{x}) = \frac{1}{N}\sum_{j=1}^N \delta_{\hat{\mathbf{x}}_{n+1}^j}(\mathbf{x})$$

- *Step 4 . Use Bayes' Rule*

$$P_{(n+1)}^N(\mathbf{x}) = \frac{\frac{1}{N}\sum\limits_{j=1}^N \delta_{\hat{\mathbf{x}}_{n+1}^j}(\mathbf{x}) \cdot \Psi_{n+1}(\mathbf{x})}{\frac{1}{N}\sum\limits_{j=1}^N \delta_{\hat{\mathbf{x}}_{n+1}^j}(\hat{\mathbf{x}}_{n+1}^j) \cdot \Psi_{n+1}(\hat{\mathbf{x}}_{n+1}^j)}$$

- *Step 5 . Resample*
  - ◇ *Sample* $\mathbf{x}_{n+1}^1, \cdots, \mathbf{x}_{n+1}^N$ *according to* $P_{n+1|n+1}^N(\mathbf{x})$
- *Step 6 .* $n \leftarrow n+1$*; go to Step (2).*

where $\delta_{\mathbf{v}}(\mathbf{w}) = 1$ if $\mathbf{w} = \mathbf{v}$ and 0 otherwise, and $\Psi_n(\mathbf{x})$ is the conditional density of the observation $\mathbf{y}_n$ given the state $\mathbf{x}$.

It is customary to call $\mathbf{x}_n^1, \cdots, \mathbf{x}_n^N$ particles. In the next few lines, we try to explain in words the evolution of these particles using the above algorithm.

Let $\hat{\mathbf{x}}_n^1, \cdots, \hat{\mathbf{x}}_n^N$ be the distinct particles at time n before incorporating the observation at time n. The probability of each particle is $\frac{1}{N}$, that is, is uniformly distributed. After using the observations, the conditional probability of each particle changes. Some will have small, and some large probabilities. Therefore, in the process of resampling, it is very likely that some particles will never be used and instead some other particles (with high probabilities) will be sampled more than once. Therefore, after resampling, some particles have repeated versions, but in the diffusion phase they go through different paths and at the end of the diffusion phase, it is very likely, we would have N distinct particles. This automatically makes the approximation one of better resolution in the areas where the probability is higher.

In [7] it is proved under some conditions that

$$\lim_{N\to\infty} E\left(\left| \frac{1}{N}\sum_{i=1}^{N} f(\hat{\mathbf{x}}_n^i) - E_{P_n}(f(\mathbf{x})) \right| \right) = 0 \tag{15}$$

for every bounded Borel test function, $f(\cdot)$.

One problem in using the PaF method is the computational cost. For a high dimensional system, getting reasonable accuracy means using a large N, which results in a heavy computational cost. In the next section, we propose a method that can reduce the number of particles for a certain class of problems.

5 Particle Filtering for Exponential Families of Densities

In the previous sections, we saw two approximation methods for nonlinear filtering. In the PaF method, we saw that the conditional distribution is approximated by the empirical distribution. In most cases, the actual conditional distribution is smooth, unlike the empirical distribution. Intuition suggests that if we have prior knowledge of some properties of the distribution, we should make effective use of such knowledge. In the following, we assume that the conditional density lies in a parametric family of densities. We will see that with this assumption, we can show the convergence of the approximated density to the actual one. In this section, after introducing our algorithm, we present some convergence results, and after that, we compare our method with the methods introduced in the previous sections.

For System (3), we assume that the probability density of $\mathbf{x}_t$, given the observation, is in a family of exponential densities $\mathcal{S}$. This assumption is rather strong. We will drop this assumption later, and we will only assume that there exists a known family of densities that approximates the real density with acceptable accuracy.

With this assumption, the proposed algorithm is as follows:

Algorithm 2 *Particle Filtering for an Exponential Family of Densities.*

- *Step 1 . Initialization*
  - ◇ *Sample* $\mathbf{x}_0^1, \cdots, \mathbf{x}_0^N$, *N i.i.d. random vectors with the density,* $p_0(\mathbf{x})$.
- *Step 2 . Diffusion*
  - ◇ *Find* $\hat{\mathbf{x}}_{n+1}^1, \cdots, \hat{\mathbf{x}}_{n+1}^N$ *from the given* $\mathbf{x}_n^1, \cdots, \mathbf{x}_n^N$, *using the dynamic rule:*

$$d\mathbf{x}_t = \mathbf{f}_t(\mathbf{x}_t)dt + G_t(\mathbf{x}_t)d\mathbf{w}_t, \quad i\tau \leq t < (i+1)\tau$$

- *Step 3 . Find the MLE of* $\hat{\theta}_{(n+1)^-}$ *given* $\hat{\mathbf{x}}_{n+1}^1, \cdots, \hat{\mathbf{x}}_{n+1}^N$

$$\hat{\theta}_{(n+1)^-} = \arg\max_{\theta} \prod_{i=1}^{N} \exp(\theta^T \mathbf{c}(\hat{\mathbf{x}}_{n+1}^i) - \Upsilon(\theta))$$

- *Step 4 . Use Bayes' Rule*

$$p(\mathbf{x}, \hat{\theta}_{(n+1)}) = \frac{\exp\left(\hat{\theta}_{(n+1)^-}^T \mathbf{c}(\mathbf{x}) - \Upsilon(\hat{\theta}_{(n+1)^-})\right) \Psi_{n+1}(\mathbf{x})}{\int \exp\left(\hat{\theta}_{(n+1)^-}^T \mathbf{c}(\mathbf{x}) - \Upsilon(\hat{\theta}_{(n+1)^-})\right) \Psi_{n+1}(\mathbf{x}) d\mathbf{x}}$$

- *Step 5 . Resample*
  - ◇ *Sample* $\mathbf{x}_{n+1}^1, \cdots, \mathbf{x}_{n+1}^N$ *according to* $p(\mathbf{x}, \hat{\theta}_{n+1})$.
- *Step 6 .* $n \leftarrow n+1$*; go to Step (2).*

To generate $\mathbf{x}_{n+1}^1, \cdots, \mathbf{x}_{n+1}^N$, a Gibbs sampler can be used [21]. This brings an extra computational cost, which should be taken into account when choosing Algorithm 2 over Algorithm 1.

It is constructive to discuss the structure of the ML estimator [22]. We are going to use this structure for the proof of convergence.

Let $\hat{\mathbf{x}}_n^1, \cdots, \hat{\mathbf{x}}_n^N$ be the values of the particles right before the measurement at time n. The MLE of θ_n, $\hat{\theta}_n$, satisfies the first order necessary condition

$$\sum_{i=1}^{N} c_j(\hat{\mathbf{x}}_n^i) - N \frac{\int_{\mathbf{x}} c_j(\mathbf{x}) \exp(\hat{\theta}_n^T \mathbf{c}(\mathbf{x})) d\mathbf{x}}{\int_{\mathbf{x}} \exp(\hat{\theta}_n^T \mathbf{c}(\mathbf{x})) d\mathbf{x}} = 0.$$

Therefore, we get

$$\frac{1}{N} \sum_{i=1}^{N} c_j(\hat{\mathbf{x}}_n^i) = E_{\hat{\theta}_n}(c_j(\mathbf{x})), \quad \text{for} \quad j = 1, \cdots, p \ . \tag{16}$$

Equation (16) says that the sample average of $c_j(\mathbf{x})$ and its probabilistic average, evaluated at $\hat{\theta}_n$, should be equal. Therefore, the MLE of θ is the solution to the system of equations in (16). Let $F_j(\theta)$ be as follows:

$$F_j(\theta) = \frac{1}{N}\sum_{i=1}^{N} c_j(\hat{\mathbf{x}}_n^i) - \frac{\int c_j(\mathbf{x})\exp(\theta^T\mathbf{c}(\mathbf{x}))d\mathbf{x}}{\int \exp(\theta^T\mathbf{c}(\mathbf{x}))d\mathbf{x}} \quad , \quad j = 1, \cdots, p.$$

For simplicity we drop the index n from θ_n. It is easy to see that

$$-\frac{\partial F_i}{\partial \theta_j} = E_\theta(c_i(\mathbf{x})c_j(\mathbf{x})) - E_\theta(c_i(\mathbf{x}))E_\theta(c_j(\mathbf{x})).$$

This shows that $(-\frac{\partial F_i}{\partial \theta_j})_{i,j} = g(\theta)$, where $g(\theta)$ is the Fisher information matrix of the exponential density at θ. Since $c_i(\mathbf{x}), i = 1, \cdots, p$ are affinely independent $g(\theta)$ is positive definite $\forall\theta \in \Theta$. Therefore (16) is the necessary and sufficient condition for optimality.

In the next few pages, we prove the convergence of the MLE of θ_n, $\hat{\theta}_n$, to θ_n in the mean square sense. This results in the convergence of the density in the weak sense.

In each iteration the proposed algorithm starts from the density $p_{\hat{\theta}_t}(\mathbf{x}_t|\mathbf{y}^t)$, $t = \tau n$, where $\hat{\theta}_t$ is the best estimate θ_t according to the algorithm. After a full iteration the algorithm yields $\hat{\theta}_{t+1}$ which is the best estimate of θ_{t+1}. The error in $\hat{\theta}_{t+1}$ is a combination of a series of possible errors for which we want to find upper bounds. The first source of error is the error in $\hat{\theta}_t$, which will propagate even if no other error is considered. The other source comes from the fact that in each iteration new particles are resampled based on the estimated density which is different from the actual density. Finally, the last source of error comes from the discretization of the stochastic dynamics of the system. We want to emphasize that here we assume $\Psi_n(\mathbf{x}) = \exp(-\frac{1}{2}(\mathbf{y}_{n\tau} - \mathbf{h}_n(\mathbf{x}_{n\tau}))^T R_n^{-1}(\mathbf{y}_{n\tau} - \mathbf{h}_n(\mathbf{x}_{n\tau})))$ lies in the family of densities. Therefore, no other error is added to the estimate because of the Bayes' correction.

We recall the following fact [22]:

Fact 1 *For the family of densities $\mathcal{S}$ with probability density $p(\mathbf{x},\theta) = \exp(\theta^T\mathbf{c}(\mathbf{x}) - \Upsilon(\theta))$, the Fisher information matrix $g(\theta) = (E(c_i(\mathbf{x})c_j(\mathbf{x})) - E(c_i(\mathbf{x}))E(c_j(\mathbf{x})))_{i,j}$ is positive definite. Also the likelihood function*

$$l(\theta) = \theta^T\mathbf{C}(\mathbf{x}) - \Upsilon(\theta),$$

is strictly concave. Therefore, for

$$c_j(\mathbf{x}) = E_\theta[c_j(\mathbf{x})], \qquad j = 1, \cdots, p,$$

if a solution exists[2]*, it is unique. In addition if $\mathbf{x}_1, \cdots, \mathbf{x}_N$ are N i.i.d. random variables distributed according to $p(\mathbf{x},\theta)$, then the MLE of θ, $\hat{\theta}_N$, is asymptotically normal, i.e.*

$$\begin{aligned} \hat{\theta}_N &= \arg\max_\theta \prod_{i=1}^{N} p(\mathbf{x}_i,\theta) \ , \\ \sqrt{N}(\hat{\theta}_N - \theta) &\sim \mathcal{N}(0, g^{-1}(\theta)). \end{aligned}$$

[2]In [18] it is shown that if $N > p$, the solution to (16) exists almost surely.

Using this fact, it is easy to see that

$$E\left(\left\|\hat{\theta}_N-\theta\right\|^2\right)=\frac{1}{N}\ trace(g^{-1}(\theta)),$$

therefore, when $N\longrightarrow\infty$, $\hat{\theta}_N\longrightarrow\theta$ in the m.s. sense. On the other hand, $\hat{\theta}_N$ is the solution to (16). Using the strong law of large numbers [23], when $N\to\infty$ the left hand side in (16) goes to $E_\theta(c_j(\mathbf{x}))$, $j=1,\cdots,p$, with probability one. In other words, the solution to (16) when the left hand side is the exact $E_\theta(c_j(\mathbf{x}))$, $j=1,\cdots,p$, gives the exact solution for θ. Using this argument, one can expect that by finding a good estimate of the left hand side of (16), a good estimate of θ is accessible. In each iteration of the algorithm presented in this section the estimate of the left hand side of (16) is found by using the Monte Carlo method and the approximate solution for the stochastic differential equation (3).

To approximate the solution to the stochastic differential equation (3), we employ a method used in [24]. In the following, we review this method briefly. The stochastic differential equation in (3) can be rewritten as follows:

$$d\mathbf{x}_t=\mathbf{f}_t\left(\mathbf{x}_t\right)dt+\sum_{r=1}^{q}\mathbf{g}_t^r\left(\mathbf{x}_t\right)dw_t^r, \tag{17}$$

where $\mathbf{g}_t^r(\cdot)$ is the r^{th} column of the matrix $G_t(\cdot)$, and w_t^r is the r^{th} component of $\mathbf{w}_t$. We introduce the operators

$$\Lambda_r u=\left(\mathbf{g}_r\ ,\ \frac{\partial}{\partial\mathbf{x}}\right)u,$$

$$Lu=\left(\frac{\partial}{\partial t}+\left(\mathbf{f}\ ,\ \frac{\partial}{\partial\mathbf{x}}\right)+\frac{1}{2}\sum_{r=1}^{q}\sum_{i=1}^{n}\sum_{j=1}^{n}g_i^r g_j^r\frac{\partial^2}{\partial\mathbf{x}_i\partial\mathbf{x}_j}\right)u,$$

where $\left(\mathbf{a}\ ,\ \frac{\partial}{\partial\mathbf{x}}\right)=\sum\limits_{i=1}^{n}a_i\frac{\partial}{\partial\mathbf{x}_i}$. Then, an approximate solution for (17) can be written as follows:

$$\begin{aligned}\mathbf{x}_{k+1}\ &=\ \mathbf{x}_k+\sum_{r=1}^{q}g_{t_k}^r\xi_k^r h^{\frac{1}{2}}+\mathbf{f}_{t_k}h+\sum_{r=1}^{q}\sum_{i=1}^{q}\left(\Lambda_r\mathbf{g}^r\right)_{t_k}\xi_k^{ir}h+\\ &\quad\ \frac{1}{2}\sum_{r=1}^{q}\left(L\mathbf{g}^r+\Lambda_r\mathbf{f}\right)_{t_k}\xi_k^r h^{\frac{3}{2}}+\left(L\mathbf{f}\right)_{t_k}\frac{h^2}{2},\end{aligned} \tag{18}$$

where h is the step size and the coefficients $\mathbf{g}_{t_k}^r$, $\mathbf{f}_{t_k}$, $\left(\Lambda_i\mathbf{g}^r\right)_{t_k}$, etc., are computed at the point $(t_k,\mathbf{x}_k)$. Also, the sets of random variables ξ_k^r, ξ_k^{ir} are independent for distinct k and can, for each k, be modelled as follows:

$$\xi^{ij}=\frac{1}{2}\xi^i\xi^j-\frac{1}{2}\gamma_{ij}\zeta^i\zeta^j,\qquad\gamma_{ij}=\begin{cases}-1 & ,\ i<j\\ 1 & ,\ i\geq j\end{cases}.$$

and ξ^i and ζ^j are independent random variables satisfying

$$\begin{array}{lll}E\xi_i=E\xi_i^3=E\xi_i^5=0, & E\xi_i^2=1, & E\xi_i^4=3,\\ E\zeta_j=E\zeta_j^3=0, & E\zeta_j^2=\zeta_j^4=1. & \end{array}$$

In particular, ξ_i can be modelled by the law $P\left(\xi=0\right)=\frac{2}{3}$, $P\left(\xi=\sqrt{3}\right)=P\left(\xi=-\sqrt{3}\right)=\frac{1}{6}$, and ζ_j can be modelled by $P\left(\zeta=-1\right)=P\left(\zeta=1\right)=\frac{1}{2}$.

Definition 2 *We say that a function $u(\cdot)$ belongs to the class $\mathcal{F}$, written as $u \in \mathcal{F}$, if we can find constants, $k > 0$, and $\kappa > 0$, such that for all $\mathbf{x} \in \mathcal{R}^n$, the following inequality holds:*

$$\|u(\mathbf{x})\| \leq k\left(1 + \|\mathbf{x}\|^{\kappa}\right).$$

Before we present the convergence results we need to define the probability space in which the random variables are defined. As we mentioned before, the stochastic determined by the dynamics and the observation equation are defined on a fixed probability space (Ω, F, P), the expectation associated to this probability space is denoted as E. In Algorithm 2 the generated particles form a Markov process. We denote the probability space associated to this process by $(\Omega', F', P'_{[Y]})$. The subindex Y is used to emphasize that this Markov process is driven by the observation $\mathbf{y}_n$. The expectation associated to this process is denoted as $E'_{[Y]}$. A set of random variables, ξ^i, ζ^i, are defined for the numerical approximation of the stochastic differential equation (17). We denote the probability space associated to these random variables by (Ω'', F'', P''). The expectation associated to this process is shown by E''. Finally we define $(\tilde{\Omega}, \tilde{F}, \tilde{P})$ and $\tilde{E}$ to be the probability space and the expectation associated to joint distribution of random variables defined in the three mentioned probability space, respectively.

The following theorem summarizes the weak approximation results for (18).

Theorem 2 [Milstein [24]] *Suppose (A1) from Section (2), and suppose that the functions $\mathbf{f}(\cdot)$, $\mathbf{g}^r(\cdot)$, $r = 1, \cdots, q$ together with the partial derivatives of sufficiently high order, belong to class $\mathcal{F}$. Also, suppose that the functions $\Lambda_i \mathbf{g}^r$, $L\mathbf{g}^r$, $\Lambda_r \mathbf{f}$, and $L\mathbf{f}$ grow at most as a linear function in $\|\mathbf{x}\|$. Then, if the function $u(\cdot)$ and all its derivatives up to order 6 belong to class $\mathcal{F}$, the approximation (18) has the order of accuracy 2, in the sense of weak approximation, i.e.,*

$$\|\tilde{E}u\left(\mathbf{x}_{0,\mathbf{x}_0}(t_k)\right) - \tilde{E}u\left(\hat{\mathbf{x}}_{0,\mathbf{x}_0}(t_k)\right)\| \leq Kh^2, \quad t_k \in [0, T],$$

where K is a constant (which depends on T) and $\mathbf{x}_{0,\mathbf{x}_0}(\cdot)$ and $\hat{\mathbf{x}}_{0,\mathbf{x}_0}(\cdot)$ are the exact and approximate solutions for the stochastic differential equation, respectively.

The Monte Carlo approximation of $\tilde{E}u\left(\mathbf{x}_{0,\mathbf{x}_0}(t_k)\right)$ brings in another error term. The combination of these errors can be expressed as follows:

$$\left\|\tilde{E}u\left(\mathbf{x}_{0,\mathbf{x}_0}(t_k)\right) - \tfrac{1}{N}\sum_{i=1}^{N} u\left(\hat{\mathbf{x}}_{0,\mathbf{x}_0^i}(t_k)\right)\right\| \leq$$
$$\left\|\tilde{E}u\left(\mathbf{x}_{0,\mathbf{x}_0}(t_k)\right) - \tilde{E}u\left(\hat{\mathbf{x}}_{0,\mathbf{x}_0}(t_k)\right)\right\| + \left\|\tilde{E}u\left(\hat{\mathbf{x}}_{0,\mathbf{x}_0}(t_k)\right) - \tfrac{1}{N}\sum_{i=1}^{N} u\left(\hat{\mathbf{x}}_{0,\mathbf{x}_0^i}(t_k)\right)\right\|.$$

If the variance of $u\left(\hat{\mathbf{x}}_{0,\mathbf{x}_0}(t_k)\right)$ is bounded, we have

$$\tilde{E}\left\|\tilde{E}u\left(\mathbf{x}_{0,\mathbf{x}_0}(t_k)\right) - \frac{1}{N}\sum_{i=1}^{N} u\left(\hat{\mathbf{x}}_{0,\mathbf{x}_0^i}(t_k)\right)\right\| \leq Kh^2 + \frac{k'}{N^{1/2}}, \tag{19}$$

where K and k' are constants, and h is the step size for the numerical approximation of the solution of the stochastic differential equation.

The next lemma relates the approximate solution to the stochastic differential equation and the estimate of the parameter θ. This lemma is the main building block for our result in this section.

Lemma 1 *For the SDE*

$$d\mathbf{x}_t = \mathbf{f}_t(\mathbf{x}_t)\,dt + G_t(\mathbf{x}_t)\,d\mathbf{w}_t, \quad \mathbf{x}_0, \quad t \in [0, t_f],$$

assume that $\mathbf{f}_t(\cdot)$, $G_t(\cdot)$ *are such that for the Brownian motion,* $\mathbf{w}_t$, *the probability density of the state* $\mathbf{x}_t$ *lies in the family* $\mathcal{S}$ *for* Θ *bounded, with* $g(\theta) \geq \vartheta I$ *for some* $\vartheta > 0$. *We also assume the conditions in* `Fact 1` *and in* `Theorem 2` *with* $\mathbf{c}(\mathbf{x})$ *replacing* $u(\mathbf{x})$. *Then, there exist* k_1 *and* k_2 *such that*

$$\tilde{E}[\|\theta_t - \hat{\theta}_t\|] \leq k_1 h^2 + \frac{k_2}{N^{1/2}}, \quad t \in [0, t_f] \tag{20}$$

where $\hat{\theta}_t$ *is the estimate of* θ_t, *and* N *and* h *are the number of particles and the time step, respectively.*

Proof: Let θ_0 be the initial condition for θ. At $t = 0$, N independent initial conditions are generated based on the density $p(\mathbf{x}, \theta_0)$, and the approximation method (18) is applied. From (19) we know that:

$$\tilde{E}\|E_{\theta_t}\mathbf{c}(\mathbf{x}_t) - \frac{1}{N}\sum_{i=1}^{N}\mathbf{c}\left(\hat{\mathbf{x}}_t^i\right)\| \leq Kh^2 + \frac{k'}{N^{1/2}}.$$

On the other hand, from (16), we know that $\hat{\theta}$ is a solution to the system of equations

$$\frac{1}{N}\sum_{i=1}^{N} c_j(\hat{\mathbf{x}}_t^i) = E_{\hat{\theta}_t}(c_j(\mathbf{x}_t)), \quad \text{for} \quad j = 1, \cdots, p.$$

From `Fact 1`, the solution is exact if we replace $\frac{1}{N}\sum_{i=1}^{N} c_j(\hat{\mathbf{x}}_t^i)$ by $E_{\theta_t}(c_j(\mathbf{x}_t))$. Subtracting the term $E_{\theta_t}(c_j(\mathbf{x}))$ from both sides of the above equation and using the vector form for it, we get

$$\frac{1}{N}\sum_{i=1}^{N}\mathbf{c}(\hat{\mathbf{x}}_t^i) - E_{\theta_t}(\mathbf{c}(\mathbf{x}_t)) = E_{\hat{\theta}_t}(\mathbf{c}(\mathbf{x}_t)) - E_{\theta_t}(\mathbf{c}(\mathbf{x}_t)).$$

On the other hand, we know that $E_\theta(\mathbf{c}(\mathbf{x}))$ is a differentiable and one to one function of θ (see `Fact 1`). The derivative of this function, $g(\theta)$, is positive definite and by assumption $g(\theta) \geq \vartheta I$. Furthermore since Θ is compact, $\exists \alpha > 0$ such that

$$\begin{aligned} \|\theta_t - \hat{\theta}_t\| &\leq \alpha\|E_{\theta_t}(\mathbf{c}(\mathbf{x}_t)) - E_{\hat{\theta}_t}(\mathbf{c}(\mathbf{x}_t))\| \\ &= \alpha\|E_{\theta_t}(\mathbf{c}(\mathbf{x}_t)) - \tfrac{1}{N}\sum_{i=1}^{N}\mathbf{c}(\hat{\mathbf{x}}_t^i)\|. \end{aligned}$$

Taking the expectation on both sides of the inequality we have

$$\begin{aligned} \tilde{E}\|\theta_t - \hat{\theta}_t\| &\leq \alpha\ \tilde{E}\|\tfrac{1}{N}\sum_{i=1}^{N}\mathbf{c}(\hat{\mathbf{x}}_t^i) - E_{\theta_t}(\mathbf{c}(\mathbf{x}_t))\| \\ &\leq \alpha\left(Kh^2 + \tfrac{k'}{N^{1/2}}\right) \\ &= k_1 h^2 + \tfrac{k_2}{N^{1/2}} \end{aligned}$$

$\diamond$

Now we are ready to present the main result of this section.

Theorem 3 *For System (3) assume that* $\mathbf{f}_t(\cdot)$, $G_t(\cdot)$, *and* $\mathbf{h}(\cdot)$ *are such that for the Brownian motion* $\mathbf{w}_t$, *and the Gaussian noise* $\mathbf{v}_n$, *the conditional probability density of the state* $\mathbf{x}_t$, *conditioned on the observations* $\mathbf{y}^t$, *lies in the family* $\mathcal{S}$ *for* Θ *bounded and for* $t \in [0,T]$. *Also assume the conditions in* `Fact 1` *and in* `Theorem 2` *with* $\mathbf{c}(\mathbf{x})$ *replacing* $u(\mathbf{x})$. *Then, if* $g^{-1}(\theta_t) E_{\theta_t}(\mathcal{L}_t \mathbf{c}(\mathbf{x}))$ *is Lipschitz with the Lipschitz constant* L *and* $g(\theta) \geq \vartheta I$, *there exist* l_1 *and* l_2 *such that*

$$\tilde{E}\|\theta_n - \hat{\theta}_n\| \leq \sum_{i=0}^{n-1} \exp(Li\tau)\left(l_1 h^2 + \frac{l_2}{N^{1/2}}\right), \qquad n\tau \in [0,T],$$

where $\hat{\theta}_n$ *is the estimate of* θ_n, *and* N *and* h *are the number of particles and the time step, respectively. This inequality implies convergence of the parameter estimate,* $\hat{\theta_n}$, *to the true parameter,* θ_n, *as* $h \longrightarrow 0$ *and* $N \longrightarrow \infty$.

Proof: Let θ_t and $\hat{\theta}_t$ be the actual and the estimated values of the parameter of the density at time $t = n\tau$, respectively. At time $t^{'} = (n+1)\tau$ the error in the estimate of $\theta_{t'}$ is a combination of the error in the estimate in $\hat{\theta}_t$ and the error added in the time interval $[t, t^{'}]$.

If the conditional density stays in the exponential family of densities, θ_t has to satisfy the following differential equation (see Theorem 1):

$$\frac{d\theta}{ds} = g^{-1}(\theta) E_{\theta_s}(\mathcal{L}_s \mathbf{c}(\mathbf{x})), \qquad t = n\tau \leq s < t' = (n+1)\tau, \quad \theta(t) = \hat{\theta}_t \ .$$

Let $\tilde{\theta}_{t'}$ be the solution of the differential equation evaluated at $s = t'$. Then

$$\|\theta_{t'} - \hat{\theta}_{t'}\| \leq \|\theta_{t'} - \tilde{\theta}_{t'}\| + \|\tilde{\theta}_{t'} - \hat{\theta}_{t'}\|.$$

By the assumptions of the theorem, $g^{-1}(\theta) E_{\theta_s}(\mathcal{L}_s \mathbf{c}(\mathbf{x}))$ is Lipschitz with Lipschitz constant L, hence by continuity of the solution of the ordinary differential equation with respect to the initial condition [25], we know that

$$\left\|\theta_{t'} - \tilde{\theta}_{t'}\right\| \leq \left\|\theta_t - \hat{\theta}_t\right\| e^{L(t'-t)},$$

therefore,

$$\tilde{E}\left\|\theta_{t'} - \tilde{\theta}_{t'}\right\| \leq \tilde{E}\left\|\theta_t - \hat{\theta}_t\right\| e^{L(t'-t)}.$$

Also from the `Lemma 1`, $\exists k_1(t^{'})$ and $k_2(t^{'})$ such that

$$\tilde{E}[\|\tilde{\theta}_{t'} - \hat{\theta}_{t'}\|] \leq k_1(t^{'})h^2 + \frac{k_2(t^{'})}{N^{1/2}},$$

therefore,

$$\tilde{E}\|\theta_{t'} - \hat{\theta}_{t'}\| \leq \tilde{E}\left\|\theta_t - \hat{\theta}_t\right\| e^{L(t'-t)} + k_1(t^{'})h^2 + \frac{k_2(t^{'})}{N^{1/2}}.$$

The observation noise $\mathbf{v}_n$ and the function $\mathbf{h}(\cdot)$ are such that Bayes' Rule does not introduce any further error in the estimate of $\hat{\theta}_{t'}$. More precisely, $\Psi_n(\mathbf{x})$ is assumed to be a member of $\mathcal{S}$. This

implies that after applying Bayes' Rule to $p(\mathbf{x}, \theta_{t'})$ and $p(\mathbf{x}, \hat{\theta}_{t'})$ parameters $\theta_{t'}$ and $\hat{\theta}_{t'}$ are shifted with the same vector and therefore, $\|\theta_{t^{+'}} - \hat{\theta}_{t^{+'}}\| = \|\theta_{t'} - \hat{\theta}_{t'}\|$ (c.f. Theorem 1, formula for $\theta_{n\tau}$). Here $t^{+'}$ represents the time right after Bayes' correction. Therefore, starting from the initial condition θ_0 we get

$$\tilde{E}\|\theta_n - \hat{\theta}_n\| \leq \sum_{i=0}^{n-1} \exp(Li\tau) \left(l_1 h^2 + \frac{l_2}{N^{1/2}} \right), \qquad n\tau \in [0, T]$$

where

$$l_i = \max_n k_i(n\tau), \qquad n\tau \in [0, T], \qquad i = 1, 2.$$

$\diamond$

Here, we would like to make a few remarks:

- The result of Theorem 3 can be easily extended to convergence in the mean square sense.
- If $\|u(\mathbf{x})\| \leq k(1 + \|\mathbf{x}\|^{\kappa})$ for $k > 0$ and $\kappa > 0$, then

$$\lim_{\substack{N \longrightarrow \infty \\ h \longrightarrow 0}} \tilde{E} \left\| E_{\theta} u(\mathbf{x}) - E_{\theta^*} u(\mathbf{x}) \right\| = 0.$$

  This is the notion of convergence used in [7].

- In [4] the observation equation is considered to be time invariant. Here, the time-varying nature of $\mathbf{h}_n(\mathbf{x})$ does not complicate the algorithm.
- In the next section we will drop the assumption that the conditional density lies in the exponential family of densities. There, we project the density into the exponential family at each measurement epoch.

6 Projection Particle Filtering for Exponential Families of Densities

In this section, we drop the assumption that the conditional density of the state given the observation (6) lies in the exponential family of densities, $\mathcal{S}$. Also, we do not require that $\Psi_n(\mathbf{x})$ is a member of $\mathcal{S}$. Instead we make other assumptions. First we need the following definition:

Definition 3 *We say that a function $u(\cdot)$ belongs to the class $\mathcal{F}_{k\kappa}$, written as $u \in \mathcal{F}_{k\kappa}$, for fixed $k > 0$ and κ, such that for all $\mathbf{x} \in \mathcal{R}^n$, the following inequality holds:*

$$\|u(\mathbf{x})\| \leq k\left(1 + \|\mathbf{x}\|^{\kappa}\right).$$

The next two assumptions are to guarantee the existence of an exponential density close to the true conditional density.

A 3 *The density in (6) stays close to the given exponential family $\mathcal{S}$ in a weak sense:*

$$\forall t \in [0,T], \quad \forall u \in \mathcal{F}_{k\kappa} \quad \exists \theta_t^* \in \Theta^* \qquad s.t. \qquad \tilde{E}\|E_{p_t}(u(\mathbf{x})) - E_{\theta_t^*}(u(\mathbf{x}))\| \leq \epsilon \tag{21}$$

where $t-1 < n\tau \leq t$ and Θ^* is convex [3] *and compact.*

A 4 *For $\theta^*_{n^-}$ in (A3) and $\Psi_n(\mathbf{x})$, $\exists \Psi_n^*(\mathbf{x})$ such that $p(\mathbf{x},\theta) = \frac{p(\mathbf{x},\theta^*_{n^-})\Psi_n^*(\mathbf{x})}{\int p(\mathbf{x},\theta^*_{n^-})\Psi_n^*(\mathbf{x})d\mathbf{x}}$ is in the family $\mathcal{S}$ for some $\theta \in \Theta^*$ and we have:*

- $\forall \theta \in \Theta^*$ *and* $\forall u(\cdot) \in \mathcal{F}_{k\kappa}$, $\exists \epsilon > 0$ *such that*

$$\tilde{E}\|\frac{E_\theta \Psi_n(\mathbf{x})u(\mathbf{x})}{E_\theta \Psi_n(\mathbf{x})} - \frac{E_\theta \Psi_n^*(\mathbf{x})u(\mathbf{x})}{E_\theta \Psi_n^*(\mathbf{x})}\| \leq \epsilon.$$

- $\forall u(\cdot) \in \mathcal{F}_{k\kappa}$, $\exists \epsilon > 0$ *such that*

$$\tilde{E}\|\frac{E_{\theta^*_{n^-}} \Psi_n^*(\mathbf{x})u(\mathbf{x})}{E_{\theta^*_{n^-}} \Psi_n^*(\mathbf{x})} - \frac{E_{p_{n^-}} \Psi_n(\mathbf{x})u(\mathbf{x})}{E_{p_{n^-}} \Psi_n(\mathbf{x})}\| \leq \epsilon.$$

From Assumption (A4) it is clear that if $\Psi_n^*(\cdot)$ satisfies the requirements of the assumption then $c\Psi_n^*(\cdot)$ satisfies the same requirements, where c is a positive constant. It is further clear that $\Psi_n^*(\cdot) = \exp(\overline{\alpha}^T \mathbf{c}(\cdot))$ for some $\overline{\alpha} \in \mathcal{R}^p$. Using Assumption (A3), we can state the following fact.

Fact 2 $\forall \theta_1, \theta_2 \in \Theta^*$ *and* $\forall u \in \mathcal{F}_{k\kappa}$, $\exists K_1, K_2$ *positive such that*

$$\|E_{\theta_1} u(\mathbf{x}) - E_{\theta_2} u(\mathbf{x})\| \quad \leq \quad K_1 \|\theta_1 - \theta_2\| \tag{22}$$

$$\|\theta_1 - \theta_2\| \quad \leq \quad K_2 \|E_{\theta_1}\mathbf{c}(\mathbf{x}) - E_{\theta_2}\mathbf{c}(\mathbf{x})\| \ . \tag{23}$$

Proof: To prove (22), define $f_u(\theta) = E_\theta u(\mathbf{x})$ for $u(\cdot) \in \mathcal{F}_{k\kappa}$. Then

$$\frac{d}{d\theta_i} f_u(\theta) = E_\theta c_i(\mathbf{x})u(\mathbf{x}) - E_\theta c_i(\mathbf{x}) E_\theta u(\mathbf{x}).$$

Since $\|u(\mathbf{x})\| \leq k(1+\|\mathbf{x}\|^\kappa)$ and $\theta \in \Theta^*$, where Θ^* is compact, then there exists a constant A such that

$$\|\frac{df_u(\theta)}{d\theta}\| \leq A \qquad \forall u(\cdot) \in \mathcal{F}_{k\kappa} \quad \text{and} \quad \forall \theta \in \Theta^*.$$

[3] It is easy to see that the assumption of convexity is very natural. Assume $\theta_1, \theta_2 \in \Theta^*$ then $\int \exp(\theta_i^T \mathbf{c}(\mathbf{x}))d\mathbf{x} < \infty$ for $i = 1, 2$. Therefore, using Holder inequality we have

$$\begin{aligned}
\int \exp((\mu\theta_1^T + (1-\mu)\theta_2^T)\mathbf{c}(\mathbf{x}))d\mathbf{x} &= \int (\exp(\theta_1^T\mathbf{c}(\mathbf{x})))^\mu (\exp(\theta_2^T\mathbf{c}(\mathbf{x})))^{(1-\mu)} d\mathbf{x} \\
&\leq \left(\int \left(\left(\exp(\theta_1^T\mathbf{c}(\mathbf{x}))\right)^\mu\right)^{1/\mu} d\mathbf{x}\right)^\mu \left(\int \left(\left(\exp(\theta_2^T\mathbf{c}(\mathbf{x}))\right)^{1-\mu}\right)^{1/1-\mu} d\mathbf{x}\right)^{1-\mu} \\
&= \left(\int \exp(\theta_1^T\mathbf{c}(\mathbf{x}))d\mathbf{x}\right)^\mu \left(\int \exp(\theta_2^T\mathbf{c}(\mathbf{x}))d\mathbf{x}\right)^{1-\mu} \\
&< \infty
\end{aligned}$$

where $0 \leq \mu \leq 1$.

Since Θ^* is convex and compact, it is clear that $\exists K_1$ independent of $u(\cdot)$ such that $f_u(\mathbf{x})$ is Lipschitz over Θ^* with the Lipschitz constant K_1 [25].

Inequality (23) follows from the fact that Θ^* is compact and the Fisher information matrix $g(\theta) > \vartheta I$ over Θ^*.

$\diamond$

Denote the interior of the set Θ^* by Θ^*_{int}. For Θ^*_{int} we can state the following fact.

Fact 3 *The set*

$$\mathcal{A} = \left\{ \alpha : \int \exp(\alpha^T \mathbf{c}(\mathbf{x})) \exp(\theta^T \mathbf{c}(\mathbf{x})) < \infty, \forall \theta \in \Theta^*_{int} \;\; and \;\; \alpha \in \mathcal{R}^p \right\}$$

is closed.

Proof: Assume $\mathcal{A}$ is not closed. Therefore, there exists a converging sequence $\{\alpha_i\} \subset \mathcal{A}$ and $\overline{\alpha} \notin \mathcal{A}$, and $\exists \overline{\theta} \in \Theta^*_{\text{int}}$ such that

$$\int \exp(\overline{\alpha}^T \mathbf{c}(\mathbf{x})) \exp(\overline{\theta}^T \mathbf{c}(\mathbf{x})) d\mathbf{x} > M, \qquad \forall M \in \mathcal{R}.$$

Since Θ^*_{int} is an open set, $\exists \epsilon > 0$ such that $\mathcal{N}_\epsilon(\overline{\theta}) \in \Theta^*_{\text{int}}$. Also, since $\{\alpha_i\}$ is a converging sequence, $\exists k > 0$ such that $\alpha_k \in \mathcal{N}_\epsilon(\overline{\alpha})$. This implies that $\theta_1 \in \Theta^*_{\text{int}}$ where $\theta_1 = \overline{\theta} + \overline{\alpha} - \alpha_k$. Therefore,

$$\int \exp(\alpha_k^T \mathbf{c}(\mathbf{x})) \exp(\theta_1^T \mathbf{c}(\mathbf{x})) d\mathbf{x} < \infty.$$

On the other hand, we know $\exp(\alpha_k^T \mathbf{c}(\mathbf{x})) \exp(\theta_1^T \mathbf{c}(\mathbf{x})) = \exp(\overline{\alpha}^T \mathbf{c}(\mathbf{x})) \exp(\overline{\theta}^T \mathbf{c}(\mathbf{x}))$. This is a contradiction, therefore, $\mathcal{A}$ is closed.

$\diamond$

The following lemma is one of the building blocks of the results of this section.

Lemma 2 *For $\theta^*_{n^-}$ and $\Psi^*_n(\mathbf{x})$ defined in (A4) and $\forall u(\cdot) \in \mathcal{F}_{k\kappa}$ and $\forall \theta_1, \theta_2 \in \Theta^*$, $\exists$ positive numbers k_1, k_2, k_3, k_4 independent of $\theta^*_{n^-}$, $\Psi^*_n(\mathbf{x})$, θ_1, and θ_2 such that*

(a) $\quad k_1 \leq \|E_\theta \Psi^*_n(\mathbf{x})\| \leq k_2 \quad \forall \theta \in \Theta^*.$

(b) $\quad \|E_\theta \Psi^*_n(\mathbf{x}) u(\mathbf{x})\| \leq k_3 \quad \forall \theta \in \Theta^*.$

(c) $\quad \|E_{\theta_1} \Psi^*_n(\mathbf{x}) u(\mathbf{x}) - E_{\theta_2} \Psi^*_n(\mathbf{x}) u(\mathbf{x})\| \leq k_4 \|\theta_1 - \theta_2\|.$

Proof: Let $\mathcal{M}$ be a set defined as follows

$$\mathcal{M} = \{\mathbf{m} : \mathbf{m} = \theta_1 - \theta_2, \;\; \forall \theta_1, \theta_2 \in \Theta^*\}.$$

We claim that $\mathcal{M}$ is compact. To prove this claim, assume $\{\mathbf{m}_i\}$ to be a sequence in $\mathcal{M}$, i.e $\mathbf{m}_i \in \mathcal{M}$ such that $\lim_{i \longrightarrow \infty} \mathbf{m}_i = \overline{\mathbf{m}}$. We know that there exist sequences $\{\theta_{1,i}\}$ and $\{\theta_{2,i}\}$ such that $\mathbf{m}_i = \theta_{1,i} - \theta_{2,i}$ and $\theta_{1,i}, \theta_{2,i} \in \Theta^*$. Since Θ^* is compact there exist converging subsequences $\{\overline{\theta}_{1,i}\}$ and $\{\overline{\theta}_{2,i}\}$ in Θ^*. This implies that $\overline{\mathbf{m}} = \overline{\theta}_1 - \overline{\theta}_2$, where $\overline{\theta}_1$ and $\overline{\theta}_2$ are the limits of the subsequences $\{\overline{\theta}_{1,i}\}$ and $\{\overline{\theta}_{2,i}\}$. Since $\overline{\theta}_1$ and $\overline{\theta}_2 \in \Theta^*$, then $\overline{\mathbf{m}} \in \mathcal{M}$, therefore $\mathcal{M}$ is closed. Since Θ^* is bounded, $\mathcal{M}$ is bounded and therefore, it is compact.

We define set $\mathcal{A}_1$ as follows:

$$\mathcal{A}_1 = \left\{ \alpha : \int \exp(\alpha^T \mathbf{c}(\mathbf{x})) \exp(\theta^T \mathbf{c}(\mathbf{x})) < \infty, \forall \theta \in \Theta^* \text{ and } \alpha \in \mathcal{R}^p \right\}$$

It is clear that $\mathcal{A}_1 \subset \mathcal{A}$. From (A4), $\Psi_n^*(\mathbf{x}) = \exp(\overline{\alpha}^T \mathbf{c}(\mathbf{x}))$ and $\overline{\alpha} \in \mathcal{A} \bigcap \mathcal{M}$. Since $\mathcal{A} \bigcap \mathcal{M}$ and Θ^* are compact we have

$$\min_{\theta \in \Theta^*} \min_{\alpha \in \mathcal{A} \bigcap \mathcal{M}} \|E_\theta \Psi_n^*(\mathbf{x})\| \leq \|E_\theta \Psi_n^*(\mathbf{x})\| \leq \max_{\theta \in \Theta^*} \max_{\alpha \in \mathcal{A} \bigcap \mathcal{M}} \|E_\theta \Psi_n^*(\mathbf{x})\|.$$

In other words (a) is true with $k_1 = \min_{\theta \in \Theta^*} \min_{\alpha \in \mathcal{A} \bigcap \mathcal{M}} \|E_\theta \Psi_n^*(\mathbf{x})\|$ and $k_2 = \max_{\theta \in \Theta^*} \max_{\alpha \in \mathcal{A} \bigcap \mathcal{M}} \|E_\theta \Psi_n^*(\mathbf{x})\|$. Similarly, since $u(\cdot) \in \mathcal{F}_{k\kappa}$, (b) is true.

Using the above argument and the argument in Fact 2, we can show that $\|\frac{d}{d\theta} E_\theta \Psi_n^*(\mathbf{x}) u(\mathbf{x})\|$ is uniformly bounded and since Θ^* is convex and compact. Therefore, (c) is true [25].

$\diamond$

In the following we go through the proof of the theorem that we state later precisely. Assume $\hat{\theta}_n$ is calculated according to Algorithm (2) and assume $p(\mathbf{x}, \hat{\theta}_n)$ is such that $\forall u \in \mathcal{F}_{k\kappa}$

$$\tilde{E} \|E_{\hat{\theta}_n} u(\mathbf{x}) - E_{\theta_n^*} u(\mathbf{x})\| \quad \leq \quad \delta \ , \tag{24}$$

where θ_n^* (see (A3)) satisfies

$$\tilde{E} \|E_{p_n} u(\mathbf{x}) - E_{\theta_n^*} u(\mathbf{x})\| \quad \leq \quad \epsilon. \tag{25}$$

Using the density $p(\mathbf{x}, \hat{\theta}_n)$, new particles $\mathbf{x}_n^1, \cdots, \mathbf{x}_n^N$ are generated. The approximate solution for the SDE at time $(n+1)\tau$ maps these particles to $\hat{\mathbf{x}}_{n+1}^1, \cdots, \hat{\mathbf{x}}_{n+1}^N$. From these new particles $\hat{\theta}_{n+1}$ is calculated. From (24) and (25) we have

$$\tilde{E} \|E_{p_n} u(\mathbf{x}) - E_{\hat{\theta}_n} u(\mathbf{x})\| \quad \leq \quad \delta + \epsilon. \tag{26}$$

We define the function $\mathbf{r}(\mathbf{x})$ as follows:

$$\mathbf{r}(\mathbf{x}) = E'' \mathbf{c}(\hat{\mathbf{x}}_{n,\mathbf{x}}((n+1)))$$

where $\hat{\mathbf{x}}_{n,\mathbf{x}}((n+1)\tau)$ is the approximate solution of SDE (17) at time $(n+1)\tau$ with the given initial condition $\mathbf{x}$ at time $n\tau$ using the method in (18). Since according to our assumption $\mathbf{c} \in \mathcal{F}_{k\kappa}$, then by using lemma 9.1 in [24], we have

$$\|\mathbf{r}(\mathbf{x})\| \leq K_3(1 + \|\mathbf{x}\|^\mu)$$

where K_3 and μ only depend on the function $\mathbf{c}(\cdot)$ and the dimension of $\mathbf{x}$. Therefore, k and κ in Definition 3 can be chosen such that $\mathbf{r} \in \mathcal{F}_{k\kappa}$. If the argument of $\mathbf{r}(\cdot)$ is a random variable, then using (26) we have

$$\tilde{E} \|E_{p_n} \mathbf{r}(\mathbf{x}) - E_{\hat{\theta}_n} \mathbf{r}(\mathbf{x})\| \quad \leq \quad \delta + \epsilon. \tag{27}$$

More explicitly,

$$\tilde{E} \|E_{p_n} E''[\mathbf{c}(\hat{\mathbf{x}}_{n,\mathbf{x}}((n+1)\tau))] - E_{\hat{\theta}_n} E''[\mathbf{c}(\hat{\mathbf{x}}_{n,\mathbf{x}}((n+1)\tau)]\| \quad \leq \quad \delta + \epsilon. \tag{28}$$

From Theorem (2) we have

$$\tilde{E}\|E_{p_n}\mathbf{c}(\mathbf{x}_{n,\mathbf{x}}((n+1)\tau)) - E_{p_n}E''\mathbf{c}(\hat{\mathbf{x}}_{n,\mathbf{x}}((n+1)\tau))\| \quad \leq \quad K_4 h^2, \tag{29}$$

for some $K_4 > 0$. Using the Monte Carlo method to calculate the $E_{p_n}\mathbf{c}(\hat{\mathbf{x}}_{n,\mathbf{x}}((n+1)\tau))$ brings another error term that is due to the finite number of particles as the initial conditions for method (18). The expectation of this error is bounded, i.e. $\exists K_5 > 0$ s.t.

$$\tilde{E}\|E_{\hat{\theta}_n}E''\mathbf{c}(\hat{\mathbf{x}}_{n,\mathbf{x}}((n+1)\tau)) - \frac{1}{N}\sum_{i=1}^{N}\mathbf{c}(\hat{\mathbf{x}}_{n,\hat{\mathbf{x}}_i^i}((n+1)\tau))\| \quad \leq \quad \frac{K_5}{N^{\frac{1}{2}}}, \tag{30}$$

where $\hat{\mathbf{x}}^i$ are distributed according to $p(\mathbf{x}, \hat{\theta}_n)$. Combining (28), (29), and (30) we get

$$\tilde{E}\|E_{p_n}\mathbf{c}(\mathbf{x}_{n,\mathbf{x}}((n+1)\tau)) - \frac{1}{N}\sum_{i=1}^{N}\mathbf{c}(\hat{\mathbf{x}}_{n,\hat{\mathbf{x}}^i}((n+1)\tau))\| \leq \delta + \epsilon + K_4 h^2 + \frac{K_5}{N^{\frac{1}{2}}}. \tag{31}$$

Based on (A3), we know that $\exists \theta^*_{(n+1)^-}$ such that

$$\tilde{E}\|E_{p_{(n+1)^-}}\mathbf{c}(\mathbf{x}) - E_{\theta^*_{(n+1)^-}}\mathbf{c}(\mathbf{x})\| \leq \epsilon. \tag{32}$$

We know that, if $\mathbf{x}$ (initial condition at time $n\tau$) is distributed according to $p_n(\mathbf{x})$, then $E_{p_{(n+1)^-}}\mathbf{c}(\mathbf{x}) = E_{p_n}\mathbf{c}(\mathbf{x}_{n,\mathbf{x}}((n+1)\tau))$, therefore, from (31) and (32) we get

$$\tilde{E}\|E_{\theta^*_{(n+1)^-}}\mathbf{c}(\mathbf{x}) - \frac{1}{N}\sum_{i=1}^{N}\mathbf{c}(\hat{\mathbf{x}}_{n,\hat{\mathbf{x}}^i}((n+1)\tau))\| \leq \delta + 2\epsilon + K_4 h^2 + \frac{K_5}{N^{\frac{1}{2}}}. \tag{33}$$

Then $\hat{\theta}_{(n+1)^-}$ given by Algorithm (2) satisfies the following inequality

$$\tilde{E}\|E_{\theta^*_{(n+1)^-}}\mathbf{c}(\mathbf{x}) - E_{\hat{\theta}_{(n+1)^-}}\mathbf{c}(\mathbf{x})\| \leq \delta + 2\epsilon + K_4 h^2 + \frac{K_5}{N^{\frac{1}{2}}}. \tag{34}$$

From (A4) we know that $\exists \theta \in \Theta^*$ such that

$$\tilde{E}\left\|\frac{E_{\theta^*_{(n+1)^-}}\Psi^*_{n+1}(\mathbf{x})u(\mathbf{x})}{E_{\theta^*_{(n+1)^-}}\Psi^*_{n+1}(\mathbf{x})} - \frac{E_{p_{(n+1)^-}}\Psi_{n+1}(\mathbf{x})u(\mathbf{x})}{E_{p_{(n+1)^-}}\Psi_{n+1}(\mathbf{x})}\right\| = \tilde{E}\left\|E_\theta u(\mathbf{x}) - E_{p_{(n+1)}}u(\mathbf{x})\right\| \leq \epsilon.$$

Since θ satisfies the inequality in (A3), we can choose $\theta^*_{(n+1)}$ to be θ, i.e. $\theta^*_{(n+1)} = \theta$.

On the other hand we have

$$\left\|E_{\theta^*_{(n+1)}}u(\mathbf{x}) - E_{\hat{\theta}_{(n+1)}}u(\mathbf{x})\right\| \quad = \quad \left\|\frac{E_{\theta^*}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})u(\mathbf{x})}{E_{\theta^*}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})} - \frac{E_{\hat{\theta}}{}_{(n+1)^-}\Psi_{n+1}(\mathbf{x})u(\mathbf{x})}{E_{\hat{\theta}}{}_{(n+1)^-}\Psi_{n+1}(\mathbf{x})}\right\|$$

$$\leq \left\|\frac{E_{\theta^*}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})u(\mathbf{x})}{E_{\theta^*}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})} - \frac{E_{\theta^*}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})u(\mathbf{x})}{E_{\hat{\theta}}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})}\right\| + \left\|\frac{E_{\theta^*}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})u(\mathbf{x})}{E_{\hat{\theta}}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})} - \frac{E_{\hat{\theta}}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})u(\mathbf{x})}{E_{\hat{\theta}}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})}\right\| +$$

$$\left\|\frac{E_{\hat{\theta}}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})u(\mathbf{x})}{E_{\hat{\theta}}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})} - \frac{E_{\hat{\theta}}{}_{(n+1)^-}\Psi_{n+1}(\mathbf{x})u(\mathbf{x})}{E_{\hat{\theta}}{}_{(n+1)^-}\Psi_{n+1}(\mathbf{x})}\right\|$$

$$\leq \quad \frac{\|E_{\theta^*}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})u(\mathbf{x})\|}{\|E_{\theta^*}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})\|\|E_{\hat{\theta}}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})\|} \quad \left\|E_{\theta^*}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x}) - E_{\hat{\theta}}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})\right\| +$$

$$\frac{1}{\|E_{\hat{\theta}}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})\|}\left\|E_{\theta^*}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})u(\mathbf{x}) - E_{\hat{\theta}}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})u(\mathbf{x})\right\| \quad +$$

$$\left\|\frac{E_{\hat{\theta}}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})u(\mathbf{x})}{E_{\hat{\theta}}{}_{(n+1)^-}\Psi^*_{n+1}(\mathbf{x})} - \frac{E_{\hat{\theta}}{}_{(n+1)^-}\Psi_{n+1}(\mathbf{x})u(\mathbf{x})}{E_{\hat{\theta}}{}_{(n+1)^-}\Psi_{n+1}(\mathbf{x})}\right\|$$

Using Lemma 2 and (A4) we get

$$\tilde{E}\|E_{\theta^*_{(n+1)}}u(\mathbf{x}) - E_{\hat{\theta}_{(n+1)}}u(\mathbf{x})\| \leq \frac{k_3k_4 + k_1k_4}{k_1^2}\tilde{E}\|\theta^*_{(n+1)^-} - \hat{\theta}_{(n+1)^-}\| + \epsilon$$

Therefore, from (34) and Fact 2 we get

$$\tilde{E}\|\theta^*_{(n+1)^-} - \hat{\theta}_{(n+1)^-}\| \leq K_2\left(\delta + 2\epsilon + K_4h^2 + \frac{K_5}{N^{\frac{1}{2}}}\right).$$

This implies that, $\exists \iota_1, \iota_2, \iota_3, \iota_4 > 0$ such that

$$\tilde{E}\|E_{\theta^*_{(n+1)}}u(\mathbf{x}) - E_{\hat{\theta}_{(n+1)}}u(\mathbf{x})\| \leq \iota_1\delta + \iota_2\epsilon + \iota_3h^2 + \iota_4N^{-\frac{1}{2}}.$$

The next theorem summarizes our result in this section.

Theorem 4 *For the system (3) assume (A1), (A2), (A3), and (A4). We also assume the conditions in* `Fact 1` *and in* `Theorem 2` *with* $\mathbf{c}(\mathbf{x})$ *replacing* $u(\mathbf{x})$*, and we assume* $\mathbf{r} \in \mathcal{F}_{k\kappa}$*. Then in* `Algorithm 2` *with approximation (18), if* $\forall u(\cdot) \in \mathcal{F}_{k\kappa}$

$$\tilde{E}\|E_{\hat{\theta}_n}u(\mathbf{x}) - E_{\theta^*_n}u(\mathbf{x})\| \quad \leq \quad \delta$$

then

$$\tilde{E}\|E_{\theta^*_{(n+1)}}u(\mathbf{x}) - E_{\hat{\theta}_{(n+1)}}u(\mathbf{x})\| \leq \iota_1\delta + \iota_2\epsilon + \iota_3h^2 + \iota_4N^{-\frac{1}{2}},$$

for some $\iota_1, \iota_2, \iota_3, \iota_4 > 0$.

In Theorem (4) only one step of Algorithm (2) is considered; it is straightforward to then use Theorem(4) repeatedly for the time interval $[0, T]$, where $T = M\tau$. In that case, $\|E_{\hat{\theta}_0} u(\mathbf{x}) - E_{\theta_0^*} u(\mathbf{x})\| \leq \delta_0$, then $\exists \alpha_1, \alpha_2, \alpha_3$, and α_4 positive such that

$$\tilde{E}\|E_{\theta_n^*} u(\mathbf{x}((n)\tau)) - E_{\hat{\theta}_n} u(\mathbf{x}((n)\tau))\| \leq \alpha_1^n \delta_0 + \sum_{i=0}^{n-1} \alpha_1^i (\alpha_2 \epsilon + \alpha_3 h^2 + \alpha_4 N^{-1/2}),$$

for $0 \leq n \leq M$.

7 Applications of Projection Particle Filtering to Integrated INS/GPS

GPS provides world wide positioning with acceptable accuracy, if four or more satellites are in view of the receiver. Although the satellite constellation guarantees availability of four or more (sometimes even nine) satellites world wide, natural or man-made obstacles can block the satellite signals easily. To overcome this vulnerability, one might think of integrating dead reckoning or INS with GPS [26][27][28][29][30][15]. In this case, INS or the dead reckoning provides positioning that is calibrated by the GPS. In this section we consider the case of an integrated INS/GPS. We show that using nonlinear filtering for positioning is essential, and we compare the proposed Projection PaF with the regular PaF and EKF. One application of accurate INS/GPS navigation is in formation flight of unmanned aerial vehicle to reduce drag [31]

7.1 GPS Observation Equation

The GPS signal consists of a clock signal and a navigation message that are amplitude modulated. Each satellite sends the clock signal in two different bands, L_1 and L_2[32]. The GPS receiver receives the signal corrupted by noise and other sources of error. The raw measurements of the code and the carrier phase can be presented as follows [33]:

$$P^i(t_k) = \rho^i(t_k) + c[dT(t_k) - dt^i(t_k)] + T^i(t_k) + I^i(t_k) + E^i(t_k) + \epsilon^i(t_k) \ ,$$

$$\lambda\Phi^i(t_k) = \rho^i(t_k) + c[dT(t_k) - dt^i(t_k)] + T^i(t_k) - I^i(t_k) + E^i(t_k) + \lambda N^i + \eta^i(t_k) \ ,$$

where

- t_k : GPS time at epoch k (s)
- P : code observation(m)
- i : satellite number
- ρ : distance between the receiver and the satellite position(m)

  $$\rho^i(r_x, r_y, r_z) = \sqrt{(r_x - s_x^i)^2 + (r_y - s_y^i)^2 + (r_z - s_z^i)^2}$$

  where $[r_x, r_y, r_z]^T$ and $[s_x^i, s_y^i, s_z^i]^T$ are the coordinate of the receiver and the i^{th} satellite, respectively.
- c : speed of light (m/s)

- dT : receiver clock bias(s)
- dt : satellite clock bias(s)
- E : effect of ephemeris error(m)
- I : ionospheric delay(m)
- T : tropospheric delay(m)
- ϵ: code observation noise(m)
- λ: carrier wavelength(m)
- Φ : carrier phase observation(cycles)
- N : integer ambiguity(cycles)
- η : carrier observation noise (m)

Access to the above observation depends on the type of user and the quality of the receiver. State-of-the-art receivers can have access to code and carrier phase measurements of 12 satellites in 2 frequencies. With this kind of receiver a significant portion of the ionospheric delay can be corrected and removed [34][35][36]. If it is possible to mount a GPS receiver at a known location, or base, and broadcast the received GPS data to the mobile receiver, or rover, one can subtract the received signal at the base from the received signal at the rover; this is called single differencing. Over short distances, ionospheric and tropospheric errors are highly correlated, and can be eliminated by the single differencing method. The definition of short distance depends on sunspot activities [34]. When sunspot activities are low, distances less than 100 Km are considered short. It can be shown that single differencing reduces ephemeris error by a factor of d/r [36], where d and r are the distances from the rover to the base and to the satellite, respectively. Also by single differencing the error due to satellite clock bias is completely eliminated.

For the rest of this paper we assume that the integer ambiguity resolution problem is resolved (see [37] for example). Therefore, we consider the observation equation provided by i^{th} GPS satellite to have the following form:

$$y_i = \rho^i(r_x, r_y, r_z) - \rho^i(b_x, b_y, b_z) + c\delta + v^i \ , \tag{35}$$

where $[b_x, b_y, b_z]^T$ is the known base coordinate, δ is the combination of the receiver clock bias in the base and the rover, and v^i is the measurement noise for the i^{th} satellite signal.

Here we would like to mention that the nonlinearity in measurement is not only due to the function ρ. As we explain later integrated INS/GPS requires coordinate transformations between INS parameters and GPS parameters, which contributes to the nonlinearity of the measurement.

7.2 Coordinate Systems

Parameters of an integrated INS/GPS are expressed in different coordinate systems. In this subsection we intend to introduce these different coordinate systems and the transformation from one to another [38].

parameter	value	Description
a	6378137.0 m	semi major axis
b	6356752.3142 m	semi minor axis
ω_{ie}	7.292115×10^{-5}	angular velocity of the Earth
f	$f = \frac{a-b}{a}$	flatness of the ellipsoid
e	$\sqrt{f(1-f)}$	eccentricity of the ellipsoid

Table 1: Definition of the parameters for WGS84 reference frame

7.2.1 ECEF frame

The GPS measurements are given in an Earth Centered Earth Fixed (ECEF) frame. Two different coordinate systems are commonly used for describing the location of a point in ECEF frame.

The usual rectangular coordinate system $[p_x, p_y, p_z]^T$ for the point p, herein referred to as the ECEF coordinate system, has its x axis extended through the intersection of the prime meridian (0° longitude) and the equator (0° latitude). The z axis extends through the true north pole(i.e. parallel to the Earth's spin axis). The y axis completes the right-handed coordinate system.

The geodetic coordinate system is defined according to the familiar latitude, longitude, and hight coordinate system and is shown by $[p_\lambda, p_\phi, p_h]^T$. For this system of coordinates, the Earth's geoid is approximated by an ellipsoid. The defining parameters for the geoid according to the WGS84 reference frame are given in Table 1.

The transformation from the ECEF geodetic to the ECEF rectangular coordinate systems is given as follows

$$\begin{aligned} p_x &= (N + p_h)cos(p_\lambda)cos(p_\phi) \\ p_y &= (N + p_h)cos(p_\lambda)sin(p_\phi) \\ p_z &= (N(1-e^2) + p_h)sin(p_\lambda), \end{aligned} \tag{36}$$

where $N = \frac{a}{\sqrt{1-e^2 sin^2(p_\lambda)}}$. The inverse transformation can be derived from (36).

7.2.2 Local Geographical frame

It is convenient to express the navigation-frame velocity in the local coordinate system. This coordinate system is rectangular, and it has the x axis, y axis, and the z axis extended through the north, the east, and the down, respectively. With this definition for the local geographic coordinate system, the navigation-frame velocity, $[v_N, v_E, v_D]^T$, is related to the geodetic rate vector according to

$$\begin{pmatrix} v_N \\ v_E \\ v_D \end{pmatrix} = \begin{pmatrix} R_\lambda + p_h & 0 & 0 \\ 0 & (R_\phi + p_h)\cos(p_\lambda) & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \dot{p_\lambda} \\ \dot{p_\phi} \\ \dot{p_h} \end{pmatrix}, \tag{37}$$

where $R_\lambda = \frac{a(1-e^2)}{(1-e^2 \sin^2(p_\lambda))^{\frac{3}{2}}}$, and $R_\phi = \frac{a}{(1-e^2 \sin^2(p_\lambda))^{\frac{1}{2}}}$.

7.2.3 Platform and Body frames

Measurements by accelerometers and gyros are expressed in the platform frame. For simplicity we assume that the gyros and the accelerometers are aligned with the axis in the platform frame. Also, we assume that the body frame and the platform frame are aligned, and the center of the coordinate system is the same for both frames. The transformation from body frame to local geographical frame is calculated at every moment. It depends on the angular rate change measured by the gyros, the rotation of the Earth, and the rotation of the local frame with respect to an inertial frame, all expressed in the body frame. The transform matrix from the platform frame to the local frame is expressed as follows

$$\frac{d}{dt}R_{b2g} = R_{b2g}\Omega^b_{gb}, \tag{38}$$

where

$$\Omega^b_{gb} = \begin{pmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{pmatrix}, \tag{39}$$

and $\omega^b_{gb} = [p, q, r]^T$ is the inertial angular rate expressed in the body frame. ω^b_{gb} can be expressed as follows

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \tilde{p} \\ \tilde{q} \\ \tilde{r} \end{pmatrix} + \begin{pmatrix} b_p \\ b_q \\ b_r \end{pmatrix} - R_{g2b}\left(w_{ie} \begin{pmatrix} \cos(p_\lambda) \\ 0 \\ -\sin(p_\lambda) \end{pmatrix} + \begin{pmatrix} v_E/(R_\phi + p_h) \\ -v_N/(R_\lambda + p_h) \\ v_E \tan(p_\lambda)/(R_\phi + p_h) \end{pmatrix} \right), \tag{40}$$

where $[\tilde{p}, \tilde{q}, \tilde{r}]^T$ is the measured angular rate, and $[b_p, b_q, b_r]^T$ is the bias in the angular rate measurement.

If we assume that in the time interval $[t, t + \delta t]$, Ω^b_{gb} is a constant matrix then we have

$$R_{g2b}(t + \delta t) = \exp(-\Omega^b_{gb}(t)\delta t)R_{g2b}(t).$$

Since Ω^b_{gb} is a skew symmetric matrix, then $\exp(-\Omega^b_{gb}(t)\delta t)$ has a simple form:

$$\exp(-\Omega^b_{gb}\delta t) = [I + \frac{\sin(\|\omega^b_{gb}(t)\delta t\|)}{\|\omega^b_{gb}(t)\|}\Omega^b_{gb} + \frac{1 - \cos(\|\omega^b_{gb}(t)\delta t\|)}{\|\omega^b_{gb}(t)\|^2}(\Omega^b_{gb})^2].$$

The transformation from the body frame to the local frame, R_{b2g}, is simply the transpose of R_{g2b}, i.e. $R_{b2g} = R^T_{g2b}$.

7.3 GPS Clock Drift and INS Dynamics

The GPS clock drift and the INS equations constitute key dynamics in integrated INS/GPS.

The INS dynamic equation can be expressed as follows.

$$
\begin{aligned}
d\begin{pmatrix} p_\lambda \\ p_\phi \\ p_h \end{pmatrix} &= \begin{pmatrix} \frac{1}{R_\lambda + p_h} & 0 & 0 \\ 0 & \frac{1}{(R_\phi + p_h)\cos(p_\lambda)} & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_N \\ v_E \\ v_D \end{pmatrix} dt \\
d\begin{pmatrix} v_N \\ v_E \\ v_D \end{pmatrix} &= \left[\begin{pmatrix} -\frac{v_E^2}{R_\phi + p_h}\tan(p_\lambda) - 2\omega_{ie}\sin(p_\lambda)v_E + \frac{v_N v_D}{R_\lambda + p_h} \\ \frac{v_E v_N}{R_\lambda + p_h}\tan(\lambda) + \omega_{ie}\sin(p_\lambda)v_N + \frac{v_E v_D}{R_\phi + p_h} + 2\omega_{ie}\cos(p_\lambda)v_D \\ -\frac{v_N^2}{R_\lambda + p_h} - \frac{v_E^2}{R_\phi + p_h} - 2\omega_{ie}\cos(p_\lambda)v_E \end{pmatrix} + \right. \\
&\qquad \left. R_{b2g}\left(\begin{pmatrix} \tilde{a}_u \\ \tilde{a}_v \\ \tilde{a}_w \end{pmatrix} + \begin{pmatrix} b_u \\ b_v \\ b_w \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix} \right] dt + d\mathbf{w}_t^v,
\end{aligned} \tag{41}
$$

where $g = 9.780327 m/s^2$ is the gravitational acceleration, $[\tilde{a}_u, \tilde{a}_v, \tilde{a}_w]^T$ is the accelerometer measurement expressed in the body frame, $[b_u, b_v, b_w]^T$ is the accelerometer measurement bias again expressed in the body frame, and $\mathbf{w}^v$ is a vector valued Brownian motion process with zero mean and known covariance matrix. The measurement bias is assumed to have the following dynamics

$$
d\begin{pmatrix} b_u \\ b_v \\ b_w \end{pmatrix} = -a_b \begin{pmatrix} b_u \\ b_v \\ b_w \end{pmatrix} dt + d\mathbf{w}_t^b, \tag{42}
$$

where $\mathbf{w}_t^b$ is a vector valued Brownian motion with zero mean and known covariance matrix, and a_b is a small positive constant.

The receiver clock drift, δ_t, is represented by the integration of an exponentially correlated random process ϱ_t [15]

$$
\begin{aligned}
d\varrho_t &= -a_\varrho \varrho_t dt + dw_t^\varrho \\
d\delta_t &= \varrho_t dt,
\end{aligned} \tag{43}
$$

with $a_\varrho = 1/500$ and w_t^ϱ is a process of Brownian motion with zero mean and variance $\sigma_\varrho^2 = 10^{-24}$. This dynamic model is typical for a quartz TCXO with frequency drift rate of $10^{-9} s/s$ [15].

7.4 Simulation and Results

In this section we present simulation results for an integrated INS/GPS. Here we apply three different filtering methods, EKF, PaF, and projection Particle Filtering for a specified exponential density. We assumed that R_{g2b} is perfectly known, i.e. the estimation problem regarding the gyro measurements is solved. Therefore, with our assumptions in the previous sections, the dimension of the dynamical system in this simulation is eleven. The state of the dynamical system is

$$
\mathbf{x} = [p_\lambda, p_\phi, p_h, v_N, v_E, v_D, b_u, b_v, b_w, \varrho, \delta]^T.
$$

The dynamics of the system is the set of differential equation in (41), (42), and (43). Here, we assume that $a_b = 0.001$, and the covariance matrices for the Brownian motions in INS dynamic equations, Σ_b and Σ_v, to be diagonal. To be more specific, $\Sigma_b = 10^{-6} I$ and $\Sigma_v = 10^{-4} I$, where I is the identity matrix of the right size. The observation equation is given in (35), where y_i is one component of the observation vector. The dimension of the observation vector is the same as the

number of available satellites. In (35) the observation is given as a function of the position in ECEF rectangular coordinate system. Therefore, to be able to write down the observation equation as a function of the state of the system, one needs to use the transform in (36).

For this simulation we simply chose an 11 dimensional Gaussian density for the projection PaF. This choice of density makes random vector generation easy and computationally affordable. To be able to use projection PaF, we used maximum likelihood estimates of the parameters of the Guassian density before and after Bayes' correction.

In this simulation, we used two Novatel GPS receivers to collect the navigation data on April 2, 2000. From the collected data, we extracted position information for the satellites, the pseudo range, and the carrier phase measurement noise powers for the L1 frequency. Using the collected information we generated the pseudo range and the carrier phase data for one static and one moving receiver (base and rover, respectively). Here we assume that for the carrier phase measurement the integer ambiguity problem is already solved. The movement of the INS/GPS platform was simulation based and the measurement data measured by the accelerometers, the gyros, the GPS pseudo range, and the GPS carrier phase data were generated according to that movement.

In the simulation we assumed that the GPS receiver starts with 6 satellites. At time $t = 100$, the receiver loses lock with 3 satellites, and it gains one satellite at $t = 400$. We want to emphasize that for instantaneous stand alone positioning GPS requires at least 4 satellites. Figures 1-3 show the actual and estimated x, y, and z components of the position of the platform in ECEF rectangular system of coordinate. The estimates are given by three different methods, EKF, PaF, and projection PaF. The errors in these three methods are plotted in Figure 4. From this figure, it can easily be seen that EKF fails to give an acceptable estimate of the position when the number of satellites is below 4. It is worth mentioning that whenever the number of satellites is more than 4, EKF can provide a very good estimate of the position. Unlike EKF, PaF and projection PaF are successful in providing a reasonable estimate of position. Figure 5 is a version of Figure 4 with an emphasis on the comparison of the errors between PaF and projection PaF. It is seen that for the same number of particles, here 500, the error of the estimate given by the projection PaF is smaller than the error for PaF.

8 Concluding Remarks

Bringing together the distinct advantages of simulation-and-sampling based density computation and analytic propagation in parameterized families of densities, we introduce in this paper a new approach to the problem of nonlinear filtering - the projection particle filter. We develop the filter for exponential families of densities. We provide bounds for estimation errors accounting for dependence on population size of particles, and step size used in numerical integration of dynamics. We illustrate the effectiveness of this new filter in the technologically important problem of integrating INS and GPS. We aim to investigate in future work application of projection particle filtering to stochastic control.

References

[1] M.H.A. Davis and S.I. Marcus. An Introduction to Nonlinear Filtering. *In M. Hazewinkel and J.C. Willems edit. Stochastic Systems: The Mathematics of Filtering and Identification and Applications*, pages 53–75, 1981.

[2] P.M. Maybeck. *Stochastic Models, Estimation, and Control*, volume 2. Academic Press, 1982.

[3] D. Brigo, B. Hanzon, and F. LeGland. A Differential Geometric Approach to Nonlinear Filtering : the Projection Filter. *Proceedings of the 34th IEEE Conference on Decision and Control*, pages 4006–4011, December 1995.

[4] D. Brigo. *Filtering by Projection on the Manifold of Exponential Densities.* PhD thesis, Department of Economics and Econometrics, Vrije Universiteit, Amsterdam, October 1996.

[5] D. Brigo and F. LeGland. A Finite Dimensional Filter with Exponential Conditional Density. *Proceedings of the 36th IEEE Conference on Decision and Control*, pages 1643–1644, December 10–12 1997.

[6] R. Beard, J. Gunther, J. Lawton, and W. Stirling. Nonlinear Projection Filter Based on Galerkin Approximation. *Journal of Guidance Control, and Dynamics*, 22(2):258–266, March–April 1999.

[7] P. Del Moral. Non Linear Filtering: Interacting Particle Solution. *Markov Processes and Related Fields*, 2(4):555–580, 1996.

[8] F. LeGland, C. Musso, and N. Oudjane. An Analysis of Regularized Interacting Particle Methods in Nonlinear Filtering,. In J. Rojicek, M. Valeckova, M. Karry, and K. Warwick, editors, *Proceedings of the 3rd IEEE European Workshop on Computer-Intensive Methods in Control and Signal Processing*, volume 1, pages 167–174. Prague, September 7–9 1998.

[9] M. Isard and A. Blake. Contour Tracking by Stochastic Propagation of Conditional Density. *Proceedings of European Conference on Computer Vision*, pages 343–356, 1996.

[10] P. Fearnhead. *Sequential Monte Carlo Methods in Filter Theory.* Ph.D thesis, Merton College, University of Oxford, 1998.

[11] N.J. Gordon, D.J. Salmond, and A.F.M. Smith. Novel Approach to Nonlinear/NonGaussian Bayesian State Estimation. *IEE Proceedings-F (Radar and Signal Processing)*, 140(2):107–113, 1993.

[12] G. Kitagawa. Monte Carlo Filter and Smoother for Non-Gaussian Nonlinear State Models. *Journal of Computational and Graphical Statistics*, 5:1–25, 1996.

[13] D.B. Rubin. Using the SIR Algorithm to Simulate Posterior Distributions. In J.M. Bernardo, M.H. DeGroot, D.V. Lindley, and A.F.M. Smith, editor, *Bayesian Statistics*, pages 395–402. Oxford University Press, Oxford, 1988.

[14] A.F.M. Smith and A.E. Gelfand. Bayesian Statistics Without Tears: A Sampling-Resampling Perspective. *The American Statistician*, 46:84–88, 1992.

[15] H. Carvalho, P. Del Moral, A. Monin, and G. Salut. Optimal Nonlinear Filtering in GPS/INS Integration. *IEEE Trans. on Aerospace and Electronic Systems*, 33(3):835–850, July 1997.

[16] K. Sobczyk. *Stochastic Differential Equations With Application to Physics and Engineering.* Kluwer Academic Publishers, 1991.

[17] R.Z. Hasminskii. *Stochastic Stability of Differential Equations.* SIJTHOFF & NOORDHOFF Alphen aan den, The Netherlands, Rockville, Maryland, USA, 1980.

[18] B.R. Crain. Exponential Models, Maximum Likelihood Estimation, and the Haar Condition. *Journal of the American Statistical Association*, 71(355):737–740, 1976.

[19] S. Amari. *Differential-Geometrical Methods in Statistics*. Springer-Verlag, 1985.

[20] C.R. Rao. Information and Accuracy Attainable in the Estimation of Statistical Parameters. *Bulletin of the Calcutta Mathematical Society*, 37:81–91, 1945.

[21] A.E. Gelfand and A.F.M. Smith. Sampling-Based Approaches to Calculating Marginal Densities. *Journal of the American Statistical Association*, 85(410):398–409, 1990.

[22] E.L. Lehmann and G. Casella. *Theory of Point Estimation*. Springer-Verlag, New York, second edition, 1998.

[23] P. Billingsley. *Probability and Measure*. John Wiley, New York, third edition, 1995.

[24] G.N. Milstein. *Numerical Integration of Stochastic Differential Equations*. Kluwer Academic Publishers, New York, 1995.

[25] H.K. Khalil. *Nonlinear Systems*. Printice-Hall, upper Saddle River, second edition, 1996.

[26] E. Abbott and D. Powell. Land-Vehicle Navigation Using GPS. *Proceedings of the IEEE*, 87(1):145–162, Jan. 1999.

[27] N. El-Sheimy and K.P. Schwarz. Navigating Urban Areas by VISAT-A Mobile Mapping System Integrating GPS/INS/Digital Camera for GIS Applications. *Navigation*, 45(4):275–285, Winter 1999.

[28] G. Elkaim, M. O'Connor, T. Bell, and B. Parkinson. System Identification of a Farm Vehicle Using Carrier-Phase Differential GPS. *ION Conference*, pages 485–494, Sept. 1996.

[29] M. O'Connor, G. Elkaim, T. Bell, and B. Parkinson. Automatic Steering of Farm Vehicles Using GPS. *3rd International Conference on Precision Agriculture*, pages 767–778, June 1996.

[30] R. Zickel and N. Nehemia. GPS Aided Dead Reckoning Navigation. *Proc. ION Nat. Tech. Meeting*, pages 577–586, 1994.

[31] B. Iannotta. Vortex Draws Flight. *Aerospace America*, pages 26–30, 2002.

[32] J.J. Spilker. GPS Signal Structure and Performance Characteristics. *Navigation, Journal of the Institute of Navigation*, 25(2):121–146, 1978.

[33] X.X. Jin. Algorithm for Carrier-Adjusted DGPS Positioning and Some Numerical Results. *Journal of Geodesy*, pages 411–423, 1997.

[34] J.A. Klobuchar. Ionospheric Effects on GPS. In B.W. Parkinson and J.J. Spilker Jr., editors, *Global Positioning System: Theory and Applications Volume I*, volume 163, pages 485–515. American Institute of Aeronautics and Astronautics, Inc., Washington D.C., 1996.

[35] B. Hofmann-Wellenhof, H. Lichtenegger, and J. Collins. *Global Positioning System: Theory and Practice*. Springer-Verlag, New York, Second edition, 1993.

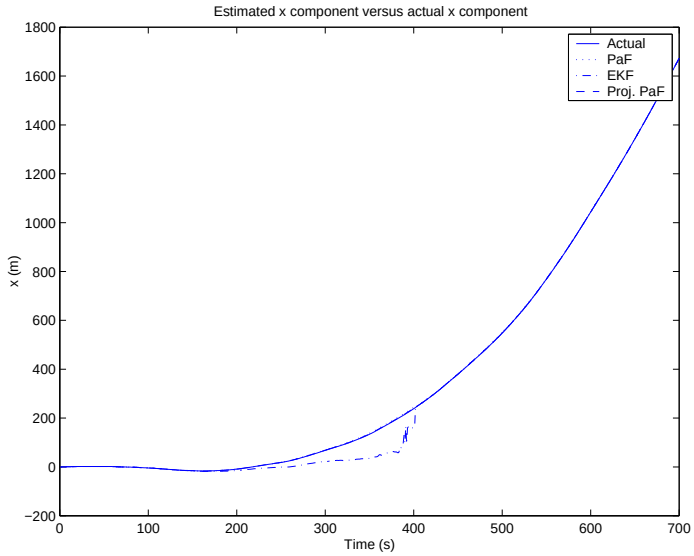

Figure 1: Comparison of the estimated and actual x component. The estimated x component is given with three different methods, EKF, PaF, and projection PaF. For $t < 100$, the number of satellites is 6, for $100 \leq t \leq 400$, the number of satellites is 3, and for $t > 400$, the number of satellites is 4.

[36] P.J.G. Teunissen and A. Kleusberg. GPS Observation Equations and Positioning Concepts. In A. Kleusberg and P.J.G. Teunissen, editors, *GPS for Geodesy*, pages 175–217. Springer-Verlag, New York, 1996.

[37] B. Azimi-Sadjadi and P.S. Krishnaprasad. Integer Ambiguity Resolution in GPS using Particle Filtering. *Proceedings of American Control Conference, ACC 2001*, pages 3761–3766, June 2001.

[38] J.A. Farrell and M. Barth. *The Global Positioning System and Inertial Navigation.* Mc Graw Hill, New York, 1998.

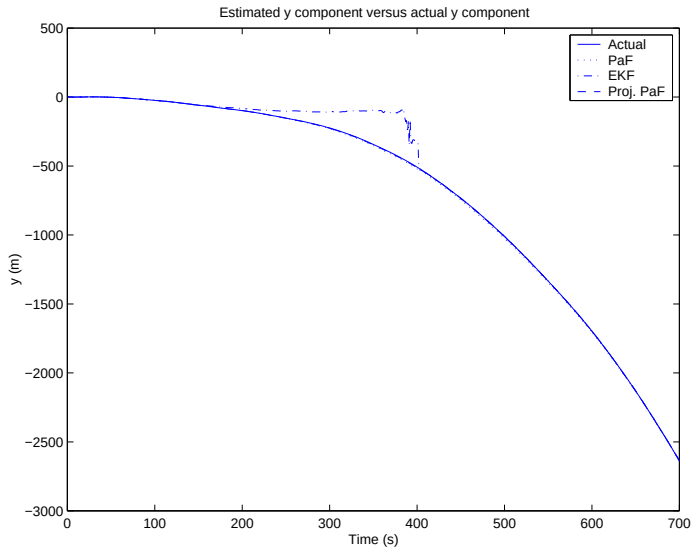

Figure 2: Comparison of the estimated and actual y component. The estimated y component is given with three different methods, EKF, PaF, and projection PaF. For $t < 100$, the number of satellites is 6, for $100 \leq t \leq 400$, the number of satellites is 3, and for $t > 400$, the number of satellites is 4.

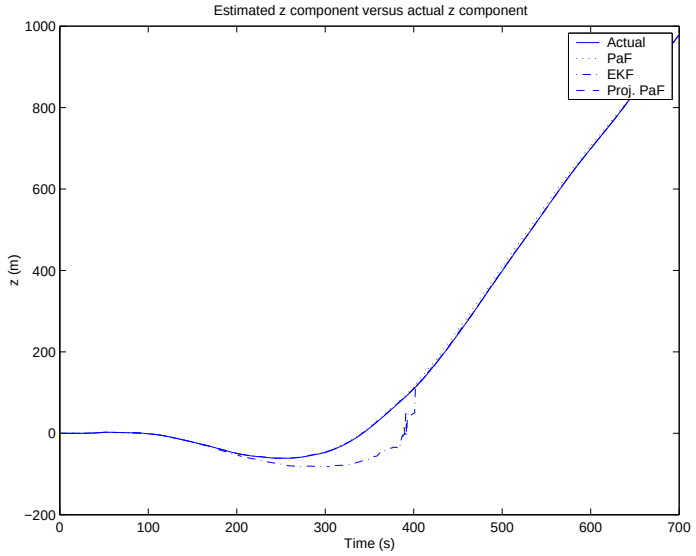

Figure 3: Comparison of the estimated and actual z component. The estimated z component is given with three different methods, EKF, PaF, and projection PaF.For $t < 100$, the number of satellites is 6, for $100 \leq t \leq 400$, the number of satellites is 3, and for $t > 400$, the number of satellites is 4.

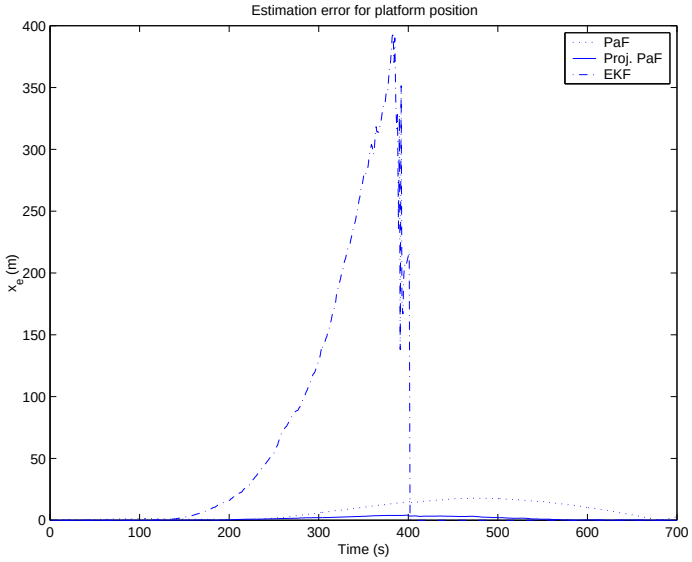

Figure 4: The estimation error for platform position for three different methods, EKF, PaF, and projection PaF.For $t < 100$, the number of satellites is 6, for $100 \leq t \leq 400$, the number of satellites is 3, and for $t > 400$, the number of satellites is 4.

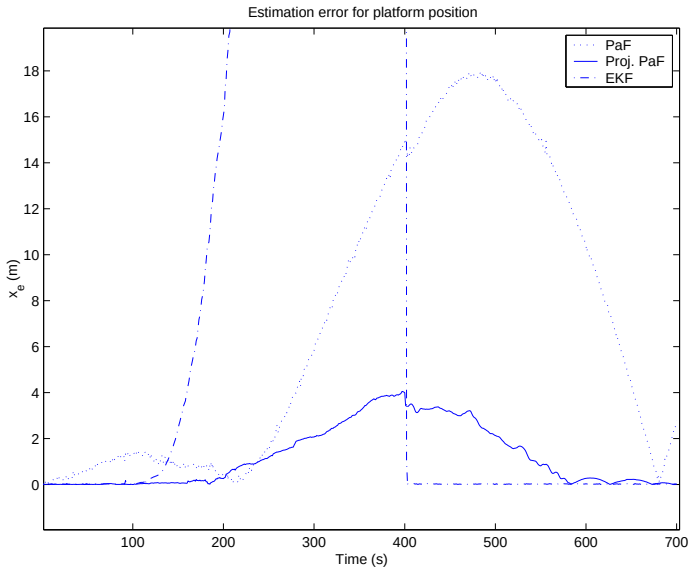

Figure 5: The estimation error for platform position for three different methods, EKF, PaF, and projection PaF.For $t < 100$, the number of satellites is 6, for $100 \leq t \leq 400$, the number of satellites is 3, and for $t > 400$, the number of satellites is 4.