Estimation and Pricing under Long-Memory Stochastic Volatility

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Abstract

We treat the problem of option pricing under a stochastic volatility model that exhibits long-range dependence. We model the price process as a Geometric Brownian Motion with volatility evolving as a fractional Ornstein-Uhlenbeck process. We assume that the model has long-memory, thus the memory parameter H in the volatility is greater than 0.5. Although the price process evolves in continuous time, the reality is that observations can only be collected in discrete time. Using historical stock price information we adapt an interacting particle stochastic filtering algorithm to estimate the stochastic volatility empirical distribution. In order to deal with the pricing problem we construct a multinomial recombining tree using sampled values of the volatility from the stochastic volatility empirical measure. Moreover, we describe how to estimate the parameters of our model, including the long-memory parameter of the fractional Brownian motion that drives the volatility process using an implied method. Finally, we compute option prices on the S&P 500 index and we compare our estimated prices with the market option prices.

Keywords: Option pricing; stochastic volatility; long memory; particle filtering; multinomial tree.

JEL classification code: C14, C5, C6, G13

1 Introduction

In the celebrated model in option pricing theory, due to Black and Scholes (1973, [3]) and Merton (1973, [21]), the dynamics of the underlying asset evolve according to a geometric

Brownian motion. However, option prices in practice are not conformable with the theoretical assumptions of this model. One of the main points of discussion is the method used for determining the volatility parameter. The empirical option pricing literature deals with this problem extensively suggesting several techniques, such as historical or implied volatility. Black-Scholes implied volatility does not capture the structure of reported option prices and as a result we observe the smile effect, well documented in the literature, that is the U-shaped pattern of implied volatility when plotted against different strike prices.

These inconsistencies indicate that the movements of stock prices are not entirely explained by the basic Black-Scholes (BS) model. One direction to understand them better is to withdraw the constant volatility assumption, and assume instead that volatility is described by a stochastic process. Models in this framework are called stochastic volatility models.

Let $\{S_t; t \in [0,T]\}$ denote the stock price process with dynamics given by

$$dS_t = \mu S_t + \sigma(Y_t)\, S_t\, dW_t, \tag{1}$$

where $\sigma(\cdot)$ is a deterministic function and Y is a stochastic process which one can call the *volatility process.* Under typical stochastic volatility models, Y is the solution of a stochastic differential equation (SDE) driven by a noise Z that could be independent of W, or correlated with it. For example, Y can be described by a

- Log-Normal process: $dY_t = c_1\, Y_t\, dt + c_2\, Y_t\, dZ_t$,
- Mean Reverting Ornstein-Uhlenbeck (OU) process: $dY_t = \alpha\, (m - Y_t)\, dt + \beta\, dZ_t$,
- Feller or Cox–Ingersoll–Ross (CIR) process: $dY_t = k\, (\nu - Y_t)\, dt + v\sqrt{Y_t}\, dZ_t$.

Depending on the volatility process Y and the function $\sigma(\cdot)$, we have a variety of models. For instance, the Hull-White model (1987, [18]) has $\sigma(y) = \sqrt{y}$ and Log-Normal Y, the Scott model (1987, [24]) uses $\sigma(y) = e^y$ and a mean-reverting OU process for Y, the model suggested by Stein and Stein (1991, [26]) sets $\sigma(y) = |y|$ and Y again a Mean-Reverting OU process, or the Heston model (1993, [17]) assumes that $\sigma(y) = \sqrt{y}$ and Y_t is a CIR process, but with noise correlated with the Brownian motion that drives the asset prices. More details about these models can be found in Fouque et al., [14].

In this article, we consider a long-memory stochastic volatility model. That is, we describe Y by a long-memory process. The idea of long-memory stochastic volatility is not new in the literature. It has been empirically observed that the autocorrelation function of the squared high frequency returns is usually characterized by its slow decay towards zero. This decay is neither exponential, as in short-memory processes, nor implies a unit root, as in integrated processes; see for example Ding et al. (1993, [12]), Dacorogna et al. (1993, [9]) and Lobato and Savin (1998, [19]), among others. Consequently, it has been suggested that squared returns may be modeled as a long memory process, whose autocorrelations decay at a hyperbolic rate. In this direction, in discrete time, Breidt et al. (1998, [4]) and Harvey (1998, [16]) proposed independently the Long-Memory SV (LMSV) model, where the log-volatility is modeled as an ARFIMA(p; d; q) process.

Comte and Renault (1998, [7]) introduced a continuous-time mean reverting process in the Hull-White setting. In this way, they were able to explain the persistence of the stochastic feature of implied volatilities in the Black-Scholes setting, when the time to maturity increases. Under their fractional stochastic volatility model, the volatility process is described by a fractional Ornstein-Uhlenbeck process; that is the standard Ornstein-Uhlenbeck process where the Brownian motion is replaced by a fractional Brownian motion.

Following the approach of Comte and Renault, [7], we work in continuous time, and assume that the dynamics of the volatility are described by

$$dY_t = \alpha\, Y_t\, dt + \beta\, dB_t^H$$

where $\{B_t^H; t \geq 0\}$ is a standard fractional Brownian motion with $H > 1/2$. Under this model the volatility exhibits long-memory, meaning intuitively that the volatility today is correlated to past volatility values with a dependence that decays very slowly. In this way we introduce long-memory in our model, but not directly in the returns, as was suggested by Cheridito (2003, [5]), Rogers (1997, [23]) and Sottinen (2001, [25]). On the contrary, we assume that the returns of the stock price are independent, which is an assumption that is supported by empirical findings, and also allows us to remain in an arbitrage-free context in continuous time.

The stochastic volatility model (1), with μ being the mean rate of return, is the stock price under the objective (market) probability measure, which we denote by $\mathbb{P}$. This is the model that we use when we deal with historical prices, for statistical inference purposes for example. However, in order to deal with the pricing problem we are in need of an arbitrage-free pricing scheme. Thus, we will introduce an equivalent martingale measure denote by $\mathbb{Q}$, under which the stochastic volatility model is as in (1), but with μ replaced by r, the risk-free interest rate. The two measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent by Girsanov's theorem. In the sequel we switch between these two measures appropriately, depending on the context (estimation or pricing). We will also see that $\mathbb{Q}$ is not necessarily unique, i.e. the market is incomplete, which is consistent with the idea that a model with one risky asset and more than one source of noise cannot be complete.

Our goal in this article is to suggest an option pricing algorithm as well as a parameter estimation procedure for our long-memory stochastic volatility model. In preparation for the task of option pricing, we adjust a genetic-type particle filtering algorithm by Del Moral et al. (2001, [10]) adapted by Florescu and Viens (2008, [13]) in a stochastic volatility setting with no memory, in order to estimate the empirical distribution of the volatility. In the sequel, following the approach in [13], we use a multinomial recombining tree algorithm for option pricing. Using simulated data we show that this algorithm performs well and captures the underlying memory of the system.

Before applying our method to real market data, we also treat the statistical inference problem for the suggested model. We observe that the crucial issue is to estimate the long-memory parameter, H, properly. A parametric approach for estimating H is not an easy task, since the volatility process is not observed and the long-memory effect can only be seen in the squared log-returns. We test a variety of non-parametric techniques which turn out to be quite disappointing. Therefore, we suggest an implied method to compute H by calibrating our computed option prices for different values of H with the realized option prices.

We apply this procedure using market data from the S&P 500 index IBM stock and we compare our results with the Binomial model and the BS model using implied volatility. At the same time, we are able to investigate the existence of long-memory in our data. In general, it seems that in the S&P 500 there is evidence of long-memory, while there is lack of it for IBM stock. Moreover, we work with data during the 2008 financial crisis, which started on Wall Street and propagated to the larger financial system: we study whether market exhibits long-memory after the successive failiures of Bear Stearns Inc., Lehmann Brothers, and AIG.

The structure of the paper is as follows. In Section 2, we describe the proposed model and discuss its main properties. Section 3 is devoted to the description of a pricing algorithm for the proposed model based on a combination of a particle filtering technique with a multinomial recombining tree algorithm. In Section 4, we treat the estimation problem for the various parameters of the model, while in the last section we apply our methodology to pricing european call options written on the S&P 500.

2 Long-Memory Stochastic Volatility Model

We choose to work with the log-returns of the stock, i.e. equivalently, the logarithm $X_t = \log S_t$ of the price process. Under a given martingale measure, these returns are driven by the following equations

$$\begin{cases} dX_t = \left(r - \frac{\sigma^2(Y_t)}{2}\right) \, dt + \sigma(Y_t) \, dW_t, \\ dY_t = \alpha \, Y_t \, dt + \beta \, dB_t^H, \end{cases} \tag{2}$$

where r is the short-term risk-free rate of interest, $\{W_t; t \geq 0\}$ is a standard Wiener process and $\{B_t^H; t \geq 0\}$ is a standard fractional Brownian motion with $H > 1/2$, both under the martingale measure. We assume that W and B^H are *independent*, although this assumption could be relaxed, in order to account for leverage effects, which will be the topic of future investigations.

2.1 Fractional Brownian motion

Before discussing the properties of the fractional SDE (2), we discuss some of the properties of the fractional Brownian motion, which is the driving noise of this process.

Definition 1 *A fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1]$ is a centered Gaussian process $\{B_t^H; t \in \mathbb{R}_+\}$ whose distribution is defined by its covariance*

$$Cov(B_t^H, B_s^H) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \ t, s \in \mathbb{R}_+,$$

and the fact that its paths are continuous with probability 1.

The covariance of fBm immediately implies that it has H-self-similar increments; for every $c > 0$ the processes $\{B_{ct}^H; t \in \mathbb{R}_+\}$ and $\{c^H B_t^H; t \in \mathbb{R}_+\}$ have the same distribution. The mean square of the increments of fBm computes as

$$\mathbf{E}\left(|B_t - B_s|^2\right) = |t-s|^{2H}, \tag{3}$$

which directly indicates that the increments are stationary. For $H = \frac{1}{2}$, the process is standard Brownian motion (the Wiener process). However, contrary to standard Brownian motion, fBm is not a semimartingale nor a Markov process when $H \neq \frac{1}{2}$.

The fBm has one additional very important property for certain values of H: its long-range dependence (a.k.a. long-memory). Indeed, when $H \neq \frac{1}{2}$, the increments of fBm over disjoint intervals are not independent; their correlation function is

$$\rho_H(n) = \frac{1}{2}\Big((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\Big).$$

We observe that when $H < \frac{1}{2}$ then $\rho_H(n) < 0$ and the increments over disjoint intervals are negatively correlated. When $H > \frac{1}{2}$ then $\rho_H(n) > 0$ and the increments over disjoint intervals are positively correlated. More specifically in the case that $H > \frac{1}{2}$ the stationary sequence X_n exhibits *long-range dependence* (or *long memory*) in the sense that $\sum_{n=1}^{\infty} \rho_H(n) = \infty$, which follows immediately from the asymptotics $\rho_H(n) = H(2H-1)n^{2H-2} + o\left(n^{2H-2}\right)$. When $H < \frac{1}{2}$, then $\sum_{n=1}^{\infty} |\rho_H(n)| < \infty$ and one often says that the process has short memory, although it might be preferable to call it "medium" memory, since exponentially decaying correlations might better describe "short" memory.

More details on fBm can be found in Nualart's textbook [20].

2.2 Fractional Ornstein-Uhlenbeck Process

The fractional Ornstein-Uhlenbeck process is the fractional analogue of the well-known Ornstein-Uhlenbeck process; it is a continuous-time first-order autoregressive process $X = \{X_t; t \geq 0\}$ which is the solution of an one-dimensional homogeneous linear stochastic differential equation driven by a fBm $\{B_t^H; t \in \mathbb{R}_+\}$ with Hurst parameter $H \in [1/2, 1)$. Specifically, it is the unique Gaussian process satisfying the following linear stochastic integral equation

$$Y_t = \alpha \int_0^t Y_s \, ds + \beta \, B_t^H, \tag{4}$$

where α and β are constant drift and variance parameters, respectively. The solution to Equation (4) is stationary, almost surely continuous, and H-self-similar. The decay of the autocovariance function of $\{Y_t; t \in \mathbb{R}_+\}$ is similar to that of the increments of the fBm, and thus it exhibits long-range dependence. See Cheridito et al. [6] for more details.

3 Option Pricing under the long memory stochastic volatility model

Although the model we consider is in continuous-time, in practice what we have available are discrete-time observations of historical stock prices $S_{t_1}, \ S_{t_2}, \ \ldots, \ S_{t_K}$, where t_K is the time today. In particular, we cannot hope to obtain observations of S of arbitrarily high frequency, and to make matters worse, the volatility process Y is unobserved. In order to deal with the option pricing problem under these circumstances, we start by estimating the filtered stochastic volatility distribution given historical stock price observations; we do this by using the so-called stochastic volatility particle filter. Then, we use a multinomial recombining tree to compute the price of the option.

3.1 Empirical Distribution of the Volatility Process

In this section we deal with historical stock prices, thus we work under the objective probability measure $\mathbb{P}$, which is the measure that corresponds to all past observations. Let

$$p_i(dy) := \mathbb{P}[Y_{t_i} \in dy | X_{t_1}, \ldots, X_{t_i}].$$

This is a probability measure for each $i = 1, \ldots, K$, where K is the number of historical observations available. Since $X_{t_1}, \ldots, X_{t_i}$ are random variables, it is customary to consider that $p_i(dy)$ is random since it depends on these, and a common appellation for it consistent with this point of view is the *stochastic filter of Y given discrete values of X*. However, when $X_{t_1}, \ldots, X_{t_i}$ are observed, it is better to consider that p_i is non-random at time i, as it depends entirely on these i known values at that time.

We now explain how to construct n time-varying particles $\{Y_{t_i}^j : i = 1, \ldots, K,\ j = 1, \ldots, n\}$ together with their corresponding probabilities $\{p_i^j : i = 1, \ldots, K,\ j = 1, \ldots, n\}$, such that given the observations $X_{t_1}, \ldots, X_{t_i}$, the empirical distribution of the particles converges to the probability measure $p_i(dy)$ as n increases to infinity.

To this end, we apply a genetic-type algorithm with two steps: a mutation step and a selection step. We refer to the article [10] by Del Moral et al. for a mathematical analysis of the method, and to [2] and [13] for its implementation in option pricing and portfolio optimization problems with no memory. The following is a description of the algorithm adapted to our context:

Step 0 :

Assume that we have $K+1$ past price observations at times $t_0 < t_1 < \ldots < t_K$, where t_K is the time today (the time of the most recent observation). For convenience we assume that these times are equidistant: $t_i - t_{i-1} = \Delta t = h$. Pick a bounded integrable function ϕ from $\mathbb{R}^q$ into $[0, \infty)$ such that

$$\int \phi(x)dx = 1, \quad \text{and} \quad \int |x|\phi(x)dx < +\infty.$$

We choose to work with $\phi(x) = e^{-2|x|}$. For $n > 0$, define also the contraction $\phi_n(x) = \sqrt[3]{n}\phi(x\sqrt[3]{n})$. The power 1/3 is taken to be consistent with the convergence theorem of del Moral et al., but in practice we have not noticed any sensitivity to other moderate choices of powers. Start by dividing the first interval $[t_0, t_1]$ into M pieces as follows:

$$t_0 = s_0 < s_1 < \ldots < s_M = t_1.$$

In the sequel, M will be the number of Euler steps in each time interval $[t_{i-1}, t_i]$.

Iteration # 1 :

We initialize our iteration scheme for the particles representing X and Y:

- $X_{t_0} = x_0$, where x_0 is the observed value of X from the historical prices at time t_0.
- $Y_{t_0} = y_0$, where y_0 can be an arbitrary point or a value sampled from the stationary distribution of Y or an estimate of Y based on the historical data, such as the historical volatility or the implied volatility.

<u>Mutation Step:</u>

1. We simulate M values of fBm, denoted by

$$\left(B_{t_0}^{H,j}(s_0), B_{t_0}^{H,j}(s_1), \ldots, B_{t_0}^{H,j}(s_M)\right).$$

See comment in Remark (i) below on how to construct these values.

2. We simulate recursively the SDEs in (2) using an Euler scheme for $k = 0, \ldots, M-1$. For convenience denote $Y_{t_0}^j(s_k) = Y_{t_0,\,k}^j$ and $X_{t_0}^j(s_k) = X_{t_0,\,k}^j$; we set

$$\begin{aligned} Y_{t_0,\,k+1}^j &= Y_{t_0,\,k}^j + \alpha\, Y_{t_0,\,k}^j(s_{k+1} - s_k) + \beta\, \left(B_{t_0}^{H,j}(s_0) - B_{t_0}^{H,j}(s_0)\right) \\ X_{t_0,k+1}^j &= X_{t_0,\,k}^j + \left(\mu - \frac{\sigma^2\left(Y_{t_0,k}^j\right)}{2}\right)(s_{k+1} - s_k) + \sigma\left(Y_{t_0,k}^j\right) Z_k \sqrt{s_{k+1} - s_k} \end{aligned}$$

where Z_k are independent standard normal random variables.

3. We keep only the final values from the previous recursive step

$$\begin{aligned} Y_{t_1}^j &:= Y_{t_0,M}^j = Y_{t_0}^j(s_M) \\ X_{t_1}^j &:= X_{t_0,M}^j = X_{t_0}^j(s_M) \end{aligned}$$

We repeat Substeps 1 to 3 above, n times independently for each j, so in the end of the mutation step we have constructed the pairs $\{X_{t_1}^j, Y_{t_1}^j\}_{j=1,\ldots,n}$.

<u>Selection Step:</u>

Introduce the following discrete probability measure constructed from the pairs at the end of the mutation step

$$\Phi_1^n(dy) = \begin{cases} \frac{1}{C}\sum_{j=1}^n \phi_n\left(X_{t_1}^j - x_1\right)\delta_{\{Y_{t_1}^j\}}, & \text{if } C > 0 \\ \delta_{\{0\}}, & \text{otherwise} \end{cases}$$

Those simulated particles which are closer to the observed return value of $x_1 = \log S(t_1)$ will have a higher weight, because ϕ_n acts as an approximation of the dirac delta function. The constant C is chosen such that $\Phi_1^n(dy)$ is a probability measure: $C := \sum_{j=1}^n \phi_n \left(X_{t_1}^j - x_1 \right)$.

Iterations # 2 to K :

For each iteration $i = 2, 3, \ldots, K$ we apply the Mutation step above with initial values for the Euler scheme, the observation $x_{t_{i-1}}$, which is known at time t_{i-1} and the volatility value $Y_{t_{i-1}}^j$ sampled from the distribution Φ_{i-1}^n; each sample is done independently for each j, but the sampling distribution depends on the entire particle system at the previous step, which is why this algorithm is considered an interacting particle system. Thus, in the end of the mutation step we obtain n pairs $\{X_{t_i}^j, Y_{t_i}^j\}_{j=1,\ldots,n}$. Then, we apply the Selection step and we obtain

$$\Phi_i^n(dy) = \begin{cases} \frac{1}{C} \sum_{j=1}^n \phi_n \left(X_{t_i}^j - x_i \right) \delta_{\{Y_{t_i}^j\}}, & \text{if } C > 0 \\ \delta_{\{0\}}, & \text{otherwise} \end{cases}$$

where $C := \sum_{j=1}^n \phi_n \left(X_{t_i}^j - x_i \right)$.

Output :

At each step i, the discrete distribution Φ_i^n is the estimate of $p_i(dy)$, i.e. of the distribution of Y_t at t_i, given the observed stock prices $x_1, x_2, \ldots, x_i$.

<u>SOME REMARKS:</u>

(i) In order to produce the fBm observations needed in the Euler Mutation step we can choose any simulation method from the existing ones. We refer to [11] for a detailed exposition of all the techniques. Here, we choose to work with the circulant method, since it is faster than others (of order $n \log n$).

(ii) In order to simulate the SDE that describes X, it is well-known that an equidistant Euler discretization converges to the continuous-time SDE. For Y this is not trivial. However, based on the work by A. Neuenkirch (2006, [22]), we can apply an equidistant Euler scheme, in order simulate the SDE recursively and get convergence to the continuous-time fractional SDE. However, the convergence of the Euler scheme of the fractional SDE is slower than that of geometric Brownian motion and depends on the value of H; the higher the value of H the slower the convergence to the continuous-time model. Thus, unless H is very close to 1/2, it is better to work with a relatively large number of Euler steps to guarantee convergence, for example $M = 600$.

3.2 Multinomial Recombining Tree

From the previous section, using past information of the stock, we have constructed the discrete distribution Φ_K^n of the empirical volatility and this will serve as our estimate of the present volatility. We now abandon the notation t_K for today's time stamp, in favor of using time 0. Today's stock price is thus S_0, which is observed. We wish to price an option with maturity T; thus our time interval is $[0, T]$.

Our tree consists of N periods, which means that we divide $[0, T]$ into N equidistant subintervals: $\Delta t = \frac{T}{N}$. The tree is branching at times $i\Delta t$. The nodes of the tree correspond to the return values of the stock, i.e. $X_t = \log S_t$.

The procedure below is described for the i^{th} timestep. Let the return be denoted by x. Then successors together with their corresponding probability weights are computed following Florescu and Viens, [13], as follows:

Step 1: Sample a volatility value from the particle filter distribution, Φ_K^n; denoted by Y_i.

Step 2: Let $j = \left[\frac{x}{\sigma(Y_i)\sqrt{\Delta t}}\right] + 1$, where $[\cdot]$ denotes the integer part, and take the four successors of x to be

$$\begin{cases} x_1 = (j+1)\sigma(Y_i)\sqrt{\Delta t} + \left(r - \frac{\sigma^2(Y_i)}{s}\right)\Delta t \\ x_2 = j\sigma(Y_i)\sqrt{\Delta t} + \left(r - \frac{\sigma^2(Y_i)}{s}\right)\Delta t \\ x_3 = (j-1)\sigma(Y_i)\sqrt{\Delta t} + \left(r - \frac{\sigma^2(Y_i)}{s}\right)\Delta t \\ x_4 = (j-2)\sigma(Y_i)\sqrt{\Delta t} + \left(r - \frac{\sigma^2(Y_i)}{s}\right)\Delta t \end{cases}$$

In order to compute the corresponding weights of the successors, let

$$\delta = \begin{cases} x - j\sigma(Y_i)\sqrt{\Delta t}, & \text{if } |(x - j\sigma(Y_i)\sqrt{\Delta t})| < |x - (j-1)\sigma(Y_i)\sqrt{\Delta t}| \\ x - (j-1)\sigma(Y_i)\sqrt{\Delta t}, & \text{otherwise} \end{cases} \tag{5}$$

and let $q = \delta / \left(\sigma(Y_i)\sqrt{\Delta t}\right)$ be the standardized distance. Now depending on the value of q we have two different sets of probability weights, as follows:

- If $q \in [-\frac{1}{2}, 0]$ the probability weights are given by

$$\begin{array}{ll} p_1 = \frac{1}{2}(1 + q + q^2) - p, & p_2 = 3p - q^2 \\ p_3 = \frac{1}{2}(1 - q + q^2) - 3p, & p_4 = p. \end{array} \tag{6}$$

- If $q \in [0, \frac{1}{2}]$ then

$$\begin{array}{ll} p_1 = p, & p_2 = \frac{1}{2}(1 - q + q^2) - 3p \\ p_3 = 3p - q^2, & p_4 = \frac{1}{2}(1 + q + q^2) - p. \end{array} \tag{7}$$

REMARK 1: If we construct a one-step quadrinomial tree with the successors given above where p is the probability of the successor furthest away from x, then $p \in \left[\frac{1}{12}, \frac{1}{6}\right]$ and in both cases above, the relations (6) and (7) define an equivalent martingale measure. Since p is not uniquely determined, this implies that our martingale measure is not unique. We refer to [13], and to point (i) in Remark 3 below, on how to choose a value of p which is consistent with the market.

Step 3: To construct the multi-period model we start by sampling n values from the discrete volatility distribution Φ_i^n. Then, with an initial value x_0 we start by computing the four successors of x_0 and their corresponding weights, as in Steps 1 and 2 for the first sampled value Y_1. After this, for each of one of the four successors we compute their respective successors for the second sampled volatility value Y_2 and so on.

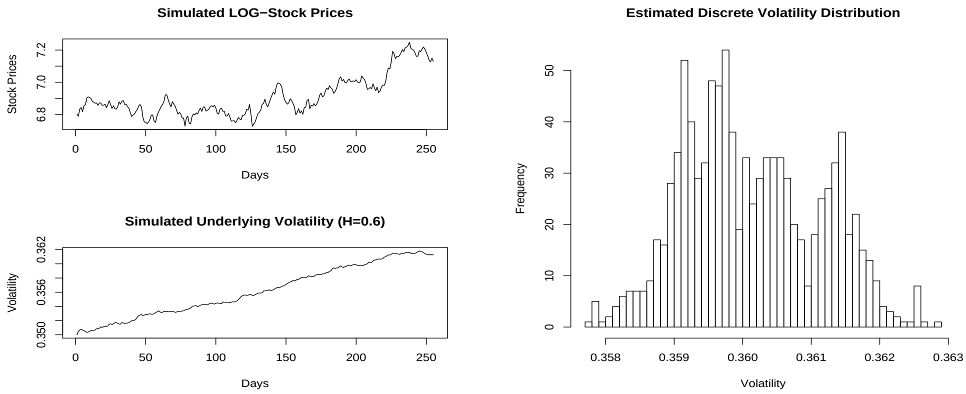
Step 4: Once we have constructed the tree we can use a standard pricing technique that is consistent with the no-arbitrage condition: we compute the value of the payoff function at the terminal nodes and we continue using a backward induction algorithm in order to compute the option price at time 0, as the discounted expecation of the final nodes using the martingale measure computed in Step 2. Since the tree is recombining by construction, the level of computation is polynomial, and we have found in practice that is of order no greater than N^α for some $\alpha < 3$.

Step 5: We can iterate this entire procedure by using n' repeated samples $\{Y^1, \ldots, Y^{n'}\}$ from Φ_K^n, constructing a different tree with each sample, and then using a Monte-Carlo approach to average all prices obtained from each generated tree using each sample. This last step reduces variability due to the single sampling used to construct the tree, and is consistent with the fact that the arbitrage-free pricing can be represented using expected values. It results in an algorithm which approximates the standard arbitrage-free price of the claim under the martingale measure chosen in step 2.

REMARK 2: Since we price the option using a tree, it works for both American and European options.

3.3 Simulated Data Example

We simulate 255 datapoints (corresponding to one year of data) with initial values $x_0 = 6.802$ and $y_0 = 0.35$. The chosen parameters for the model are: $H = 0.6$, $\alpha = 0.027$, $\beta = 0.076$, $\mu = -0.0014$ and $\sigma(y) = y$. The simulated model is shown in Figure 1 (i).



(i) (ii)

Figure 1: (i) Simulated model, (ii) Empirical Discrete Volatility Distribution of the simulated model.

Using the particle filter algorithm described in Section 3.1 we construct the discrete empirical distribution of Y, using $M = 600$ Euler steps and $n = 1000$ particles. The particle filter is shown in Figure 1 (ii).

We use the multinomial tree algorithm described above in order to compute the price of a European call option with initial stock price $S_0 = 1248.413$, risk-free interest rate $r = 0.02$ and maturity $T = 100$ days. In the tree we use $N = 50$ time steps and we choose $p = 0.16$. The option prices computed for a range of strike prices are shown on Table 1 together with the corresponding prices computed using a Binomial and a Black-Scholes model.

Remark 3:

(i) We compute option prices for various values of the free parameter p and we plot these prices with respect to p. As shown in Figure 2 (i) the computed option prices are robust with respect to the choice of p.

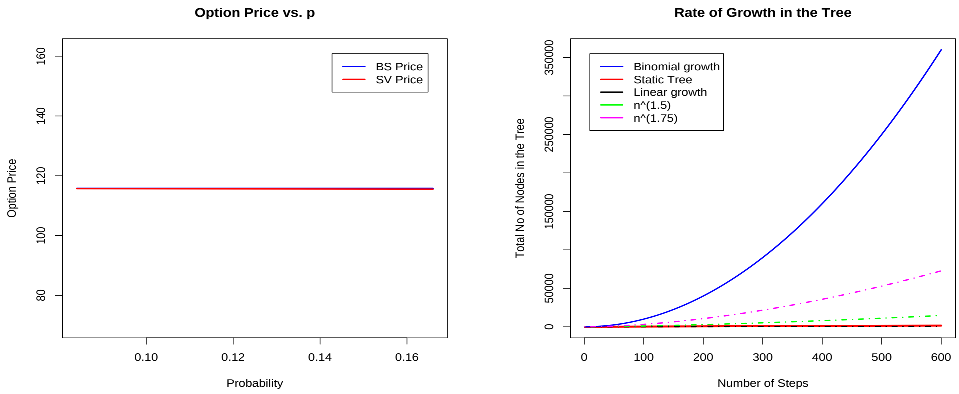
(ii) In Figure 2 (ii) we can visualize the level of recombination of the tree which is quite high. Actually, there is a linear growth of nodes for each additional step.

4 Inference for the Long-Memory Stochastic Volatility Model

Before applying our technique to real-time data, it is crucial to discuss how we can estimate the parameters in the model. These are: the mean rate of return μ, the parameters α and β of the fractional SDE and the memory-parameter H.

Strike Price	BS Price	Binomial Price	SV Price
700	554.1863303	554.1849152	554.1057391
750	505.0754056	505.0731263	505.0367213
800	456.5147933	456.5091821	456.5332716
850	408.8715749	408.8625159	408.9530273
900	362.6076577	362.604282	362.7570372
950	318.2430249	318.2361917	318.439265
1000	276.3039653	276.2815377	276.5298347
1050	237.2679239	237.2904777	237.4716566
1100	201.5163164	201.5084491	201.6655051
1150	169.3031044	169.3418959	169.3787487
1200	140.7421204	140.7419948	140.7180868
1250	115.8118004	115.7890584	115.6567938
1300	94.37315462	94.38510372	94.09626885
1350	76.19566378	76.21586201	75.81583452
1400	60.98603152	60.99854268	60.51299286
1450	48.41578386	48.40507526	47.86828359
1500	38.14508725	38.10619578	37.57190267
1550	29.84146784	29.84695049	29.24575048
1600	23.19315611	23.20979316	22.60585766
1650	17.91748433	17.90359371	17.34007038
1700	13.76515254	13.75997314	13.23094349
1750	10.52132021	10.52142996	10.03311333
1800	8.004455588	7.988044783	7.565121012
1850	6.063753329	6.060815433	5.674675782

Table 1: European call option prices computed for the simulated model for a range of strike prices for the simulated model stochastic volatility, the Black-Scholes and the Binomial model.



(i) (ii)

Figure 2: For the simulated model: (i) Robustness with respect to p , (ii) Growth with respect to N.

It would be optimal to estimate these parameters jointly. Unfortunately, such a problem is highly non-trivial and to our knowledge has not been solved even in the case in which we actually observe a fractional Ornstein-Uhlenbeck process. In the majority of the state-space models with long-memory in the literature, a rigorous way to jointly estimate this vector of parameters is unknown. The difficulty is mainly due to the fact we do not know H. If H were known, then using high-frequency data we could adjust standard techniques in the literature for non-fractional mean-reverting stochastic volatility models in order to estimate α and β. For example, one can use the variogram approach as discussed in Section 4.2 in Fouque et al., [14]. Therefore we begin with a discussion of how to find H.

4.1 Inference for the long-memory parameter H

If we want to estimate H, the first issue that arises is that the Hurst index appears in the volatility process which is unobserved. However, as discussed in several articles in the literature [9], [19], evidence of long-memory appears in the squared log-returns. More specifically, if $X_t = \log S_{t+1} - \log S_t$, then $Cov(X_{t+1}, X_t) = 0$ (which complies with our hypothesis of uncorrelated returns), but

$$Cov(X_{t+1}^2, X_t^2) \approx t^{2H-2}$$

Therefore, we could apply a non-parametric technique in order to estimate H. The best statistic to estimate H is the Rescaled-Range statistic (R/S statistic); it is non-parametric. We implement it in the simulated example discussed in the previous section. The true value of

Method	$\hat{\mathbf{H}}$ (*true* $H = 0.6$)	**st. error**($\hat{\mathbf{H}}$)
R/S Statistic	0.88	0.02
Periodogram	1.59	0.23
Boxed (or modified) Periodogram	1.43	0.05
Peng's (or Variance of residuals)	1.29	0.04
Higuchi's (or fractal dimension)	0.94	0.04

Table 2: Various non-parametric methods for estimating H from the squared log-returns and their corresponding standard errors.

H in our simulated model is 0.6; using the R/S statistic in the squared returns, we obtain is $\hat{H} = 0.88$ with corresponding standard error $se(\hat{H}) = 0.02$. We also applied the same procedure for various simulated models and the results were similarly unsatisfactory. The situation is the same when we estimate H using other non-parametric techniques such as the aggregated variance or the periodogram method. The estimated values of H using various non-parametric techniques are summarized in Table 2.

This leads to the following natural question: how important is it to estimate H properly? That is, if a rough estimate of H does not affect significantly the discrete distribution of Y and consequently the option price, then its estimation is of little consequence. To check the effect of H in the particle filter as well as the option price, we need perform a *sensitivity analysis* with respect to H. Before doing so, we study the effect of the number of particles chosen on the discrete volatility distribution estimated from the particle filtering algorithm.

4.1.1 Empirical Volatility Distribution & Number of Particles

In [13], $H = 0.5$ was used (that is the volatility is driven by a standard Brownian motion and thus the model is Markovian), the number of particles needed for the particle filtering algorithm was not large. Obviously, by using a large number of particles, for example $n = 1000$, we generate a finer mesh for the discrete volatility distribution, finer than the 20 or so particles needed to get satisfactory results as in [13], but the empirical range of the volatility filter or the resulting option price remains of the same magnitude whether one uses 20 or 1000 particles in this short memory case. However, this is not the case when H is greater than 0.5. One can easily see (for example in Figure 3) that if we keep the number of particles constant, then the very nature of the generated particle filter depends on the value of H, in the sense that the higher H is the smaller the spread of the empirical distribution is.

The plots in Figure 3 imply that we need to use a significantly larger number of particles

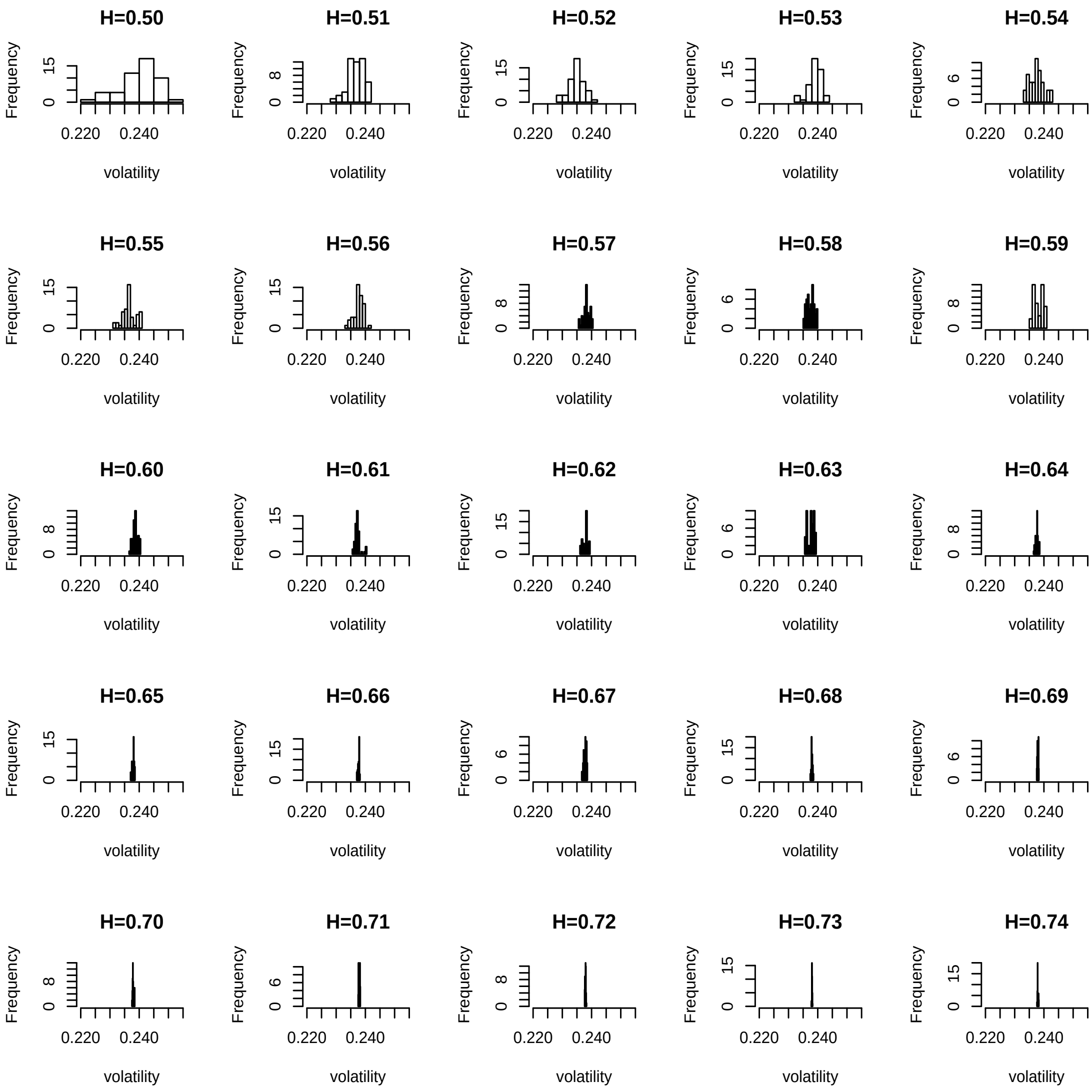
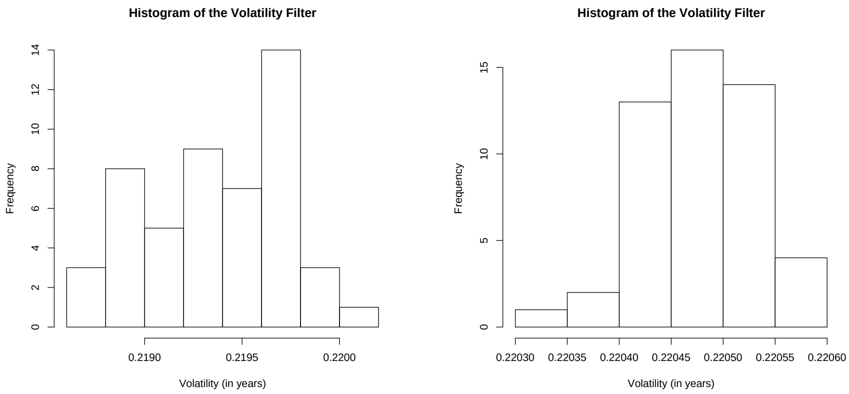


Figure 3: Histogram of the estimated discrete volatility distribution for the IBM stock (07/22/2009 - 08/24/2009) for different values of H keeping constant the number of particles and equal to $n = 50$.



$H = 0.6$ $\qquad$ $H = 0.72$

Figure 4: Histogram of the estimated discrete volatility distribution of the simulated model for different values of H.

in order to keep the spread of the empirical distribution roughly constant. By performing several simulations using different models, we observe that if we use 1000 particles or more, then the spread of the particle filter for all the values of H is robust (it remains roughly the same for all values of H and for all filters with over 1000 particles). The only disadvantage now is that the time required to generate such a filter increases significantly. Note however that for pricing problems, the filter can be computed off-line and updated as the last observations arrive thereby significantly reducing the computational time at the time of pricing.

4.1.2 Sensitivity Analysis with respect to H

We use the same simulated model as in Section 3.3, in which the true value of H was 0.6, and we plot the price of a European call option computed from values of H between 0.5 and 0.95 with step 0.05. For the remaining parameters, we use the true values, but we increase the number of particles as H is closer to 0.95, so that we keep the range of the generated histogram roughly constant. Then, for each particle filter corresponding to different values of H we use the multinomial tree algorithm in order to compute the value of the option and then we plot it against H.

In general, in the different particle filters generated for the various values of H for the simulated model, we observe that as H gets closer to 1, the histogram is shifted to the right, meaning that the volatility tends to obtain higher values. Furthermore, in certain cases there is no overlap between the range of the volatility values.

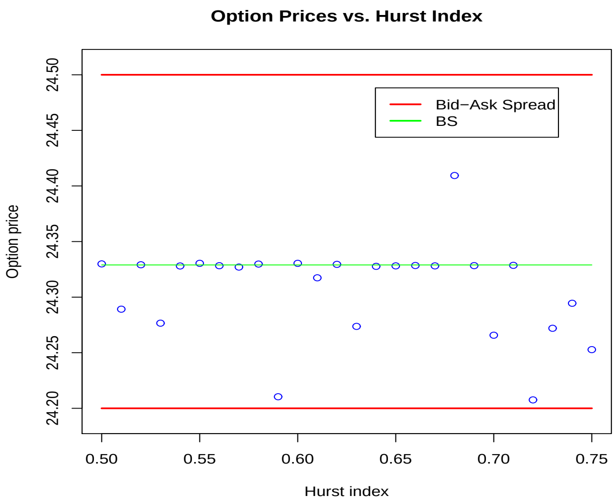


Figure 5: European call option prices with respect to different values of H for the simulated model and comparison with BS.

To continue our sensitivity analysis we plot the option prices obtained for against different values of H; this is shown in Figure 5.

In the graph we also compare the option price obtained using our long-memory stochastic volatility model with a standard Black-Scholes option price, in which the volatility used is the historical volatility. Moreover, we price an option which is at the money (i.e. Strike price $K = 1250$). We observe that the option price that is closer to the BS price is the one for $H = 0.6$ while the rest diverge significantly from it. In general, whether or not we are comparing our output with BS prices, we see that the effect of H on the option price is significant. We must conclude that its proper estimation is crucial.

4.2 Implied H approach

Section 4.1.2 shows it is very important to estimate H properly, since the value of the option depends on the true value of H. In addition, from what we discussed in Section 4.1, there is no known parametric or non-parametric approach to estimate H in a satisfactory way. However, a closer inspection of Figure 5 shows that our data can give us information on the underlying value of H and this can be done using an implied approach similar to that of the implied volatility technique.

The idea is to use our data and generate various volatility filters for a range of values of H and then use the realized option prices in order to obtain a value of H that most closely matches our data. The range of values of H should be between 0.5 and 0.95 and the step should be 0.05 or smaller. If we use a step size of 0.01, then the differences in the computed option prices will be quite small and several values of H will be close to the true option price (in the sense that they are inside the bid-ask spread). Therefore we need a standardized way to choose H. Towards this direction, we suggest using the Mean Square Error (MSE) between the computed option price and the center of the bid-ask spread. Then, the chosen value of implied H would be the one that corresponds to the smallest MSE. We formalize our technique in the steps:

1. For values of H varying from 0.5 to 0.95 (with increments of 0.01):
   (a) Estimate the filtered stochastic volatility distribution.
   (b) Compute the corresponding option prices for various strike prices.
   (c) Compute the mean squared error of the option price from the center of the market's bid-ask spread for that same option.
2. Choose the value of H that corresponds to the smallest MSE.

Once we have chosen the implied value of H, i.e. once our model has been calibrated to the market data, we can use it to generate a volatility particle filter which we then use in our multinomial recombining tree algorithm. One drawback is the computational time needed to generate all these different particle filters (especially for higher values of H). However, this is not a major issue since this calibration procedure does not have to be repeated every time we compute an option price, but only once, for at the beginning of each trading day.

5 Application to Market Data

We apply our methodology to pricing European call options on the S&P 500, investigating at the same time whether there is long-memory (i.e. $H > 0.5$) in the volatility or not.

Remark 4: The historical stock prices we use to estimate the empirical volatility ditribution are obtained from Yahoo Finance. The corresponding option prices are obtained from the "Delta Neutral" database, and the interest rates from the US Department of the Treasury.

For the first example we use 1 month's data, between December 29, 2008 and January 29, 2009 to estimate the volatility particle filter. The day "today" is January 30, 2009 and we wish to price a European call option on the S&P 500 with expiration on March 20, 2009. The underlying stock price is \$825.88 and the strike price we consider is $K = \$820$. The implied volatility (for an at-the-money option) is $\sigma_{implied} = 0.405$, while the historical volatility is computed as $\sigma_{historical} = 0.387$. The risk-free interest rate is $r = 0.24\%$ (we consider as interest rate the yield of a 3-month maturity T-bond).

The first task is to estimate the parameters of the model. Using the approach described in the previous section we find:

$$\hat{\alpha} = 0.0585, \quad \hat{\mu} = -0.0054.$$

The next task is to compute the implied value of H in order to price the option we are interested in. As discussed above, we generate various volatility filters. We choose to work with $M = 600$ Euler steps and $n = 1000$ particles. In order to compute the theoretical option prices that will be used for calibration we consider a tree of $N = 100$ steps and we use a Monte-Carlo approach by generating $m = 1000$ trees and averaging over the different option prices.

In Table 3 we can see a sample from the computed option prices using the long-memory stochastic volatility that we just described and the corresponding bid-ask spreads. It turns out that the value of H that gives us the smallest MSE is $H = 0.53$.

A first observation from our results in Table 3, where our prices are compared to the bid-ask spread, is that our model tends to underestimate options that are deep in the money and overestimate those that are out of the money, while for options at the money the computed option price is very close to the center of the bid-ask spread.

REMARK 5: One crucial issue to discuss is the time horizon over which we consider H to be constant. This is important because it determines the range of the historical data that we are going to use in order to generate the volatility filter as well as the maturity of the options we wish to price. We use market data from the S&P 500 from different time periods and of various time ranges. More specifically we consider data from January 2008 until December 2008 and from January 2009 until August 2009. The time horizons we study, and the maturity of the options we price, are 1 month, 2 months, 3 months, 6 months and 1 year. It turns out that it is quite safe to consider H to be constant for a 2-month period. Therefore, using 1 month historical data, we are able to price option that expire in one month from today.

From the first example we observe that there is evidence of long-memory in the S&P 500 data. However, it would be interesting to investigate whether this observation is something systematic or not. We choose to work with data from the 2008 Wall Street crisis for two different periods: (a) between March 7^{th} and May 9^{th} 2008 and (b) between September 12^{th} and December 19^{th} 2008. the first period corresponds to the collapse of the Bear Stearns

Strike Price	Bid	Ask	H=0.5	H=0.51	H=0.53	H=0.55	H=0.57	H=0.59
700	138.2	140.8	131.73	131.46	131.33	131.43	131.63	131.20
710	129.8	132.3	123.45	123.18	123.06	123.15	123.36	122.93
720	121.4	123.9	115.41	115.14	115.03	115.12	115.31	114.90
730	113.3	115.8	107.63	107.35	107.25	107.33	107.53	107.12
740	105.3	107.8	100.11	99.84	99.75	99.82	100.02	99.62
750	97.5	100	92.88	92.61	92.53	92.60	92.79	92.41
760	89.9	92.4	85.95	85.68	85.60	85.67	85.86	85.49
770	82.6	85	79.32	79.05	78.99	79.05	79.23	78.88
780	75.5	77.8	73.02	72.74	72.69	72.74	72.92	72.58
790	68.6	71	67.02	66.75	66.70	66.76	66.93	66.60
800	61.9	64.4	61.35	61.09	61.05	61.10	61.26	60.95
810	55.6	58.2	56.01	55.75	55.71	55.76	55.92	55.62
820	49.5	52.3	50.98	50.74	50.71	50.75	50.90	50.62
830	43.8	46.7	46.28	46.04	46.02	46.06	46.20	45.94
840	38.6	41.4	41.89	41.66	41.65	41.68	41.82	41.57
850	33.5	36.4	37.82	37.60	37.59	37.61	37.74	37.51
860	28.9	31.8	34.04	33.83	33.82	33.85	33.97	33.76
870	24.7	27.5	30.56	30.36	30.36	30.38	30.49	30.29
880	20.7	23.5	27.36	27.16	27.16	27.18	27.29	27.11
890	17.8	20.0	24.42	24.24	24.24	24.26	24.36	24.19
900	14.7	16.8	21.74	21.57	21.57	21.59	21.68	21.52

Table 3: Implied H: Option prices for different values of H.

Companies and the second one to the crash of September 2008 with the bankruptcy of Lehman Brothers, Inc. and the "bailout" of American International Group, Inc.

Bear Stearns Companies, Inc. In this example we want to study whether there is evidence of long-memory in the market after the crash of the Bear Stearns.

We price European call options on the S&P 500 index with maturity approximately one month and using historical data of one month's range. Using a rolling window of a week, we price a European call option on Friday, March 7^{th} 2008 with expiration in a month (based on historical data between 02/01/08 until 03/06/08) and we continue until Friday, May 9^{th} 2008 with an option that expires in a month (based on historical data between 03/01/08 until 04/08/08).

Figure 6 has the values of H that we computed using our implied technique for each week. We observe that that there is no structural long-memory in the stock and options market before the crash, since the value of $H_{implied}$ is 0.5. However, the week after the crash, it appears that a model with long-memory in the volatility is more suitable for the S&P 500, since $H_{implied}$ is higher than 0.5. The memory in the market is not persistent, since one month after the investment bank's collapse the implied value of H returns to 0.5 and remains there.

Lehman Brothers Holdings, Inc. & American International Group, Inc. (AIG) One might expect that the full-blown financial crisis of September 2008 had a stronger effected on the memory of the market than the failiure of a single investment bank. Indeed, Lehman Brothers declares bankruptcy on Monday, September 15, in one of the largest bankruptcy filings in the U.S. history. One day later, on September 16 the American International Group, Inc. (AIG) receives a $85 billion credit facility from the U.S. Federal Reserve Bank to meet increased collateral obligations consequent to the credit rating downgrade in the beginning of same month. This crisis created instability, resulting in the Dow Jones reaching a 6-year low on November 20.

As in the previous example, we price European call options on the S&P 500 index with maturity approximately one month and using one-month historical data. We compute prices beginning on Friday, September 9, 2008, based on historical data from 09/08/08 going back to 08/01/08. We continue pricing until Friday, December 19, 2008 for an option that expires in a month (based on historical data between 11/01/08 until 12/18/08). In Figure 7 we plot the values of H for each week.

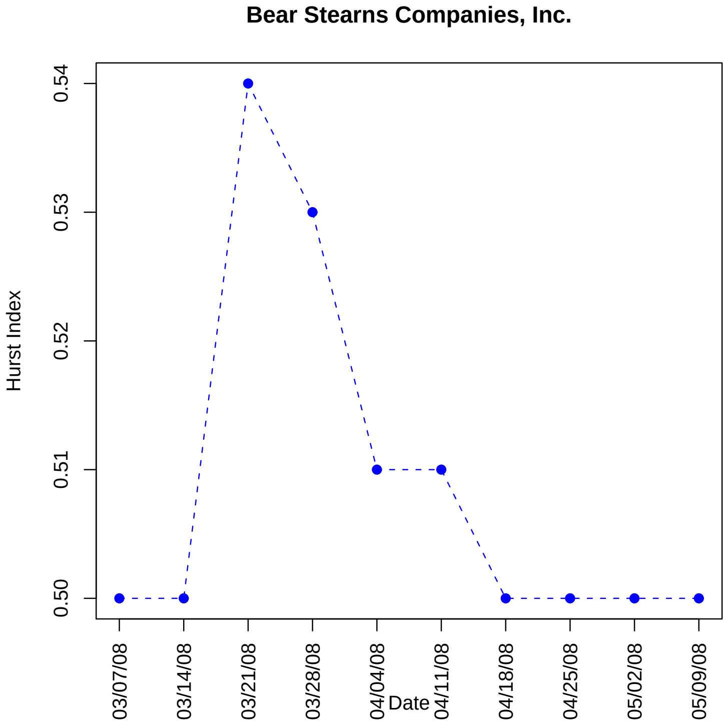


Figure 6: *Bear Stearns Companies, Inc.* was a global investment bank, securities trading and brokerage firm. On *March 16*, *2008*, Bear Stearns signed a merger agreement with JP Morgan Chase in a stock swap worth $2 a share or less than 10% of its market value just two days earlier.

Although the market is quite unstable during this period of time, there is no evidence of long-memory before the crash, since the value of $H_{implied}$ is 0.5. However, after the crash, the value of H is higher than 0.5 and remains high for an extended period (more than three months). This is evidence that the amplitude of the information hitting the market is related to its persistence over time; it also suggests that long memory may be an aspect of market instability.

6 Conclusion

In this article we studied a long-memory stochastic volatility model: we introduced a methodology for option pricing under such a model as well as a parameter estimation procedure. The determination of the market's current volatility structure is based on the construction of largely non-parametric particle filter for the empirical distribution of the volatility using historical prices of the underlying. This empirical distribution is the basis of a multinomial recombining tree, used for arbitrage-free option pricing. The option pricing algorithm has the advantage of employing a tree structure, which implies that it can be used for pricing any path-dependent option. In addition, since the tree is highly recombining, by construction, the computational time needed for pricing is minimized.

The construction of the particle filter is time-consuming; and this is compounded by the fact that we we need to compute the implied value of H, since several particle filters (for different values of H) need to be computed. However, this is not practical issue, since the filters can be computed offline based on historical data, for instance at the start of each trading day; H may need to be calibrated even less frequently, since one observes that it remains constant for a month at a time.

Since the entire option pricing scheme is found to be sensitive to the value of H, and standard long memory estimation techniques are found to be ineffective for stock price data, even in simulations, we suggest an implied technique to extract H by minimizing the Mean Square Error of the distance between the computed option price (using our algorithm) and the center of the bid-ask spread given by the options market. The fact that in the market data examples the MSE is minimized for values of H higher than 0.5 also implies that the prices computed under the long-memory volatility model were closer to the center of the bid-ask spread than under the "classical" memoryless volatility model. This is of some practical importance, since we chose our examples from periods of high market instablility.

The question of the construction of a hedging strategy to accompany our pricing scheme is an ongoing project and will be discussed in a future article.

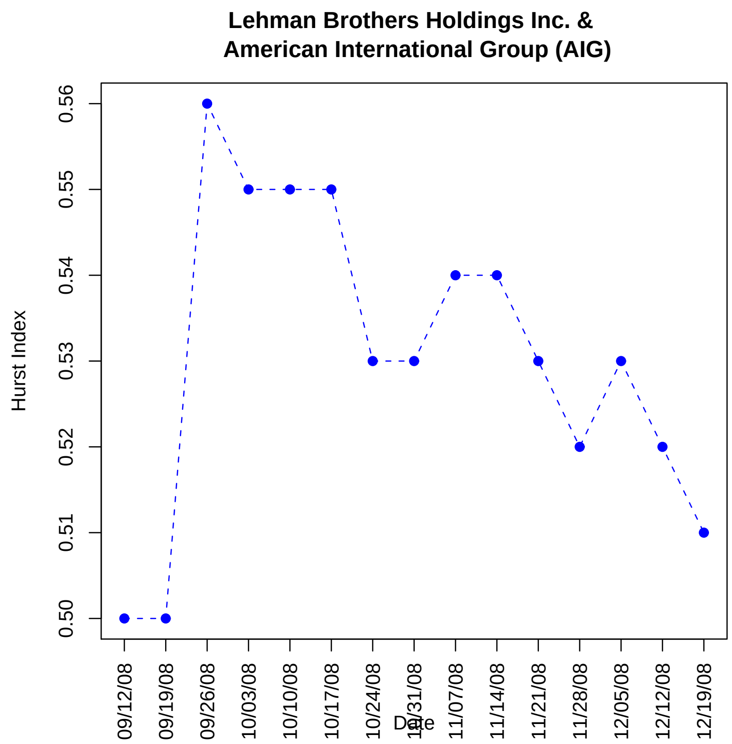


Figure 7: *Lehman Brothers Holdings Inc.* was a global financial services firm which, until declaring bankruptcy in 2008, participated in business in investment banking, equity and fixed-income sales, research and trading, investment management, private equity, and private banking. It was a primary dealer in the U.S. Treasury securities market. *American International Group, Inc. (AIG)* is an American insurance corporation. AIG was once the 18th-largest public company in the world. It was listed on the Dow Jones Industrial Average from April 8, 2004 to September 22, 2008.

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