Particle methods in stochastic engineering

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↪ Feynman-Kac formulae. Genealogical and interacting particle systems, Springer (2004), + Ref.
↪ DM, Doucet, Jasra. SMC Samplers. *JRSS B (2006).*
↪ Bercu, DM, Doucet. Fluctuations of Interacting Markov Chain Monte Carlo Models (2008) + *Ref.*.

http - references & Web links resources

- Master lecture notes on Stochastic engineering with scilab programs (in french)
- A pedagogical book on simulation and stochastic algorithms (in french)
- A series of selected research articles on Feynman-Kac models and particle algorithms : convergence, performance analysis, fluctuations, large deviations, propagations of chaos properties, exponential estimates,...
- Some web-links to Feynman-Kac and Interacting particle application model areas : particle filtering, robotics, image processing, audio signal, tracking, GPS, fluid mechanics, financial math, biology, chemistry, rare event, optics, hybrid systems,...

Outline

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Particle Interpretation models

- **Mathematical physics and molecular chemistry** ($\geq 1950's$) : Particle/microscopic interpretation models, particle absorption, macro-molecular chains, quantum and diffusion Monte Carlo.
- **Environmental studies and biology** ($\geq 1950's$): Population, gene evolutions, species genealogies, branching/birth and death models.
- **Evolutionary mathematics and engineering sciences** ($\geq 1970's$): Adaptive stochastic search method, evolutionary learning models, interacting stochastic grids approximations, genetic algorithms.
- **Applied Probability and Bayesian Statistics** ($\geq 1990's$): Approximating simulation technique (recursive acceptance-rejection model), Sequential Monte Carlo, `http-ref : interacting Monte Carlo Markov chains (Andrieu, Bercu, DM, Doucet, Jasra).`
- **Pure mathematics ($\geq 1960's$ for fluid models, $\geq 1990's$ for discrete time and interacting jump models)**: Stochastic linearization tech., mean field particle interpretations of nonlinear PDE and measure valued equations.

- **Central idea of particle/SMC in stochastic engineering :**

$$\left\{ \begin{array}{l} \textbf{Physical and Biological intuitions} \\ \textit{[learning, adaptation, optimization,...]} \end{array} \right\} \in \textbf{Engineering problems}$$

Sequential Monte Carlo	Sampling	Resampling
Particle Filters	Prediction	Updating
Genetic Algorithms	Mutation	Selection
Evolutionary Population	Exploration	Branching
Diffusion Monte Carlo	Free evolutions	Absorption
Quantum Monte Carlo	Walkers motions	Reconfiguration
Sampling Algorithms	Transition proposals	Acceptance-rejection

More botanical names : spawning, cloning, pruning, enrichment, go with the winner, replenish, and many others.

- **Pure mathematical point of view :**
  = Mean field particle interpretation of Feynman-Kac measures

Some application areas of Feynman-Kac formulae

- **Physics** :
  - Feynman-Kac-Schroedinger semigroups $\in$ nonlinear integro-differential equations ($\sim$ generalized Boltzmann models).
  - Spectral analysis of Schrödinger operators and large matrices with nonnegative entries.
  - Particle evolutions in disordered/absorbing media.
  - Multiplicative Dirichlet problems with boundary conditions.
  - Microscopic and macroscopic interacting particle interpretations.
- **Chemistry and Biology**:
  - Self-avoiding walks, macromolecular simulation, directed polymers.
  - Spatial branching and evolutionary population models.
  - Coalescent and Genealogical tree based evolutions.

Some application areas of Feynman-Kac formulae

- **Rare events analysis**:
  - Multisplitting and branching particle models (Restart type methods).
  - Importance sampling and twisted probability measures.
  - Genealogical tree based simulations (default tree sampling models).
- **Advanced Signal processing**:
  - Optimal filtering, prediction, smoothing.
  - Open loop optimal control, optimal regulation.
  - Interacting Kalman-Bucy filters.
  - Stochastic and adaptative grid approximation-models
- **Statistics/Probability**:
  - Restricted Markov chains (w.r.t terminal values, visiting regions, constraints simulation problems,...)
  - Analysis of Boltzmann-Gibbs type distributions (simulation, partition functions, localization models...).
  - Random search evolutionary algorithms, interacting Metropolis/simulated annealing algo, combinatorial counting.

Outline

The filtering problem $\subset$ Bayesian statistics

- X_t**:=Signal=Stochastic process**

Engineering/physics/biology/economics :

- Non cooperative targets (defense : missile, boat, plane,...).
- Physics (Fluids : twisters, cyclones, ocean models, pressure/temperature/diffusion coefficients,...).
- Finance (assets, portfolios, volatilities, default indexes,...).
- Signal (speech, codes, informations transmissions, waves,...).

Dynamics and sources of randomness :

- Physical evolution equations (example : $\sum_i u_i \overrightarrow{F}_i = \overrightarrow{A}$)
- Perturbations and random sources:
  - Model uncertainties $\oplus$ External perturbations.
  - **Unknown controls and related model parameters.**

$\rightsquigarrow$ **A Priori Law/Knowledge** (unknown quantities=random samples.)

The filtering model

- Y_t=**Partial and Noisy observations of the signal** X_t **:**

Engineering/physics/biology/economics :

- Engineering : Radar, Sonar, GPS, ...
- Physics (sensors : pressure/temperature/...).
- Finance (assets, portfolios,...).
- Statistics (real data: medecine, pharmacology, politics, economics,...).

Dynamics and sources of randomness :

- Partial observations : complex mixtures, partial coordinates.
- Perturbations et random sources :
  - Noisy sensor measures (thermal noise).
  - External/environmental perturbations.
  - Model uncertainties.

Objectives

Compute/Sample/Estimate inductively the flow of measures

$$t \in \mathbb{R}_+ \quad \text{or} \quad t = n \in \mathbb{N} \longrightarrow \eta_t = \mathrm{Law}(X_t \mid Y_0, \ldots, Y_t)$$

Note

- Filtering the trajectories : $X_t = (X'_0, \ldots, X'_t) \in E_t$

$$\Updownarrow \textbf{[State space enlargement]}$$

$$\eta_t = \mathrm{Law}((X'_0, \ldots, X'_t) \mid (Y_0, \ldots, Y_t)) = \mathrm{Law}(X_t \mid Y_0, \ldots, Y_t)$$

Equivalent terminologies :

- Data Assimilation (forecasting, fluids/ocean models).
- Hidden Markov Chains Models (HMM).
- A Posteriori Law=Law($X|Y$) (A Priori=$\mathrm{Law}(X)$).

Heuristic particle filters

Sample a population of N "individuals"/particles" s.t. at **any time**

$$(\widehat{\xi}_t^1,\ldots,\widehat{\xi}_t^N)\in E_t^N \rightsquigarrow \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \delta_{\widehat{\xi}_t^i} = \mathrm{Law}(X_t \mid (Y_0,\ldots,Y_t))$$

Heuristic learning/filtering scheme :

- Prediction/Exploration $\rightsquigarrow$ sampling N local transitions of the signal.
- Updating/Correction $\rightsquigarrow$ birth and death process = branching particle algo (fixed size N).
  - Kill/stop individuals/proposal with poor likelihood value.
  - Multiply/increase individuals with high likelihood value.

$\rightsquigarrow$ Path space models : $X_t = (X_0',\ldots,X_t')$
$\Rightarrow$ **Genealogical tree based learning algorithm :**

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \delta_{\text{i-th ancestral line(t)}} = \mathrm{Law}((X_0',\ldots,X_t') \mid (Y_0,\ldots,Y_t))$$

Some typical rare events

- Physical/biological/economical stochastic process : atomic/molecular configurations fluctuations, queueing evolutions, communication network, portfolio and financial assets, ...
- Potential function-Event restrictions : Energy/Hamiltonian potential functions, overflows levels, critical thresholds, epidemic propagations, radiation dispersion, ruin levels.

Objectives

Rare event probabilities & the law of the process $\in$ critical regime

Particle heuristic model

Default tree model = Branching particle genealogical tree model
(Branching on "more likely" gateways to critical regimes)

Event restrictions and confinements

- **Non intersecting simple random walks on $\mathbb{Z}^d$**

$$\mathbb{P}\left(\forall p < q \leq n,\ X_p \neq X_q\right) = \frac{1}{(2d)^n} \times \#\{\text{not} \cap \text{walks length n}\}$$
$$\simeq \exp\left(c\ n\right)$$

$$\text{Law}((X_0, \ldots, X_n) \mid \forall p < q \leq n \quad X_p \neq X_q)$$

- **Confinement model/Lyap. exp. and top eigenval.**

$$\mathbb{P}(\forall 0 \leq p \leq n \ \ X_p \in A) \simeq \exp\left(-\lambda(A)\ n\right)$$

$$\text{Law}((X_0, \ldots, X_n) \mid \forall 0 \leq p \leq n \quad X_p \in A)$$

- **Tube confinement :** as above with $(X_p \in A) \rightsquigarrow (X_p \in A_p)$

Heuristic particle model :

$\rightsquigarrow$ Accept-Reject interacting X-motions

Terminal levels conditioning and excursion models

1. **Terminal level set conditioning :**

$$\mathbb{P}(V_n(X_n) \geq a) \quad \& \quad \mathrm{Law}((X_0,\ldots,X_n) \mid V_n(X_n) \geq a)$$

2. **Fixed terminal value :** $\mathrm{Law}_{\pi,K}((X_0,\ldots,X_n) \mid X_n = x_n)$.
3. **Critical excursion behavior :**
$\mathbb{P}(X \text{ hits } B \text{ before } C) \quad \& \quad \mathrm{Law}(X \mid X \text{ hits } B \text{ before } C)$

Heuristic particle models :

1. Interacting X-transitions increasing the potential V_n.
2. Interacting M-transitions increasing the Metropolis type potential ratio $\frac{\pi(dx_2)K(x_2,dx_1)}{\pi(dx_1)M(x_1,dx_2)}$
3. Interacting X-excursions on gateways levels $\rightsquigarrow B$.

A pair of target Boltzmann-Gibbs measures

1. $\eta_n(dx) \propto e^{-\beta_n V(x)}\ \lambda(dx)$ with $\beta_n \uparrow$
2. $\eta_n(dx) \propto 1_{A_n}(x)\ \lambda(dx)$ with $A_n \downarrow$
3. Normalizing constants $\lambda(e^{-\beta_n V})$ and $\lambda(A_n)$

Heuristic particle models :

1. $e^{-(\beta_{n+1}-\beta_n)V}$-interacting MCMC moves with local targets η_n
2. A_{n+1}-interacting MCMC moves with local targets η_n
3. Time product of the empirical interaction potential functions.

Previous heuristic type models $\subset$ A single (sequential) Feynman-Kac/Boltzmann-Gibbs formulation:

$$
\begin{aligned}
d\eta_n &= \frac{1}{\mathcal{Z}_n}\left\{\prod_{0\leq p<n} G_p(X_p)\right\} d\mathbb{P}_n^X \\
&\overset{G_n=1_{A_n}}{=} \mathrm{Law}((X_0,\ldots,X_n)\mid X_0\in A_0,\ldots,X_n\in A_n) \\
\text{and } \mathcal{Z}_n &= \mathbb{P}(X_0\in A_0,\ldots,X_n\in A_n)
\end{aligned}
$$

Note : η_n = "nonlinear" transformation of the proba. meas. η_{n-1}

$$\left\{\prod_{0\leq p\leq n} G_p(X_p)\right\} = \left\{\prod_{0\leq p\leq (n-1)} G_p(X_p)\right\} \times G_n(X_n)$$

Same heuristic $\sim$ multiplicative structure :

$\rightsquigarrow$ (Accept-Reject) G-interacting X-motions [and inversely!]

Outline

Standard notation

E measurable space, $\mathcal{P}(E)$ proba. on E, $\mathcal{B}(E)$ bounded meas. functions.

- $(\mu, f) \in \mathcal{P}(E) \times \mathcal{B}(E) \quad \longrightarrow \quad \mu(f) = \int \mu(dx)\ f(x)$
- $M(x, dy)$ **integral operator on E**

$$\begin{aligned} M(f)(x) &= \int M(x, dy) f(y) \\ [\mu M](dy) &= \int \mu(dx) M(x, dy) \qquad (\Longrightarrow [\mu M](f) = \mu[M(f)]\ ) \end{aligned}$$

- **Bayes-Boltzmann-Gibbs transformation :** $G : E \to [0, \infty[$ with $\mu(G) > 0$

$$\Psi_G(\mu)(dx) = \frac{1}{\mu(G)}\ G(x)\ \mu(dx)$$

Note

$$\text{If} \qquad \mu = \mathrm{Law}(X) \quad \text{and} \quad M(x, dy) := \mathbb{P}(Y \in dy \mid X = x)$$

Then

- **Expectation operators**

$$\begin{aligned} \mu(f) &= \int \mathbb{P}(X \in dx)\ f(x) = \mathbb{E}(f(X)) \\ M(f)(x) &= \int \mathbb{P}(Y \in dy \mid |X = x)\ f(y) = \mathbb{E}(f(Y) \mid X = x) \\ [\mu M](dy) &= \int \mathbb{P}(Y \in dy \mid X = x)\ \mathbb{P}(X \in dx) = \mathbb{P}(Y \in dy) \end{aligned}$$

- **Bayes rule ($Y = y$ fixed observation) :**

$$\mu(dx) := p(x)\ dx \quad \text{and} \quad G(x) = p(y \mid x)$$

$$\Downarrow$$

$$\Psi_G(\mu)(dx) = \frac{1}{\mu(G)}\ G(x)\ \mu(dx) = p(x \mid y)\ dx$$

Only 3 Ingredients

- **A state space :**

  E_n with $n =$ time/level index **[transitions, paths, excursions,...].**

$$X_n := (X'_{n-1}, X'_n), \quad X'_{[0,n]}, \quad X'_{[t_{n-1},t_n]}, \quad X'_{[T_{n-1},T_n]}, \cdots$$

- **A Markov Proposal/Exploration/Mutation transition :**

$$M_n(x_{n-1}, dx_n) := \mathbb{P}\left(X_n \in dx_n \mid X_{n-1} = x_{n-1}\right)$$

- **A potential/likelihood/fitness/weight function on E_n :**

$$G_n \ : \ x_n \in E_n \ \longrightarrow \ G_n(x_n) \in [0, \infty[$$

Running Examples :

- [Confinement] X_n=Simple random walk (SRW) on $E_n = \mathbb{Z}$ and $G_n = 1_A$.
- [Filtering] M_n=signal transitions, G_n=Likelihood weight function.

SMC/Genetic type branching particle model :

$$\begin{bmatrix} \xi_n^1 \\ \vdots \\ \xi_n^i \\ \vdots \\ \xi_n^N \end{bmatrix} \xrightarrow{\text{Updating/Branching}} \begin{bmatrix} \widehat{\xi}_n^1 \\ \vdots \\ \widehat{\xi}_n^i \\ \vdots \\ \widehat{\xi}_n^N \end{bmatrix} \begin{array}{c} \xrightarrow{M_{n+1}} \\ \vdots \\ \xrightarrow{\text{Prediction/Exploration}} \\ \vdots \\ \longrightarrow \end{array} \begin{bmatrix} \xi_{n+1}^1 \\ \vdots \\ \xi_{n+1}^i \\ \vdots \\ \xi_{n+1}^N \end{bmatrix}$$

Selection/Branching : $(\forall \epsilon_n \geq 0 \text{ s.t. } \epsilon_n(x^1, \ldots, x^N) \times G_n(x^i) \in [0,1])$

- **Acceptance probability:**

$$\widehat{\xi}_n^i = \xi_n^i \quad \text{with probability} \quad \epsilon_n(\xi_n^1, \ldots, \xi_n^N)\ G_n(\xi_n^i)$$

- **Otherwise :**

$$\widehat{\xi}_n^i = \xi_n^j \quad \text{with probability} \quad \frac{G_n(\xi_n^j)}{\sum_{k=1}^N G_n(\xi_n^k)}$$

Running examples: [Confinement & Filtering] = [($G_n = 1_A$) & (G_n=Likelihood)].

Some remarks :

- $\epsilon_n = 0 \implies$ *Simple Mutation-Selection Genetic model.*
- $G_n = \exp\{-V_t \Delta t\}$ & $\epsilon_n = 1 \implies$ *V_t-expo-clocks* $\oplus$ uniform selection
- $G_n \in [0,1]$ & $\epsilon_n = 1 \Rightarrow$ *Interacting Acceptance-Rejection Sampling.*
- Better fitted individuals acceptance :

$$\text{For} \qquad \epsilon_n(x^1, \ldots, x^N) G_n(x^i) = G_n(x^i) / \sup_{1 \leq j \leq N} G_n(x^j)$$

- Related branching rules:

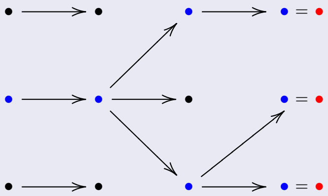
  [DM-Crisan-Lyons MPRF 99, DM 04] (Given $\xi_n = (\xi_n^i)_i$)

$$P_n^i := \text{Proportion of offsprings of the individual } \xi_n^i$$

  - Unbiasedness property : $\mathbb{E}\left(P_n^i\right) = G_n(\xi_n^i) / \sum_{k=1}^N G_n(\xi_n^k)$
  - Local mean error : $\mathbb{E}\left(\left[\sum_{i=1}^N \left[P_n^i - \mathbb{E}\left(P_n^i\right)\right] f(\xi_n^i)\right]^2\right) \leq \frac{Cte}{N}$

Interacting-Branching proc. $\hookrightarrow$ 3 Particle/SMC occupation measures:

$(N = 3)$



- Current population $\hookrightarrow \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_n^i}$ $\leftarrow$ *i*-th individual at time *n*

- Historical/genealogical tree $\hookrightarrow \frac{1}{N}\sum_{i=1}^{N} \delta_{(\xi_{0,n}^i,\xi_{1,n}^i,\ldots,\xi_{n,n}^i)}$ $\leftarrow$ *i*-th ancestral line

- **Complete genealogical tree** $\hookrightarrow \frac{1}{N}\sum_{i=1}^{N} \delta_{(\xi_0^i,\xi_1^i,\ldots,\xi_n^i)}$

- $\oplus$ *Mean potential values [Success proportions $(G_n = 1_A)$]* $\hookrightarrow \frac{1}{N}\sum_{i=1}^{N} G_n(\xi_n^i)$

Limiting measures ("Test" functions $f : E_n \rightarrow \mathbb{R}$)

- **Occupation measures of the Current population**

$$\eta_n^N(f) := \frac{1}{N}\sum_{i=1}^{N} f(\xi_n^i) \longrightarrow_{N\uparrow\infty} \eta_n(f) := \frac{\gamma_n(f)}{\gamma_n(1)}$$

with the Feynman-Kac measures (X_n Markov with transitions M_n):

$$\gamma_n(f) := \mathbb{E}\left(f_n(X_n) \prod_{0\leq p<n} G_p(X_p)\right)$$

First rigorous convergence result :

↪Markov Processes and Related Fields, vol. 2, no; 4, pp. 555-580 (1996).

More recent developments :

↪ Feynman-Kac formulae. Genealogical and interacting particle systems, Springer (2004), + References

Limiting measures ("Test" functions $f : E_n \to \mathbb{R}$)

$$\eta_n(f) := \frac{\gamma_n(f)}{\gamma_n(1)} \quad \text{with} \quad \gamma_n(f) := \mathbb{E}\left(f_n(X_n) \prod_{0 \leq p < n} G_p(X_p)\right)$$

Running examples :

- *Confinement $G_n = 1_A$:*

$$\gamma_n(1) = \mathbb{P}\left(\forall 0 \leq p < n \quad X_p \in A\right) \quad \& \quad \eta_n = \mathrm{Law}\left(X_n \mid \forall 0 \leq p < n \quad X_p \in A\right)$$

- *Filtering: G_n=Likelihood function :*

$$\gamma_n(1) = p_n(y_0, \ldots, y_{n-1}) \quad \& \quad \eta_n = \mathrm{Law}\left(X_n \mid Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1}\right)$$

Limiting measures
("Test" function on path space $f_n : E_n = (E'_0 \times \ldots \times E'_n) \to \mathbb{R}$)

- **Occupation measures of the historical/genealogical tree**

$$\eta_n^N(f_n) := \frac{1}{N}\sum_{i=1}^{N} f_n\left(\xi^i_{0,n}, \xi^i_{1,n}, \ldots, \xi^i_{n,n}\right) \longrightarrow_{N\uparrow\infty} \eta_n(f_n) := \frac{\gamma_n(f_n)}{\gamma_n(1)}$$

with the Feynman-Kac measures on path space :

$$\gamma_n(f_n) := \mathbb{E}\left(f_n(X'_0, \ldots, X'_n) \prod_{0\leq p<n} G_p(X'_0, \ldots, X'_p)\right)$$

Genealogical tree based algorithms :

$\hookrightarrow$DM, Miclo L. Annals of Applied Probability , vol. 11, No. 4, pp. 1166-1198 (2001).

More recent developments :

$\hookrightarrow$ Feynman-Kac formulae. Genealogical and interacting particle systems, Springer (2004), + References

Running examples

Confinement

$$X_n = (X'_0, \ldots, X'_n) \quad \text{SRW} \qquad G_n(X_n) = 1_A(X'_n)$$

$$\downarrow$$

$$\eta_n \quad = \quad \text{Law}\left((X'_0, \ldots, X'_n) \mid \forall 0 \leq p < n \quad X'_p \in A\right)$$

Filtering :

$$X_n = (X'_0, \ldots, X'_n) = \text{Path signal} \quad \text{and} \quad G_n(X_n) = \text{Likelihood functions}$$

$$\downarrow$$

$$\eta_n = \text{Law}\left((X'_0, \ldots, X'_n) \mid Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1}\right)$$

Limiting measures
("Test" function on path space $F_n : (E_0 \times \ldots \times E_n) \to \mathbb{R}$)

- **Occupation measures of the complete genealogical tree ($\epsilon_n = 0$)**

$$\frac{1}{N}\sum_{i=1}^{N} F_n\left(\xi_0^i, \xi_1^i, \ldots, \xi_n^i\right) \longrightarrow_{N\uparrow\infty} (\eta_0 \otimes \ldots \otimes \eta_n)(F_n)$$

with the Feynman-Kac tensor product measures :

$$(\eta_0 \otimes \ldots \otimes \eta_n)(F_n) = \int_{E_0} \ldots \int_{E_n} \eta_0(dx_0) \ldots \eta_n(dx_n)\; F_n(x_0, \ldots, x_n)$$

- Acceptance parameter $\epsilon_n \neq 0 \rightsquigarrow$ Limiting McKean measures.

$$\eta_n = \mathrm{Law}(\overline{X}_n) \quad \text{with Markov transition} \quad \overline{X}_n \overset{\eta_n}{\rightsquigarrow} \overline{X}_{n+1}$$

Interacting-Branching model $=$ Mean-field interpretation of $\overline{X}_n$

Limiting mean potential/success proportions ($G_n = 1_A$)

$$\eta_n^N(G_n) := \frac{1}{N}\sum_{i=1}^{N} G_n(\xi_n^i) \longrightarrow_{N\uparrow\infty} \eta_n(G_n) \stackrel{def.}{=} \frac{\gamma_n(G_n)}{\gamma_n(1)} = \frac{\gamma_{n+1}(1)}{\gamma_n(1)} \qquad (1)$$

$\Rightarrow$**Unbiased estimate of the normalizing cts/partition functions :**

$$\gamma_n^N(1) := \prod_{0\leq p<n} \eta_p^N(G_p) \longrightarrow_{N\uparrow\infty} \gamma_n(1) = \prod_{0\leq p<n} \eta_p(G_p)$$

with the key product formula :

$$(1) \Longrightarrow \gamma_n(1) := \mathbb{E}\left(\prod_{0\leq p<n} G_p(X_p)\right) = \prod_{0\leq p<n} \eta_p(G_p)$$

Running ex. : [X_n SRW & $G_n = 1_A$]

$$\prod_{0\leq p<n} \textbf{(Success proportion time p)} \simeq \mathbb{P}\left(\forall 0 \leq p < n \quad X_p \in A\right)$$

Equivalent Sequential Monte Carlo formulation

Preliminary observations

Updated Feynman-Kac models

$$\widehat{\gamma}_n(f_n) := \mathbb{E}\left(f_n(X'_0,\ldots,X'_n) \prod_{0\leq p\leq n} G_p(X'_0,\ldots,X'_p) \right)$$

$\Updownarrow$ **[Path space models]** $x_n = (x'_0,\ldots,x'_n)$

$$\widehat{\gamma}_n(dx_n) = \left\{ \eta'_0(dx'_0) \prod_{p=1}^{n} M'_p(x'_{p-1},dx'_p) \right\} \times \left\{ \prod_{0\leq p\leq n} G_p(x'_0,\ldots,x'_p) \right\}$$

$$\Updownarrow$$

$$\widehat{\gamma}_n(dx_n) = \widehat{\gamma}_{n-1}(dx_{n-1}) \times M'_n(x'_{n-1},dx'_n) \times G_n(x_n)$$

Preliminary observations

Prediction Feynman-Kac models

$$\gamma_n(f_n) := \mathbb{E}\left(f_n(X'_0,\ldots,X'_n) \prod_{0\leq p<n} G_p(X'_0,\ldots,X'_p) \right)$$

$\Updownarrow$ **[Path space models]** $x_n = (x'_0,\ldots,x'_n)$

$$\widehat{\gamma}_n(dx_n) = \widehat{\gamma}_{n-1}(dx_{n-1}) \times M'_n(x'_{n-1}, dx'_n)$$

SMC formulation

Goal = Sample from the target measures :

$$\text{[unnormalized recursions]} \quad \widehat{\gamma}_n(dx_n) = \widehat{\gamma}_{n-1}(dx_{n-1}) \times M'_n(x'_{n-1}, dx'_n)$$

on the path space sequence

$$x_n := (x'_0, \ldots, x'_n) \in E_n := (E'_0 \times \ldots \times E'_n)$$

1. Sampling N Local explorations :

$$x_{n-1} \rightsquigarrow x_n = (x_{n-1}, x'_n) \quad \text{with} \quad x'_n \sim M'_n(x'_{n-1}, dx'_n)$$

2. Compute the weight of each sample :

$$G_n(x_n) = \frac{\widehat{\gamma}_n(dx_n)}{\widehat{\gamma}_{n-1}(dx_{n-1}) \times M'_n(x'_{n-1}, dx'_n)}$$

   and resample N random paths with weights $G_n(x_n)$.

Summary-Conclusions

SMC/Genetic type branching/particle model

[M_n-**free exploration** $\oplus$ G_n-**weighted branchings/adaptation**]

$$\Downarrow \quad \& \quad \Uparrow$$

Feynman-Kac measures

[M_n-**free motion** $\oplus$ G_n-**potential functions**]

Outline

1 Introduction

2 Some heuristic like particle algorithms

3 A simple mathematical model

4 Some Feynman-Kac sampling recipes
- Exploration/Branching rules
- Tuning parameters
- Some "wrong" approximation ideas
- A nonlinear approach
- Some key advantages

5 A series of applications

6 Interacting sampling techniques

7 Mean field particle methods

8 Some theoretical aspects

9 Interacting MCMC models

10 Fluctuations & comparisons

Some evolutionary sampling recipes

Nonlinear Feynman-Kac measures $\sim (G_n, M_n)$

$$\eta_n(f) = \gamma_n(f)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f) = \mathbb{E}\left(f_n(X_n) \prod_{0 \leq p < n} G_p(X_p)\right)$$

- $\rightsquigarrow$ **Interacting stochastic algorithm :**

$$\text{accept/reject/select/branch/prune/clone/spawn/enrich} \quad \rightsquigarrow \quad G_n$$
$$\text{exploration/proposition/prediction/mutation/free evolution} \quad \rightsquigarrow \quad M_n$$

And Inversely !

- Normalizing constants $\rightsquigarrow$ key multiplicative formula.
- Path space models $\rightsquigarrow$ path-particles=ancestral lines

Occupation meas. of genealogical trees $\simeq \eta_n \in$ path-space

Tuning parameters

$$(G_n, M_n) \longleftrightarrow (\widehat{G}_n, \widehat{M}'_n)$$

Change of ref. measures, path/excursion spaces, selection periods, weights interpretations,...

An elementary illustration of a change of probability measure

Updated FK-models $\quad\hookrightarrow\quad \widehat{\gamma}_n(f_n) := \mathbb{E}\left(f_n(X_n) \prod_{0\leq p\leq n} G_p(X_p)\right)$

$$\eta_0(dx_0)G_0(x_0) \times \prod_{p=1}^n \{M_p(x_{p-1}, dx_p)G_p(x_p)\}$$

$$= \eta_0(G_0) \left(\frac{\eta_0(dx_0)G_0(x_0)}{\eta_0(G_0)}\right) \times \prod_{p=1}^n M_p(G_p)(x_{p-1}) \left\{\frac{M_p(x_{p-1},dx_p)G_p(x_p)}{M_p(G_p)(x_{p-1})}\right\}$$

with

$$M_p(G_p)(x_{p-1}) := \int_{E_p} M_p(x_{p-1}, dx_p)\ G_p(x_p)$$

$(G_n, M_n) \longleftrightarrow (\widehat{G}_n, \widehat{M}'_n)$

$$\widehat{\gamma}_n(f_n) := \mathbb{E}\left(f_n(X_n) \prod_{0 \leq p \leq n} G_p(X_p) \right) \rightsquigarrow [X - \text{free motion} \ \oplus \ G - \text{branching}]$$

$$\eta_0(dx_0) G_0(x_0) \times \prod_{p=1}^n \{M_p(x_{p-1}, dx_p) G_p(x_p)\}$$

$$= \eta_0(G_0)\ \widehat{\eta}_0(dx_0) \times \left\{\prod_{p=1}^n \widehat{M}_p(x_{p-1}, dx_p)\right\} \times \left\{\prod_{0 \leq p < n} \widehat{G}_p(x_p)\right\}$$

with

$$\widehat{M}_p(x_{p-1}, dx_p) := \frac{M_p(x_{p-1}, dx_p) G_p(x_p)}{M_p(G_p)(x_{p-1})} \quad \text{and} \quad \widehat{G}_p(x_p) := M_{p+1}(G_{p+1})(x_p)$$

$$\Downarrow$$

$$\widehat{\gamma}_n(f_n) \propto \mathbb{E}\left(f_n(\widehat{X}_n) \prod_{0 \leq p < n} \widehat{G}_p(\widehat{X}_p) \right) \rightsquigarrow \left[\widehat{X} - \text{free motion} \ \oplus \ \widehat{G} - \text{branching}\right]$$

$\Longrightarrow$ 2 alternative particle interpretations

1. **Interacting acceptance-rejection algorithm**

$$\widehat{\gamma}_n(f_n) := \mathbb{E}\left(f_n(X_n) \prod_{0\leq p\leq n} G_p(X_p)\right) \rightsquigarrow X - \text{motion} \oplus G - \text{branching}$$

2. **Local Conditioning $\rightsquigarrow$ Conditional exploration algorithm**

$$\widehat{\gamma}_n(f_n) \propto \mathbb{E}\left(f_n(\widehat{X}_n) \prod_{0\leq p<n} \widehat{G}(\widehat{X}_p)\right) \rightsquigarrow \widehat{X} - \text{motion} \oplus \widehat{G} - \text{branching}$$

Running examples

$$\boxed{\widehat{\mathbf{X}} - \text{motion} \ \oplus \ \widehat{\mathbf{G}} - \text{branching}}$$

1. Confinement $G_n = 1_A \rightsquigarrow$ Local transition conditioning

$$\widehat{M}_n(x_{n-1}, dx_n) := \frac{M_n(x_{n-1}, dx_n) 1_A(x_n)}{M_n(1_A)(x_{n-1})} \quad \text{and} \quad \widehat{G}_n(x_n) := M_{n+1}(1_A)(x_n)$$

2. $G_n(x_n) = \text{Filtering Likelihood weight function} = p(y_n|x_n)$

$$\widehat{M}_n(x_{n-1}, dx_n) := \frac{p(y_n|x_n) \ p(x_n|x_{n-1})}{p(y_n|x_{n-1})} \quad \text{and} \quad \widehat{G}_n(x_n) := p(y_{n+1}|x_n)$$

$$\Updownarrow$$

$$\mathbf{G_n(x_n) = p(y_n|x_n)} \ \Leftrightarrow \ \mathbf{p(x_n|x_{n-1}, y_n)} - \text{motion} \ \oplus \ \mathbf{p(y_n|x_{n-1})} - \text{branching}$$

$\hookrightarrow$`Annals of Applied Prob, vol. 8, no. 2, 1254-1278 (1998).`

Approximation models

At each stage & from any local individual position x_{n-1}

$$\downarrow$$

N (or N') auxiliary/lookahead variables $(X_n^i(x))_{1\leq i\leq N}$ with law $M_n(x_{n-1}, dx_n)$

$$\Downarrow$$

$$M_n(x_{n-1}, dx_n) \simeq M_n^N(x_{n-1}, dx_n) := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_n^i(x)}$$

$$\Downarrow$$

$$\widehat{M}_n(x_{n-1}, dx_n) \quad \simeq \quad \widehat{M}_n^N(x_{n-1}, dx_n) := \frac{M_n^N(x_{n-1}, dx_n) G_n(x_n)}{M_n^N(G_n)(x_{n-1})}$$

$$\text{and} \quad \widehat{G}_n(x_n) \quad \simeq \quad \widehat{G}_n^N(x_n) := M_{n+1}^N(G_{n+1})(x_n)$$

↪Annals of Applied Prob, vol. 8, no. 2, 1254-1278 (1998).
↪DM, Guionnet A. SPA, vol. 78, pp. 69-95 (1998).

State space enlargements $\rightsquigarrow$ same model!

$$X_n = (X'_{n-1}, X'_n) \quad \text{or} \quad X_n = (X'_0, \ldots, X'_n) \quad \text{or} \quad \text{excursions, ...}$$

Path space models :

$$X_n = (X'_0, \ldots, X'_n)$$

$$\Downarrow$$

$$\eta_n(f_n) \propto \mathbb{E}\left(f_n(X'_0, \ldots, X'_n) \prod_{0 \leq p < n} G_p(X'_0, \ldots, X'_p) \right)$$

Alternative Boltzmann-Gibbs' formulation :

$$d\eta_n = \frac{1}{\mathcal{Z}_n} \left\{ \prod_{0 \leq p < n} G_p(X_p) \right\} d\mathbb{P}^X_n$$

An example of state space enlargements

$$\forall t_n < t_{n+1} \qquad X_n = X'_{[t_n,t_{n+1}[} := (X'_{t_n}, X'_{t_n+1}, X'_{t_n+1}, \ldots, X'_{t_{n+1}-1})$$

and

$$G_n(X_n) := \prod_{t_n \leq s < t_{n+1}} G'_s(X'_s)$$

$$\Downarrow$$

$$\begin{aligned} \eta_n(f_n) &\propto \mathbb{E}\left(f_n(X_n) \prod_{0 \leq p < n} G_p(X_p)\right) \\ &= \mathbb{E}\left(f_n(X'_{[t_n,t_{n+1}[}) \prod_{0 \leq s < t_{n+1}} G'_p(X'_s)\right) \end{aligned}$$

An example of state space enlargements

$$\forall t_n < t_{n+1} \qquad X_n = X'_{[t_n,t_{n+1}[} := (X'_{t_n}, X'_{t_n+1}, X'_{t_n+1}, \ldots, X'_{t_{n+1}-1})$$

and

$$G_n(X_n) := \prod_{t_n \leq s < t_{n+1}} G'_s(X'_s)$$

$$\Downarrow$$

$\left[\widehat{X} - \text{free motion} \; \oplus \; \widehat{G} - \text{branching}\right]$-models :

$$\widehat{M}_n(x_{n-1}, dx_n) := \frac{M_n(x_{n-1}, dx_n) G_n(x_n)}{M_n(G_n)(x_{n-1})} \quad \text{and} \quad \widehat{G}_n(x_n) := M_{n+1}(G_{n+1})(x_n)$$

with for any path sequences $x_n = x'_{[t_n,t_{n+1}[}$

$$\begin{cases} M_n(x_{n-1}, dx_n) & = & \mathbb{P}\left(X'_{[t_n,t_{n+1}[} \in dx_n \mid X'_{[t_{n-1},t_n[} = x_{n-1}\right) \\ G_n(x_n) & := & \prod_{t_n \leq s < t_{n+1}} G'_s(x'_s) \end{cases}$$

Approximation path-models

At each stage & from any local individual path sequences $x_{n-1} = X'_{[t_{n-1},t_n[}$

$$\downarrow$$

N (or N') auxiliary/lookahead path sequences

$$\forall 1 \leq i \leq N \qquad X_n^i(x) = X'^{\,i}_{[t_n,t_{n+1}[}(x)$$

with law

$$M_n(x_{n-1}, dx_n) = \mathbb{P}\left(X'_{[t_n,t_{n+1}[} \in dx_n \mid X'_{[t_{n-1},t_n[} = x_{n-1}\right)$$

$$\Downarrow$$

Approximated path-transitions :

$$M_n(x_{n-1}, dx_n) \simeq M_n^N(x_{n-1}, dx_n) := \frac{1}{N}\sum_{i=1}^{N} \delta_{X'^{\,i}_{[t_n,t_{n+1}[}(x)}$$

Approximation path-models

Approximated path-transitions :

$$M_n(x_{n-1}, dx_n) \simeq M_n^N(x_{n-1}, dx_n) := \frac{1}{N}\sum_{i=1}^{N} \delta_{X'^{\,i}_{[t_n, t_{n+1}[}(x)}$$

$\downarrow$ [Weighted empirical conditional transitions]

$$\begin{aligned}\widehat{M}_n^N(x_{n-1}, dx_n) &:= \frac{M_n^N(x_{n-1}, dx_n) G_n(x_n)}{M_n^N(G_n)(x_{n-1})} \\ &= \sum_{i=1}^{N} \frac{G_n(X'^{\,i}_{[t_n, t_{n+1}[}(x))}{\sum_{j=1}^{N} G_n(X'^{\,j}_{[t_n, t_{n+1}[}(x))}\, \delta_{X'^{\,i}_{[t_n, t_{n+1}[}(x)}(dx_n)\end{aligned}$$

and

$$\widehat{G}_n^N(x_n) := M_{n+1}^N(G_{n+1})(x_n) = \frac{1}{N}\sum_{j=1}^{N} G_{n+1}(X'^{\,j}_{[t_{n+1}, t_{n+2}[}(x))$$

Importance sampling distributions $\rightsquigarrow$ same model!

- Change of proba. : $X_n = (X'_{n-1}, X'_n) \rightsquigarrow Y_n = (Y'_{n-1}, Y'_n)$

$$\mathbb{E}\left(f_n(X_n) \prod_{0\leq p<n} G_p(X_p)\right) \propto \mathbb{E}\left(f_n(Y_n) \prod_{0\leq p<n} H_p(Y_p)\right)$$

- Related weighted meas. $G_n = G_n^{\epsilon_n} \times G_n^{1-\epsilon_n} = G_n^{(1)} \times G_n^{(2)} = \ldots$

Complexity and Sampling problems

- Path integration formulae, infinite dimensional state spaces
- Nonlinear-Nongaussian models
- Complex probability mass variations

Some "wrong" approximation ideas

- "Pure" weighted Monte Carlo methods : X^i iid copies of X

$$\frac{1}{N}\sum_{i=1}^{N} f_n(X_n^i)\left\{\prod_{0\leq p<n} G_p(X_p^i)\right\} \simeq \mathbb{E}\left(f_n(X_n)\prod_{0\leq p<n} G_p(X_p)\right)$$

$\rightsquigarrow$ bad grids X^i $\oplus$ degenerate weights (running ex $G_n = 1_A$)

$\oplus$ DM, Jacod J. : Interacting particle filtering with discrete-time observations: asymptotic behaviour in the Gaussian case. Stochastics in infinite dimensions, Trends in Mathematics, Birkhauser (2001).

- Uncorrelated MCMC for each target measure η_n ($\uparrow$ complex.).
- "Pure" branching $\rightsquigarrow$ critical random population sizes

$$G_n(x) = \mathbb{E}(g_n(x)) \quad \text{with} \quad g_n(x) \text{ r.v. } \in \mathbb{N}$$

- Harmonic/(Gaussian+linearisation) approximations.
- $G.M(H) \propto H \rightsquigarrow G \propto H/M(H) \rightsquigarrow H$-process X^H (unknown).

A nonlinear approach $\sim$ **Feynman-Kac evolution equation**

[$\eta_n \in \mathcal{P}(E_n)$ probability measures $\uparrow$ complexity].

$$\eta_{n+1} = \Phi_{n+1}(\eta_n) = \Psi_{G_n}(\eta_n) M_{n+1}$$

With only 2 transformations:

- Bayes-Boltzmann-Gibbs updating transformation :

$$\Psi_{G_n}(\eta_n)(dx) := \frac{1}{\eta_n(G_n)} \; G_n(x) \; \eta_n(dx)$$

- X-Free Markov transport/prediction eq. : [X_n Markov M_n]

$$\mu(dx) \; \rightsquigarrow \; (\mu M_n)(dy) := \int \mu(dx) \; M_n(x, dy)$$

Proof :

By the Markov property :

$$\begin{aligned}
\gamma_n(f_n) &= \mathbb{E}\left(\mathbb{E}\left(f_n(X_n) \mid (X_p)_{0\leq p<n}\right) \prod_{0\leq p<n} G_p(X_p)\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(f_n(X_n) \mid X_{n-1}\right) \prod_{0\leq p<n} G_p(X_p)\right) \\
&= \mathbb{E}\left(M_n(f_n)(X_{n-1}) \prod_{0\leq p<n} G_p(X_p)\right) \\
&= \mathbb{E}\left([G_{n-1}M_n(f_n)](X_{n-1}) \prod_{0\leq p<(n-1)} G_p(X_p)\right)
\end{aligned}$$

$$\Downarrow$$

$$\gamma_n(f_n) = \gamma_{n-1}(G_{n-1}M_n(f_n)) \quad \text{and} \quad \gamma_n(1) = \gamma_{n-1}(G_{n-1})$$

Proof :

$$\gamma_n(f_n) = \gamma_{n-1}(G_{n-1}M_n(f_n)) \quad \text{and} \quad \gamma_n(1) = \gamma_{n-1}(G_{n-1})$$

$$\Downarrow$$

$$\eta_n(f_n) = \frac{\gamma_n(f_n)}{\gamma_n(1)} = \frac{\gamma_{n-1}(G_{n-1}M_n(f_n))}{\gamma_{n-1}(G_{n-1})}$$

$$\Downarrow$$

$$\eta_n(f_n) = \frac{\gamma_{n-1}(G_{n-1}M_n(f_n))/\gamma_{n-1}(1)}{\gamma_{n-1}(G_{n-1})/\gamma_{n-1}(1)} = \frac{\eta_{n-1}(G_{n-1}M_n(f_n))}{\eta_{n-1}(G_{n-1})}$$

$$\Downarrow$$

$$\eta_n(f_n) = \Psi_{G_{n-1}}(\eta_{n-1})(M_n(f_n))$$

$$\Updownarrow$$

$$\eta_n = \Psi_{G_{n-1}}(\eta_{n-1})M_n$$

■

(Updating/Prediction)$\simeq$(Select./Mutation)=(Branching/Exploration)

$$\Updownarrow$$

$$\eta_n \xrightarrow{\text{Updating}} \Psi_{G_n}(\eta_n) \xrightarrow{\text{Prediction}} \eta_{n+1} = \Psi_{G_n}(\eta_n) M_{n+1}$$

$$\Updownarrow$$

$$\xi_n = (\xi_n^i)_{1\leq i\leq N} \xrightarrow{\text{Branching/Selection}} \widehat{\xi}_n = (\widehat{\xi}_n^i)_{1\leq i\leq N} \xrightarrow{\text{Exploration/Mutation}} \xi_{n+1}$$

2 Local sources of randomness with mean :

$$\mathbb{E}\left(\eta^N_{n+1}(f) \mid \xi_n\right) \quad = \quad \sum_{i=1}^{N} \frac{G_n(\xi^i_n)}{\sum_{j=1}^{N} G_n(\xi^j_n)} \; M_{n+1}(f)(\xi^i_n) = \Phi_{n+1}\left(\eta^N_n\right)(f)$$

$$\Updownarrow$$

The particle measures η^N_n "almost" solve the updating/prediction system :

$$\mathbb{E}(\quad \left[\eta^N_{n+1} - \Phi_{n+1}\left(\eta^N_n\right)\right](f) \quad \mid \xi_n) = 0 \quad \longleftrightarrow \quad \eta_{n+1} = \Phi_{n+1}(\eta_n)$$

Up to the local fluctuation errors :

$$\eta^N_{n+1} = \Phi_{n+1}\left(\eta^N_n\right) + \underbrace{\frac{1}{\sqrt{N}}}_{\text{Monte Carlo precision}} \times \underbrace{\left[\sqrt{N}\;\left(\eta^N_{n+1} - \Phi_{n+1}\left(\eta^N_n\right)\right)\right]}_{:=W^N_n \simeq \text{Gaussian Field}}$$

Some key advantages

- Stochastic linearization/perturbation technique :

$$\eta_n^N = \Phi_n(\eta_{n-1}^N) + \frac{1}{\sqrt{N}} \; W_n^N$$

  with $W_n^N \simeq W_n$ **independent and centered Gauss fields**.

- $\eta_n = \Phi_n(\eta_{n-1})$ **stable dynamical system**

$$\Longrightarrow \quad \text{local errors do not propagate}$$

$$\Longrightarrow \quad \text{uniform control of errors w.r.t. the time parameter}$$

- "No need" to study the cv of equilibrium of MCMC models.
- Adaptive stochastic grid approximations
- Take advantage of the nonlinearity of the system to define beneficial interactions. Non intrusive methods.
- Natural and easy to implement, etc.

Outline

Filtering models

- **Signal-Observation likelihood functions** (X_n, G_n) :

$$\begin{aligned} \eta_n &= \mathrm{Law}((X_0,\ldots,X_n) \mid (Y_0,\ldots,Y_n)) \\ L_n &= \frac{1}{n}\log\gamma_n(1) = \text{Log-likelihood function} \end{aligned}$$

- **Example :**

$$Y_n = H_n(X_n) + V_n \quad \text{with} \quad \mathbb{P}(V_n \in dv_n) = g_n(v_n)\ dv_n$$

$$\Downarrow [Y_n = y_n]$$

$$G_n(x_n) = g_n(y_n - H_n(x_n))$$

- $\rightsquigarrow$ Particle filters, sampling/resampling alg., bootstrap filter, genetic filter,...

Rare events analysis

- Confinements potentials: $G_n = 1_{A_n}$

$$\begin{aligned} \eta_n &= \mathrm{Law}((X_0,\ldots,X_n) \mid X_0 \in A_0,\ldots,X_n \in A_n) \\ \mathcal{Z}_n &= \mathbb{P}(X_0 \in A_0,\ldots,X_n \in A_n) \end{aligned}$$

- Twisted measures $\sim \mathbb{P}(V_n(X_n) \geq a)$?

$$\mathbb{E}(f_n(X_n)\ e^{\lambda V_n(X_n)}) = \mathbb{E}\left(f_n(X_n)\ \prod_{0\leq p\leq n} e^{\lambda(V_p(X_p)-V_{p-1}(X_{p-1}))}\right)$$

$\rightsquigarrow$ Interacting particle simulation of twisted measures

Hitting B before C

- *Multi-level decomposition* $B_0 \supset B_1 \supset \ldots \supset B_m$, $B_0 \cap C = \emptyset$.
- *Inter-level excursions :*

$$T_n = \inf\{p \geq T_{n-1} \ : \ Y_p \in B_n \cup C\}$$

- *Level excursions and level detection potentials:*

$$X_n = (Y_p \ ; T_{n-1} \leq p \leq T_n) \quad \text{and} \quad G_n(X_n) = 1_{B_n}(Y_{T_n})$$

$$\Updownarrow$$

$$\begin{aligned}
\mathbb{P}(Y \text{ hits } B_m \text{ before } C) &= \mathbb{E}\left(\prod_{1\leq p\leq m} G_p(X_p)\right) \\
\mathbb{E}(f(Y_0,\ldots,Y_{T_m})\, 1_{B_m}(Y_{T_m})) &= \mathbb{E}\left(f(X_0,\ldots,X_m) \prod_{1\leq p\leq m} G_p(X_p)\right)
\end{aligned}$$

$\rightsquigarrow$ Branching-multilevel splitting algorithms

Objectives - Markov processes with fixed terminal values

- X_n Markov with transitions $L(x, dy)$ on E
- $\mathrm{Law}(X_0) = \pi$ only known up to a normalizing constant.
- For a given fixed terminal value x solve/sample inductively the following flow of measures

$$n \mapsto \mathrm{Law}_\pi((X_0, \ldots, X_n) \mid X_n = x)$$

FK-formulation - Markov processes with fixed terminal values

- π "target type" measure+(K, L) pair Markov transitions

$$\text{Metropolis potential} \quad G(x_1, x_2) = \frac{\pi(dx_2)L(x_2, dx_1)}{\pi(dx_1)K(x_1, dx_2)}$$

- Theorem [Time reversal formula] :

$$\mathbb{E}^L_\pi(f_n(X_n, X_{n-1} \ldots, X_0)|X_n = x)$$

$$= \frac{\mathbb{E}^K_x(f_n(X_0, X_1, \ldots, X_n) \ \{\prod_{0\leq p<n} G(X_p, X_{p+1})\})}{\mathbb{E}^K_x(\{\prod_{0\leq p<n} G(X_p, X_{p+1})\})}$$

- $\rightsquigarrow$ time reversal genealogical tree simulation
  $\rightsquigarrow$ Interacting Metropolis-Hastings algorithms

Non intersecting random walks (& connectivity constants)

$$X_n := (X'_0, \ldots, X'_n) \quad \text{and} \quad G_n(X_n) = 1_{\not\in \{X'_p \ , \ p<n\}}(X'_n)$$

$$\Downarrow$$

$$\eta_n = \mathrm{Law}((X'_0, \ldots, X'_n) \mid \forall p < q < n \quad X'_p \neq X'_q)$$

$\rightsquigarrow$ Dynamic Pruning-Enrichment Rosenbluth Monte Carlo model

Molecular simulation $\sim$ Particle absorption models

- X_n Markov $\in (E_n, \mathcal{E}_n)$ with transitions M_n, and $G_n : E_n \longrightarrow [0,1]$

$$Q_n(x,dy) = G_{n-1}(x)\ M_n(x,dy) \quad \text{sub-Markov operator}$$

- $\rightsquigarrow E_n^c = E_n \cup \{c\}$.

$$X_n^c \in E_n^c \xrightarrow{\text{absorption } \sim G_n} \widehat{X}_n^c \xrightarrow{\text{exploration } \sim M_n} X_{n+1}^c$$

With:

- **Absorption:** $\widehat{X}_n^c = X_n^c$, with proba $G(X_n^c)$; otherwise $\widehat{X}_n^c = c$.
- **Exploration:** elementary free explorations $X_n \rightsquigarrow X_{n+1}$

Feynman-Kac integral model

- $T = \inf\{n : \widehat{X}_n^c = c\}$ **absorption time :** $\forall f_n \in \mathcal{B}_b(E_n)$

$$\begin{aligned} \mathbb{P}(T \geq n) &= \gamma_n(1) := \mathbb{E}\left(\prod_{0 \leq p < n} G(X_p)\right) \\ \mathbb{E}(f_n(X_n^c)\ ;\ (T \geq n)) &= \gamma_n(f_n) := \mathbb{E}\left(f_n(X_n) \prod_{0 \leq p < n} G_p(X_p)\right) \end{aligned}$$

- **Continuous time models :** $\Delta =$ time step

$$(M, G) = (Id + \Delta\ L, e^{-V\Delta}) \implies Q \rightsquigarrow L^V := L - V$$

$\rightsquigarrow$ L-motions $\oplus$ expo. clocks rate V $\oplus$ Uniform selection.

Spectral radius-Lyapunov exponents

- $Q(x,dy) = G(x)M(x,dy)$ sub-Markov operator on $\mathcal{B}_b(E)$
- Normalized FK-model : $\eta_n(f) = \gamma_n(f)/\gamma_n(1)$.

$$\mathbb{P}(T \geq n) = \mathbb{E}\left(\prod_{0\leq p\leq n} G(X_p)\right) = \prod_{0\leq p\leq n} \eta_p(G) \simeq e^{-\lambda n}$$

with $e^{-\lambda} \stackrel{M\ reg.}{=}$ Q-top eigenvalue or

$$\begin{aligned}\lambda &= -\mathrm{LogLyap}(Q) = \lim_{n\to\infty} -\frac{1}{n}\log \|Q^n\| \\ &= -\frac{1}{n}\log \mathbb{P}(T \geq n) = -\frac{1}{n}\sum_{0\leq p\leq n} \log \eta_p(G) = -\log \eta_\infty(G)\end{aligned}$$

Feynman-Kac-Shroedinger ground states energies

$M \quad \mu - \text{reversible :}$

$$\Rightarrow \mathbb{E}(f(X_n^c) \mid T > n) \simeq \frac{\mu(H\ f)}{\mu(H)} \quad \text{with} \quad Q(H) = e^{-\lambda} H$$

Limiting FK-measures

$$\eta_n = \Phi(\eta_{n-1}) \rightarrow_{n\uparrow\infty} \eta_\infty \quad \text{with} \quad \frac{\eta_\infty(G\ f)}{\eta_\infty(G)} = \frac{\mu(H\ f)}{\mu(H)}$$

leadsto Particle approximations :

$$\lambda \ \simeq_{n,N\uparrow} \ \lambda_n^N := \frac{1}{n} \sum_{0 \leq p \leq n} \log \eta_p^N(G) \quad \text{and} \quad \eta_\infty \ \simeq_{n,N\uparrow} \ \eta_n^N$$

$$\text{Law}((X_0^c, \ldots, X_n^c) \mid (T \geq n)) \simeq \text{Genealogical tree measures}$$

$$\Updownarrow$$

Diffusion and quantum Monte Carlo models

Boltzmann-Gibbs measures

- X r.v. $\in (E, \mathcal{E})$ with $\mu = \mathrm{Law}(X)$
- Target measures associated with $g_n : E \to \mathbb{R}_+$

$$\eta_n(dx) := \Psi_{g_n}(\mu)(dx) = \frac{1}{\mu(g_n)}\ g_n(x)\ \mu(dx)$$

Running examples :

$$\begin{array}{rcl} g_n = 1_{A_n} & \Rightarrow & \eta_n(dx) \propto 1_{A_n}(x)\ \mu(dx) \\ g_n = e^{-\beta_n V} & \Rightarrow & \eta_n(dx) \propto e^{-\beta_n V(x)}\ \mu(dx) \\ g_n = \prod_{0 \leq p \leq n} h_p & \Rightarrow & \eta_n(dx) \propto \left\{ \prod_{0 \leq p \leq n} h_p(x) \right\}\ \mu(dx) \end{array}$$

Applications : global optimization pb., tails distributions, hidden Markov chain models, etc.

Boltzmann-Gibbs distribution flows

- Target distribution flow : $\eta_n(dx) \propto g_n(x)\ \mu(dx)$
- Product hypothesis :

$$g_n = g_{n-1} \times G_{n-1} \Longrightarrow \eta_n = \Psi_{G_{n-1}}(\eta_{n-1})$$

Running Examples:

$$\begin{array}{rcl} g_n &=& 1_{A_n} \quad \text{with } A_n \downarrow \quad \Rightarrow \quad G_{n-1} = 1_{A_n} \\ g_n &=& e^{-\beta_n V} \text{ with } \beta_n \uparrow \quad \Rightarrow \quad G_{n-1} = e^{-(\beta_n - \beta_{n-1})V} \\ g_n &=& \prod_{0 \leq p \leq n} h_p \qquad \Rightarrow \quad G_{n-1} = h_n \end{array}$$

- Problem : $\eta_n = \Psi_{G_{n-1}}(\eta_{n-1})$ = unstable equation.

FK-stabilization

Choose $M_n(x,dy)$ s.t. local fixed point eq. $\rightarrow \eta_n = \eta_n M_n$

Examples (Metropolis, Gibbs,...) :

- Set restriction :

$$\eta_n(dx) = \frac{1}{\mu(A_n)}\ 1_{A_n}(x)\ \mu(dx)$$

Hyp. M reversible w.r.t μ ($\Leftrightarrow \mu(fM(g)) = \mu(M(f)g)$):

$$\rightsquigarrow \quad M_n(x,dy) = M(x,dy)\ 1_{A_n}(y) + (1 - M(x,A_n))\ \delta_x(dy)$$

Proof :

$$\begin{aligned}\mu\left(1_{A_n}\ M_n(f)\right) &= \mu\left(1_{A_n}\ M(f1_{A_n})\right) + \mu\left(1_{A_n}\ (1 - M(1_{A_n}))\ f\right)\\ &= \mu\left(M(1_{A_n})\ f1_{A_n}\right) + \mu\left(1_{A_n}\ (1 - M(1_{A_n}))\ f\right) = \mu(1_{A_n} f)\end{aligned}$$

$$\Rightarrow \eta_n(M_n(f)) = \frac{\mu\left(1_{A_n}\ M_n(f)\right)}{\mu\left(1_{A_n}\right)} = \frac{\mu(1_{A_n} f)}{\mu(1_{A_n})} = \eta_n(f)$$

■

Example of reversible Markov

For any $a \in [0,1)$, the Markov transition

$$M(x,dy) = \frac{1}{\sqrt{2\pi}}\ \exp\left(-\frac{1}{2}\ (y-ax)^2\right)\ dy$$

is reversible with respect to the gaussian measure on $E = \mathbb{R}$ given by

$$\mu(dx) = \sqrt{\frac{1-a^2}{2\pi}}\ \exp\left\{-\frac{1-a^2}{2}\ x^2\right\}\ dx$$

(**Exercise**).

FK-stabilization

- Boltzmann-Gibbs measures :

$$\eta_n(dx) = \frac{1}{\mu(e^{U_n})}\; e^{U_n(x)}\; \mu(dx)$$

Hyp. M reversible w.r.t μ ($\Leftrightarrow \mu(fM(g)) = \mu(M(f)g)$):

$$\rightsquigarrow \quad M_n(x,dy) = M(x,dy)\; e^{U_n(y)} + (1 - M(e^{U_n})(x))\; \delta_x(dy)$$

Proof :

$$\begin{aligned} \mu\left(e^{U_n}\; M_n(f)\right) &= \mu\left(e^{U_n}\; M(fe^{U_n})\right) + \mu\left(e^{U_n}\; \left(1 - M(e^{U_n})\right)\; f\right) \\ &= \mu\left(M(e^{U_n})\; fe^{U_n}\right) + \mu\left(e^{U_n}\; \left(1 - M(e^{U_n})\right)\; f\right) = \mu(e^{U_n} f) \end{aligned}$$

$$\Rightarrow \eta_n(M_n(f)) = \frac{\mu\left(e^{U_n}\; M_n(f)\right)}{\mu\left(e^{U_n}\right)} = \frac{\mu(e^{U_n} f)}{\mu(e^{U_n})} = \eta_n(f)$$

■

FK-stabilization

- Choose $M_n(x, dy)$ s.t. local fixed point eq. $\rightarrow \eta_n = \eta_n M_n$
- **Stable equation :**

$$g_n = g_{n-1} \times G_{n-1} \quad \Longrightarrow \quad \eta_n = \Psi_{G_{n-1}}(\eta_{n-1})$$
$$\Longrightarrow \quad \eta_n = \eta_n M_n = \Psi_{G_{n-1}}(\eta_{n-1}) M_n$$

- **Feynman-Kac "dynamical" formulation (X_n Markov M_n)**

$$\int f(x)\, g_n(x)\, \mu(dx) \propto \mathbb{E}\left(f(X_n) \prod_{0 \leq p < n} G_p(X_p) \right)$$

- $\rightsquigarrow$ Interacting Metropolis/Gibbs/... stochastic algorithms.

Outline

Stochastic sampling problems

- "Nonlinear" distribution flow with $\uparrow$ level of complexity.

$$\eta_n(dx_n) = \frac{\gamma_n(dx_n)}{\gamma_n(1)} \quad \textit{Time index} \;\; n \in \mathbb{N} \quad \textit{State var.} \;\; x_n \in E_n$$

- Two objectives :

  1. $\sim$ "Sampling independent " random variables w.r.t. η_n
  2. Computation of the normalizing constants $\gamma_n(1)$ ($= \mathcal{Z}_n$ Partition functions).

- **Examples :** Prediction/Updating filtering equation, series of condition events, decreasing temperature schedule,...

Two simple ingredients

- Find or Understand the probability mass transformation

$$\eta_n = \Phi_n(\eta_{n-1})$$

$\sim$ *Cooling schemes, temp. variations, constraints sequences, subset restrictions, observation data, conditional events,...*

- Natural interacting sampling idea :

Use η_{n-1} or <u>its empirical approx.</u> to sample w.r.t. η_n

1. Monte-Carlo/ Mean Field models :

$$\eta_n = \mathrm{Law}(\overline{X}_n) \quad \text{with} \quad \text{Markov} : \ \overline{X}_{n-1} \xrightarrow{\sim\ \eta_{n-1}} \overline{X}_n$$

2. Interacting MCMC models :

$$\left\{ \begin{array}{l} \text{Use the occupation measures} \\ \text{of an MCMC with target } \eta_{n-1} \end{array} \right\} \rightsquigarrow \text{MCMC target } \eta_n$$

Outline

Mean field interpretation

- **Nonlinear Markov models :** Always $\exists K_{n,\eta}(x, dy)$ Markov s.t.

$$\eta_n = \Phi_n(\eta_{n-1}) = \eta_{n-1} K_{n,\eta_{n-1}} = \text{Law}\left(\overline{X}_n\right)$$

  i.e. :

$$\mathbb{P}(\overline{X}_n \in dx_n \mid \overline{X}_{n-1}) = K_{n,\eta_{n-1}}(\overline{X}_{n-1}, dx_n)$$

- **McKean measures :**

$$\text{Law}\left(\overline{X}_0, \overline{X}_1, \ldots, \overline{X}_n\right)$$

$$\downarrow$$

$$\mathbb{P}\left((\overline{X}_0, \overline{X}_1, \ldots, \overline{X}_n) \in d\left(x_0, x_1, \ldots, x_n\right)\right)$$

$$= \eta_0(dx_0) K_{1,\eta_0}(x_0, dx_1) \ldots K_{n,\eta_{n-1}}(x_{n-1}, dx_n)$$

Mean field particle interpretation

- Markov chain $\xi_n = (\xi_n^1, \ldots, \xi_n^N) \in E_n^N$ s.t.

$$\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq_{N \uparrow \infty} \eta_n$$

- Particle approximation transitions ($\forall 1 \leq i \leq N$)

$$\xi_{n-1}^i \rightsquigarrow \xi_n^i \sim K_{n,\eta_{n-1}^N}(\xi_{n-1}^i, dx_n)$$

Discrete generation mean field particle model

Schematic picture : $\xi_n \in E_n^N \rightsquigarrow \xi_{n+1} \in E_{n+1}^N$

$$
\begin{array}{ccc}
\xi_n^1 & \xrightarrow{K_{n+1,\eta_n^N}} & \xi_{n+1}^1 \\
\vdots & & \vdots \\
\xi_n^i & \longrightarrow & \xi_{n+1}^i \\
\vdots & & \vdots \\
\xi_n^N & \longrightarrow & \xi_{n+1}^N
\end{array}
$$

Rationale :

$$
\begin{array}{rcl}
\eta_n^N \simeq_{N\uparrow\infty} \eta_n & \Longrightarrow & K_{n+1,\eta_n^N} \simeq_{N\uparrow\infty} K_{n+1,\eta_n} \\
& \Longrightarrow & \xi_n^i \text{ almost iid copies of } \overline{X}_n
\end{array}
$$

An elementary mean field interpretation model

- **Nonlinear Markov models :** Always $\exists K_{n,\eta}(x,dy)$ Markov s.t.

$$\eta_n = \Phi_n(\eta_{n-1}) = \eta_{n-1} K_{n,\eta_{n-1}} = \text{Law}\left(\overline{X}_n\right)$$

$$\downarrow \quad \text{Example}$$

$$K_{n,\eta_{n-1}}(x_{n-1}, dx_n) = \Phi_n(\eta_{n-1})(dx_n)$$

- **McKean measures :**

$$\text{Law}\left(\overline{X}_0, \overline{X}_1, \ldots, \overline{X}_n\right) = \eta_0 \otimes \eta_1 \otimes \ldots \otimes \eta_n$$

$$\Updownarrow$$

$$\mathbb{P}\left(\left(\overline{X}_0, \overline{X}_1, \ldots, \overline{X}_n\right) \in d\left(x_0, x_1, \ldots, x_n\right)\right) = \eta_0(dx_0)\eta_1(dx_1)\ldots\eta_n(dx_n)$$

Mean field particle interpretation

- Markov chain $\xi_n = (\xi_n^1, \ldots, \xi_n^N) \in E_n^N$ s.t.

$$\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq_{N \uparrow \infty} \eta_n$$

- Particle approximation transitions ($\forall 1 \leq i \leq N$)

$$\xi_{n-1}^i \rightsquigarrow \xi_n^i \sim \Phi_n \left(\frac{1}{N} \sum_{j=1}^N \delta_{\xi_{n-1}^j} \right) (dx_n)$$

$$\Updownarrow \text{[Given the past occupation measure]}$$

$$(\xi_n^1, \ldots, \xi_n^N) \quad \text{i.i.d.} \quad \sim \quad \Phi_n \left(\frac{1}{N} \sum_{j=1}^N \delta_{\xi_{n-1}^j} \right)$$

Ex.: Feynman-Kac distribution flows

- FK-Nonlinear Markov models :

$\epsilon_n = \epsilon_n(\eta_n) \geq 0$ s.t. η_n-a.e. $\epsilon_n G_n \in [0,1]$ ($\epsilon_n = 0$ not excluded)

$$K_{n+1,\eta_n}(x, dz) = \int S_{n,\eta_n}(x, dy)\; M_{n+1}(y, dz)$$

$$S_{n,\eta_n}(x, dy) := \epsilon_n G_n(x)\; \delta_x(dy) + (1 - \epsilon_n G_n(x))\; \Psi_{G_n}(\eta_n)(dy)$$

- Mean field genetic type particle model :

$$\xi_n^i \in E_n \xrightarrow{\text{accept/reject/selection}} \widehat{\xi}_n^i \in E_n \xrightarrow{\text{proposal/mutation}} \xi_{n+1}^i \in E_{n+1}$$

- **Examples :**
  - $G_n = 1_A \rightsquigarrow$ killing with uniform replacement.
  - M_n-Metropolis/Gibbs moves $\rightsquigarrow$ G_n-interaction function (subsets fitting or change of temperatures)

Mean field genetic type particle model :

$$
\left.\begin{array}{c} \xi_n^1 \\ \vdots \\ \xi_n^i \\ \vdots \\ \xi_n^N \end{array}\right] \xrightarrow{S_{n,\eta_n^N}} \left[\begin{array}{ccc} \widehat{\xi}_n^1 & \xrightarrow{M_{n+1}} & \xi_{n+1}^1 \\ \vdots & & \vdots \\ \widehat{\xi}_n^i & \longrightarrow & \xi_{n+1}^i \\ \vdots & & \vdots \\ \widehat{\xi}_n^N & \longrightarrow & \xi_{n+1}^N \end{array}\right]
$$

Accept/Reject/Selection transition :

$$S_{n,\eta_n^N}(\xi_n^i, dx)$$

$$:= \epsilon_n G_n(\xi_n^i)\ \delta_{\xi_n^i}(dx) + \left(1 - \epsilon_n G_n(\xi_n^i)\right)\ \sum_{j=1}^N \frac{G_n(\xi_n^j)}{\sum_{k=1}^N G_n(\xi_n^k)} \delta_{\xi_n^j}(dx)$$

Ex. : $G_n = 1_A,\ \epsilon_n = 1 \rightsquigarrow G_n(\xi_n^i) = 1_A(\xi_n^i)$

$\hookrightarrow$ FK-Mean field particle models = *sequential Monte Carlo, population Monte Carlo, genetic algorithms, particle filters, pruning, spawning, reconfiguration, quantum Monte carlo, go with the winner...*

Path space models

- $X_n = (X'_0, \ldots, X'_n) \rightsquigarrow$ **genealogical tree/ancestral lines**

$$\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{(\xi_{0,n}^i, \xi_{1,n}^i, \ldots, \xi_{n,n}^i)} \simeq_{N \uparrow \infty} \eta_n$$

- **Unbias particle approximations :**

$$\gamma_n^N(1) = \prod_{0 \leq p < n} \eta_p^N(G_p) \simeq_{N \uparrow \infty} \gamma_n(1) = \prod_{0 \leq p < n} \eta_p(G_p)$$

- **Ex.** $G_n = 1_A$ **:**

$$\Rightarrow \gamma_n^N(1) = \prod_{0 \leq p < n} (\text{success \% at p})$$

- **Complete genealogical tree $\simeq$ McKean measures**

$$\frac{1}{N} \sum_{1 \leq i \leq N} \delta_{(\xi_0^i, \xi_1^i, \ldots, \xi_n^i)} \simeq_{N \uparrow \infty} \eta_0 \times K_{1,\eta_0} \times \ldots \times K_{n,\eta_{n-1}}$$

$\epsilon_n = 0$

- "Elementary" FK-Nonlinear Markov models :

$$K_{n+1,\eta_n}(x, dz) \;=\; \int \Psi_{G_n}(\eta_n)(dy)\; M_{n+1}(y, dz) = \Phi_{n+1}(\eta_n)(dy)$$

- Simple genetic particle model :

$$\xi_n^i \in E_n \xrightarrow{\text{accept/reject/selection}} \widehat{\xi}_n^i \in E_n \xrightarrow{\text{proposal/mutation}} \xi_{n+1}^i \in E_{n+1}$$

with

$$\widehat{\xi}_n^1, \ldots, \widehat{\xi}_n^N \quad \text{i.i.d.} \quad \sim \Psi_{G_n}\left(\eta_n^N\right)$$

$$\Updownarrow$$

$$\xi_{n+1}^1, \ldots, \xi_{n+1}^N \quad \text{i.i.d.} \quad \sim \Phi_{n+1}\left(\frac{1}{N}\sum_{i=1}^N \delta_{\xi_n^i}\right)$$

$\epsilon_n = 0 \Rightarrow$ Simple genetic particle model :

$$\left.\begin{array}{c} \xi_n^1 \\ \vdots \\ \xi_n^i \\ \vdots \\ \xi_n^N \end{array}\right] \xrightarrow{\textit{Selection}} \left[\begin{array}{ccc} \widehat{\xi}_n^1 & \xrightarrow{M_{n+1}} & \xi_{n+1}^1 \\ \vdots & & \vdots \\ \widehat{\xi}_n^i & \longrightarrow & \xi_{n+1}^i \\ \vdots & & \vdots \\ \widehat{\xi}_n^N & \longrightarrow & \xi_{n+1}^N \end{array}\right]$$

Elementary selection transition :

$$\widehat{\xi}_n^1, \ldots, \widehat{\xi}_n^N \quad \text{i.i.d.} \quad \sim \Psi_{G_n}\left(\eta_n^N\right) = \sum_{j=1}^{N} \frac{G_n(\xi_n^j)}{\sum_{k=1}^{N} G_n(\xi_n^k)} \delta_{\xi_n^j}$$

Path space models with $\epsilon_n = 0$

- $X_n = (X'_0, \ldots, X'_n) \rightsquigarrow$ **genealogical tree/ancestral lines**

$$\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{(\xi_{0,n}^i, \xi_{1,n}^i, \ldots, \xi_{n,n}^i)} \simeq_{N\uparrow\infty} \eta_n$$

- **Unbias particle approximations :**

$$\gamma_n^N(1) = \prod_{0 \leq p < n} \eta_p^N(G_p) \simeq_{N\uparrow\infty} \gamma_n(1) = \prod_{0 \leq p < n} \eta_p(G_p)$$

- **Ex.** $G_n = 1_A$ **:**

$$\Rightarrow \gamma_n^N(1) = \prod_{0 \leq p < n} (\text{success \% at p})$$

- **Complete genealogical tree $\simeq$ McKean measures**

$$\frac{1}{N} \sum_{1 \leq i \leq N} \delta_{(\xi_0^i, \xi_1^i, \ldots, \xi_n^i)} \simeq_{N\uparrow\infty} \eta_0 \otimes \eta_1 \otimes \ldots \otimes \eta_n$$

Outline

Non asymptotic results

- **Weak estimates** $\leftrightarrow$ **Bias estimates** ($\leftrightarrow$ **Propagations of chaos)**

Law(q particles among N at time n) $\simeq_{N\uparrow\infty}$ **Law(q iid r.v. w.r.t.** η_n)

1. Total variation $= \frac{q^2}{N} c(n)$, Boltzmann entropy $= \frac{q}{N} c(n)$.
2. Stable models: uniform estimates w.r.t. time $\sup_n c(n) < \infty$.
3. Path space and genealogical tree models $c(n) = c \times n$.
4. Explicit weak decompositions at any order $\frac{1}{N^k}$.

$\hookrightarrow$ http-ref : DM-Patras-Rubenthaler, Coalescent tree based functional representations for some Feynman-Kac particle models, Hal-INRIA (2006).

- $\mathbb{L}_p$**-mean error bounds** [(2),(3) as above]

$$\sup_{N\geq 1} \sqrt{N}\, \mathbb{E}\left(\sup_{f_n\in\mathcal{F}_n} \left|\eta_n^N(f_n) - \eta_n(f_n)\right|^p\right) \leq b(p)\ c(n)$$

- **Exponential estimates** [(2) as above & empirical processes $\sim \mathcal{F}_n$]

$$\mathbb{P}(|\eta_n^N(f_n) - \eta_n(f_n)| > \epsilon) \leq c(n)\ \exp\left\{-\epsilon^2 N/c(n)\right\}$$

A stochastic perturbation model $\Leftrightarrow$ Uniform estimates w.r.t. time

Feynman-Kac (nonlinear) dynamical semigroup : $\boxed{\eta_p \rightsquigarrow \Phi_{p,n}(\eta_p) := \eta_n}$

A local transport formulation (works $\forall$ approximation scheme $\eta_n^N \simeq \eta_n$!)

$$
\begin{array}{ccccccccc}
\eta_0 & \rightarrow & \eta_1 = \Phi_1(\eta_0) & \rightarrow & \eta_2 = \Phi_{0,2}(\eta_0) & \rightarrow & \cdots & \rightarrow & \eta_n = \Phi_{0,n}(\eta_0) \\
\Downarrow & & & & & & & & \\
\eta_0^N & \rightarrow & \Phi_1(\eta_0^N) & \rightarrow & \Phi_{0,2}(\eta_0^N) & \rightarrow & \cdots & \rightarrow & \Phi_{0,n}(\eta_0^N) \\
 & & \Downarrow & & & & & & \\
 & & \eta_1^N & \rightarrow & \Phi_2(\eta_1^N) & \rightarrow & \cdots & \rightarrow & \Phi_{1,n}(\eta_1^N) \\
 & & & & \Downarrow & & & & \\
 & & & & \eta_2^N & \rightarrow & \cdots & \rightarrow & \Phi_{2,n}(\eta_2^N) \\
 & & & & & & \Downarrow & & \vdots \\
 & & & & & & \eta_{n-1}^N & \rightarrow & \Phi_n(\eta_{n-1}^N) \\
 & & & & & & & & \Downarrow \\
 & & & & & & & & \eta_n^N
\end{array}
$$

$\rightsquigarrow$**Key decomposition formula**

$$
\eta_n^N - \eta_n = \sum_{q=0}^{n} \left[\Phi_{q,n}(\eta_q^N) - \Phi_{q,n}(\Phi_q(\eta_{q-1}^N))\right] \text{ `` } \simeq \text{ '' } \sum_{q=0}^{n} \frac{1}{\sqrt{N}} \, e^{-\lambda\,(n-q)}
$$

Semigroup structure

$$\begin{aligned}\gamma_n(f_n) &= \mathbb{E}[f_n(X_n) \prod_{0\leq k<n} G_k(X_k)] \\ &= \mathbb{E}\left[\mathbb{E}\left[f_n(X_n) \prod_{p\leq k<n} G_k(X_k) \mid X_0,\ldots,X_p\right] \prod_{0\leq k<p} G_k(X_k)\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[f_n(X_n) \prod_{p\leq k<n} G_k(X_k) \mid X_p\right] \prod_{0\leq k<p} G_k(X_k)\right]\end{aligned}$$

$$\Downarrow$$

$$\gamma_n(f_n) = \mathbb{E}\left[Q_{p,n}(f_n)(X_p) \prod_{0\leq k<p} G_k(X_k)\right] = \gamma_p\left(Q_{p,n}(f_n)\right)$$

with

$$Q_{p,n}(f_n)(x_p) = \mathbb{E}\left[f_n(X_n) \prod_{p\leq k<n} G_k(X_k) \mid X_p = x_p\right]$$

Semigroup structure

$$\forall f_n \quad \gamma_n(f_n) = \gamma_p\left(Q_{p,n}(f_n)\right) \Longleftrightarrow \gamma_n = \gamma_p Q_{p,n}$$

Note :

$$Q_{p,n}(f_n)(x_p) = \mathbb{E}\left[f_n(X_n) \prod_{p \leq k < n} G_k(X_k) \mid X_p = x_p\right]$$

$$\Downarrow$$

$$Q_{n-1,n}(f_n)(x_{n-1}) = G_{n-1}(x_{n-1})\ M_n(f_n)(x_{n-1})$$

$$\Downarrow$$

$$\begin{aligned}
\gamma_n &= \gamma_{n-1} Q_{n-1,n} \\
&= \gamma_{n-2} Q_{n-2,n-1} Q_{n-1,n} \\
&= \ldots \\
&= \gamma_p \underbrace{Q_{p,p+1} \ldots Q_{n-2,n-1} Q_{n-1,n}}_{Q_{p,n}}
\end{aligned}$$

Semigroup structure

$$\gamma_n = \gamma_p Q_{p,n}$$

$$\Downarrow$$

$$\Phi_{p,n}(\eta_p)(f_n) \quad = \quad \frac{\gamma_p Q_{p,n}(f_n)}{\gamma_p Q_{p,n}(1)} = \frac{\gamma_p Q_{p,n}(f_n)/\gamma_p(1)}{\gamma_p Q_{p,n}(1)\gamma_p(1)}$$

$$\Downarrow$$

$$\Phi_{p,n}(\eta_p)(f_n) \quad = \quad \frac{\eta_p Q_{p,n}(f_n)}{\eta_p Q_{p,n}(1)} = \frac{\eta_p(G_{p,n}P_{p,n}(f_n))}{\eta_p(G_{p,n})} = \Psi_{G_{p,n}}(\eta_p)P_{p,n}(f_n)$$

with the Potential function $G_{p,n}$ and the Markov transition $P_{p,n}$

$$G_{p,n}(x_p) = Q_{p,n}(1)(x_p) \quad \text{and} \quad P_{p,n}(f_n)(x_p) = \frac{Q_{p,n}(f_n)(x_p)}{Q_{p,n}(1)(x_p)}$$

Semigroup structure & Stability properties

$$\Phi_{p,n}(\eta_p) \quad = \quad \Psi_{G_{p,n}}(\eta_p) P_{p,n}(f_n)$$

Important observation :

$$\begin{aligned} P_{p,n}(f_n) &= \frac{Q_{p,n}(f_n)(x_p)}{Q_{p,n}(1)(x_p)} = \frac{Q_{p,p+1}Q_{p+1,n}(f_n)}{Q_{p,p+1}Q_{p+1,n}(1)} = \frac{M_{p+1}[Q_{p+1,n}(f_n)]}{M_{p+1}[Q_{p+1,n}(1)]} \\ &= \frac{M_{p+1}[G_{p+1,n} \ P_{p+1,n}(f_n)]}{M_{p+1}[G_{p+1,n}]} \\ &= R^{(n)}_{p+1} P_{p+1,n}(f_n) \quad \Rightarrow \quad P_{p,n} = R^{(n)}_{p+1} R^{(n)}_{p+2} \ldots R^{(n)}_{n-1} R^{(n)}_{n} \end{aligned}$$

with the **Markov transitions** $\quad R^{(n)}_p(f_p) = \dfrac{M_p(G_{p,n} f_p)}{M_p(G_{p,n})}$

$$\Downarrow$$

$$\Phi_{p,n}(\eta_p) = \Psi_{G_{p,n}}(\eta_p) \ \underbrace{R^{(n)}_{p+1} R^{(n)}_{p+2} \ldots R^{(n)}_{n-1} R^{(n)}_{n}}_{(n-p) \text{ Markov transitions}}$$

Semigroup structure & Stability properties

$$\Phi_{p,n}(\eta_p) = \Psi_{G_{p,n}}(\eta_p) \underbrace{R^{(n)}_{p+1} R^{(n)}_{p+2} \ldots R^{(n)}_{n-1} R^{(n)}_n}_{(n-p) \text{ Markov transitions}}$$

Under some mixing conditions on the Markov transitions $R^{(n)}_p$

$$\Phi_{p,n}(\eta_p) = \Psi_{G_{p,n}}(\eta_p) R^{(n)}_{p+1} R^{(n)}_{p+2} \ldots R^{(n)}_{n-1} R^{(n)}_n$$

$$\simeq_{(n-p)\uparrow\infty}$$

$$\Phi_{p,n}(\eta'_p) = \Psi_{G_{p,n}}(\eta'_p) R^{(n)}_{p+1} R^{(n)}_{p+2} \ldots R^{(n)}_{n-1} R^{(n)}_n$$

Stability prop. of the s.g. $\Phi_{p,n} \Longleftrightarrow$ Stability prop. of non homogeneous Markov chains $\sim$ $(R^{(n)}_p)_{0\leq p\leq n}$

↪ On the stability of interacting proc. DM P. and Guionnet A. Annales de l'IHP, Vol. 37, No. 2, 155-194 (2001).
↪ Feynman-Kac formulae. Genealogical and interacting particle systems, Springer (2004), Chap 4 & 5

Some crude uniform estimates w.r.t. time

Hypothesis : (Time homogeneous models) $\exists (m, r)$ s.t. for any (x, y)

$$M^m(x, .) \geq \epsilon \, M^m(y, .) \quad \text{and} \quad G_n(x) \leq r \, G_n(y)$$

- **Limiting system stability properties :**

$$\|\Phi_{p,p+nm}(\eta) - \Phi_{p,p+nm}(\mu)\|_{tv} \leq (1 - \epsilon^2/r^{m-1})^n$$

  and w.r.t. Csiszár's H-entropy criteria

$$H(\Phi_{p,p+nm}(\mu), \Phi_{p,p+nm}(\eta)) \leq \alpha_H(r^m/\epsilon) \, (1 - \epsilon^2/r^{m-1})^n \, H(\mu, \eta)$$

- **Examples :**

  $\alpha_H(t) = t$ (tv norm & Boltzmann entropy), $\alpha_H(t) = t^{1+p}$ (Havrda-Charvat & Kakutani-Hellinger p-integrals, $\alpha_H(t) = t^3$ ($\mathbb{L}_2$-norm),...

Some crude uniform estimates w.r.t. time

Hypothesis : (Time homogeneous models) $\exists (m,r)$ s.t. for any (x,y)

$$M^m(x,.) \;\geq\; \epsilon\; M^m(y,.) \quad \text{and} \quad G_n(x) \;\leq\; r\; G_n(y)$$

- $\mathbb{L}_p$**-mean error bounds**

$$\sup_{n\geq 0}\; \sup_{N\geq 1} \sqrt{N}\; \mathbb{E}\left(\left|\left[\eta_n^N-\eta_n\right](f)\right|^p\right)^{\frac{1}{p}} \leq 2\; b(p)\; m\; r^{2m-1}/\epsilon^3$$

with $\quad b(2p)^{2p}=(2p)_p\, 2^{-p} \quad$ and $\quad b(2p+1)^{2p+1}=\dfrac{(2p+1)_{(p+1)}}{\sqrt{p+1/2}}\; 2^{-(p+1/2)}$

- **Uniform concentration estimates :**

$$\sup_{n\geq 0} \mathbb{P}\left(\left|\left[\eta_n^N-\eta_n\right](f)\right| \geq\; \delta\right) \leq 6\; \exp\left(-N\; \delta^2\; \epsilon^5/(32 m r^{4m-1})\right)$$

- **Extensions to Zolotarev's seminorms** $\|\eta_n^N-\eta_n\|_{\mathcal{F}}=\sup_{f\in\mathcal{F}}|[\eta_n^N-\eta_n](f)|$

Fluctuations

- **Central Limit Theorems** [Sharp $\mathbb{L}_p$ estimates]

{http-ref : 1999⇝2004 : DM, Guionnet, Jacod, Ledoux, Tindel}

$$V_n^N(f) := \sqrt{N}\ \left[\eta_n^N(f) - \eta_n(f)\right] \Longrightarrow V_n(f) = \text{Centered Gaussian r.v.}$$

1. **Functional Central Limit Theorems.** $[\forall d,\ \forall (f^i)_{1\leq i\leq d}]$

$$\left(V_n^N(f^1), \ldots, V_n^N(f^d)\right) \Longrightarrow \left(V_n(f^1), \ldots, V_n(f^d)\right)$$

2. **Unbounded $\mathbb{L}_2$-functions $\oplus$ algebra sets of functions with some growth conditions.**
   ↪ (Path space models) DM, Guionnet. Annals of Applied Probability, Vol. 9, No. 2, 275-297 (1999).
   ↪ (Donsker+explicit variance) DM, Ledoux, Journal of Theoret. Probability, Vol. 13, No. 1, 225-257 (2000).
   ↪ (marginal approx. models) DM, Jacod, The Fields Institute Communications, Ed. T.J. Lyons, T.S. Salisbury, American Mathematical Society, (2002).
3. **Donsker type theorems, Berry Esseen type theorems, path spaces,...**

Large deviations

- **Large deviations principles** [Sharp asymptotic expo estimates]

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(\eta_n^N \not\in \mathcal{V}(\eta_n)\right)$$

Example : $\mathcal{V}(\eta_n) = \{\mu \ : \ |\eta_n^N(f) - \eta_n(f)| \leq \epsilon\}$ (weak and strong τ-topo).

$\{$`http-ref 1998⇝2004 :  DM, Dawson, Guionnet, Zajic`$\}$

Feynman-Kac (nonlinear) semigroup $\eta_p \longrightarrow \Phi_{p,n}(\eta_p) := \eta_n$

LOCAL FLUCTUATION THEOREM : $W_n^N := \sqrt{N}\left[\eta_n^N - \Phi_n\left(\eta_{n-1}^N\right)\right] \simeq W_n$ **Centered and Independent Gaussian field**

Local transport formulation :

$$
\begin{array}{ccccccccc}
\eta_0 & \rightarrow & \eta_1 = \Phi_1(\eta_0) & \rightarrow & \eta_2 = \Phi_{0,2}(\eta_0) & \rightarrow & \cdots & \rightarrow & \Phi_{0,n}(\eta_0) \\
\Downarrow & & & & & & & & \\
\eta_0^N & \rightarrow & \Phi_1(\eta_0^N) & \rightarrow & \Phi_{0,2}(\eta_0^N) & \rightarrow & \cdots & \rightarrow & \Phi_{0,n}(\eta_0^N) \\
 & & \Downarrow & & & & & & \\
 & & \eta_1^N & \rightarrow & \Phi_2(\eta_1^N) & \rightarrow & \cdots & \rightarrow & \Phi_{1,n}(\eta_1^N) \\
 & & & & \Downarrow & & & & \\
 & & & & \eta_2^N & \rightarrow & \cdots & \rightarrow & \Phi_{2,n}(\eta_2^N) \\
 & & & & & & & & \vdots \\
 & & & & & & \Downarrow & & \\
 & & & & & & \eta_{n-1}^N & \rightarrow & \Phi_n(\eta_{n-1}^N) \\
 & & & & & & & & \Downarrow \\
 & & & & & & & & \eta_n^N
\end{array}
$$

$\rightsquigarrow$ **Key decomposition formula entering the stability of the limiting system:**

$$
\begin{aligned}
\eta_n^N - \eta_n &= \sum_{q=0}^{n} \left[\Phi_{q,n}(\eta_q^N) - \Phi_{q,n}(\Phi_q(\eta_{q-1}^N))\right] \\
&\simeq \frac{1}{\sqrt{N}} \sum_{q=0}^{n} W_q^N D_{q,n} \longleftrightarrow \text{First order decomp. } \Phi_{p,n}(\eta) - \Phi_{p,n}(\mu) \simeq (\eta-\mu)D_{p,n} + (\eta-\mu)^{\otimes 2} \ldots
\end{aligned}
$$

$\Rightarrow$ **Two lines proof of a Functional CLT :** $\sqrt{N}\left[\eta_n^N - \eta_n\right] \simeq \sum_{q=0}^{n} W_q D_{q,n}$

Outline

Objective

- Find a series of MCMC models $X^{(n)} := (X_k^{(n)})_{k\geq 0}$ s.t.

$$\begin{aligned}\eta_k^{(n)} &= \frac{1}{k+1}\sum_{0\leq l\leq k}\delta_{X_l^{(n)}}\\ &\simeq_{k\uparrow\infty} \eta_n\\ &\Rightarrow \text{Use } \eta_k^{(n)} \simeq \eta_n \text{ to define } X^{(n+1)} \text{ with target } \eta_{n+1}\end{aligned}$$

Advantages

- Using η_n the sampling η_{n+1} is often easier.
- Improve the proposition step in any Metropolis type model with target η_{n+1} ($\rightsquigarrow$ enters the stability prop. of the flow η_n)
- Increases the precision at every time step.

  But CLT variance often $\geq$ CLT variance mean field models.
- Easy to combine with mean field stochastic algorithms.

Interacting Markov chain Monte Carlo models

- Find M_0 and a collection of transitions $M_{n,\mu}$ s.t.

$$\eta_0 = \eta_0 M_0 \quad \text{and} \quad \Phi_n(\mu) = \Phi_n(\mu) M_{n,\mu}$$

- $(X_k^{(0)})_{k\geq 0}$ Markov chain $\sim M_0$.
- Given $X^{(n)}$, we let $X_k^{(n+1)}$ with Markov transtions $M_{n+1,\eta_k^{(n)}}$

Rationale :

$$\eta_k^{(n)} \simeq \eta_n \quad \Longrightarrow \quad \left\{ \begin{array}{rcl} \Phi_{n+1}(\eta_k^{(n)}) & \simeq & \Phi_{n+1}(\eta_n) = \eta_{n+1} \\ M_{n+1,\eta_k^{(n)}} & \simeq & M_{n+1,\eta_n} \quad \text{with fixed point } \eta_{n+1} \end{array} \right.$$
$$\Longrightarrow \quad \eta_k^{(n+1)} \simeq \eta_{n+1}$$

Example : $M_{n,\mu}(x,dy) = \Phi_n(\mu)(dy) \rightsquigarrow X_k^{(n+1)}$ r.v. $\sim \Phi_{n+1}\left(\eta_k^{(n)}\right)$

$((n-1)\text{-th chain})$ $\qquad\qquad$ $(n\text{-th chain})$

$$
\begin{array}{ccc}
X_0^{(n-1)} & & X_0^{(n)} \\
\downarrow & & \downarrow \\
X_1^{(n-1)} & & \vdots \\
\downarrow & & \vdots \\
\vdots & & \downarrow \\
\downarrow & & X_k^{(n)} \\
X_k^{(n-1)} & \xrightarrow{\eta_k^{(n-1)} \simeq \eta_{n-1}} & \downarrow \\
& & M_{n,\eta_k^{(n-1)}} \simeq M_{n,\eta_{n-1}} \\
\downarrow & & \downarrow \\
\vdots & & X_{k+1}^{(n)}
\end{array}
$$

Outline

[MEAN FIELD PARTICLE MODEL] **Nonlinear semigroup** $\longrightarrow \Phi_{p,n}(\eta_p) := \eta_n$

Local fluctuation theorem : $W_n^N := \sqrt{N}\left[\eta_n^N - \Phi_n\left(\eta_{n-1}^N\right)\right] \simeq W_n \perp$ Centered Gaussian field

Local transport formulation :

$$
\begin{array}{ccccccccc}
\eta_0 & \rightarrow & \eta_1 = \Phi_1(\eta_0) & \rightarrow & \eta_2 = \Phi_{0,2}(\eta_0) & \rightarrow & \cdots & \rightarrow & \Phi_{0,n}(\eta_0) \\
\Downarrow & & & & & & & & \\
\eta_0^N & \rightarrow & \Phi_1(\eta_0^N) & \rightarrow & \Phi_{0,2}(\eta_0^N) & \rightarrow & \cdots & \rightarrow & \Phi_{0,n}(\eta_0^N) \\
 & & \Downarrow & & & & & & \\
 & & \eta_1^N & \rightarrow & \Phi_2(\eta_1^N) & \rightarrow & \cdots & \rightarrow & \Phi_{1,n}(\eta_1^N) \\
 & & & & \Downarrow & & & & \\
 & & & & \eta_2^N & \rightarrow & \cdots & \rightarrow & \Phi_{2,n}(\eta_2^N) \\
 & & & & & & & & \vdots \\
 & & & & & & \Downarrow & & \\
 & & & & & & \eta_{n-1}^N & \rightarrow & \Phi_n(\eta_{n-1}^N) \\
 & & & & & & & & \Downarrow \\
 & & & & & & & & \eta_n^N
\end{array}
$$

$\rightsquigarrow$ **Key decomposition formula :**

$$
\begin{aligned}
\eta_n^N - \eta_n &= \sum_{q=0}^{n} \left[\Phi_{q,n}(\eta_q^N) - \Phi_{q,n}(\Phi_q(\eta_{q-1}^N))\right] \\
&\simeq \frac{1}{\sqrt{N}} \sum_{q=0}^{n} W_q^N D_{q,n} \hookleftarrow \text{First order decomp. } \Phi_{p,n}(\eta) - \Phi_{p,n}(\mu) \simeq (\eta-\mu)D_{p,n} + (\eta-\mu)^{\otimes 2} \ldots
\end{aligned}
$$

$\Rightarrow$ Example Functional CLT : $\sqrt{N}\left[\eta_n^N - \eta_n\right] \simeq \sum_{q=0}^{n} W_q D_{q,n}$

[i-MCMC] **Nonlinear sg** $\Phi_{p,n}(\eta_p) = \eta_n$ with a first order decomp. :

$$\Phi_{p,n}(\eta) - \Phi_{p,n}(\mu) \simeq (\eta - \mu) D_{p,n} + (\eta - \mu)^{\otimes 2} \ldots$$

$$\Downarrow$$

Functional CLT for correlated/interacting MCMC models :

$$\sqrt{k}\ \left[\eta_k^{(n)} - \eta_n\right] \simeq \sum_{q=0}^{n} \frac{\sqrt{(2(n-q))!}}{(n-q)!}\ V_q D_{q,n}$$

with $\left(V_q\right)_{q\geq 0} \perp$ Centered Gaussian field

$$\mathbb{E}\left(V_q(f)^2\right) = \eta_q\left[(f - \eta_q(f))^2\right] + 2\sum_{m\geq 1} \eta_q\left[(f - \eta_q(f))\ M^m_{q,\eta_{q-1}}(f - \eta_q(f))\right]$$

"Comparisons" : [Mean field case] $\left(W_q\right)_{q\geq 0} \perp$ Centered Gaussian field

$$\mathbb{E}\left(W_q(f)^2\right) = \eta_{q-1}\left\{K_{q,\eta_{q-1}}(f - K_{q,\eta_{q-1}}(f))^2\right\}$$

Case : $K_{q,\eta}(x,dy) = M_{q,\eta}(x,dy) = \Phi_q(\eta)(dy) \Longrightarrow (V_q = W_q) \Longrightarrow$ [Mean field] > [i-MCMC]

Outline

Interacting stochastic simulation algorithms

- **Mean field and Feynman-Kac particle models :**
  - Feynman-Kac formulae. Genealogical and interacting particle systems, Springer (2004) ⊕ Refs.
  - joint work with L. Miclo. A Moran particle system approximation of Feynman-Kac formulae. *Stochastic Processes and their Applications*, Vol. 86, 193-216 (2000).
  - joint work with L. Miclo. Branching and Interacting Particle Systems Approximations of Feynman-Kac Formulae. *Séminaire de Probabilités XXXIV, Lecture Notes in Mathematics, Springer-Verlag Berlin*, Vol. 1729, 1-145 (2000).
- **Sequential Monte Carlo models :**
  - joint work with Doucet A., Jasra A. Sequential Monte Carlo Samplers. JRSS B (2006).
  - joint work with A. Doucet. On a class of genealogical and interacting Metropolis models. Sém. de Proba. 37 (2003).

Interacting stochastic simulation algorithms

- **i-MCMC algorithms :**
  - joint work with A. Doucet. `Interacting Markov Chain Monte Carlo Methods For Solving Nonlinear Measure-Valued Eq.`, HAL-INRIA RR-6435, (Feb. 2008).
  - joint work with B. Bercu and A. Doucet. `Fluctuations of Interacting Markov Chain Monte Carlo Models`. HAL-INRIA RR-6438, (Feb. 2008).
  - joint work with C. Andrieu, A. Jasra, A. Doucet. *Non-Linear Markov chain Monte Carlo via self-interacting approximations.* Tech. report, Dept of Math., Bristol Univ. (2007).
  - joint work with A. Brockwell and A. Doucet. *Sequentially interacting Markov chain Monte Carlo.* Tech. report, Dept. of Statistics, Univ. of British Columbia (2007).