Genetic Genealogical Models in Rare Event Analysis

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Abstract

We present in this article a genetic type interacting particle systems algorithm and a genealogical model for estimating a class of rare events arising in physics and network analysis. We represent the distribution of a Markov process hitting a rare target in terms of a Feynman-Kac model in path space. We show how these branching particle models described in previous works can be used to estimate the probability of the corresponding rare events as well as the distribution of the process in this regime.

1 Introduction

Let $X = \{X_t \; ; \; t \in I\}$ be a Markov process taking values in some metric state space (S, d) with discrete or continuous time index

$$I = \mathbb{R}_+ = [0, \infty) \quad \text{or} \quad I = \mathbb{N} = \{0, 1, 2, \dots\}$$

We suppose X is defined on the canonical filtered probability space

$$(\Omega = D(I, S), \mathcal{G} = (\mathcal{G}_t)_{t \in I}, (\mathbb{P}_x)_{x \in S})$$

of left continuous and right limited paths $D(I, S)$ from I into S. The process X is assumed to have a recurrent Borel set $A_\infty \subset S$ and to start in some Borel set $A_0 \subset S$ with $A_0 \cap A_\infty = \emptyset$ with a given probability distribution η_0 on S. In other words if we write

$$\mathbb{P} = \int_S \eta_0(dx) \; \mathbb{P}_x \quad \text{and} \quad R = \inf\{t \geq 0 \; ; \; X_t \in A_\infty\}$$

the distribution of X and the entrance time into A_∞ then we have that

$$\mathbb{P}(R < \infty) = 1 \quad \text{and} \quad \mathbb{P}(X_0 \in A_0) = \eta_0(A_0) = 1$$

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During its excursion from A_0 to A_∞ the process visit different regions of the state space. For a given target Borel set $B \subset S$ we define the first time the process X hits B before visiting the recurrent set A_∞, namely

$$T = \inf\{0 \leq t < R \; : \; X_t \in B\}$$

We always take the convention $\inf_\infty = R$ so that $T = R$ if X never succeed to reach the desired target B before visiting A_∞. In may happens that most of the realizations of X never reach the set B before visiting A_∞. These rare sets are extremely difficult to analyze. In particular one would like to estimate the quantities

$$\mathbb{P}(T < R) \quad \text{and} \quad \mathrm{Law}(X_t \; ; \; 0 \leq t \leq T \mid T < R) \tag{1}$$

These two quantities have two "dual" interpretations.

- Suppose the state $S = A \cup A_\infty$ is decomposed into two separate regions A and A_∞. The process X starts in A and we want to estimate the probability of the entrance time into a target $B \subset A$ before exiting A. In this context the conditional distribution (1) represents the law of the process in this "ballistic" regime.
- Suppose the $S = B \cup C$ is decomposed into two separate regions B and C. The process X evolves in the region C which contains a collection of "hard obstacles" represented by a subset $A_\infty \cup C$. The particle is killed as soon as it enters the "hard obstacle" set A_∞. In this context the quantities (1) represent respectively the probability of exiting the pocket of obstacles C without been killed and the distribution of the process which succeed to escape this region.

The estimation of these quantites arise in many research areas such as in physics and engineering problems. In network analysis such as in advanced telecommunication systems studies X traditionally represents the length of service centers in an open/closed queueing network processing jobs. In this contect these two quantities represent repectively the probability of buffer-overflows and the distribution of the queueing process in this overflow regime.

Several numerical methods have been proposed in the literature to estimate the entrance probability into a rare set. We refer the reader to the excellent paper [8] which contains a precise review on these methods as well as a detailed list of references. For the convenience of the reader we present hereafter a brief description of the two leading ideas.

The first one is based on changing the reference probability so that rare sets becomes less rare. This probabilistic approach often requires the finding of the right change of measure. This step is usually done using large deviations techniques. Another more physical approach consists in splitting the state space into a sequence of sub-levels the particle need to pass before to reach the rare target. This splitting stage is based on a precise physical description of the evolution of the process between each level leading to the rare set. The next step is to introduce a system of particles evolving in this level decomposition of the state. Each particle branches as soon as they enter into a more important level.

The purpose of the present article is to connect the multilevel splitting techniques with the branching and interacting particle systems approximations of Feynman-kac distributions studied in previous articles. This work has been influenced by the three referenced papers [6, 7] and [8].

Our objective is two folds. First we propose a Feynman-Kac representation of the quantities (1). The general idea behind our construction is to consider the level crossing Markov chain in path space and associated to the splitting of the state space. The concatenation of the corresponding states will contain all information on the way the process passes each level before entering into the final and target rare set. Based on this modeling we introduce a natural mean field type genetic particle approximation of the desired quantities (1). More interestingly we also show how the genealogical structure of the particle at each level can be used to find the distribution of the process during its excursions to the rare target.

When the state space is splitted into m levels the particle model evolve into m steps. At time $n = 0$ the start with N independent copies of the process X. The particles which enter the recurrent set A_∞ are killed and instantly a particle having reached the first level produces an offspring. If the whole system is killed the algorithm is stopped. Otherwise by construction of the birth and death rule we obtain N particles at the first level. At time $n = 1$ the N particles in the first level evolve according to the same rule as the process X. Here again particles which enter the recurrent set A_∞ are killed and instantly a particle having reached the second level produces an offspring and so on.

From this brief description we see that the former particle scheme follows the same splitting strategies as the one discussed in [8]. The new mathematical models presented here allows to qualibrate with precision the asymptotic behavior of this particle techniques as the size of the systems tends to infinity. In addition and in contrast to previous articles on the subject the Feynman-Kac analysis in path space presented hereafter allows to study the genealogical structure of these splitting particle models. We will show that the empirical measures associated to the corresponding historical processes converge as $N \to \infty$ to the distribution of the whole path process between each levels.

An empirical method called RESTART [11, 10] can also be used to compute rare transient events and the probability of rare events in steady state, not only the probability to reach the target before coming back to a recurrent set. It is an advantage with respect to splitting, but this method requires some mathematical approximations. Nevertheless this method is not taking into account by the previous formalism. So, a further work could be an extension of the former particle scheme for covering RESTART.

A short description of the paper is as follows. Section 2 of this paper sets out the Feynman-Kac representation of the quantities (1). Section 3 provides the description of the corresponding genetic-type interacting particle system approximating model. Section 4 introduces a path-valued interacting particle systems model for the historical process associated to the previous genetic algorithm. Finally, Section 5 deals with a numerical example, based on the Ornstein-Uhlenbeck process. Estimation of exit time for diffusions controlled by potentials are suggested also, since the lack of exact calculations, even if some heuristics may be applied.

2 Multi-level Feynman-Kac formulae

In practice the process X, before visiting A_∞ or entering into the desired set B, passes through a decreasing sequence of Borel level sets $(B_n)_{n=0,\ldots,m+1}$ with

$$B = B_{m+1} \subset B_m \subset \ldots \subset B_1 \subset B_0$$

The parameter m and the sequence of level set depend on the problem at hand but it is important to choose these sets such that

$$A_0 = B_0 - B_1 \quad \text{and} \quad B_0 \cap A_\infty = \emptyset$$

and such that the process X cannot enter B_n before visiting B_{n-1}. The choice of the recurrent set A_∞ depends on the nature of the underlying process X. To visualize these level sets we propose hereafter the two "dual" constructions corresponding to the two "dual" interpretations presented in the introduction.

- In the ballistic recursion the state $S = A \cup A_\infty$ is decomposed into two disjoint Borel sets A and A_∞. The target set B is a subset of A. In this context the decreasing sequence $(B_n)_{n=0,\ldots,m+1}$ represents the physical levels the process X needs to pass before to reach B. When $S = \mathbb{R}^d$ is the euclidian space we can think of a sequence of centered decreasing balls with radius $1/(n+1)$

  $$B_n = \mathcal{B}(0, 1/(n+1)) \subset \mathbb{R}^d$$

  and for some $\epsilon \in (0,1)$
  $$A_\infty = S - \mathcal{B}(0, 1+\epsilon)$$

  When the process X we consider reaches this set in finite time then $\mathbb{P}(T < R)$ is the probability that X enters the centered ball with radius $1/(m+2)$ starting in some $1/2 < |X_0| \leq 1$ and before exiting the ball of radius $(1+\epsilon)$. The distribution (1) represents the conditional distribution of the process X in this ballistic regime. When $S = \mathbb{R}_+$ we can choose for instance the intervals $B_n = [n+1, \infty)$ and $A_\infty = [0, 1-\epsilon]$.

- In the case of a particle X evolving in a pocket C of the space S containing "hard obstacles" $A_\infty \subset C$ the target set is $B = S - C$. In this context we want to estimate the probability that the particle find the way out of the pocket of obstacles C before beeing killed. The sequence $(B_n)_{n=0,\ldots,m+1}$ represents the exit level sets the process need to reach to get out of C before beeing killed by an obstacle.

In connection with the case $S = \mathbb{R}^d$ we examined previously we take here for each $0 \leq n \leq (m+1)$

$$B_n = S - \mathcal{B}(0, 1/((m-n)+2)) \quad \text{and} \quad A_\infty = B(0, 1/(m+2)) = S - B_0$$

In this choice the process starts in $A_0 = B_0 - B_1$, that is

$$1/(m+2) < |X_0| \leq 1/(m+1)$$

and it is killed as soon as it hits the hard obstacle $B(0, 1/(m+2))$. The conditional distribution (1) represents the probability of exiting the unit ball before beeing killed.

To capture the behavior of X between the different levels $(B_n)_{n=0,\ldots,m+1}$ we introduce the discrete stochastic sequence

$$\mathcal{X}_n = (X_t \ ; \ T_{n-1} \leq t \leq T_n) \in E = \cup_{s \leq t} D([s,t], S)$$

where T_n, $1 \leq n \leq (m+1)$, represents the first time X reaches B_n before visiting A_∞, that is

$$T_n = \inf \{0 \leq t < R \ : \ X_t \in B_n\}$$

with the convention $\inf_\infty = R$. With this convention we can alternatively define T_n as the first time X enters into $A_\infty \cup B_n$, that is

$$T_n = \inf \{0 \leq t \ : \ X_t \in A_\infty \cup B_n\}$$

By construction we also notice that

$$T_0 = 0 \leq T_1 \leq \ldots \leq T_m \leq T_{m+1} = T$$

and for each n

$$T_n = R \Longrightarrow \forall p > n \quad T_p = R \quad \text{and} \quad \mathcal{X}_p = X_R \ (\in A_\infty \subset S)$$

By a direct inspection we also see that if $T_n < R$ then $\mathcal{X}_n$ represents the path of X between the successive levels B_{n-1} and B_n. From these observation we can alternatively define the times T_n by the inductive formula

$$T_n = \inf \{T_{n-1} \leq t < R \ : \ X_t \in B_n\}$$

with the convention $\inf_\infty = R$ so that $T_n = R$ if $T_{n-1} = R$ or if starting at B_{n-1} at time T_{n-1} the process never reaches B_n before visiting A_∞. We also observe that

$$(T < R) = (T_{m+1} < R) = (T_1 < R, \ldots, T_{m+1} < R)$$

By the strong Markov property we check that the stochastic sequence $(\mathcal{X}_n)_{0 \leq n \leq m+1}$ forms an E-valued Markov chain. One way to check whether the path has succeed to reach the desired n-th level is to consider the potential functions g_n on E defined for each $x = (x_r)_{s \leq r \leq t} \in D([s,t], S)$ with $s \leq t$ by

$$g_n(x) = 1_{B_n}(x_t)$$

In this notation we have for each n

$$\begin{array}{rcl}(T_n < R) & = & (T_1 < R, \ldots, T_n < R) = (g_1(\mathcal{X}_1) = 1 \ldots, g_n(\mathcal{X}_n) = 1) \\ (\mathcal{X}_0, \ldots, \mathcal{X}_n) & = & (X_0, (X_t \; ; \; 0 \leq t \leq T_1), \ldots, (X_t \; ; \; T_{n-1} \leq t \leq T_n))\end{array}$$

In the further development we slight abuse notations we write

$$(X_0, (X_t \; ; \; 0 \leq t \leq T_1), \ldots, (X_t \; ; \; T_{n-1} \leq t \leq T_n)) = [X_t \; ; \; 0 \leq t \leq T_n]$$

Thus we have proved the following Feynman-Kac representation of the quantities (1).

Theorem 1 (Multilevel Feynman-Kac formula)
For any n and for any $f \in \mathcal{B}_b(E)$ we have

$$\mathbb{E}(f(X_t \; ; \; T_{n-1} \leq t \leq T_n) \mid T_n < R) = \frac{\mathbb{E}(\; f(\mathcal{X}_n) \; \prod_{p=0}^n g_p(\mathcal{X}_p) \;)}{\mathbb{E}(\; \prod_{p=0}^n g_p(\mathcal{X}_p) \;)}$$

and

$$\mathbb{P}(T_n < R) = \mathbb{E}(\; \prod_{p=0}^n g_p(\mathcal{X}_p) \;)$$

In addition for any $f \in \mathcal{B}_b(E^{n+1})$ we have that

$$\mathbb{E}(f([X_t \; ; \; 0 \leq t \leq T_n]) \mid T_n < R) = \frac{\mathbb{E}(\; f(\mathcal{X}_0, \ldots, \mathcal{X}_n) \; \prod_{p=0}^n g_p(\mathcal{X}_p) \;)}{\mathbb{E}(\; \prod_{p=0}^n g_p(\mathcal{X}_p) \;)}$$

3 Genetic approximating models

In previous studies [6, 7] we design a collection of branching and interacting particle systems approximating models for solving a general class of Feynman-Kac models. These particle techniques can be used to solve the formulae presented in theorem 1. Next we have chosen to focus on a simple mutation/selection genetic algorithm. To describe this particle approximating model we first recall that the Feynman-Kac distribution flow $\eta_n \in \mathcal{P}(E)$ defined by

$$\eta_n(f) = \frac{\gamma_n(f)}{\gamma_n(1)} \quad \text{with} \quad \gamma_n(f) = \mathbb{E}(f(\mathcal{X}_n) \prod_{p=0}^{n-1} g_p(\mathcal{X}_p))$$

is solution of the following measure valued dynamical system

$$\eta_{n+1} = \Phi_{n+1}(\eta_n) \tag{2}$$

The mappings Φ_{n+1} from the set of measures

$$\mathcal{P}_n(E) = \{\eta \in \mathcal{P}(E) \; ; \; \eta(g_n) > 0\}$$

into $\mathcal{P}(E)$ are defined by

$$\Phi_{n+1}(\eta) = \Psi_n(\eta)\mathcal{K}_{n+1} = \int_E \Psi_n(\eta)(du) \; \mathcal{K}_{n+1}(u, .)$$

The Markov kernels $\mathcal{K}_n(u, dv)$ represent the Markov transitions of the chain $\mathcal{X}_n$. The updating mappings Ψ_n are defined from $\mathcal{P}_n(E)$ into $\mathcal{P}_n(E)$ and for any $\eta \in \mathcal{P}_n(E)$ and $f \in \mathcal{B}_b(E)$ by the formula

$$\Psi_n(\eta)(f) = \eta(f g_n)/\eta(g_n)$$

Thus we see that the recursion (2) involves two separate updating/prediction transitions

$$\eta_n \in \mathcal{P}(E) \xrightarrow{\text{Updating}} \widehat{\eta}_n = \Psi_n(\eta_n) \in \mathcal{P}(E) \xrightarrow{\text{Prediction}} \eta_{n+1} \in \mathcal{P}(E) \tag{3}$$

It is also conventient to recall that the finite and positive measures γ_n on E can be expressed in terms of the flow $\{\eta_p\ ;\ p < n\}$, using the easily checked formula

$$\gamma_n(f) = \eta_n(f) \prod_{p=0}^{n-1} \eta_p(g_p)$$

In these notations we readily observe that

$$\gamma_n(g_n) = \mathbb{P}(T_n < R)$$

and

$$\widehat{\eta}_n(f) =_{def.} \Psi_n(\eta_n)(f) = \mathbb{E}(f(X_t\ ;\ T_{n-1} \leq t \leq T_n) \mid T_n < R)$$

The genetic type N-particle system associated to an abstract measure valued process of the form (2) is the Markov chain $\xi_n = (\xi_n^1, \ldots, \xi_n^N)$ taking values at each time n in the product state spaces $E^N \cup \{\Delta\}$ where Δ stands for a cemetery or cofin point. Its transitions are defined as follows. For any configuration $x = (x^1, \ldots, x^N) \in E^N$ such that $\frac{1}{N}\sum_{i=1}^N \delta_{x^i} \in \mathcal{P}_n(E)$ we set

$$\mathbb{P}(\xi_{n+1} \in dy \mid \xi_n = x) = \prod_{p=1}^{N} \Phi_{n+1}\left(\frac{1}{N}\sum_{i=1}^{N} \delta_{x^i}\right)(dy^p) \tag{4}$$

where $dy = dy^1 \times \cdots \times dy^N$ is an infinitesimal neighborhood of $y = (y^1, \ldots, y^N) \in E^N$. When the system arrives in some configuration $\xi_n = x$ such that

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x^i} \notin \mathcal{P}_n(E)$$

the particle algorithm is stopped and we set $\xi_{n+1} = \Delta$. The initial system of particles $\xi_0 = (\xi_0^1, \ldots, \xi_0^N)$ consists in N independent random variables with common law $\eta_0 = \text{Law}(\mathcal{X}_0) = \text{Law}(X_0)$. The superscript $i = 1, \ldots, N$ represents the label of the particle and the parameter N is the size of the systems and the precision of the algorithm. As mentioned above the choice of the N-particle approximating model of (2) is not unique. Next we propose an alternative scheme which contains in some sense less randomness [5]. The key idea is to notice that the updating mapping $\Psi_n : \mathcal{P}_n(E) \to \mathcal{P}_n(E)$ can be rewritten in the following form

$$\Psi_n(\eta)(dv) = \eta\mathcal{S}_{n,\eta}(dv) = \int_E \eta(du)\mathcal{S}_{n,\eta}(u, dv)$$

with the collection of Markov transitions $S_{n,\eta}(u,dv)$ on S defined by

$$\mathcal{S}_{n,\eta}(u,dv) = (1 - 1_{g_n^{-1}\{1\}}(u))\ \Psi_n(\eta)(dv) + 1_{g_n^{-1}\{1\}}(u)\ \delta_u(dv)$$

where $g_n^{-1}\{1\}$ stands for the set of paths in S entering the level B_n, that is

$$\begin{aligned} g_n^{-1}\{1\} &= \{u \in E\ ,\quad g_n(u) = 1\} \\ &= \{u = (u_r)_{s \leq r \leq t} \in D([s,t])\ ,\ s \leq t \quad \text{and} \quad u_t \in B_n\} \end{aligned}$$

In this notation (2) can be rewritten as

$$\eta_{n+1} = \eta_n \mathcal{K}_{n+1,\eta_n}$$

with the composite Markov transition $\mathcal{K}_{n+1,\eta_n}$ defined by

$$\mathcal{K}_{n+1,\eta_n}(u,dw) = (\mathcal{S}_{n,\eta_n} K_{n+1})(u,dw) = \int_E \mathcal{S}_{n,\eta_n}(u,dv)\ K_{n+1}(v,dw)$$

The alternative N-particle model associated to this new description is defined as before by replacing (4) by

$$\mathbb{P}(\xi_{n+1} \in dy \mid \xi_n = x) = \prod_{p=1}^{N} \mathcal{K}_{n+1,\frac{1}{N}\sum_{i=1}^N \delta_{x^i}}(x^p, dy^p) \tag{5}$$

By definition of Φ_{n+1} and $\mathcal{K}_{n+1,\eta_n}$ we have for any $x \in E^N$ with $\frac{1}{N}\sum_{i=1}^N \delta_{x^i} \in \mathcal{P}_n(E)$

$$\Phi_{n+1}\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^i}\right) = \sum_{i=1}^N \frac{g_n(x^i)}{\sum_{j=1}^N g_n(x^j)}\ K_{n+1}(x^i, .)$$

In much the same way we find that

$$\mathcal{K}_{n+1,\frac{1}{N}\sum_{i=1}^N \delta_{x^i}} = \mathcal{S}_{n,\frac{1}{N}\sum_{i=1}^N \delta_{x^i}} K_{n+1}$$

with the selection transition

$$\mathcal{S}_{n,\frac{1}{N}\sum_{i=1}^N \delta_{x^i}}(x^p, dv) = (1 - 1_{g_n^{-1}\{1\}}(x^p))\ \Psi_n(\frac{1}{N}\sum_{i=1}^N \delta_{x^i})(dv) + 1_{g_n^{-1}\{1\}}(x^p)\ \delta_{x^p}(dv)$$

where

$$\Psi_n(\frac{1}{N}\sum_{i=1}^N \delta_{x^i}) = \sum_{i=1}^N \frac{g_n(x^i)}{\sum_{j=1}^N g_n(x^j)}\ \delta_{x^i}$$

Thus, we see that the transition $\xi_n \to \xi_{n+1}$ of the former Markov models splits up into two separate genetic type mechanisms

$$\xi_n \in E^N \cup \{\Delta\} \xrightarrow{\text{Selection}} \widehat{\xi}_n = (\widehat{\xi}_n^i)_{1 \leq i \leq N} \in E^N \cup \{\Delta\} \xrightarrow{\text{Mutation}} \xi_{n+1} \in E^N \cup \{\Delta\}$$

By construction we notice that

$$\xi_n = \Delta \implies \forall\ p \geq n \quad \xi_p = \Delta \quad \text{and} \quad \widehat{\xi}_p = \Delta$$

By definition of the path valued Markov chain $\mathcal{X}_n$ this genetic model consists in N-path valued particles

$$\begin{aligned} \xi_n^i &= (\xi_n^i(t) \; ; \; T_{n-1}^i \leq t \leq T_n^i) \in D([T_{n-1}^i, T_n^i], S) \\ \widehat{\xi}_n^i &= (\widehat{\xi}_n^i(t) \; ; \; \widehat{T}_{n-1}^i \leq t \leq \widehat{T}_n^i) \in D([\widehat{T}_{n-1}^i, \widehat{T}_n^i], S) \end{aligned}$$

The random time-pairs (T_{n-1}^i, T_n^i) and $(\widehat{T}_{n-1}^i, \widehat{T}_n^i)$ represent the length of the corresponding paths.

Next we describe in more details the genetic evolution of the path-particles.

At the time $n = 0$ the initial configuration consists in N independent and identically distributed S-valued random variables ξ_0^i with common law η_0. Since we have $g_0(u) = 1$ for η_0-almost every $u \in S$, there is no updating transition at time $n = 0$ and we set $\widehat{\xi}_0^i = \xi_0^i$ for each $1 \leq i \leq N$. If we use the convention $T_{-1}^i = \widehat{T}_{-1}^i = 0$ and if we set $T_0^i = \widehat{T}_0^i = 0$ we notice that the single states ξ_0^i and $\widehat{\xi}_0^i$ can be written in the path-form

$$\xi_0^i = \xi_0^i(0) = (\xi_0^i(t) \; ; \; T_{-1}^i \leq t \leq T_0^i) \quad \text{and} \quad \widehat{\xi}_0^i = \widehat{\xi}_0^i(0) = (\widehat{\xi}_0^i(t) \; ; \; \widehat{T}_{-1}^i \leq t \leq \widehat{T}_0^i)$$

The mutation transition $\widehat{\xi}_n \to \xi_{n+1}$ at time $(n+1)$ is defined as follows. If $\widehat{\xi}_n = \Delta$ we set $\xi_{n+1} = \Delta$. Otherwise during mutation independently of each other each selected path-particle

$$\widehat{\xi}_n^i = (\widehat{\xi}_n^i(t) \; ; \; \widehat{T}_{n-1}^i \leq t \leq \widehat{T}_n^i)$$

evolves randomly according to the Markov transition $\mathcal{K}_{n+1}$ of the Markov chain X_{n+1} at time $(n+1)$ so that

$$\xi_{n+1}^i = (\xi_{n+1}^i(t) \; ; \; T_n^i \leq t \leq T_{n+1}^i)$$

is a random variable with distribution $\mathcal{K}_{n+1}(\widehat{\xi}_n^i, .)$. In other words we set $T_n^i = \widehat{T}_n^i$ and the particle $\widehat{\xi}_n^i(t)$ at that time $t = T_n^i$ evolves randomly as a copy $(\xi_{n+1}^i(s))_{s \geq T_n^i}$ of the process $(X_s)_{s \geq T_n^i}$ starting at $X_{T_n^i} = \widehat{\xi}_n^i(T_n^i)$ at time $s = T_n^i$ and up to the first time it visits B_{n+1} or returns to A_∞. The stopping time T_{n+1}^i is the first time this particle reaches B_{n+1} before visiting A_∞, that is

$$T_{n+1}^i = \inf \{T_n^i \leq t < R_{n+1}^i \; : \; \xi_{n+1}^i(t) \in B_{n+1}\}$$

with

$$R_{n+1}^i = \inf \{T_n^i \leq t \; : \; \xi_{n+1}^i(t) \in A_\infty\}$$

and the convention $\inf_\infty = R$ so that $T_{n+1}^i = R_{n+1}^i$ if it visit A_∞ before B_{n+1}.
The selection transition $\xi_{n+1} \to \widehat{\xi}_{n+1}$ is defined as follows. ¿From the previous mutation transition we obtain N path-particle

$$\xi_{n+1}^i = (\xi_{n+1}^i(t) \; ; \; T_n^i \leq t \leq T_{n+1}^i)$$

Some of these particle have succeed to reach to desired set B_{n+1} and the other ones have visited A_∞. We denote by $I^N(n+1)$ the labels of the particle having succeed to reach the $(n+1)$-th level

$$I^N(n+1) = \{1 \leq i \leq N \ : \ \xi^i_{n+1}(T^i_{n+1}) \in B_{n+1}\} = \{1 \leq i \leq N \ : \ T^i_{n+1} < R^i_{n+1}\}$$

If $I^N(n+1) = \emptyset$ then none of the particle have succed to reach the desired level. Since

$$I^N(n+1) = \emptyset \iff \frac{1}{N}\sum_{i=1}^N g_{n+1}(\xi^i_{n+1}) = 0 \iff \frac{1}{N}\sum_{i=1}^N \delta_{\xi^i_{n+1}} \notin \mathcal{P}_{n+1}(E)$$

we see that in this situation the algorithm is stopped and $\widehat{\xi}_{n+1} = \Delta$. Otherwise the selection transition of the N-particle models (4) and (5) are defined as follows. In the first situation the system $\widehat{\xi}_{n+1} = (\widehat{\xi}^i_{n+1})_{1\leq N}$ consists in N independent random variables

$$\widehat{\xi}^i_{n+1} = (\widehat{\xi}^i_{n+1}(t) \ ; \ \widehat{T}^i_n \leq t \leq \widehat{T}^i_{n+1})$$

with common distribution

$$\begin{aligned} \Psi_{n+1}(\frac{1}{N}\sum_{i=1}^N \delta_{\xi^i_{n+1}}) &= \sum_{i=1}^N \frac{g_{n+1}(\xi^i_{n+1})}{\sum_{j=1}^N g_{n+1}(\xi^j_{n+1})} \ \delta_{\xi^i_{n+1}} \\ &= \frac{1}{\mathrm{Card}(I^N(n+1))} \sum_{i\in I^N(n+1)} \delta_{(\xi^i_{n+1}(t) \ ; \ T^i_n \leq t \leq T^i_{n+1})} \end{aligned}$$

In the alternative model (5) each particle

$$\widehat{\xi}^i_{n+1} = (\widehat{\xi}^i_{n+1}(t) \ ; \ \widehat{T}^i_n \leq t \leq \widehat{T}^i_{n+1})$$

is sampled according to the selection distribution

$$\begin{aligned} &\mathcal{S}_{n+1,\frac{1}{N}\sum_{j=1}^N \delta_{\xi^j_{n+1}}}(\xi^i_{n+1}, dv) \\ &= (1 - 1_{g^{-1}_{n+1}\{1\}}(\xi^i_{n+1})) \ \Psi_n(\tfrac{1}{N}\textstyle\sum_{j=1}^N \delta_{\xi^j_{n+1}})(dv) + 1_{g^{-1}_{n+1}\{1\}}(\xi^i_{n+1}) \ \delta_{\xi^i_{n+1}}(dv) \\ &= (1 - 1_{B_{n+1}}(\xi^i_{n+1}(T^i_{n+1}))) \ \Psi_n(\tfrac{1}{N}\textstyle\sum_{j=1}^N \delta_{\xi^j_{n+1}})(dv) + 1_{B_{n+1}}(\xi^i_{n+1}(T^i_{n+1})) \ \delta_{\xi^i_{n+1}}(dv) \end{aligned}$$

More precisely we have

$$\xi^i_{n+1}(T^i_{n+1}) \in B_{n+1} \implies \widehat{\xi}^i_{n+1} = \xi^i_{n+1}$$

In the opposite we have $\xi^i_{n+1}(T^i_{n+1}) \notin B_{n+1}$ when the particle has not succeed to reach the $(n+1)$-th level. In this case $\widehat{\xi}^i_{n+1}$ is chosen randomly and uniformly in the set

$$\{\xi^j_{n+1} \ ; \ \xi^j_{n+1}(T^j_{n+1}) \in B_{n+1}\} = \{\xi^j_{n+1} \ ; \ j \in I^N(n+1)\}$$

of all particle having succeed to enter into B_{n+1}. In other words each particle which doesn't enter into the $(n+1)$-th level is killed and instantly a different particle in the B_{n+1} level splits into two offsprings.

We denote by τ^N the life-time of the N-genetic model

$$\tau^N = \inf\left\{ n \geq 0\ ;\ \frac{1}{N}\sum_{i=1}^{N}\ \delta_{\xi_n^i} \not\in \mathcal{P}_n(E)\right\}$$

For each time $n < \tau^N$ we denote by η_n^N and $\widehat{\eta}_n^N$ the particle density profiles associated to the N-particle model

$$\eta_n^N = \frac{1}{N}\sum_{i=1}^{N}\ \delta_{\xi_n^i} \quad \text{and} \quad \widehat{\eta}_n^N = \Psi_n(\eta_n^N)$$

For each time $n < \tau^N$ the N-particle approximating measures γ_n^N associated to γ_n are defined for any $f \in \mathcal{B}_b(E)$ by

$$\gamma_n^N(f) = \eta_n^N(f) \quad \prod_{p=0}^{n-1} \eta_p^N(g_p)$$

Note that

$$\gamma_n^N(g_n) = \prod_{p=0}^{n} \eta_p^N(g_p) = \frac{1}{N^n}\ \prod_{p=1}^{n} \mathrm{Card}(I^N(p))$$

and

$$\widehat{\eta}_n^N = \Psi_n(\eta_n^N) = \frac{1}{\mathrm{Card}(I^N(n))}\ \sum_{i \in I^N(n)}\ \delta_{(\xi_n^i(t)\ ;\ T_{n-1}^i \leq t \leq T_n^i)}$$

The asymptotic behavior as $N \to \infty$ of the interacting particle model we have constructed has been studied in many works. We refer the reader to the survey paper [6] in tha case of strictly positive potentials g_n and [3, 4] for non negative potentials. For the convenience of the reader we have chosen to present the impact of some exponential and $\mathbb{L}_p$-mean error estimates in the analysis of rare events.

Theorem 2 *For any time $n \geq 0$ there exists a finite constant $c(n)$ such that for any $N \geq 1$*

$$\mathbb{P}(\tau^N \leq n) \leq c(n) \exp\left(-N/c(n)\right)$$

The particle estimates $\gamma_n^N(g_n)$ are unbias

$$\mathbb{E}(\gamma_n^N(g_n) 1_{n<\tau^N}) = \mathbb{P}(T_n < R)$$

and for each $p \geq 1$ we have

$$\mathbb{E}(|\gamma_n^N(g_n) 1_{n<\tau^N} - \mathbb{P}(T_n < R)|^p)^{\frac{1}{p}} \leq a(p) b(n)/\sqrt{N}$$

for some finite constant $a(p) < \infty$ which only depend on the parameter p and some finite constant $b(n) < \infty$ which depends on the time parameter n. In addition, for any test function $f \in \mathcal{B}_b(E)$, with $\|f\| \leq 1$

$$\mathbb{E}(|\widehat{\eta}_n^N(f) 1_{n<\tau^N} - \mathbb{E}(f(X_p\ ,\ \ T_{n-1} \leq p \leq T_n)\ |\ T_n < R)|^p)^{\frac{1}{p}} \leq a(p) b(n)/\sqrt{N}$$

We illustrate the impact of this asymptotic convergence theorem by chosing some particular test functions. For each $u > 0$ we define the function $f^{(u)}$ on E by setting for each $x = (x_r)_{s \leq r \leq t} \in D([s,t], S)$ with $s \leq t$,

$$f^{(u)}(x) = \begin{cases} 1 & \text{if} \quad |t-s| \leq u \\ 0 & \text{if} \quad |t-s| > u \end{cases} \tag{6}$$

In this notation $u \to \Psi_n(\eta_n)(f^{(u)})$ is the repartition function of the intertime $T_n - T_{n-1}$ between two consecutive levels B_{n-1} and B_n, that is

$$\Psi_n(\eta_n)(f^{(u)}) = \mathbb{P}(T_n - T_{n-1} \leq u \mid T_n < R)$$

The particle approximation of this quantity is the proportion of paths having passed from B_{n-1} to B_n in time u.

4 Genealogical tree based models

The genetic particle approximating model described in the previous section can be interpretated as a birth and death particle model. The particle dies if it does not succeed to reach the desired level and it duplicates in some offsprings when it hits this level. One way to model the genealogical tree and the line of ancestors of the particles alive at some given date is to consider the stochastic sequence

$$\mathcal{Y}_n = (\mathcal{X}_0, \ldots, \mathcal{X}_n) \in E_n = \underbrace{E \times \ldots \times E}_{(n+1)-times}$$

It is not difficult to check that $\mathcal{Y}_n$ forms a time inhomogenous Markov chain with Markov transitions $\mathcal{G}_{n+1}$ from E_n into E_{n+1}

$$\mathcal{G}_{n+1}((u_0, \ldots, u_n), d(v_0, \ldots, v_n, v_{n+1}))$$

$$= \delta_{(u_0,\ldots,u_n)}(d(v_0, \ldots, v_n)) \; \mathcal{K}_{n+1}(v_n, dv_{n+1})$$

Let h_n be the mapping from E_n into $[0, \infty)$ defined by

$$h_n(x_0, \ldots, x_n) = g_n(x_n)$$

In this notation we have for any $f_n \in \mathcal{B}_b(E_n)$ the Feynman-Kac representation

$$\begin{aligned} \widehat{\mu}_n(f_n) \quad &=_{def.} \quad \frac{\mathbb{E}(\; f_n(\mathcal{Y}_n) \; \prod_{p=0}^n \; h_p(\mathcal{Y}_p) \;)}{\mathbb{E}(\; \prod_{p=0}^n h_p(\mathcal{Y}_p) \;)} \\ &= \quad \mathbb{E}(f_n(X_0, (X_t \; ; \; 0 \leq t \leq T_1), \ldots, (X_t \; ; \; T_{n-1} \leq t \leq T_n)) \mid T_n < R) \\ &= \quad \mathbb{E}(f_n([X_t \; ; \; 0 \leq t \leq T_n]) \mid T_n < R) \end{aligned}$$

Using the same lines of reasoning as above the N-particle approximating model associated to these Feynman-Kac distributions is again a genetic algorithm with mutation transitions $\mathcal{G}_n$ and potential functions h_n. Here the path-particle at time n take values in E_n and they can be written as follows

$$\zeta_n^i = (\xi_{0,n}^i, \ldots, \xi_{n,n}^i) \quad \text{and} \quad \widehat{\zeta}_n^i = (\widehat{\xi}_{0,n}^i, \ldots, \widehat{\xi}_{n,n}^i) \in E_n$$

with for each $0 \leq p \leq n$

$$\xi_{p,n}^i = (\xi_{p,n}^i(t) \ ; \ T_{p-1,n}^i \leq t \leq T_{p,n}^i) \quad \text{and} \quad \widehat{\xi}_{p,n}^i = (\widehat{\xi}_{p,n}^i(t) \ ; \ \widehat{T}_{p-1,n}^i \leq t \leq \widehat{T}_{p,n}^i) \in E$$

The selection transition consists in randomly selecting a path-sequence

$$\zeta_n^i = (\xi_{0,n}^i, \ldots, \xi_{n,n}^i)$$

proportionally to its fitness

$$h_n(\xi_{0,n}^i, \ldots, \xi_{n,n}^i) = g_n(\xi_{n,n}^i)$$

The mutation stage consists in extending the selected paths according to an elementary $\mathcal{K}_{n+1}$-transition, that is

$$\begin{aligned} \zeta_{n+1}^i &= ((\xi_{0,n+1}^i, \ldots, \xi_{n,n+1}^i), \xi_{n+1,n+1}^i) \\ &= ((\widehat{\xi}_{0,n}^i, \ldots\ldots, \widehat{\xi}_{n,n}^i), \ \xi_{n+1,n+1}^i) \in E_{n+1} = E_n \times E \end{aligned}$$

where $\xi_{n+1,n+1}^i$ is a random variable with law $\mathcal{K}_{n+1}(\widehat{\xi}_{n,n}^i, .)$. By a simple argument we see that the evolution associated to the end points of the paths

$$\xi_n = (\xi_{n,n}^1, \ldots, \xi_{n,n}^N) \quad \text{and} \quad \widehat{\xi}_n = (\widehat{\xi}_{n,n}^1, \ldots, \widehat{\xi}_{n,n}^N) \in E$$

coincide with the genetic algorithms described in section 3. We conclude that the former path-particle Markov chain models the evolution in time of the corresponding genealogical trees. For each time $n < \tau^N$ we denote by μ_n^N and $\widehat{\mu}_n^N$ the particle density profiles associated to the ancestor lines of this genealogical tree based algorithm

$$\mu_n^N = \frac{1}{N} \sum_{i=1}^N \delta_{(\xi_{0,n}^i, \ldots, \xi_{n,n}^i)} \quad \text{and} \quad \widehat{\mu}_n^N = \frac{1}{\mathrm{Card}(I^N(n))} \sum_{i \in I^N(n)} \delta_{(\xi_{0,n}^i, \ldots, \xi_{n,n}^i)}$$

with

$$I^N(n) = \{1 \leq i \leq N \ : \ \xi_{n,n}^i(T_{n,n}^i) \in B_n\}$$

The asymptotic behavior of genealogical tree based algorithm has been studied in [7] in the context of strictly positive potentials and further developped in [3] for non negative ones. In our context the path-version of the $\mathbb{L}_p$-mean error estimates presented in theorem 2 can be stated as follows.

Theorem 3 *For any $p \geq 1$, $0 \leq n \leq m+1$ and any test function $f_n \in \mathcal{B}_b(E_n)$, with $\|f\| \leq 1$ we have*

$$\mathbb{E}(|\widehat{\mu}_n^N(f_n) 1_{n<\tau^N} - \mathbb{E}(f_n([X_t \ , \ \ 0 \leq p \leq T_n]) \mid T_n < R)|^p)^{\frac{1}{p}} \leq a(p) b(n)/\sqrt{N}$$

for some finite constant $a(p) < \infty$ which only depend on the parameter p and some finite constant $b(n) < \infty$ which depends on the time parameter n.

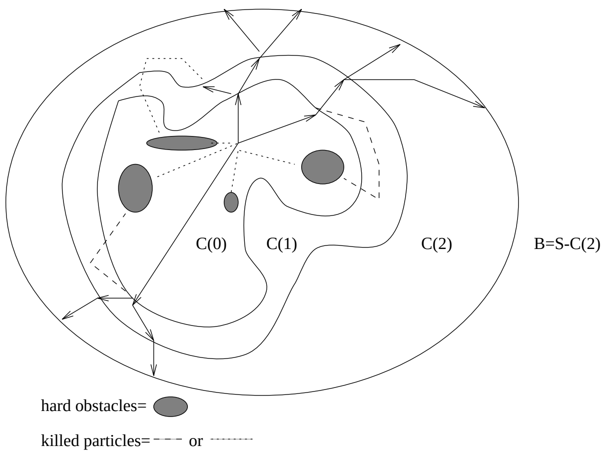

Figure 1: Genealogical model, [exit of C(2) before killing] (N=4)

Following the observations given the end of the previous section let us choose a collection of times $u_1 > 0$,..., $u_n > 0$. Let $f_n^{(u)}$, $u = (u_1, \ldots, u_n)$, be the test function on E_n defined by

$$f_n^{(u)}(x_0, \ldots, x_n) = f^{(u_1)}(x_1) \ldots f^{(u_n)}(x_n)$$

with $f^{(u_p)}$ defined in (6). In this situation we have

$$\mu_n(f_n^{(u)}) = \mathbb{P}(T_1 - T_0 \leq u_1, \ldots, T_n - T_{n-1} \leq u_n \mid T_n < R)$$

The particle approximations consists in counting at each level $1 \leq p \leq n$ the proportion of ancestral lines having succed to pass the p-th levels in time u_p.

In the figure 1 we illustrate the genealogical particle model associated to a particle X evolving in a pocket $C \subset S$ containing four "hard obstacles" A_∞. We associate to a given stratification of the pocket C

$$A_\infty \subset C(0) \subset C(1) \subset C(2)$$

the sequence of exit levels

$$B_0 = S - A_\infty \supset B_1 = S - C(0) \supset B_2 = S - C(1) \supset B_3 = S - C(2)$$

The desired target set here is $B = B(3)$.

In the figure 2 we illustrate the genealogical particle model for a particle X evolving in a set $A \subset S$ with recurrent subset $A_\infty = S - A$. To reach the desired target set B_4 the process need to pass the sequence of levels

$$B_0 \supset B_1 \supset B_2 \supset B_3 \supset B_4$$

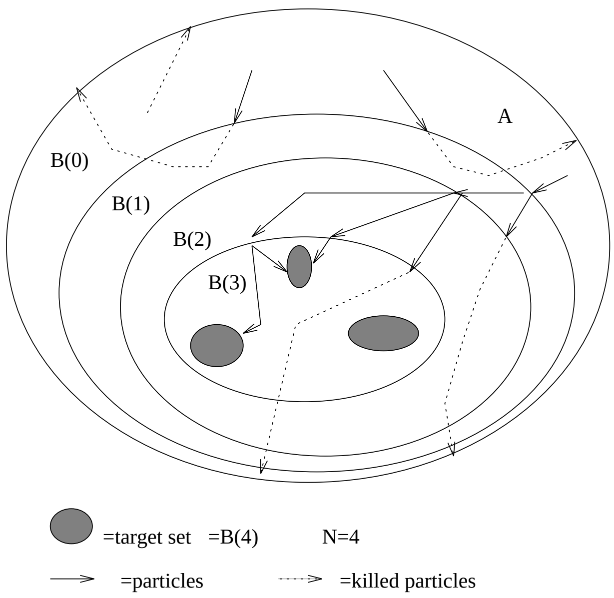

Figure 2: Genealogical model, [ballistic regime, target B(4)] (N=4)

5 Numerical example: application to Ornstein-Uhlenbeck processes

We will show in this section how the previous method to simulate rare events works in a simple case. Although this is clearly a toy model, it allows us to check the method accuracy on the computation of some quantities that have formal rigorous expressions. Moreover, this process having simple Gaussian increments, there is no numerical error due to discretization scheme.

The process X is taken to be the 1-D Ornstein-Uhlenbeck process, i.e. the solution of the SDE

$$dX_t = -aX_t\,dt + \sigma\,dW_t,$$

where a and σ are strictly positive constants and W the standard BM in $\mathbb{R}$. The reccurent set A_∞ is chosen as $]-\infty, b^-]$, and then the process X is started at some $x_0 \in A =]b^-, +\infty[$. Given some $b^+ > x_0$, we set the target $B = [b^+, +\infty[$. It is clear that if we take b large enough, the probability to hit the target can be made arbitrarily small. Let us denote by τ the stopping time

$$\tau = \inf\{t < 0,\, X_t \notin]b^-, b^+[\}.$$

In oder to check the method, we will compute $\mathbb{E}[\tau \,|\, X_\tau = b^+]$ using both a Monte–Carlo method based on our rare event analysis approach and the theoretical expression. From [2] we have

$$\mathbf{L}(\alpha) = \mathbb{E}_{x_0}\left(e^{-\alpha\tau}\mathbf{1}_{X_\tau = b^+}\right) = \frac{S(\frac{\alpha}{a}, \frac{x_0}{\tilde{\sigma}}, \frac{b^-}{\tilde{\sigma}})}{S(\frac{\alpha}{a}, \frac{b^+}{\tilde{\sigma}}, \frac{b^-}{\tilde{\sigma}})}$$

where $\tilde{\sigma} = \frac{\sigma}{\sqrt{2a}}$ and S is a special function to be define in the sequel. Using the derivative of the Laplace transform we get

$$\mathbb{E}[\tau \mid X_\tau = b^+] = -\frac{1}{P(X_\tau = b^+)} \left.\frac{d\mathbf{L}(\alpha)}{d\alpha}\right|_{\alpha=0}. \tag{7}$$

The probability in the denominator is given by

$$P(X_\tau = b^+) = \frac{u(x_0) - u(b^-)}{u(b^+) - u(b^-)}, \tag{8}$$

where the function u (the scale function of the process) is in our case a primitive of $u'(x) = \exp[\frac{ax^2}{\sigma^2}]$. This function u is then easily computed using any standard numerical integration routine. The derivative of $\mathbf{L}$ is more tricky. First we write the expression of the function S, for any real x and y, and $\nu > 0$,

$$S(\nu, x, y) = \frac{\Gamma(\nu)}{\pi}\, e^{\frac{1}{4}(x^2+y^2)}(D_{-\nu}(-x)D_{-\nu}(y) - D_{-\nu}(x)D_{-\nu}(-y)),$$

where the functions D are the parabolic cylinder functions defined by

$$\begin{aligned} D_{-\nu}(x) \quad = \quad & e^{-\frac{1}{4}x^2}\, 2^{-\frac{\nu}{2}} \sqrt{\pi} \left\{ \frac{1}{\Gamma((\nu+1)/2)} \left(1 + \sum_{k=1}^{\infty} \frac{\nu(\nu+2)\dots(\nu+2k-2)}{3.5\dots(2k-1)k!} \left(\frac{x^2}{2}\right)^k\right) \right. \\ & \left. - \frac{x\sqrt{2}}{\Gamma(\nu/2)} \left(1 + \sum_{k=1}^{\infty} \frac{(\nu+1)(\nu+3)\dots(\nu+2k-1)}{3.5\dots(2k+1)k!} \left(\frac{x^2}{2}\right)^k\right)\right\}. \end{aligned}$$

These functions are computed using the numerical method and the source code provided in [9]. Now we still need to compute the derivative in equation (7). We did not want to derive formaly this quite complicated expression, and used instead a numerical appriximation from a local rate of variation:

$$\left.\frac{d\mathbf{L}(\alpha)}{d\alpha}\right|_{\alpha=0} \simeq \frac{\mathbf{L}(2\varepsilon) - \mathbf{L}(\varepsilon)}{\varepsilon},$$

where $\varepsilon > 0$ is chosen small enough.

Now we explain how the Monte-Carlo computation was carried out. For the decreasing sequence of Borel sets $\{B_j,\ j = 1 \dots M\}$ we chose an increasing squence of real numbers $\{b_j,\ j = 1 \dots M\}$, with $b^- < b_1 < \dots < b_M < b^+$ and take $B_j =]b_j, +\infty[$. In our special case, we can choose the probability for a particle started at b_j to reach b_{j+1}, and compute both he number of levels and each level accordingly. If we take these probability equal for all j to say p, then

$$M = \lfloor \frac{\log \mathbb{E}[\tau \mid X_\tau = b^+]}{\log p} \rfloor.$$

Alternatively we can choose M and compute p. Note that the probability of the N−particle cloud to be killed before reaching b^+ is $M(1-p)^N$, which can be small

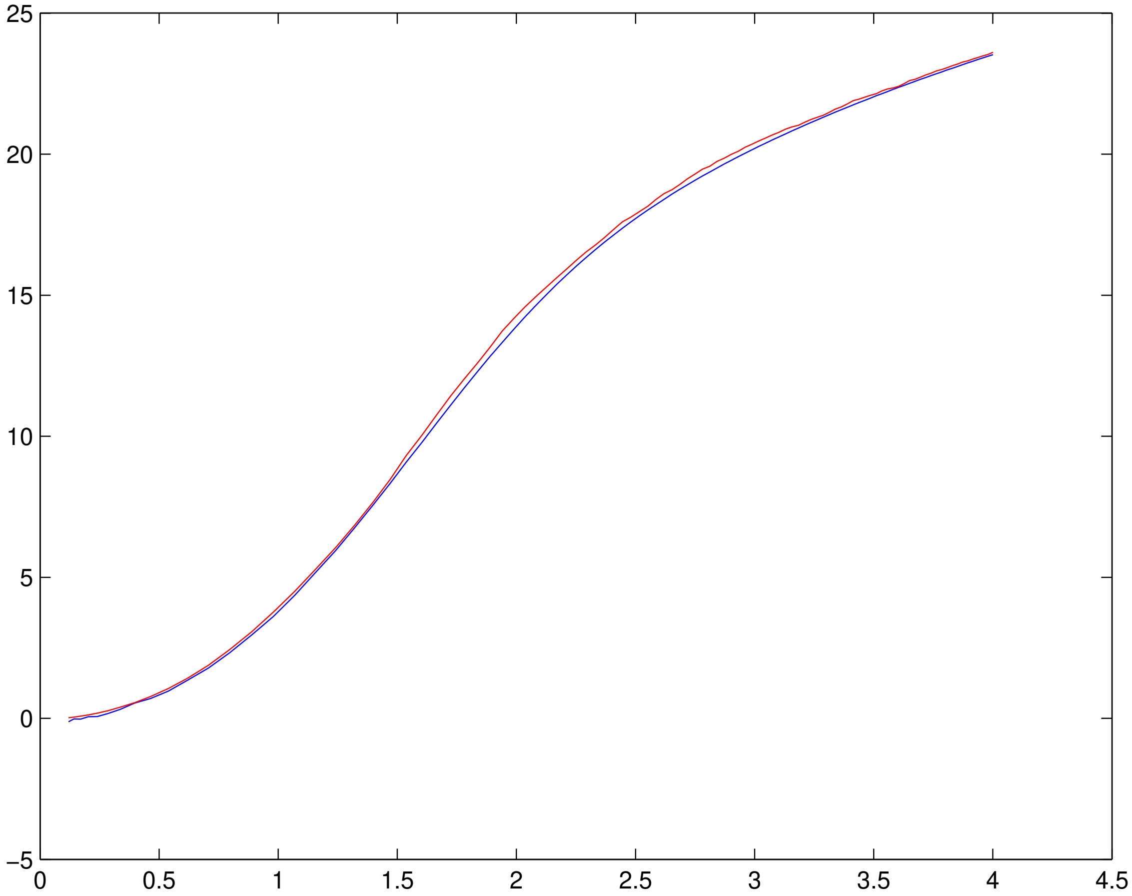

Figure 3: Theoretical and Monte–Carlo mean conditional stopping times

even with a small number N of particles when p is say larger than 1/2. From this we see that a good strategy is to make many runs of our algorithm on a small number of particles, instead of only several ones on a large number of particles (on the same run, all the generated trajectories are obviously strongly correlated). All the corresponding values b_j are easily computed using expressions as the one in equation (8).

In figure 5 we see the expectation $\mathbb{E}[\tau \mid X_\tau = b^+]$ as a function of b^+, with $b^- = 0$. The blue curve is the numerically computed theoretical value, and the red curve is the Monte–Carlo simulation result, with 880 runs of 8 particles each. The parameters of the Ornstein-Uhlenbeck process are $a = 0.1$, $\sigma = 0.3$ and $x_0 = 0.1$. The largest value of b^+ was 4.0. This means that the probability for the process started at $x_0 = 0.1$ to reach the desired level is approximatly 1.6460×10^{-08}, so there is no way of simulating trajectories by the naive approach.

Another examples of rare events for diffusions may be found in the excellent book [1], which presents a Poisson clumping heuristic as well as numerous examples and references. For instance, following [1][§I11], let consider a diffusion in $\mathbb{R}^d$ with drift $\mu(x) = -\nabla H$ and variance $\sigma(x) = \sigma_0 I$. Suppose H is a smooth convex function attaining its minimum at 0 with $H(0) = 0$ and such that $H(x) \to \infty$ as $|x| \to \infty$. Let B be a ball with center at 0 with radius r, where r is sufficiently large that $\pi(B^c)$ is small, where π is the stationary distribution

$$\pi(x) = c\exp\Big(\frac{-2H(x)}{\sigma_0^2}\Big) \approx (\sigma_0^2\pi)^{-d/2}|Q|^{1/2}\exp\Big(\frac{-2H(x)}{\sigma_0^2}\Big), \quad Q = \frac{\partial^2 H}{\partial x_i \partial x_j}(0).$$

We want an estimation of the first exit time from the ball B. There are two qualitatively situations; radially symmetric potentials ($H(x) = h(|x|)$) and non-symmetric potentials. We presents here only the second one, by assuming that H attains its minimum, over the spherical surface ∂B, at a unique point $z_0 = (r, 0, 0, \cdots)$. Since the stationary distribution decreases exponentially fast as H increases, we can suppose that exits from B will likely occur near z_0 and then approximate T by T_F, the first hitting time on the $(d-1)$-dimensional hyperplane F tangent to B at z_0. The heuristic used in [1] gives that T is approximatively exponentially distributed with mean $(\pi_F|\nabla H(z_0)|)^{-1}$, where π_F designed the restriction of the measure π to F. We obtain

$$\begin{aligned}\pi_F &\approx (\sigma_0^2\pi)^{-d/2}|Q|^{1/2}\exp(-2\sigma_0^{-2}H(z_0))\int_F \exp(-2\sigma_0^{-2}(H(x)-H(z_0))dx \\ &\approx (\sigma_0^2\pi)^{-1/2}|Q|^{1/2}|Q_1|^{-1/2}\exp(-2\sigma_0^{-2}H(z_0)),\end{aligned}$$

where Q_1 is the matrix

$$\frac{\partial^2 H}{\partial x_i \partial x_j}(z_0), \qquad i, j \geq 2.$$

Thus

$$\mathbb{E}(T) \approx \sigma_0 \frac{\pi^{1/2}|Q|^{-1/2}|Q_1|^{1/2}}{-\frac{\partial H}{\partial x_1}(z_0)}\exp(2\sigma_0^{-2}H(z_0)).$$

The simplest concrete example is the Ornstein-Uhlenbeck process in wich $H(x) = \frac{1}{2}\sum \rho_i x_i^2$ whith $0 < \rho_1 < \rho_2 < \cdots$. Here H has two minima on ∂B, at $\pm z_0 = \pm(r, 0, 0, \cdots)$ and so the mean exit time is

$$\mathbb{E}(T) \approx \frac{1}{2}\sigma_0 \pi^{1/2}\Big(\prod_{i\geq 2}\rho_i \Big/ \prod_{i \geq 1}\rho_i\Big)\rho_1^{-1} r^{-1}\exp\Big(\frac{\rho_1 r^2}{\sigma_0^2}\Big).$$

To adapt this example to the formalism, introduced previously, we slightly modify it by considering the first exit time from the ball B before reaching a little ball B_ϵ centered at 0 with radius ϵ small. Thus, we suppose that $\mathbb{R}^d$ is decomposed into two separate regions B^c and B and that the process X evolves in B starting from outside B_ϵ, but near from ∂B_ϵ. The process will be killed as soon as it hits ∂B_ϵ. By considering a particle system algorithm and a genealogical model, an estimation of the first exit time before returning in the neighbourhood of the origin and of the distribution of the process during its excursions should be obtained.

It will also be interesting to study the Kramers' equation

$$\begin{cases} dX_t &= V_t dt \\ dV_t &= -H'(X_t)dt - \gamma V_t dt + \sqrt{2\gamma} dB_t \end{cases}$$

In [1], the heuristic may be applied for small and large coefficients γ, but it is a hard problem to say which of these behaviors dominates in a specific non-asymptotic case; hence the simulation approach.

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