Simulation of Rare Event in Queueing Systems

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Abstract *The paper is devoted on computer simulation of rare event probability, which is a critical issue in areas such as reliabilities, telecommunications, aircraft management, etc. This means that it is necessary to evaluate probabilityties less than* 10^{-9}*, which is unreasonable to be simulated with standard Monte Carlo approach. M/M/1/N queuing system and tandem queuing system are under investigation where the overflow probabilities are estimated with rare event simulation. For queuing systems performance were used discrete-time Markov chains models. An algorithm with RESTART approach and Limited Relative Error was developed to speed-up the rare event simulation. This algorithm has been successfully applied to M/M/1/N queuing system and tandem queuing system. Simulation results for two servers tandem queuing system with Poisson arrival and service rates are shown.*

Keywords: *Queuing System, Rare Event Simulation, Importance Splitting, Limited Relative Error.*

1. INTRODUCTION

Estimating the probability of a rare event has several applications in reliability, telecommunications, insurance, and several other areas. This means that such events appear very rare but if appear they can completely destroy complex systems. In this case it is necessary to evaluate probabilities less than 10^{-9}, which is unreasonable to be done via standard Monte Carlo simulation [1]. For complex models, when it is necessary to evaluate not very low probabilities of appearance of certain events in principle this can be done by Monte Carlo simulation. When the event of interest is really rare, straightforward simulation would usually require an excessive number of runs for reaching rare event to happen enough frequently so that the estimator is meaningful [2].

RESTART (Repetitive Simulation Trials After Reaching Thresholds) is an accelerated simulation method, which belongs to the so-called importance splitting methods, is used to the speed-up the rare event simulation [3].

The Limited Relative Error (LRE) measures the complementary distribution function of the queue occupancy distribution and performs the Run Time Control (RTC) of the simulation. The LRE performs the Run Time Control with two conditions: first the Large Sample Conditions and second the Relative Error Condition. The first condition assures that the queuing system has reached the steady state. The second one - Relative Error Condition, represents a measure to estimate the relative error at the current state of the simulation [4].

Queuing systems and Markov chains are very often used for describing the processes in communication systems and their simulation. In such models under investigation are all inter-arrival times and service times and their distribution, and the quantity of interest is, for example, an overflow probability. Some other performance measures, like delays, cannot be obtained from the discrete-time Markov chain description [5].

2. QUEUEING SYSTEM

2.1. Queueing system M/M/1/N

A basic reference model for rare event simulation is the single server queuing system M/M/1/N - FIFO with a finite buffer size N.

The arrival rate is λ and the service rate is μ. We are interested in the probability that the buffer content reaches a high level *k* during one busy period (i.e., the time interval between two successive periods in which the buffer is empty). The maximum occupancy is $L=N+1$, and the traffic load is $\rho=\lambda/\mu$. The discrete random variable is describe with stationary complementary distribution function (*c.d.f.*) $G(x)=1- F(x)$, for the loss probability P_B and the local correlation co-efficient Cor_i for the interval $i-1\le x<i$, $i=1,\ldots,k$ [3].

2.2. Overflow probability of tandem queues

Consider the overflow probability of the total population in a network consisting of two queues in tandem.

In this example customers arrive at the first queue according to a Poisson-process with rate λ. Both servers have exponentially-distributed service times.

The purpose is to receive an estimation of the overflow probabilities in simple Jackson networks with rates μ_1 and μ_2. The state of the system at any time is given by the two integer values n_1 and n_2, which are the number of customers in the first and second queues, as is shown on Fig. 1.

The difficulty of applying accelerated simulation techniques arises when the first queue is the bottleneck and the rare set definition is related to the value of n_2. If splitting or RESTART is applied, a more careful choice of the importance function has to be made in this case.

The importance splitting methods allow the evaluation of extremely low probabilities, e.g., 10^{-10} or 10^{-40}, by simulation. This type of rare event simulation was developed for evaluation of different processes in telecommunication systems with very low probabilities [2,4].

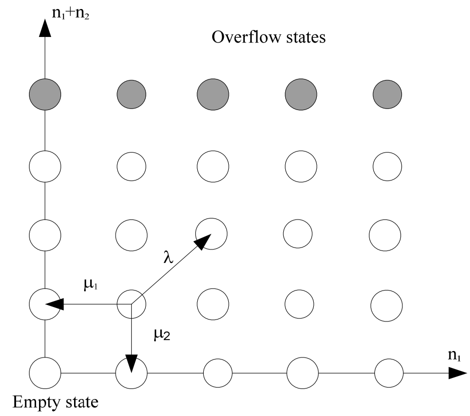

Fig. 1 The state space of two queues in tandem network

Let consider the two-queue tandem network presented as reference model for the following three definitions of the rare set A: The numbers of retrials were chosen to have values of n_i as close as possible to those given by (1).

$$n_1 + n_2 \geq L,$$
$$n_2 \geq L, \tag{1}$$
$$\mathrm{Min}(n_1, n_2) \geq L$$

2.2.1. Rare State Defined as $n_1 + n_2 \geq L$

For this definition of the rare set, the most natural importance function is $\Phi = n_1 + n_2$. The possible states at event B_i are $(0,n_1+n_2)$ or $(0,(n_1+n_2-1))$. The importance of these states is different. The higher the value of n_1 (for n_1+n_2), the higher the importance of the state, given that customer at n_1 has to be served by both servers before leaving the system, while a customer at n_2 has to be served only at the second one. Here the bottleneck is the first queue, so the states with high value n_1 and low value of n_2 have the highest probability.

2.2.2. Rare State Defined as $n_1 \geq L$

For this definition of the rare set $n_1 \geq L$, the most natural importance function is not $\Phi = n_2$. To obtain states with closer importance, it seems that weight, albeit smaller than the weight given to n_2, must be given to n_1, so the most natural importance function is not $\Phi = an_1 + n_2$, with $0 \leq a \leq 1$.

2.2.3. Rare State Defined as $Min(n_1, n_2) \geq L$

For $n_1 \leq L$ and $n_2 \leq L$, the definition of Φ is $\Phi = a\Phi_1 + \Phi_2$. The definition of the importance function Φ_i is (2).

$$\Phi i = \begin{cases} n_i, if \quad n_i \leq L \\ L + b(n_1 - L), if \quad n_i > L \end{cases} \tag{2}$$

The values of coefficient a are smaller than 1.

3. ALGORITHM WITH RESTART

The RESTART and Limited Relative Error are described in [3,4]:

Step 1 Initialization

Specification of the rare event L, number of levels m, the values of the thresholds $L_{0}, \ldots, L_{m-1}$, and the maximal relative error RE_{max};

Specification of β the values for β_{max} and β_{min};

Further parameter for arrival and service process;

Calculate max step error $RE_{max\,i}$ from RE_{max};

Generate model objects (traffic sources, queues, network nodes);

Choose:

For **M/M/1/N** system *N* or

For **Tandem queue**:

Case 1: Rare State Defined as $n_1 + n_2 \geq L$

Case 2: Rare State Defined as $n_2 \geq L$

Case 3: Rare State Defined as $\mathrm{Min}(n_1 + n_2) \geq L$

Definition of the rare set of the model:

Initialization of number of runs n:=0;the time of simulation s:=1;

Set scaling factor U:=1;

Start simulation;
Step 2 Outer loop
If (n≥10^3 AND (l_i, d_i) ≥10^2 AND (a_i, l_i-a_i, d_i-a) ≥10)
Step 3 For *i*=0
While not (error< RE_{max} for L_0-1 to L_0)

Simulation continues generating data: new value of β examples with exponential distribution. And if the value of β is estimated: β is in the left part as L_0-1 and in the right part as $L_{0;}$

New value of β is accepted if it is in the right part:
Evaluation Counter=Counter+1,
Then store state for RESTART;
End;
Step 4 Inner loop
For *i*=1 to *m*
Set the thresholds $L_0,\ldots,L_m$ for the rare event L;
While not (error< RE_{max} for $L_{i\text{-}1}$ to L_i)
Restore one random state from $L_{i\text{-}1}$;
Calculation Relative Error for Complementary Cumulative Distribution Function for $G_{L_{i-1}|\beta \ge L_{i-1}}$ to $G_{L|\beta \ge L_{i-1}}$. If the RE>RE_{max} the simulation stop and new random state is re-stored;

Simulation continues generating data: new value of β examples with exponential distribution. And if the value of β is estimated: β is in the left part as L_0-1 and in the right part as L_0 ;

New value of β is accepted if it is in the right part:
Evaluation Counter=Counter+1;
Then store state for RESTAR;
Continue simulation until the last value of n;
End;
Multiplication of G-values with U for $L_{i\text{-}1},\ldots, L_i$;

Set scaling factor U:= $G_{L|\beta \ge L_{i-1}}$;
Step 5 Output results
$\hat{G}_L$ and $\hat{\rho}_i$ **:** $\tilde{G}_i = \frac{v_i}{n}$; $\hat{\rho}(x) = \hat{\rho}_i = 1 - \frac{c_i / d_i}{1 - d_i / n}$; $Cor[x] = (1 + \hat{\rho}_i)(1 - \hat{\rho}_i)$
Estimation the probability of rare events.
END.

4. SIMULATION AND RESULTS

All the simulation examples were executed on Personal computer with properties: Pentium 2, 1.4 GHz, 312Mb RAM, operating system Windows XP Professional.

In order to be able to compare our measures with known results, we study the model of tandem queue and queuing system with finite buffers. The distribution of the relative error of several simulations runs is shown in respect to the prescribed error. We observed that the prescribed error limit was exceeded several times by the RE-START/LRE.

The algorithm with RESTART was implemented for single server queuing system M/M/1/220 FIFO and for two servers' tandem queuing system (Fig. 2) as a part of simulation system. RESTART simulation runs were performed and checked against theoretical results.

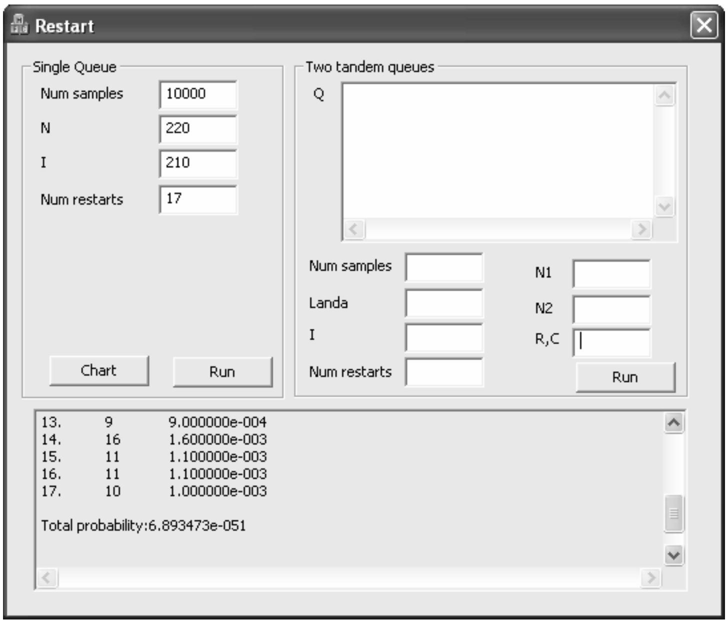

Figure 2 Program realization

The algorithm with RESTART was implemented for one server queuing system. The buffer space at each queue is assumed to be finite. Consider example with ρ=0.8 and ρ=0.7. The prescribed error is in the range of 5% and 10%. The number of simulation samples is n=10 000, number of restarts are 2, 3 and 5. The analytical and simulation results are shown in Tab.1.

Tab. 1: M/M/1/220 system: Comparison of Simulated and Theoretical Values for d_s=5% and d_s=10%, n=10 000, R- number restarts..".

λ	μ	L	Analytical G_L	G_L	Time,ms	R
0.6	0.9	200	3.756785×10^{-36}	4.111528e-036	57.397	15
0.6	0.9	210	6.464279×10^{-38}	2.37e-038	67.712	13
0.7	0.8	200	5.927943×10^{-13}	7.334912e-013	9.561	5
0.7	0.8	210	1.553475×10^{-13}	7.318969e-013	20.686	4

The algorithm was implemented for two servers tandem queuing system. Customers with Poisson arrival enter the first queue and, after being served, enter the second one. The load at each queue is $\rho_i = \lambda/\mu_i$ (i = 1,2) The analytical and simulation results are shown in Table 2.

The buffer space at each queue is assumed to be finite. Consider example with different arrival rates λ=0.6, λ=0.7, λ=0.8 and λ=0.9, μ_1=0.7 μ_1=0.8 and μ_2=0.9. The number of simulation samples is n=100 000. Assume that values for are n_1=9 and n_2=3. Then the possible states at event B are (0,26),(1,26), (2,26), (26,6), (26,1) and (26,0). The importance of each of these states is different. The threshold was chosen as L=0.8.

TABLE 2 Results for two-queue tandem network with n=100 000 simulation samples, relative error 10%.

λ	G_L theoretical	G_L	Time, ms	Number re-starts
0.6	8,38139158e-7	9.292000e-5	2.836	2
0.7	2,01795417e-5	8.912656e-5	0.314	2
0.8	4,16646409e-6	8.849558e-6	2.349	2
0.9	1,78426699e-4	2.380952e-4	0.159	10

5. CONCLUSIONS

Modeling and simulation of M/M/1/N queuing system and tandem queuing system were investigated where the overflow probabilities were estimated with rare event simulation. An algorithm with RESTART approach and Limited Relative Error was developed to speed-up the rare event simulation. This algorithm has been successfully applied to M/M/1/N queuing system and tandem queuing system with Poisson arrival and service rates.

6. ACKNOWLEDGEMENTS

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