Schrödinger and Sinkhorn bridges

P. Del Moral, INRIA Bordeaux - Sud Ouest

ESSEC, Data Analytics Seminar, May 15th 2025.

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Synthesis of some (hyper)-refs:

- *Gaussian models (+Akyildiz-Miguez), Arxiv 2412.18432 (2024).*
- *Lyap./contraction (+Akyildiz-Miguez), Arxiv 2503.09887 (2025).*
- *Log-concave models, Arxiv 2503.15963 (2025).*
- *Conditional covariances, Arxiv 2504.18822 (2025).*
- *Riccati eq./Floquet form (+Horton), $\rightsquigarrow$ Arxiv (2021)/SIAM 2022*

Meas., transition, function $= \mu(dx)$, $\mathcal{K}(x, dy)$, $f(y)$

Integral/matrix operations $[(\mu, \mathcal{K}, f) \stackrel{finite-state}{=}$ (row,matrix,column)]

$$
\begin{aligned}
\mu(f) &:= \int f(x)\ \mu(dx) \\
(\mu\mathcal{K})(dy) &= \int \mu(dx)\mathcal{K}(x, dy) \qquad \mathcal{K}(f)(x) = \int \mathcal{K}(x, dy) f(y)
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Composition

$$(\mathcal{K}_1\mathcal{K}_2)(x, dz) = \int \mathcal{K}_1(x, dy)\mathcal{K}_2(y, dz)$$

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$$(\mu\times\mathcal{K})(d(x,y)) := \mu(dx)\ \mathcal{K}(x,dy)$$

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Disintegration/Set of couplings

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Relative entropy

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- $\nu_{m,\sigma} = \mathbb{R}^d$**-valued Gaussian with mean** "m" **and cov.** "σ"
- G = Centered unit variance Gaussian $\sim \nu_{0,I}$

Bayes' rule = dual/backward transition

Bayes' rule (Bayesian notation)

$$p(x, y) = p(x)\ p(y|x) = p(y)\ p(x|y)$$

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Reverting coordinates $(x, y) \rightsquigarrow (y, x)$

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Bayes' rule for linear-Gauss models: $X \sim \mu = \nu_{m,\sigma}$

$$x \overset{K_\theta}{\rightsquigarrow} Z_\theta(x) = \alpha + \beta x + \tau^{1/2} G \quad \text{with} \quad \theta = (\alpha, \beta, \tau)$$

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$\rightsquigarrow$ **Regression/Dual/Backward/Conjugate/Update formula**

$$\left\{ \begin{array}{l} \mu \times K_\theta = ((\mu K_\theta) \times K_{\mathbb{B}_{m,\sigma}(\theta)})^\flat \\ \\ \end{array} \right.$$

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with the gain/regression matrix

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$\Longleftrightarrow$ **Bayes' map**

$$\theta \mapsto \mathbb{B}_{m,\sigma}(\theta) = (m - \kappa_\theta(\alpha + \beta m), \kappa_\theta, \varsigma_\theta)$$

Entropic optimal transport - (Static) Schrödinger bridge

Target/Marginals proba (μ, η) **& Reference** $\mathcal{P} := \mu \times \mathcal{K}$

$$\mathcal{P}_{\mu,\eta} := \underset{\mathcal{Q} \in \Pi(\mu,\eta)}{\arg\min}\ \mathcal{H}(\mathcal{Q} \mid \mathcal{P})$$

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Theo 5.1 Arxiv 2503.15963 (2025): For "good ref. $\mathcal{P}$" $\exists \varepsilon > 0$ s.t.

$$\mathcal{H}(\eta\mathcal{K} \mid \mu\mathcal{K}) \leqslant \mathcal{H}(\mathcal{P}_{\mu,\eta\mathcal{K}} \mid \mathcal{P}_{\mu,\mu\mathcal{K}}) \leqslant \varepsilon\ \mathcal{H}(\eta \mid \mu)$$

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$\sim$ *For good ref.* $\mathcal{P}$

Ent. marginals-$\mathcal{K}$ $\leqslant$ Ent. bridges $\leqslant$ $\varepsilon \times$ Ent. marginals

Sinkhorn (half) bridges, start from the ref. $\mathcal{K}_0 = \mathcal{K}$

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// Filtering // Bayes stats $\rightsquigarrow$ sequential Bayes updates

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$/\!\!/$ **Filtering** $/\!\!/$ **Bayes stats** $\rightsquigarrow$ **sequential Bayes updates**

Finite state $\rightsquigarrow$ *matrix operations* ($=$ *Iterative proportional fitting*)

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Finite state $\rightsquigarrow$ matrix operations (= Iterative proportional fitting)

Non finite/Non gaussian: Sinkhorn eq. <u>UNSOLVABLE</u> + Algo.

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Choose $\mathcal{K}_2 := \mathcal{K}_1^\sharp$ and "to correct" $(\eta\mathcal{K}_1)$ set

$$\mathcal{P}_2 := \mu \times \mathcal{K}_2 = ((\mu\mathcal{K}_2) \times \mathcal{K}_2^\sharp)^\flat \in \Pi(\mu, \mu\mathcal{K}_2)$$

and so on...

// Filtering // Bayes stats $\rightsquigarrow$ sequential Bayes updates

Finite state $\rightsquigarrow$ matrix operations (= Iterative proportional fitting)

Non finite/Non gaussian: Sinkhorn eq. <u>UNSOLVABLE</u> + Algo.
// Nonlinear filtering equation + filtering algorithm

2-blocks Gibbs samplers

Even targets:

$$\mathcal{P}_{2n} := \mu \times \mathcal{K}_{2n}$$

2-blocks Gibbs samplers

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$$\mathcal{P}_{2n} := \mu \times \mathcal{K}_{2n} = (\pi_{2n} \times \mathcal{K}_{2n+1})^\flat \quad \text{with} \quad \pi_{2n} = \mu\mathcal{K}_{2n}$$

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$$\Longrightarrow \forall n \geqslant 0 \quad \text{fixed point eq.:} \quad \mu[\mathcal{K}_{2n}\mathcal{K}_{2n+1}] = \mu$$

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$$\mathcal{P}_{2n-1} := (\eta \times \mathcal{K}_{2n-1})^\flat = \pi_{2n-1} \times \mathcal{K}_{2n} \quad \text{with} \quad \pi_{2n-1} = \eta\mathcal{K}_{2n-1}$$

2-blocks Gibbs samplers

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2-blocks Gibbs samplers - Gibbs loop process

$$\mu[\mathcal{K}_{2n}\mathcal{K}_{2n+1}] = \mu \overset{(1)}{=} \pi_{2n}\mathcal{K}_{2n+1} \quad \text{and} \quad \eta\,[\mathcal{K}_{2n-1}\mathcal{K}_{2n}] = \eta \overset{(2)}{=} \pi_{2n-1}\mathcal{K}_{2n}$$

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Example: $n = 0$, $\mu = e^{-U}$ **and** $\mathcal{K}_0$ **symmetric-Gaussian:**

$$\mathcal{P}_0(d(x,y)) \propto \exp\left(-U(x) - \frac{1}{2t}\,\|x-y\|_2^2\right)\,dxdy$$

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$$\begin{aligned} \Rightarrow \pi_{2(n+1)} &:= \mu\mathcal{K}_{2(n+1)} \overset{(1)}{=} \pi_{2n}\left[\mathcal{K}_{2n+1}\ \mathcal{K}_{2(n+1)}\right] \longrightarrow_{n\to\infty} \eta \\ \pi_{2n+1} &:= \eta\mathcal{K}_{2n+1} \overset{(2)}{=} \pi_{2n-1}\left[\mathcal{K}_{2n}\ \mathcal{K}_{2n+1}\right] \longrightarrow_{n\to\infty} \mu \end{aligned}$$

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"Auxiliary/Proximal sampler" with a "restricted **oracle** *sampling"* $\mathcal{K}_0^\sharp$...

2-blocks Gibbs samplers - Gibbs loop process

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Oxford Language/Dictionary: **Priest/Medium**

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"Auxiliary/Proximal sampler" with a "restricted **oracle** *sampling"* $\mathcal{K}_0^{\sharp}\ldots$

Oxford Language/Dictionary: **Priest/Medium**

"prophecy/response/message **(from the gods) especially an ambiguous one.**"

In terms of half Schrödinger bridges

Even $\rightsquigarrow$ Odd

$$\mathcal{P}_{2n+1} := \underset{\mathcal{Q}\in\Pi(\pi_{2n+1},\eta)}{\arg\min}\ \mathcal{H}(\mathcal{Q} \mid \mathcal{P}_{2n})$$

In terms of half Schrödinger bridges

Even $\rightsquigarrow$ Odd

$$\mathcal{P}_{2n+1} := \operatorname*{arg\,min}_{\mathcal{Q} \in \Pi(\pi_{2n+1}, \eta)} \mathcal{H}(\mathcal{Q} \mid \mathcal{P}_{2n})$$

Sketched proof:

$$\mathcal{H}((\eta \times \mathcal{L})^\flat \mid \mu \times \mathcal{K}_{2n}) = \mathcal{H}(\eta \mid \mu\mathcal{K}_{2n}) + \mathcal{H}(\eta \times \mathcal{L} \mid \eta \times \mathcal{K}_{2n+1}).$$

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$$\Longrightarrow \text{optimal transition } \mathcal{L} = \mathcal{K}_{2n+1} \Longrightarrow \mathcal{P}_{2n+1} = (\eta \times \mathcal{K}_{2n+1})^\flat$$

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Odd $\rightsquigarrow$ Even

$$\mathcal{P}_{2(n+1)} := \underset{\mathcal{Q}\in\Pi(\mu,\pi_{2(n+1)})}{\arg\min}\ \mathcal{H}(\mathcal{Q} \mid \mathcal{P}_{2n+1})$$

In terms of half Schrödinger bridges

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"Conclusion":

$$(\pi_{2n+1}, \pi_{2(n+1)}) \longrightarrow_{n\to\infty} (\mu, \eta)$$

In terms of half Schrödinger bridges

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"Conclusion":

$$(\pi_{2n+1}, \pi_{2(n+1)}) \longrightarrow_{n\to\infty} (\mu, \eta) \implies \mathcal{P}_{2n+1} \;\&\; \mathcal{P}_{2(n+1)} \longrightarrow_{n\to\infty} \mathcal{P}_{\mu,\eta}$$

In terms of Schrödinger potentials $(\mu, \eta) = (e^{-U}, e^{-V})$

Reference $\mathcal{P}_0 = \mu \times \mathcal{K} \rightsquigarrow \mathcal{P}_{2n} = \mu \times \mathcal{K}_{2n}$:

$$\mathcal{K}(x, dy) = Q(x, dy) := q(x, y)dy \quad \text{and set} \quad R(y, dx) := q(x, y)dx$$

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$$\mathcal{P}_{2n}(d(x, y)) \overset{n=0}{=} e^{-U_{2n}(x)} \; q(x, y) \; e^{-V_{2n}(y)} \; dxdy \quad \text{with} \quad (U_0, V_0) = (U, 0)$$

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$$\Longleftrightarrow \mathcal{K}_{2n}(x, dy) = \frac{Q(x, dy)e^{-V_{2n}(y)}}{Q(e^{-V_{2n}})(x)} \quad \text{and} \quad e^{-U} =: e^{-U_{2n}} Q(e^{-V_{2n}})$$

In terms of Schrödinger potentials $(\mu, \eta) = (e^{-U}, e^{-V})$

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$\mathcal{P}_{2n}$-**Conjugate** $dx|y = \mathcal{K}_{2n+1}(y, dx)$

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Schrödinger system $(V_{2(n+1)} = V_{2n+1})$**:**

$$U_{2n+1} = U + \log Q(e^{-V_{2n}}) \quad \text{and} \quad V_{2(n+1)} := V + \log R(e^{-U_{2n+1}}).$$

Log-concave at ∞ marginals + lin-gauss ref. transition

Markov transport $\downarrow$ entropy $\rightsquigarrow$ sandwich inequalities

$$\mathcal{H}(\pi_{2n} \mid \eta) \leqslant \mathcal{H}(\mu \mid \pi_{2n-1}) \leqslant \mathcal{H}(\pi_{2(n-1)} \mid \eta)$$

Proof: $(\pi_{2n}, \pi_{2n-1}) = (\mu\mathcal{K}_{2n}, \eta\mathcal{K}_{2n-1})$

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Entropy bridge decays

$$\mathcal{H}(\eta \mid \pi_{2n}) \leqslant \mathcal{H}(P_{\mu,\eta} \mid \mathcal{P}_{2n}) = \mathcal{H}(P_{\mu,\eta} | \mathcal{P}_{2n-1}) - \mathcal{H}(\mu \mid \pi_{2n-1})$$

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Markov transport $\downarrow$ entropy $\rightsquigarrow$ sandwich inequalities

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Entropy bridge decays

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By Theo 7.9 Arxiv 2503.15963 (2025) [$\sim$ *for good references*]

Entropy bridges $\leqslant \varepsilon \times$ entropy marginals

$$\Downarrow$$

$$\mathcal{H}(P_{\mu,\eta} \mid \mathcal{P}_{2n}) \leqslant (1+\varepsilon^{-1})^{-n}\ \mathcal{H}(P_{\mu,\eta} \mid \mathcal{P}_0)$$

Linear decays in a few lines ($\mathcal{P}_{n+1}$ "corrects marg of " $\mathcal{P}_n$)

$$\frac{d\mathcal{P}_{2q}}{d\mathcal{P}_0}(x,y) \quad = \quad \left[\prod_{0\leqslant n<q} \frac{d\mathcal{P}_{2n+1}}{d\mathcal{P}_{2n}}(x,y)\right] \left[\prod_{0\leqslant n<q} \frac{d\mathcal{P}_{2(n+1)}}{d\mathcal{P}_{2n+1}}(x,y)\right]$$

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$$\begin{aligned}\frac{d\mathcal{P}_{2q}}{d\mathcal{P}_0}(x,y) &= \left[\prod_{0\leqslant n<q} \frac{d\mathcal{P}_{2n+1}}{d\mathcal{P}_{2n}}(x,y)\right]\left[\prod_{0\leqslant n<q} \frac{d\mathcal{P}_{2(n+1)}}{d\mathcal{P}_{2n+1}}(x,y)\right]\\ &= \left[\prod_{0\leqslant n<q} \frac{d\eta}{d\pi_{2n}}(y)\right]\left[\prod_{0\leqslant n<q} \frac{d\mu}{d\pi_{2n+1}}(x)\right].\end{aligned}$$

Linear decays in a few lines ($\mathcal{P}_{n+1}$ "corrects marg of" $\mathcal{P}_n$)

$$
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&= \left[\prod_{0\leqslant n<q} \frac{d\eta}{d\pi_{2n}}(y)\right]\left[\prod_{0\leqslant n<q} \frac{d\mu}{d\pi_{2n+1}}(x)\right].
\end{aligned}
$$

For any $\mathcal{Q}\in\Pi(\mu,\eta)$ **by** *sandwich* $\downarrow$ *inequalities we have*

$$
\mathrm{Ent}(\mathcal{Q}\mid\mathcal{P}_0) \geqslant \mathrm{Ent}(\mathcal{Q}\mid\mathcal{P}_0)-\mathrm{Ent}(\mathcal{Q}\mid\mathcal{P}_{2(q+1)}) = \mathcal{Q}\left(\log\frac{d\mathcal{P}_{2(q+1)}}{d\mathcal{P}_0}\right)
$$

Linear decays in a few lines ($\mathcal{P}_{n+1}$ "corrects marg of" $\mathcal{P}_n$)

$$
\begin{aligned}
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$$
\begin{aligned}
\mathrm{Ent}(\mathcal{Q} \mid \mathcal{P}_0) &\geqslant \mathrm{Ent}(\mathcal{Q} \mid \mathcal{P}_0) - \mathrm{Ent}(\mathcal{Q} \mid \mathcal{P}_{2(q+1)}) = \mathcal{Q}\left(\log \frac{d\mathcal{P}_{2(q+1)}}{d\mathcal{P}_0}\right) \\
&= \sum_{0\leqslant n\leqslant q} \left(\mathrm{Ent}\left(\eta \mid \pi_{2n}\right) + \mathrm{Ent}\left(\mu \mid \pi_{2n+1}\right)\right)
\end{aligned}
$$

Linear decays in a few lines ($\mathcal{P}_{n+1}$ "corrects marg of" $\mathcal{P}_n$)

$$\frac{d\mathcal{P}_{2q}}{d\mathcal{P}_0}(x,y) = \left[\prod_{0\leqslant n<q} \frac{d\mathcal{P}_{2n+1}}{d\mathcal{P}_{2n}}(x,y)\right]\left[\prod_{0\leqslant n<q} \frac{d\mathcal{P}_{2(n+1)}}{d\mathcal{P}_{2n+1}}(x,y)\right]$$
$$= \left[\prod_{0\leqslant n<q} \frac{d\eta}{d\pi_{2n}}(y)\right]\left[\prod_{0\leqslant n<q} \frac{d\mu}{d\pi_{2n+1}}(x)\right].$$

For any $\mathcal{Q} \in \Pi(\mu,\eta)$ **by** *sandwich* $\downarrow$ *inequalities we have*

$$\begin{aligned}\mathrm{Ent}(\mathcal{Q} \mid \mathcal{P}_0) &\geqslant \mathrm{Ent}(\mathcal{Q} \mid \mathcal{P}_0) - \mathrm{Ent}(\mathcal{Q} \mid \mathcal{P}_{2(q+1)}) = \mathcal{Q}\left(\log \frac{d\mathcal{P}_{2(q+1)}}{d\mathcal{P}_0}\right)\\ &= \sum_{0\leqslant n\leqslant q} \left(\mathrm{Ent}\left(\eta \mid \pi_{2n}\right) + \mathrm{Ent}\left(\mu \mid \pi_{2n+1}\right)\right)\\ &\geqslant (q+1)\left(\mathrm{Ent}\left(\eta \mid \pi_{2q}\right) + \mathrm{Ent}\left(\mu \mid \pi_{2q+1}\right)\right)\end{aligned}$$

Gaussian models

Marginals $(\mu, \eta) = (\nu_{m,\sigma}, \nu_{\overline{m},\overline{\sigma}})$ **and reference transition:**

$$\mathcal{K}(x, dy) = K_\theta(x, dy) = \mathbb{P}(Z_\theta(x) \in dy)$$

with the random map

$$\theta = (\alpha, \beta, \tau) \;\mapsto\; Z_\theta(x) := \alpha + \beta x + \tau^{1/2} G$$

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Some advantages: *Closed-form Schrödinger & dual potential bridges,*

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Gaussian Schrödinger map = Riccati steady-state

$$Z_{\mathbb{S}(\theta)}(x) = \overline{m} + \varsigma_\theta \ \chi_\theta \ (x - m) + \varsigma_\theta^{1/2} \ G \quad \text{with} \quad \chi_\theta := \tau^{-1}\beta$$

$\Longleftrightarrow$ **fixed point of Riccati matrix equation:**

$$(\varsigma_\theta \ \chi_\theta) \ \sigma \ (\varsigma_\theta \ \chi_\theta)' + \varsigma_\theta = \overline{\sigma}$$

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$$(\varsigma_\theta \; \chi_\theta) \; \sigma \; (\varsigma_\theta \; \chi_\theta)' + \varsigma_\theta = \overline{\sigma} \Longleftrightarrow r_\theta \; \varpi_\theta^{-1} \; r_\theta + r_\theta = I$$

with the rescaled matrices

$$\varsigma_\theta =: \overline{\sigma}^{1/2} \; r_\theta \; \overline{\sigma}^{1/2} \quad \text{and} \quad \varpi_\theta^{-1} := \overline{\sigma}^{1/2} \; (\chi_\theta \sigma \chi_\theta') \; \overline{\sigma}^{1/2}$$

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Closed form Discrete Algebraic Riccati equation (DARE)

$$r_\theta = -\frac{\varpi_\theta}{2} + \left(\varpi_\theta + \left(\frac{\varpi_\theta}{2}\right)^2\right)^{1/2}$$

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+Sharp rates $\subset$ Riccati eq./Floquet form $\subset$ Arxiv (2021)/SIAM 2022

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+Sharp rates ⊂ Riccati eq./Floquet form ⊂ Arxiv (2021)/SIAM 2022

⊕ Brief review ⊂ Appendix B in Arxiv 2412.18432 (2024)

Gaussian Sinkhorn $(\theta_0 \stackrel{ref}{=} \theta)$ & $\theta_{2n} \rightsquigarrow \theta_{2n+1} \rightsquigarrow \theta_{2(n+1)}$

$$\left\{ \begin{array}{ccccc} \mathcal{P}_{2n} & = & \mu \times K_{\theta_{2n}} & = & ((\mu K_{\theta_{2n}}) \times K_{\theta_{2n+1}})^{\flat} \\ & & & & \end{array} \right.$$

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with the Bayes'/Regression/Dual/Conjugate maps

$$\theta_{2n+1} = \mathbb{B}_{m,\sigma}(\theta_{2n}) \quad \text{and} \quad \theta_{2(n+1)} = \mathbb{B}_{\overline{m},\overline{\sigma}}(\theta_{2n+1})$$

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$\rightsquigarrow \theta_n := (\alpha_n, \beta_n, \tau_n)$ **& Sinkhorn Gauss marginals**

$$\mu K_{\theta_{2n}} = \nu_{m_{2n},\sigma_{2n}} \quad \text{and} \quad \eta K_{\theta_{2n+1}} = \nu_{m_{2n+1},\sigma_{2n+1}}$$

Gaussian Schrödinger map = Riccati steady state

$$Z_{\mathbb{S}(\theta)}(x) = \overline{m} + \varsigma_\theta\ \chi_\theta\ (x - m) + \varsigma_\theta{}^{1/2}\ G \quad \text{with} \quad \chi_\theta := \tau^{-1}\beta$$

with the rescaled matrices

$$\varsigma_\theta =: \overline{\sigma}^{1/2}\ r_\theta\ \overline{\sigma}^{1/2} \quad \text{and} \quad r_\theta = \mathrm{Ricc}_{\varpi_\theta}(r_\theta)$$

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Sinkhorn map $=$ Riccati difference equation:

$$Z_{\theta_{2n}}(x) = m_{2n} + \tau_{2n} \ \chi_\theta \ (x - m) + \tau_{2n}^{1/2} \ G$$

with the rescaled matrices

$$\tau_{2n} = \overline{\sigma}^{1/2} \ \upsilon_{2n} \ \overline{\sigma}^{1/2} \quad \text{and} \quad \upsilon_{2n} := \mathrm{Ricc}_{\varpi_\theta}(\upsilon_{2(n-1)}) < I$$

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and the expected values

$$m_{2n} - \overline{m} = \overline{\sigma}^{1/2}\ (I - \upsilon_{2n})\ \overline{\sigma}^{-1/2}\ (m_{2(n-1)} - \overline{m})$$

Brief review of $v_{n+1} = \mathrm{Ricc}_{\varpi}(v_n) := (I + (\varpi + v_n)^{-1})^{-1}$

Uniform estimates $1 \leqslant p \leqslant n$

$$\mathrm{Ricc}_{\varpi}^{p}(0) \leqslant \mathrm{Ricc}_{\varpi}^{n}(0) \leqslant \mathrm{Ricc}_{\varpi}^{n}(v) \leqslant \mathrm{Ricc}_{\varpi}^{n-1}(I) \leqslant \mathrm{Ricc}_{\varpi}^{p-1}(I) \leqslant I$$

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Unique (closed-form) positive fixed point

$$r_{\varpi} = \mathrm{Ricc}_{\varpi}(r_{\varpi}) = -\frac{\varpi}{2} + \left(\varpi + \left(\frac{\varpi}{2}\right)^2\right)^{1/2}$$

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Sharp exp. $\downarrow$ *(Floquet form & 1d-closed-form),* ℓ_{min}*:=min eigenvalue*

$$\|v_n - r_\varpi\| \leqslant c_{1,\varpi}\ \delta^n_\varpi\ \|v_0 - r_\varpi\| \quad \text{with} \quad \delta_\varpi := (1 + \ell_{min}(\varpi + r_\varpi))^{-2}$$

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Matrix products

$$\|(I - v_n)\dots(I - v_1)(I - v_0)\| \leqslant c_{2,\varpi}\ \delta^{n/2}_\varpi$$

Some convergence theorems (reference $\theta = \theta_0$)

$$\mathcal{P}_{2n} := P_{\theta_{2n}} := \mu \times K_{\theta_{2n}} \longrightarrow_{n\to\infty} P_{\mathbb{S}(\theta)} := \mu \times K_{\mathbb{S}(\theta)}$$

In terms of the parameter

$$\rho_\theta := (1 + \ell_{min}(\varpi_\theta + r_{\varpi_\theta}))^{-2} \quad \text{with} \quad \varpi_\theta := \overline{\sigma}^{-1/2}\,\tau\,(\beta\sigma\beta')^{-1}\,\tau\,\overline{\sigma}^{-1/2}$$

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Entropy estimates *(sharper than any log-concave marg. studies)*

$$\mathrm{Ent}\left(P_{\theta_{2n}} \mid P_{\mathbb{S}(\theta)}\right) \leqslant c_\theta\ \rho_\theta^n\ \left(\|\tau - \varsigma_\theta\| + \|m_0 - \overline{m}\|^2\right).$$

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***p*-Wasserstein distance**

$$\mathbb{W}_p\left(P_{\theta_{2n}}, P_{\mathbb{S}(\theta)}\right) \leqslant c_{1,\theta}(p)\ \rho_\theta^n\ \|\tau - \varsigma_\theta\| + c_{2,\theta}\ \rho_\theta^{n/2}\ \|m_0 - \overline{m}\|$$

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p-**Wasserstein distance**

$$\mathbb{W}_p\left(P_{\theta_{2n}}, P_{\mathbb{S}(\theta)}\right) \leqslant c_{1,\theta}(p)\ \rho_\theta^n\ \|\tau - \varsigma_\theta\| + c_{2,\theta}\ \rho_\theta^{n/2}\ \|m_0 - \overline{m}\|$$

Same Riccati eq. $\subset$ *Log-concave (vs Cramer-Rao & Brascamp-Lieb)*

- *Log-concave marginals Arxiv 2503.15963 (2025)*
- *Conditional covariances Arxiv 2504.18822 (2025)*

Example: Linear-diffusion flow $(X_{s,s}(x) = x)$

$$t \in [s, \infty[\rightsquigarrow d_t X_{s,t}(x) = (A_t X_{s,t}(x) + b_t)\ dt + dW_t$$

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Semigroup & Marginals = Internal states distributions :

$$P_{s,t}(x, dy) := \mathbb{P}(X_{s,t}(x) \in dy) \quad \text{and} \quad \nu_{\mathbf{t}} = \nu_{\mathbf{0}} \mathbf{P}_{\mathbf{0},\mathbf{t}} =: \mathrm{Law}(\mathbf{X}_{\mathbf{t}})$$

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On any given final time t:

$$\mu \times \mathcal{K} = \nu_0 \times P_{0,t} = \nu_0 \times K_{\theta[t]} \quad \text{with} \quad \theta[t] = (\alpha[t], \beta[t], \tau[t])$$

and the parameters

$$\alpha[t] \quad := \quad \int_0^t \mathcal{E}_{s,t}(A)\ b_s\ ds \quad \text{with} \quad \partial_t\, \mathcal{E}_{s,t}(A) := A_t\, \mathcal{E}_{s,t}(A)$$

$$\beta[t] \quad := \quad \mathcal{E}_{0,t}(A) \quad \text{and} \quad \tau[t] := \int_0^t \mathcal{E}_{s,t}(A)\ \Sigma_s\ \mathcal{E}_{s,t}(A)'\ ds.$$

Backward/Dual semigroup on $[0,t] \ni s$

$$P_{t,s}(y,dx) := \mathbb{P}(X_{t,s}(y) \in dx) \Longleftrightarrow \nu_s \times P_{s,t} = ((\nu_s P_{s,t}) \times P_{t,s})^\flat$$

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Example $\nu_s(dx) = p_s(x)dx$ **and** $X_{t,t}(x) = x$**:**

$$-d_s X_{t,s}(x) = (-(A_s X_{t,s}(x) + b_s) + (\nabla \log p_s)(X_{t,s}(x)))\, ds + dW_s$$

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On any given final/terminal time t:

$$\mu \times \mathcal{K} = \nu_0 \times P_{0,t} \quad \text{and} \quad (\mu \times \mathcal{K})^\flat = (\mu\mathcal{K}) \times \mathcal{K}^\sharp = \nu_t \times P_{t,0}$$

Backward/Dual semigroup on $[0, t] \ni s$

$$P_{t,s}(y, dx) := \mathbb{P}(X_{t,s}(y) \in dx) \Longleftrightarrow \nu_s \times P_{s,t} = ((\nu_s P_{s,t}) \times P_{t,s})^\flat$$

Example $\nu_s(dx) = p_s(x)dx$ and $X_{t,t}(x) = x$:

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Notes:

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Notes:

- **Pb with continuous time $\rightsquigarrow$ Sol. discretize SDE**
- **Pb with unknown p_s $\rightsquigarrow$ Sol. with scores/neural networks**

Static vs **Dynamic bridges** $\mathbf{Q}(d\omega)$ on $\Omega := \mathcal{C}([0,T],\mathbb{R}^d)$

$$
\begin{aligned}
\mathbf{Q}^{x,y}(d\omega) &:= \mathbf{Q}(d\omega \mid (\omega_0,\omega_T)=(x,y)) \\
\mathcal{Q}(d(x,y)) &:= \eta(dx)\,\mathcal{L}(x,dy) \quad \text{with} \quad \mathcal{L}(x,dy) := \mathbf{Q}(\omega_T \in dy \mid \omega_0 = x).
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$$\mathrm{Ent}(\mathbf{Q} \mid \mathbf{P}) = \mathrm{Ent}(\mathcal{Q} \mid \mathcal{P}) + \int \underbrace{\mathrm{Ent}(\ \mathbf{Q}^{x,y} \mid \mathbf{P}^{x,y})}_{=0 \Leftarrow \mathbf{Q}^{x,y} = \mathbf{P}^{x,y}} \ \mathcal{Q}(d(x,y)).$$

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Dynamic Schrödinger bridge $\mathbf{C}(\mu,\eta) = \{\mathbf{Q} \ : \ \mathcal{Q} \in \Pi(\mu,\eta)\}$

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Solution = **Static bridge** + **Pinned process :**

$$\mathbf{P}_{\mu,\eta}(d\omega) := \int \mathbf{P}^{x,y}(d\omega)\mathcal{P}_{\mu,\eta}(d(x,y))$$

c-Optimal vs Entropic transport $(\mu, \eta) = (e^{-U}, e^{-V})$

$\mathcal{Q} = \mu \times \mathcal{L}$ s.t. $\mu\mathcal{L} = \eta$ **and ref.** $\mathcal{P}(\mathbf{d}(\mathbf{x}, \mathbf{y})) = \mu(\mathbf{dx})\ \mathbf{e}^{-\mathbf{c}(\mathbf{x},\mathbf{y})/\mathbf{t}}\ \mathbf{dy}$

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Entropic cost

$$\mathcal{Q}(\log \tfrac{d\mathcal{Q}}{d\mathcal{P}}) = \mathcal{Q}(\log \tfrac{d\mu\otimes\eta}{d\mathcal{P}}) + \mathcal{Q}(\log \tfrac{d\mathcal{Q}}{d\mu\otimes\eta})$$

$$\Longrightarrow \mathcal{H}\left(\mathcal{Q} \mid \mathcal{P}\right) + \eta(V) = \frac{\mathbf{1}}{\mathbf{t}} \int \mathbf{c}(\mathbf{x},\mathbf{y})\ \mathcal{Q}(\mathbf{d}(\mathbf{x},\mathbf{y})) + \mathcal{H}\left(\mathcal{Q} \mid \mu \otimes \eta\right)$$

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t-regularized entropic optimal transport:

$$\mathcal{P}_{\mu,\eta} := \underset{\mathcal{Q} \in \Pi(\mu,\eta)}{\arg\min} \; \mathcal{H}(\mathcal{Q} \mid \mathcal{P}) \;\; \simeq_{t\downarrow 0} \;\; \underset{\mathcal{Q} \in \mathbf{\Pi}(\mu,\eta)}{\arg\min} \int \mathbf{c}(\mathbf{x}, \mathbf{y}) \; \mathcal{Q}(\mathbf{d}(\mathbf{x}, \mathbf{y}))$$

Ex.: Gauss targets + Lin-Gauss ref (= Quadratic cost)

Riccati ref. matrix:

$$\tau = t\ I \Longrightarrow \frac{\varpi_\theta}{t^2} = \overline{\sigma}^{-1/2}\ \sigma_\beta^{-1}\ \overline{\sigma}^{-1/2} \quad \text{with} \quad \sigma_\beta := \beta\sigma\beta'$$

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Riccati fixed point $\simeq$ Geometric mean covariance:

$$\frac{r_\theta}{t} = -\frac{\varpi_\theta}{2t} + \left(\frac{\varpi_\theta}{t^2} + \left(\frac{\varpi_\theta}{2t}\right)^2\right)^{1/2} \simeq \left(\overline{\sigma}^{-1/2}\ \sigma_\beta^{-1}\ \overline{\sigma}^{-1/2}\right)^{1/2}$$

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Limiting transport map from $\nu_{m,\sigma}$ to $\nu_{\overline{m}\overline{\sigma}}$

$$Z_{\mathbb{S}(\theta)}(x) = \overline{m} + \frac{\varsigma_\theta}{t}\ \beta\ (x-m) + \varsigma_\theta^{1/2}\ G \simeq \overline{m} + (\sigma_\beta^{-1} \sharp\ \overline{\sigma})\ \beta\ (x-m)$$

Parametric class $\{K_\theta, \theta \in \Theta\}$ & projections

$\theta_{2n} \rightsquigarrow \theta_{2n+1}$:

$$\mathrm{Ent}((\eta \times K_{\theta_{2n+1}})^\flat \mid \mu \times K_{\theta_{2n}})$$

$$:= \inf_{\theta \in \Theta} \mathrm{Ent}((\eta \times K_\theta)^\flat \mid \mu \times K_{\theta_{2n}})$$

$$= \mathrm{Ent}(\eta \mid \mu K_{\theta_{2n}}) + \inf_{\theta \in \Theta} \mathrm{Ent}(\eta \times K_\theta \mid \eta \times K^\sharp_{\theta_{2n}}).$$

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Note:

$$\inf_{\theta \in \Theta} \mathrm{Ent}(\eta \times K_\theta \mid \eta \times K^\sharp_{\theta_{2n}}) = 0 \Longleftrightarrow \exists \theta \in \Theta \ \text{ such that } K_\theta = K^\sharp_{\theta_{2n}}.$$

Linear-Gaussian = unbiasedness property.