A Berry-Esseen theorem for Feynman-Kac and interacting particle models

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Abstract

In this paper we investigate the speed of convergence of the fluctuations of a general class of Feynman-Kac particle approximation models. We design an original approach based on new Berry-Esseen type estimates for abstract martingale sequences combined with original exponential concentration estimates of interacting processes. These results extend the corresponding statements in the classical theory and apply to a class of branching and genealogical path-particle models arising in non linear filtering literature as well as in statistical physics and biology.

1 Introduction

Feynman-Kac distribution flows and their particle interpretations arise in the modeling and the numerical solving of a variety of problems includign directed polymer simulations in biology and industrial chemistry, non linear filtering in advanced signal processing and Bayesian statistics methodology, rare event estimation in telecommunication and computer systems analysis as well as physics in the spectral analysis of Schrödinger operators and in the study of particle absorbtions. Their asymptotic behavior as the size of the systems and/or the time parameter tend to infinity has been the subject of various research articles. For more details on both the theoretical and applied aspects of the topic we refer the reader to the review article [5] and references therein.

To better connect this study with existing and related articles in the literature we give a brief discussion on the fluctuation analysis of these models:
The first "local" central limit theorems were presented in [1]. These fluctuations were restricted to local sampling errors of an abstract

class of genetic type particle model. This study was extended in [3] in the spirit of Shiga-Tanaka's celebrated article [9] to particle and McKean path-measures. This approach to fluctuations in path space was centered around Girsanov type change of measures techniques and a theorem of Dynkin-Mandelbaum on symmetric statistics [7]. This strategy entirely relies on appropriate regularity conditions on the Markov kernels which are not satisfied for genealogical tree evolution models as the ones described in [1]. Another drawback of this approach is that the description of resulting limiting variance is not explicit but expressed in term of the inverse of an $\mathbb{L}_2$ integral operator.
Donsker's type theorems and an explicit computation of the limiting variance in terms of Feynman-Kac semi-groups were further developed in [4] in the context of particle density profile models. These explicit functional formulations were the starting point of a new approach to central limit theorems based on judicious martingale decompositions and Feynman-Kac semi-group techniques [2, 5].
The main objective of the current article is to complete and further extend these studies with the analysis of the speed of convergence of fluctuations.

The article is organized as follows:

In section 1.1 we describe the Feynman-Kac and the particle models discussed in this article. In section 1.2 we present our main results and specify the set of regularity conditions needed in the sequel. Section 2 is concerned with a precise Berry-Esseen type estimate for abstract martingale sequences. In section 3 we show how these martingale fluctuations apply to a sufficiently regular class of McKean particle interpretations.

We end this section with some rather standard and classical notation that will be of current use in the article:

By $\mathcal{M}(E)$ we denote the set of all bounded and positive measures on a measurable space $(E, \mathcal{E})$, by $\mathcal{P}(E) \subset \mathcal{M}(E)$ we denote the subset of probability measures on $(E, \mathcal{E})$ and by $\mathcal{B}_b(E)$ the Banach space of all bounded $\mathcal{E}$-measurable functions f on E equipped with the uniform norm $\|f\| = \sup_{x \in E} |f(x)|$. We also let $\mathrm{Osc}(E) \subset \mathcal{B}_b(E)$ be the subset of all bounded measurable functions with oscillations $\mathrm{osc}(f) = \sup_{(x,y)} |f(x) - f(y)| \leq 1$.
We finally recall that an integral positive and bounded operator Q from $(E, \mathcal{E})$ into another measurable space $(E', \mathcal{E}')$ generates two operators. One acting on functions $f \in \mathcal{B}_b(E')$ and taking values in $\mathcal{B}_b(E)$, the other acting on measures $\mu \in \mathcal{M}(E)$ into $\mathcal{M}(E')$ and de-

fined by

$$Q(f)(x) = \int_{E'} Q(x, dx')f(x') \ , \quad \mu Q(dx') = \int_{E} \mu(dx) \ Q(x, dx')$$

To clarify the presentation we shall slightly abuse the notations, and we often write $Q(f-Q(f))^2$ for the function $x \to Q(f-Q(f)(x))^2(x)$. Finally we shall use the letter c to denote any non negative and universal constant whose values may vary from line to line but does not depend on the time parameter nor on the Feynman-Kac models.

1.1 Description of the models

We consider some collections of measurable spaces $(E_n, \mathcal{E}_n)_{n\in\mathbb{N}}$, of Markov transitions $M_{n+1}(x_n, dx_{n+1})$ from E_n to E_{n+1}, and bounded $\mathcal{E}_n$-measurable and strictly positive functions G_n on E_n. We assume that the latter are chosen so that for any $n \in \mathbb{N}$ we have

$$r_n = \sup_{(x_n, y_n)\in E_n^2} (G_n(x_n)/G_n(y_n)) < \infty \tag{1}$$

We associate to the pair (G_n, M_n) the Bolzmann-Gibbs transformation Ψ_n on $\mathcal{P}(E_n)$ and the mapping Φ_{n+1} from $\mathcal{P}(E_n)$ into $\mathcal{P}(E_{n+1})$ given for any $(x_n, \mu_n) \in (E_n, \mathcal{P}(E_n))$ by

$$\begin{array}{rcl} \Psi_n(\mu_n)(dx_n) & = & G_n(x_n)\mu_n(dx_n)/\mu_n(G_n) \\ \Phi_{n+1}(\mu_n) & = & \Psi_n(\mu_n)M_{n+1} \end{array}$$

For any $\eta_0 \in \mathcal{P}(E_0)$ we denote by $\mathbb{E}_{\eta_0}(.)$ the expectation operator with respect to the distribution of a Markov chain X_n with initial distribution η_0 and elementary transitions M_n. We consider the distribution flow $\eta_n \in \mathcal{P}(E_n)$, $n \in \mathbb{N}$, defined for any $f_n \in \mathcal{B}_b(E_n)$ by the Feynman-Kac formulae

$$\eta_n(f_n) = \gamma_n(f_n)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f_n) = \mathbb{E}_{\eta_0}\Big[f_n(X_n) \prod_{0\leq p<n} G_p(X_p)\Big] \tag{2}$$

with the convention $\prod_\emptyset = 1$. Using the multiplicative structure of the Feynman-Kac model and the Markov property one readily checks that the flow η_n satisfies the non linear equation

$$\eta_{n+1} = \eta_n K_{n+1,\eta_n} \tag{3}$$

where $(K_{n+1,\mu_n})_{n\in\mathbb{N},\mu_n\in\mathcal{P}(E_n)}$, is a *non unique* collection of Markov transitions satisfying the compatibility condition

$$\forall n \in \mathbb{N} \ , \quad \forall \mu_n \in \mathcal{P}(E_n) \, , \qquad \mu_n K_{n+1,\mu_n} = \Phi_{n+1}(\mu_n). \tag{4}$$

These collections of transitions are often called the McKean interpretations of the equation (3). Notice that the compatibility relation (4) is satisfied if we take

$$K_{n+1,\mu_n}(x_n,\cdot) = \varepsilon_n\ G_n(x_n)\ M_{n+1}(x_n,\cdot) + (1-\varepsilon_n\ G_n(x))\ \Phi_{n+1}(\mu_n) \tag{5}$$

for any non negative constant ε_n such that $\varepsilon_n\ G_n(x_n) \in [0,1]$. We finally notice that the random variables X_n may represent the path of an auxiliary Markov chain X'_p from the origin up to time n and taking values in some Hausdorff topological spaces E'_p, that is we have

$$X_n = (X'_0,\ldots,X'_n) \in E_n = (E'_0 \times \ldots \times E'_n) \tag{6}$$

For each $N \geq 1$ we denote by m_N the mapping from the product space E^N into $\mathcal{P}(E)$ which associates to each configuration $x = (x^i)_{1\leq i\leq N} \in E^N$ the empirical measure $m_N(x) = \frac{1}{N}\sum_{i=1}\delta_{x^i}$. The interacting particle system associated to a given McKean interpretation is defined as a Markov chain $\xi_n^{(N)} = (\xi_n^{(N,i)})_{1\leq i\leq N}$ taking values in the product spaces E_n^N with initial distribution $\eta_0^{\otimes N}$ and elementary transitions

$$\text{Prob}\left(\xi_n^{(N)} \in dx_n \mid \xi_{n-1}^{(N)}\right) = \prod_{i=1}^N K_{n,m_N(\xi_{n-1}^{(N)})}(\xi_{n-1}^{(N,i)}, dx_n^i) \tag{7}$$

where $dx_n = \times_{1\leq i\leq N} dx_n^i$ stands for an infinitesimal neighborhood of the point $x_n = (x_n^i)_{1\leq i\leq N} \in E_n^N$.

Under appropriate regularity conditions on the McKean transitions kernels K_{n,μ_n} it is known that in some sense the particle measures

$$\eta_n^N = m_N(\xi_n^{(N)})$$

converge as N tends to infinity to the desired distributions η_n.

To illustrate this model we note that the particle interpretation of the Feynman-Kac flow associated to McKean transitions (5) forms a two step selection/mutation genetic algorithm. The particular situation where $\varepsilon_n = 0$ corresponds to a simple genetic model with an overlapping mutation/selection transition. In the same vein the corresponding particle interpretation model of the Feynman-Kac path measures associated to the chain (6) forms is a genetic type algorithm taking values in path space. Note that in this situation the path-particles have the form

$$\xi_n^{(N,i)} = (\xi_{0,n}^{(N,i)}, \xi_{1,n}^{(N,i)}, \ldots, \xi_{n,n}^{(N,i)}) \in E_n = (E'_0 \times \ldots \times E'_n)$$

In addition if the potential functions only depend on terminal values in the sense that $G_n(x'_0, \ldots, x'_n) = G'_n(x'_n)$ for some potential function G'_n on E'_n then the resulting path-particle model can be interpreted as a genealogical tree evolution model.

As traditionally to clarify the presentation we slightly abuse the notation, by suppressing the size index N and we write $(m(x), \xi_n, \xi_n^i)$ instead of $(m_N(x), \xi_n^{(N)}, \xi_n^{(N,i)})$.

1.2 Statement of some results

For any sequence of $\mathcal{F}_n^N$-adapted random variables Z_n^N defined on some filtered probability spaces $(\Omega^N, (\mathcal{F}_n^N)_{n\geq 0}, \mathbb{P}^N)$ we denote by ΔZ_n^N the difference process $\Delta Z_n^N = Z_n^N - Z_{n-1}^N$, with the convention $\Delta Z_0^N = Z_0^N$ for $n = 0$. If ΔM_n^N is a given $\mathcal{F}_n^N$-martingale difference, then we denote by M_n^N the $\mathcal{F}_n^N$-martingale defined by $M_n^N = \sum_{p=0}^n \Delta M_p^N$. We recall that its increasing process $\langle M^N \rangle_n$ is given by

$$\langle M^N \rangle_n = \sum_{p=0}^n \mathbb{E}^N \left[(\Delta M_p^N)^2 \mid \mathcal{F}_{p-1}^N \right],$$

with the convention $\mathcal{F}_{-1}^N = \{\emptyset, \Omega^N\}$ for $p = 0$. It is also convenient to introduce the increasing process $C_n^N = N \langle M^N \rangle_n$ of the normalized martingale $L_n^N = \sqrt{N} M_n^N$.

The example we have in mind is the situation where $(\mathcal{F}_n^N)_{n\geq 0}$ is the natural filtration associated to the particle model (7) and the $\mathcal{F}^N$-martingale difference $\Delta M_n^N = \Delta \mathcal{M}_n^N(f_n)$, with $f_n \in \mathcal{B}_b(E_n)$, is given by the particle n-th sampling error

$$\Delta \mathcal{M}_n^N(f_n) = \eta_n^N(f_n) - \eta_{n-1}^N K_{n,\eta_{n-1}^N}(f_n) \tag{8}$$

with the convention $\eta_{-1}^N K_{0,\eta_{-1}^N} = \eta_0$ for $n = 0$. In this situation the increasing processes $\langle \mathcal{M}^N(f) \rangle_n$ and $\mathcal{C}_n^N(f) = \langle \mathcal{L}^N(f) \rangle_n$ of the corresponding martingales $\mathcal{M}_n^N(f)$ and $\mathcal{L}_n^N(f) = \sqrt{N} \mathcal{M}_n^N(f)$ are connected by the formula

$$\mathcal{C}_n^N(f) = N \langle \mathcal{M}^N(f) \rangle_n = \sum_{p=0}^n \eta_{p-1}^N K_{p,\eta_{p-1}^N}(f_p - K_{p,\eta_{p-1}^N}(f_p))^2 \tag{9}$$

Our first main result concerns a Berry-Esseen theorem for an abstract class of martingale sequences under the following set of conditions:

(H1) For any $n \geq 0$ there exists some constants $a_1(n) < \infty$ and $0 < c_1(n) \leq 1$ such that for any $n \geq 0$ and $\lambda^3 \leq c_1(n)\ N^{1/2}$ we have, $\mathbb{P}^N$ almost surely,

$$\left|\mathbb{E}\left[\ e^{i\lambda N^{1/2}\Delta M_n^N + \frac{\lambda^2}{2}\Delta C_n^N}\ \mid \mathcal{F}_{n-1}^N\right] - 1\right| \leq a_1(n)\lambda^3/N^{1/2}.$$

(H2) For any $n \geq 0$ there exists some finite constant $a_2(n) < \infty$ such that for any $N \geq 1$, $\lambda > 0$ and $n \geq 0$

$$\left|\mathbb{E}\left[\ e^{i\lambda N^{1/2} M_n^N}\right]\right| \leq \mathbb{E}\left[e^{-\frac{\lambda^2}{2}\Delta C_n^N}\right] \quad e^{\lambda^3 a_2(n)/N^{1/2}}$$

(H3) There exists a non-negative and strictly increasing deterministic process $C = (C_n)_{n\geq 0}$ as well as some finite constants $0 < a_3(n) < \infty$ such that for any $\varepsilon > 0$ we have

$$\mathbb{E}\left[e^{\varepsilon N^{1/2}|\Delta C_n^N - \Delta C_n|}\right] \leq (1 + \varepsilon a_3(n))\ e^{\varepsilon^2 a_3^2(n)}.$$

Theorem 1.1 *Let $M^N = (M_n^N)_{n\geq 0}$ be a sequence of $\mathcal{F}^N$-martingales satisfying conditions (Hi), $i = 1, 2, 3$, for some non negative and strictly increasing process C_n. We let F_n^N, and respectively F_n, be the distribution function of the random variable $N^{1/2} M_n^N$, and respectively the one of a centered Gaussian random variable with variance C_n. Then for any $n \geq 0$ we have*

$$\limsup_{N\to\infty} N^{1/2}\|F_n^N - F_n\| < \infty.$$

The above theorem does not apply directly to the particle martingale sequence introduced in (8). The first two conditions (Hi), $i = 1, 2$, are rather standard. They can be checked for any kind of any McKean interpretation model using simple and rather standard asymptotic expansions of characteristic functions. The third condition is an exponential continuity condition of the increasing processes introduced in (9). Next we provide a sufficient regularity condition which can be easily checked in various McKean interpretation models. If we set for any $\mu_n \in \mathcal{P}(E_n)$

$$\mu_n^{-1}(0) = \{h \in \mathrm{Osc}(E_n)\ :\ \mu_n(h) = 0\}$$

then this condition reads

(H) There exists a collection of uniformly bounded positive measures $\Gamma_{n,f}$ and $\Gamma'_{n+1,f}$ on the sets $\eta_n^{-1}(0)$ and $\eta_{n+1}^{-1}(0)$ and indexed by $n \in \mathbb{N}$ and $f \in \mathrm{Osc}(E_{n+1})$ and such that

$$\begin{aligned}
&\|K_{n+1,\mu_n}(f) - K_{n+1,\eta_n}(f)\| \\
&\leq \int |\mu_n(h)| \; \Gamma_{n,f}(dh) + \int |\Phi_{n+1}(\mu_n)(h)| \; \Gamma'_{n,f}(dh)
\end{aligned}$$

When condition (H) is met we denote by Γ the supremum of the total mass quantities $\Gamma'_{n,f}(1)$ and $\Gamma_{n,f}(1)$.

Note that (H) is related to some Lipschitz-type regularity of the increasing process and it is clearly met for the McKean transitions given (5), since we have in this case

$$K_{n+1,\mu_n}(f) - K_{n+1,\eta_n}(f) = (1 - \varepsilon_n G_n) \; [\Phi_{n+1}(\mu_n) - \Phi_{n+1}(\eta_n)](f).$$

Note that in this situation we have that (H) is met with $\Gamma_{n,f} = 0$ and $\Gamma'_{n,f} = \delta_h$ with $h = [f - \eta_{n+1}(f)]$ so that $\Gamma_{n,f}(1) = 0$ and $\Gamma'_{n,f}(1) = 1$. When the parameter $\varepsilon_n = \varepsilon_n(\mu_n)$ in (5) depends on the index measure μ_n we also find that (H) is met as soon as we have

$$|\varepsilon_n(\mu_n) - \varepsilon_n(\eta_n)| \leq \int |\mu_n(h)| \; \Lambda_n(dh)$$

for some uniformly bounded positive measures Λ_n on $\eta_n^{-1}(0)$.

To check the exponential estimates stated in condition (H3) we shall use a refined version of Burkholder's inequality recently presented by the first author with L. Miclo and F. Viens in [6]. Roughly speaking these sharp $\mathbb{L}_p$-estimates combined with some judicious error decompositions lead to the desired exponential concentrations estimates for the normalized sampling error martingale $\mathcal{L}_n^N(f)$ defined by (9) with the limiting increasing process

$$\mathcal{C}_n(f) = \sum_{p=0}^{n} \eta_{p-1} K_{p,\eta_{p-1}}(f_p - K_{p,\eta_{p-1}}(f_p))^2 \tag{10}$$

Even if this strategy would lead to the desired Berry-Essen estimates on $\mathcal{M}_n^N(f)$ we still need to transfer these rates of fluctuations to the random field sequence defined by $\sqrt{N} \; (\eta_n^N - \eta_n)$. One of the most elegant approach is probably to follow to semi-group techniques and the martingales decompositions developped in [5]. To describe these decompositions with some precision we let $Q_{p,n}$ be the linear

Feynman-Kac semi-group associated to the flow $\gamma_n(=\gamma_p Q_{p,n})$ and we associate to the latter the "normalized" semi-group $\overline{Q}_{p,n}$ defined for $f_n \in \mathcal{B}_b(E_n)$

$$\overline{Q}_{p,n}(f_n) = \frac{Q_{p,n}(f_n)}{\eta_p Q_{p,n}(1)} = \frac{\gamma_p(1)}{\gamma_n(1)}\ Q_{p,n}(f_n). \tag{11}$$

If we let $(W^N_{p,n}(f_n))_{p\leq n},\ f_n \in \mathcal{B}_b(E_n)$ be the random field sequence defined by

$$W^N_{p,n}(f_n) = \sqrt{N}\ (\eta^N_p - \eta_p)(f_{p,n}), \quad \text{with} \quad f_{p,n} = \overline{Q}_{p,n}\,(f_n - \eta_n f_n)$$

then we have the Doob's type decomposition

$$W^N_{p,n}(f_n) = \mathcal{B}^N_{p,n}(f_n) + \mathcal{L}^N_{p,n}(f_n), \tag{12}$$

with the predictable and martingale sequences given by

$$\begin{aligned}
\Delta\mathcal{B}^N_{p,n}(f_n) &= \sqrt{N}\ \left[1 - \eta^N_{q-1}(G_{q-1})/\eta_{q-1}(G_{q-1})\right] \\
&\qquad\qquad \times\ [\Phi_q(\eta^N_{q-1})(f_{q,n}) - \Phi_q(\eta_{q-1})(f_{q,n})] \\
\Delta\mathcal{L}^N_{p,n}(f_n) &= \sqrt{N}\left[\eta^N_q(f_{q,n}) - \eta^N_{q-1}K_{q,\eta^N_{q-1}}f_{q,n}\right],
\end{aligned}$$

The above decomposition is now more or less standard. For the convenience of the reader its proof is housed in the appendix.

Intuitively speaking we see from the quadratic structure of the predictable term that it should not influence the fluctuation rate. We will make precise this observation with a Stein's type approximation lemma and we will prove the following

Theorem 1.2 *Let $f_n \in \mathcal{B}_b(E_n)$, and $W^N_{p,n}$ the quantity defined by (12). For any McKean interpretation model satisfying condition (H) we have*

$$\limsup_{N\to\infty} \sqrt{N} \sup_{u\in\mathbb{R}} \left|\mathbb{P}\left(W^N_{n,n}(f_n) \leq u\ \sigma_n(f)\right) - \int_{-\infty}^{u} e^{-v^2/2}\ \frac{dv}{(2\pi)^{1/2}}\right| < \infty.$$

for any $f_n \in \mathcal{B}_b(E_n)$ and $n \geq 0$ with

$$\sigma^2_n(f) \;=\; \sum_{p=0}^{n} \eta_{p-1}K_{p,\eta_{p-1}}(f_{p,n} - K_{p,\eta_{p-1}}(f_{p,n}))^2$$

2 An estimate for martingale sequences

The central limit theorem for sequences of random variables is usually obtained by convergence of characteristic functions. Unsurprisingly, the natural question of determining the speed of convergence in the CLT can also be handled through characteristic functions considerations. The formalization of this idea is due to Berry and Esseen, and can be summarized in the following theorem:

Theorem 2.1 (Berry-Esseen) *Let (F_1, F_2) be a pair of distribution functions with characteristic functions (f_1, f_2). Also assume that F_2 has a derivative with $\|\frac{\partial F_2}{\partial x}\| < \infty$. Then for any $a > 0$ we have*

$$\|F_1 - F_2\| \leq \frac{2}{\pi} \int_0^a \frac{|f_1(x) - f_2(x)|}{x} \, dx + \frac{24}{a\pi} \left\| \frac{\partial F_2}{\partial x} \right\|.$$

In this section, we will try to apply this theorem to a sequence of martingales satisfying the general set of hypothesis (H1)-(H3) in order to get a sharp asymptotic result for its convergence towards a Gaussian martingale. In order to prepare for the proof of theorem 1.1 we start with the following technical key lemma.

Lemma 2.2 *Suppose we are given a sequence of martingale $M^N = (M_n^N)_{n\geq 0}$ with respect to some filtrations $\mathcal{F}_n^N$, satisfying the conditions (Hi), $i = 1, 2, 3$ stated in page 6. Then, for any $n \geq 0$, there exist a finite constant $a(n) < \infty$, a positive constant $b(n)$, and some $N(n) \geq 1$ such that for any $N \geq N(n)$ and $0 < \lambda \leq b(n)N^{1/2}$,*

$$\left| \mathbb{E}\left[\, e^{i\lambda N^{1/2} M_n^N} \right] - e^{-\frac{\lambda^2}{2} C_n} \right| \leq a(n) \; e^{-\frac{\lambda^2}{4} \Delta C_n} \; \frac{\lambda^2 (1 + \lambda)}{N^{1/2}}.$$

Since the proof of theorem 1.1 is a simple consequence of the above lemma we have chosen to give it first.

Proof of theorem 1.1:

By Theorem 2.1 and Lemma 2.2 we have, for any $N \geq N(n)$,

$$\begin{aligned}
& N^{1/2} \|F_n^N - F_n\| \\
\leq & \frac{2a(n)}{\pi} \int_0^{b(n)N^{1/2}} e^{-\frac{\lambda^2}{4} \; \Delta C_n} \; \lambda(1 + \lambda) \; d\lambda + \frac{24}{b(n) \, (2e\pi^3 C_n)^{1/2}} \\
\leq & \frac{2a(n)}{\pi} \int_0^{\infty} e^{-\frac{\lambda^2}{4} \; \Delta C_n} \; \lambda(1 + \lambda) \; d\lambda + \frac{24}{b(n) C_n^{1/2}},
\end{aligned}$$

for some $N(n) \geq 1$ and some finite positive constant $0 < b(n) < \infty$. Invoking the fact that $\Delta C_n > 0$, this ends the proof of the theorem.

$\square$

We now come to the proof of the lemma 2.2.

Proof of lemma 2.2: Let I_n^N be the function defined for any $\lambda \geq 0$ by

$$I_n^N(\lambda) = \mathbb{E}\left[e^{i\lambda N^{1/2} M_n^N + \frac{\lambda^2}{2} C_n} \right] - 1,$$

and notice that

$$\mathbb{E}\left[e^{i\lambda N^{1/2} M_n^N}\right] - e^{-\frac{\lambda^2}{2} C_n} = e^{-\frac{\lambda^2}{2} C_n} I_n^N(\lambda). \tag{13}$$

Furthermore, we have the easily verified recursive equations

$$\begin{aligned} &I_n^N(\lambda) - I_{n-1}^N(\lambda) \\ &= \mathbb{E}\left[e^{i\lambda N^{1/2} M_{n-1}^N + \frac{\lambda^2}{2} C_{n-1}} \left(\mathbb{E}\left[e^{i\lambda N^{1/2} \Delta M_n^N + \frac{\lambda^2}{2} \Delta C_n} | F_{n-1}^N \right] - 1 \right) \right], \end{aligned}$$

and hence

$$I_n^N(\lambda) - I_{n-1}^N(\lambda) = A(\lambda) + B(\lambda),$$

with

$$A(\lambda) = \mathbb{E}\left[e^{i\lambda N^{1/2} M_{n-1}^N + \frac{\lambda^2}{2} C_{n-1}} \left(e^{\frac{\lambda^2}{2}(\Delta C_n - \Delta C_n^N)} - 1 \right) \right],$$

and

$$\begin{aligned} B(\lambda) = \mathbb{E}\Bigg[& e^{i\lambda N^{1/2} M_{n-1}^N + \frac{\lambda^2}{2} C_{n-1}} \left(e^{\frac{\lambda^2}{2}(\Delta C_n - \Delta C_n^N)} \right) \\ & \times \left(\mathbb{E}\left[e^{i\lambda N^{1/2} \Delta M_n^N + \frac{\lambda^2}{2} \Delta C_n^N} \ \mathcal{F}_{n-1}^N \right] - 1 \right) \Bigg]. \end{aligned}$$

Using this we obtain

$$\left| I_n^N(\lambda) - I_{n-1}^N(\lambda) \right| \leq e^{\frac{\lambda^2}{2} C_{n-1}} \left(A_1(\lambda) + B_1(\lambda) \right),$$

where

$$A_1(\lambda) = \mathbb{E}\left[\ \left| \mathbb{E}\left[e^{i\lambda N^{1/2} \Delta M_n^N + \frac{\lambda^2}{2} \Delta C_n^N} \ | \mathcal{F}_{n-1}^N \right] - 1 \right| \ e^{\frac{\lambda^2}{2}(\Delta C_n - \Delta C_n^N)} \right],$$

and

$$B_1(\lambda) = \mathbb{E}\left[e^{\frac{\lambda^2}{2} \ |\Delta C_n - \Delta C_n^N|} - 1 \right].$$

Now, under conditions (H1) and (H3) applied for $\varepsilon = \frac{\lambda^2}{2N^{1/2}}$, we find that

$$
\begin{aligned}
&|I_n^N(\lambda) - I_{n-1}^N(\lambda)| \\
&\leq e^{\frac{\lambda^2}{2}C_{n-1}} \left[\frac{a_1(n)\lambda^3}{N^{1/2}} \left(1 + \frac{\lambda^2 a_3(n)}{2N^{1/2}}\right) \; e^{\frac{\lambda^4}{4N}a_3^2(n)} \right. \\
&\qquad\qquad\qquad\qquad \left. + \left(1 + \frac{\lambda^2 a_3(n)}{2N^{1/2}}\right) \; e^{\frac{\lambda^4}{4N}a_3^2(n)} - 1 \right] \\
&\leq e^{\frac{\lambda^2}{2}C_{n-1}} \left[\frac{a_1(n)\lambda^3}{N^{1/2}} \left(1 + \frac{\lambda^2 a_3(n)}{2N^{1/2}}\right) \; e^{\frac{\lambda^4}{4N}a_3^2(n)} \right. \\
&\qquad\qquad\qquad \left. + \; \left(e^{\frac{\lambda^4}{4N}a_3^2(n)} - 1\right) + \frac{\lambda^2 a_3(n)}{2N^{1/2}} \; e^{\frac{\lambda^4}{4N}a_3^2(n)} \right],
\end{aligned}
$$

for any $0 < \lambda^3 \leq c_1(n) \; N^{1/2}$. Since for these pairs of parameters (λ, N) we have $\lambda^2 \leq N^{1/2}$ (and therefore $\lambda^4 \leq N$), we find that

$$
N^{1/2} \; |I_n^N(\lambda) - I_{n-1}^N(\lambda)| \leq d(n) \; e^{\frac{\lambda^2}{2}C_{n-1}} \; \lambda^2(1+\lambda),
$$

for some finite constant $d(n)$, whose value only depend on $a_i(n)$, $i = 1, 3$, and such that

$$
d(n) \leq c \; e^{\frac{a_3^2(n)}{4}} (1 \vee a_1(n) \vee a_3(n))^2
$$

If we set

$$
c_\star(n) = \wedge_{p=0}^n c_1(p) \; (\leq 1) \quad \text{and} \quad d^\star(n) = \vee_{p=0}^n d(p),
$$

then for any $0 \leq p \leq n$ and any $0 < \lambda^3 \leq c_\star(n) \; N^{1/2}$, we have that

$$
N^{1/2} \; |I_p^N(\lambda) - I_{p-1}^N(\lambda)| \leq d^\star(n) \; e^{\frac{\lambda^2}{2}C_{n-1}} \; \lambda^2(1+\lambda)
$$

It is now easily verified from these estimates that

$$
N^{1/2} \; |I_n^N(\lambda)| \leq (n+1) d^\star(n) e^{\frac{\lambda^2}{2}C_{n-1}} \lambda^2(1+\lambda),
$$

from which we conclude that, for any $0 < \lambda^3 \leq c_\star(n) \; N^{1/2}$,

$$
\left| \mathbb{E}\left[\; e^{i\lambda N^{1/2} M_n^N} \right] - e^{-\frac{\lambda^2}{2}C_n} \right| \leq (n+1) d^\star(n) \; e^{-\frac{\lambda^2}{2} \; \Delta C_n} \frac{\lambda^2(1+\lambda)}{N^{1/2}}. \quad (14)
$$

On the other hand, we have, for any pair (λ, N),

$$\left|\mathbb{E}\left[\ e^{i\lambda N^{1/2}M_n^N}\right] - e^{-\frac{\lambda^2}{2}C_n}\right| \leq \left|\mathbb{E}\left[\ e^{i\lambda N^{1/2}M_n^N}\right]\right| + e^{-\frac{\lambda^2}{2}C_n}, \qquad (15)$$

and under condition $(H2)$,

$$\left|\mathbb{E}\left[\ e^{i\lambda N^{1/2}M_n^N}\right]\right| \ \leq \ \mathbb{E}\left[e^{-\frac{\lambda^2}{2}\Delta C_n^N}\right] \ e^{\lambda^3 a_2(n) N^{-1/2}}.$$

Using again $(H3)$ we also find that

$$\begin{aligned}\left|\mathbb{E}\left[\ e^{i\lambda N^{1/2}M_n^N}\right]\right| &\leq e^{-\frac{\lambda^2}{2}\Delta C_n} \ \left(1+\frac{\lambda^2 a_3(n)}{2N^{1/2}}\right) \ e^{\frac{\lambda^4 a_3^2(n)}{4N}} \ e^{\frac{\lambda^3 a_2(n)}{N^{1/2}}} \\ &= \left(1+\frac{\lambda^2 a_3(n)}{2N^{1/2}}\right) \exp\left[-\frac{\lambda^2}{2}\left(\Delta C_n - \frac{\lambda}{N^{1/2}}\left(2a_2(n)+a_3^2(n)\frac{\lambda}{2N^{1/2}}\right)\right)\right]\end{aligned}$$

Recall that $\Delta C_n > 0$, and observe that for any pair (λ, N) such that

$$\lambda \leq c^\star(n) N^{1/2} \ \text{ with } \ c^\star(n) = \left[2a_3^{-2}(n) \wedge \left(2^{-1}\Delta C_n \left(1+2a_2(n)\right)^{-1}\right)\right],$$

we have

$$\frac{\lambda}{N^{1/2}}\left(2a_2(n)+a_3^2(n)\frac{\lambda}{2N^{1/2}}\right) \ \leq \ \frac{\lambda}{N^{1/2}}\left(2a_2(n)+1\right) \leq \frac{\Delta C_n}{2}.$$

This yields that

$$\left|\mathbb{E}\left[\ e^{i\lambda N^{1/2}M_n^N}\right]\right| \leq \left(1 \vee \frac{a_3(n)}{2}\right) \ \left(1+\frac{\lambda^2}{N^{1/2}}\right) \ e^{-\lambda^2\frac{\Delta C_n}{4}}, \qquad (16)$$

and hence, by (15), and for any $\lambda \leq c^\star(n) N^{1/2}$, we find that

$$\left|\mathbb{E}\left[\ e^{i\lambda N^{1/2}M_n^N}\right] - e^{-\frac{\lambda^2}{2}C_n}\right| \leq e^{-\frac{\lambda^2}{4} \ \Delta C_n} \left(2 \vee a_3(n)\right) \ \left(1+\frac{\lambda^2}{N^{1/2}}\right) \qquad (17)$$

To take the final step we observe that for any

$$N \geq c_\star(n)/c^\star(n)^3 \quad \text{and} \quad c_\star^{1/3}(n) N^{1/6} \leq \lambda \leq c^\star(n) \ N^{1/2},$$

we have $1 = c_\star(n)/c_\star(n) \leq c_\star^{-1}(n)\lambda^3/N^{1/2}$, and by (17),

$$\left|\mathbb{E}\left[\ e^{i\lambda N^{1/2}M_n^N}\right] - e^{-\frac{\lambda^2}{2}C_n}\right| \leq c_\star^{-1}(n)\left(2 \vee a_3(n)\right) \ \frac{\lambda^2\left(1+\lambda\right)}{N^{1/2}} \ e^{-\lambda^2\frac{\Delta C_n}{4}}. \qquad (18)$$

In conjunction with (14) we conclude that for any $N \geq N(n) = c_\star(n)/c^\star(n)^3$ and any $\lambda \leq c^\star(n) \ N^{1/2}$,

$$\left|\mathbb{E}\left[\ e^{i\lambda N^{1/2}M_n^N}\right] - e^{-\frac{\lambda^2}{4}C_n}\right| \leq a(n) \ \frac{\lambda^2}{N^{1/2}}\left(1+\lambda\right) \ e^{-\frac{\lambda^2}{4} \ \Delta C_n},$$

with $a(n) = \left[(n+1)d^\star(n)\right] \vee \left[c_\star^{-1}(n)\left(2 \vee a_3(n)\right)\right]$. This ends the proof of the lemma.

$\square$

3 Application to interacting processes

In this section, we prove that Theorem 1.1 can be applied to our particle approximations. We shall go through a series of preliminary results leading to the proof of Theorem 1.2.

The first step is of course to provide some exponential estimates for the particle density profiles. In the next pivotal lemma we describe an original exponential concentration result in terms of the following pair of parameters

$$\begin{aligned} \beta(P_{p,n}) &= \sup_{(x_p,y_p)\in E_p^2} \|P_{p,n}(x_p,\,.\,) - P_{p,n}(y_p,\,.\,)\|_{\rm tv} \\ r_{p,n} &= \sup_{(x_p,y_p)\in E_p^2} Q_{p,n}(1)(x_p)/Q_{p,n}(1)(y_p) \end{aligned} \tag{19}$$

where $\|\mu-\nu\|_{\rm tv} = \sup_{A\in\mathcal{E}} |\mu(A)-\nu(A)|$ represents the total variation distance between probabilities and $P_{p,n}$ denotes the Markov transition from E_p into E_n defined by

$$P_{p,n}(x_p,dx_n) = Q_{p,n}(x_p,dx_n)/Q_{p,n}(x_p,E_n)$$

Lemma 3.1 *For any McKean model we have for every $n\geq 0$, $f_n\in \mathrm{Osc}_1(E_n)$ and $\varepsilon>0$*

$$\mathbb{E}\left[e^{\varepsilon N^{1/2}|\eta_n^N(f_n)-\eta_n(f_n)|}\right] \leq \left(1+\varepsilon 2^{-1/2}\ b(n)\right)\ e^{(\varepsilon b(n))^2/2}$$

for some finite constant $b(n)$ such that $b(n)\leq 2\ \sum_{q=0}^n\ r_{q,n}\ \beta(P_{q,n})$.

Remark 3.2 *The quantities $(r_{p,n},\beta(P_{p,n}))$ play an important role in the asymptotic and long time behavior of Feynman-Kac particle approximation models. The above lemma combined with the semigroups approach developped in [5] readily yields uniform exponential concentration properties. To be more precise let us suppose that $r=\vee_n r_n<\infty$. Also assume that there exist some integer parameter $m\geq 1$ and some $\rho\in(0,1]$ such that for any $(x,y)\in E_n^2$, $A\in\mathcal{E}_{n+m}$ and $n\geq 0$*

$$M_{n,n+m}(x,A)\geq \rho\ M_{n,n+m}(y,A)$$

where $M_{n,n+m}=(M_{n+1}\dots M_{n+m})$ stands for the composition of the Markov kernels M_p from $p=(n+1)$ to $p=(n+m)$. In this situation, following the arguments given in [5] one proves that

$$r_{n,n+m}\leq r^m/\rho \quad \textit{and} \quad \beta(P_{n,n+m})\leq (1-r^{m-1}\rho^2)^{[(n-p)/m]}$$

In this situation the constants $b(n)$ in lemma 3.1 can be chosen such that $\vee_n b(n)\leq 2mr^{2m-1}/\rho^3$.

The proof of lemma 3.1 being rather technical it is housed in the appendix. One consequence of lemma 3.1 is the following central estimate.

Lemma 3.3 *Suppose the McKean interpretation model satisfies condition (H) for some finite constant* $\Gamma < \infty$. *In this situation, the martingale* $\mathcal{M}_n^N(f)$ *defined by (8) satisfies conditions* $(Hi)_{i=1,2,3}$, *for some universal constants*

$$(a_1(n), a_2(n)) = (a_1, a_2)$$

with the non-negative increasing process $C_n(f)$ *defined at (10), as soon as* $n \to C_n(f)$ *is strictly increasing. In addition, the constant* $a_3(n)$ *in (H3) can be chosen such that, for any* $n \geq 0$,

$$0 < a_3(n) \leq 4\sqrt{2}\ (1+\Gamma) \sup_{q=n,n-1} \sum_{p=0}^{q} r_{p,q}\ \beta(P_{p,q})$$

Furthermore when the regularity condition stated in remark 3.2 are met for some triplet (m, r, ρ) *the constant* $a_3(n)$ *can be chosen such that* $0 < \vee_n a_3(n) \leq 8\sqrt{2}\ m\ r^{2m-1}(1+\Gamma)/\rho^3$.

The second step will be to get rid of the predictable term defined by equation (12), with the help of the following lemma:

Lemma 3.4 *Let* F_Z *be the distribution function associated to a real valued random variable* Z, *and let* W *be a centered Gaussian random variable with unit variance. For any pair of random variables* (X, Y) *we have*

$$\|F_{X+Y} - F_W\| \leq \|F_X - F_W\| + 4\mathbb{E}(|XY|) + 4\mathbb{E}(|Y|). \qquad (20)$$

Lemma 3.4 can be proved using the Stein's approach to fluctuations and it can be found for instance lemma 1.3 chapter 11 in the book of Shorack [8]. Since the proof of theorem 1.2 is now a simple consequence of lemma 3.3 and lemma 3.4 we postpone the proof of 3.3 and start with the

Proof of theorem 1.2: We first notice then that

$$W_{n,n}^{\eta,N} = \sqrt{N}(\eta_n^N - \eta_n)$$

and by (12) we have $W_{p,n}^N(f_n) = \mathcal{B}_{p,n}^N(f_n) + \mathcal{L}_{p,n}^N(f_n)$. Let us show now that the main term in the fluctuations of the cdf of $W_{p,n}^{\eta,N}(f_n)$ is due to $\mathcal{L}_{p,n}^N(f_n)$. Indeed it is easily checked that

$$\mathbb{E}(|B_{n,n}^N(f_n)|^2)^{1/2} \vee \mathbb{E}(|B_{n,n}^N(f_n)|) \leq \frac{b(n)}{\sqrt{N}}. \qquad (21)$$

By definition of the martingale term, it is also easily seen that

$$\mathbb{E}(|\mathcal{L}^N_{n,n}(f_n)|^2)^{1/2} \leq b(n). \tag{22}$$

Set now

$$X = \mathcal{L}^N_{n,n}(f_n)/(C_n(f))^{1/2} \quad \text{and} \quad Y = \mathcal{B}^N_n(f_n)/(C_n(f))^{1/2}.$$

The estimates (21) and (22) yield

$$\mathbb{E}(|XY|) \leq \frac{1}{C_n(f)}\mathbb{E}^{1/2}\left[|\mathcal{L}^N_{n,n}(f_n)|^2\right]\mathbb{E}^{1/2}\left[|B^N_{n,n}(f_n)|^2\right] \leq \frac{b(n)}{N^{1/2}},$$

and $\mathbb{E}(|Y|) \leq b(n)/N^{1/2}$. Hence, applying lemma 3.4, the proof of theorem 1.2 can be reduced to show that

$$\sup_{u\in\mathbb{R}}\left|\mathbb{P}\left(\mathcal{L}^N_n(f_{n,n}) \leq u(C_n(f))^{1/2}\right) - \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{u} e^{-v^2/2}\,dv\right| \leq \frac{b(n)}{N^{1/2}}.$$

This last estimate is now a direct consequence of lemma 3.3 and theorem 1.1.

□

We now come to the

Proof of lemma 3.3: Let us first check that the regularity condition $(H3)$ is satisfied. Since we have

$$\begin{aligned}\Delta\mathcal{C}_n(f) &= \eta_{n-1}\left[K_{n,\eta_{n-1}}((f_n - K_{n,\eta_{n-1}}f_n)^2)\right] \\ &= \Phi_n(\eta_{n-1})(f_n^2) - \eta_{n-1}((K_{n,\eta_{n-1}}f_n)^2)\end{aligned}$$

we easily prove that

$$\begin{aligned}|\Delta\mathcal{C}^N_n(f) - \Delta\mathcal{C}_n(f)| \leq 2\Big(&|\Phi_n(\eta^N_{n-1})(h_n)| + |\eta^N_{n-1}(h'_{n-1})| \\ &+ \|K_{n,\eta^N_{n-1}}(f_n) - K_{n,\eta_{n-1}}(f_n)\|\Big),\end{aligned}$$

with the pair of functions $(h_n, h'_{n-1}) \in (\mathrm{Osc}(E_n)\times\mathrm{Osc}(E_{n-1}))$ defined by

$$\begin{aligned}h_n &= (f_n^2 - \eta_n(f_n^2))/2 \\ h'_{n-1} &= ((K_{n,\eta_{n-1}}f_n)^2 - \eta_{n-1}((K_{n,\eta_{n-1}}f_n)^2))/2\end{aligned}$$

On the other hand, under condition (H), we have that

$$\|K_{n,\eta^N_{n-1}}(f_n) - K_{n,\eta_{n-1}}(f_n)\|$$

$$\leq \int |\eta^N_{n-1}(h)|\;\Gamma_{n-1,f'_n}(dh) + \int |\Phi_n(\eta^N_{n-1})(h)|\;\Gamma'_{n-1,f'_n}(dh)$$

from which we find that

$$|\Delta\mathcal{C}_n^N(f) - \Delta\mathcal{C}_n(f)|$$

$$\leq 2\left(\int \ |\eta_{n-1}^N(h)| \ \widetilde{\Gamma}_{n-1}(dh) + \mathbb{E}(\int \ |\eta_n^N(h)| \ \widetilde{\Gamma}'_{n-1}(dh) \ | \ \mathcal{F}_{n-1}^N)\right)$$

with

$$\widetilde{\Gamma}_{n-1} = \Gamma_{n-1,f'_n} + \delta_{h'_{n-1}} \quad \text{and} \quad \widetilde{\Gamma}'_{n-1} = \Gamma'_{n-1,f'_n} + \delta_{h_n}$$

Applying Jensen's inequality, we find that for any $\varepsilon > 0$

$$\mathbb{E}\left[e^{\varepsilon N^{1/2}|\Delta C_n^N(f) - \Delta C_n(f)|}\right]$$

$$\leq \mathbb{E}\left[e^{2\varepsilon N^{1/2}\left\{\int \ |\eta_{n-1}^N(h)| \ \widetilde{\Gamma}_{n-1}(dh) + \int \ |\eta_n^N(h)| \ \widetilde{\Gamma}'_{n-1}(dh)\right\}}\right].$$

Now, applying Cauchy-Schwarz's inequality, we obtain

$$\mathbb{E}\left[e^{\varepsilon N^{1/2}|\Delta C_n^N(f) - \Delta C_n(f)|}\right]$$

$$\leq \mathbb{E}^{1/2}\left[e^{4\varepsilon N^{1/2}\int |\eta_{n-1}^N(h)|\widetilde{\Gamma}_{n-1}(dh)}\right] \mathbb{E}^{1/2}\left[e^{4\varepsilon N^{1/2}\int \ |\eta_n^N(h)|\widetilde{\Gamma}'_{n-1}(dh)}\right].$$

If we set $\widetilde{\Gamma} = \Gamma + 1$ then using again Jensen's inequality we find

$$\mathbb{E}\left[e^{\varepsilon N^{1/2}|\Delta\mathcal{C}_n^N(f) - \Delta\mathcal{C}_n(f)|}\right]^2 \leq \int \mathbb{E}\left[e^{4\varepsilon N^{1/2}\widetilde{\Gamma}|\eta_{n-1}^N(h)|}\right] \frac{\widetilde{\Gamma}_{n-1}(dh)}{\widetilde{\Gamma}_{n-1}(1)}$$
$$\int \mathbb{E}\left[e^{4\varepsilon N^{1/2} \ \widetilde{\Gamma} \ |\eta_n^N(h)|}\ \right] \frac{\widetilde{\Gamma}'_{n-1}(dh)}{\widetilde{\Gamma}'_{n-1}(1)},$$

from which we conclude that

$$\mathbb{E}\left[e^{\varepsilon N^{1/2}|\Delta\mathcal{C}_n^N(f) - \Delta\mathcal{C}_n(f)|}\right] \leq \sup_{h\in \mathrm{Osc}(E_p), p=n,n-1} \mathbb{E}(e^{4\varepsilon N^{1/2} \ \widetilde{\Gamma} \ |\eta_p^N(h)|} \ )$$

Using lemma 3.7 we conclude that

$$\mathbb{E}\left[e^{\varepsilon|\Delta\mathcal{C}_n^N(f) - \Delta\mathcal{C}_n(f)|}\right] \leq \left(1 + \frac{\varepsilon \ a_3(n)}{N^{1/2}}\right) \ e^{\varepsilon^2 a_3^2(n)/N},$$

for some finite constant $a_3(n)$ such that

$$a_3(n) \leq 4\sqrt{2} \ \widetilde{\Gamma} \ \sup_{q=n,n-1} \sum_{p=0}^{q} \ r_{p,q} \ \beta(P_{p,q}).$$

To prove that $(H2)$ is met, we first recall that

$$\left|\mathbb{E}\left[\ e^{i\lambda N^{1/2}\mathcal{M}_n^N(f)}\right]\right| \leq \mathbb{E}\left[\left|\mathbb{E}\left[\ e^{i\lambda N^{1/2}\Delta\mathcal{M}_n^N(f)}\ \middle|\ F_{n-1}^N\right]\right|\right]. \qquad (23)$$

Then we use a standard symmetrization technique: given the particle model ξ_p up to time $p \leq n-1$ we let $\overline{\eta}_n^N$ an auxiliary independent copy of η_n^N. In other words $\overline{\eta}_n^N$ is the empirical measure associated to an independent copy $\overline{\xi}_n$ of the configuration of the system ξ_n at time n. With some obvious abusive notation, we readily check that

$$\left|\mathbb{E}\left[\ e^{i\lambda N^{1/2}\Delta\mathcal{M}_n^N(f)}\ \middle|\ F_{n-1}^N\right]\right|^2 = \mathbb{E}\left[\ e^{i\lambda N^{1/2}[\Delta\mathcal{M}_n^N(f)-\Delta\overline{\mathcal{M}}_n^N(f)]}\ \middle|\ F_{n-1}^N\right],$$

where $\Delta\overline{\mathcal{M}}_n^N(f) = \left[\overline{\eta}_n^N(f_n) - \Phi_n(\eta_{n-1}^N)(f_n)\right]$. We deduce from this that

$$\left|\mathbb{E}\left[\ e^{i\lambda N^{1/2}\Delta\mathcal{M}_n^N(f)}\ \middle|\ F_{n-1}^N\right]\right|^2 = \prod_{j=1}^{N}\mathbb{E}\left[e^{i\frac{\lambda}{N^{1/2}}[f_n(\xi_n^j)-f_n(\overline{\xi}_n^j)]}\ \middle|\ \mathcal{F}_{n-1}^N\right].$$

Since the random variables $[f_n(\xi_n^j) - f_n(\overline{\xi}_n^j)]$ and $-[f_n(\xi_n^j) - f_n(\overline{\xi}_n^j)]$ have the same law, their characteristic functions are real, and we have

$$\begin{aligned}\mathbb{E}&\left[e^{i\frac{\lambda}{N^{1/2}}[f_n(\xi_n^j)-f_n(\overline{\xi}_n^j)]}\ \middle|\ \mathcal{F}_{n-1}^N\right]\\ &= \mathbb{E}\left[\cos\left(\frac{\lambda}{N^{1/2}}[f_n(\xi_n^j) - f_n(\overline{\xi}_n^j)]\right)\ \middle|\ \mathcal{F}_{n-1}^N\right].\end{aligned}$$

Using the elementary inequalities

$$\cos u \leq 1 - u^2/2 + |u|^3/3!\ , \quad 1+u \leq e^u \ \text{ and } \ |u-v|^3 \leq 4(|u|^3+|v|^3),$$

we prove that

$$\begin{aligned}&\mathbb{E}\left[e^{i\frac{\lambda}{N^{1/2}}[f_n(\xi_n^j)-f_n(\overline{\xi}_n^j)]}\ \middle|\ \mathcal{F}_{n-1}^N\right]\\ &\leq 1 - \tfrac{\lambda^2}{N}K_{n,\eta_{n-1}^N}(f_n - K_{n,\eta_{n-1}^N}(f_n)))^2(\xi_n^j) + \tfrac{c\ \lambda^3}{N^{3/2}}\\ &\leq e^{-\frac{\lambda^2}{N}K_{n,\eta_{n-1}^N}(f_n-K_{n,\eta_{n-1}^N}(f_n)))^2(\xi_n^j)+\frac{c\ \lambda^3}{N^{3/2}}}.\end{aligned}$$

Multiplying over j, we obtain

$$\left|\mathbb{E}\left[\ e^{i\lambda N^{1/2}\Delta\mathcal{M}_n^N(f)}\ \middle|\ F_{n-1}^N\right]\right|^2 \leq e^{-\lambda^2\Delta\mathcal{C}_n^N(f)+\frac{c\ \lambda^3}{N^{1/2}}},$$

and by (23) we conclude that condition $(H2)$ is met with $a_2(n) = c/2$.

We now come to the proof of $(H1)$. By definition of the particle model associated to a given collection of transitions $K_{n,\eta}$ we have

$$\mathbb{E}\left[\, e^{i\lambda N^{1/2}\Delta\mathcal{M}_n^N(f)+\frac{\lambda^2}{2}\Delta C_n^N(f)} \,\Big|\, \mathcal{F}_{n-1}^N\right]$$

$$= \prod_{j=1}^N \left[K_{n,\eta_{n-1}^N}\left(e^{i\frac{\lambda}{N^{1/2}}\tilde{f}_n^j+\frac{\lambda^2}{2N}\Delta C_n^N(f)}\right)\right](\xi_{n-1}^j),$$

with the random function $\tilde{f}_n^j = (f_n - K_{n,\eta_{n-1}^N}(f_n)(\xi_{n-1}^j)$. Using the elementary inequality

$$\left|e^z - \left(1+z+\frac{z^2}{2}\right)\right| \le e^{|z|}\ \frac{|z|^3}{3!},$$

it is easily seen that, for any $\lambda \le N^{1/2}$, we have

$$e^{i\frac{\lambda}{N^{1/2}}\tilde{f}_n^j+\frac{\lambda^2}{2N}\Delta\mathcal{C}_n^N(f)} = 1 + i\tfrac{\lambda}{N^{1/2}}\tilde{f}_n^j + \tfrac{\lambda^2}{2N}[\Delta\mathcal{C}_n^N(f) - (\tilde{f}_n^j)^2] + r_{n,1}^N(f)$$

with $|r_{n,1}^N(f)| \le c\ \lambda^3 N^{-3/2}$. This clearly implies that, for any $\lambda \le N^{1/2}$,

$$\left[K_{n,\eta_{n-1}^N}\left(e^{i\frac{\lambda}{N^{1/2}}\tilde{f}_n^j+\frac{\lambda^2}{2N}\Delta C_n^N(f)}\right)\right](\xi_{n-1}^j)$$

$$= 1 + \tfrac{\lambda^2}{2N}[\Delta\mathcal{C}_n^N(f) - K_{n,\eta_{n-1}^N}(\tilde{f}_n^j)^2(\xi_{n-1}^j)] + r_{n,2}^N(f),$$

with $|r_{n,2}^N(f)| \le c\ \lambda^3 N^{-3/2}$. It is now convenient to notice that for any $\lambda \le N^{1/2}$

$$\left|\frac{\lambda^2}{2N}[\Delta\mathcal{C}_n^N(f) - K_{n,\eta_{n-1}^N}(\tilde{f}_n^j)^2(\xi_{n-1}^j)] + r_{n,2}^N(f)\right| \le \frac{c\ \lambda}{N^{1/2}}.$$

On the other hand for any $|z| \le 1/2$ and with the principal value of the logarithm we recall that

$$\log(1+z) = z - \int_0^z \frac{u}{1+u}\ du = z - z^2\int_0^1 \frac{t}{1+tz}\ dt.$$

Since for any $|z| \le 1/2$ and $t \in [0,1]$ we have $|1+tz| \ge 1/2$, we find that for any $|z| \le 1/2$ we have $|\log(1+z) - z| \le |z^2|$. The previous computations show that there exists some universal constant $c_0 \in (0,1)$ such that for any $\lambda \le c_0\ N^{1/2}$ we have

$$\log K_{n,\eta_{n-1}^N}(e^{i\frac{\lambda}{N^{1/2}}\tilde{f}_n^j+\frac{\lambda^2}{2N}\Delta\mathcal{C}_n^N(f)})(\xi_{n-1}^j)$$

$$= \tfrac{\lambda^2}{2N}[\Delta\mathcal{C}_n^N(f) - K_{n,\eta_{n-1}^N}(\tilde{f}_n^j)^2(\xi_{n-1}^j)] + r_{n,3}^N(f)$$

with $|r^N_{n,3}(f)| \leq c\ \lambda^3/N^{3/2}$. Summing over j, we see that for any $\lambda \leq c_0\ N^{1/2}$

$$\left|\sum_{j=1}^{N} \log K_{n,\eta^N_{n-1}}(e^{i\frac{\lambda}{N^{1/2}}\tilde{f}^j_n+\frac{\lambda^2}{2N}\Delta\mathcal{C}^N_n(f)})(\xi^j_{n-1})\right| \leq c\ \lambda^3/N^{1/2}.$$

Finally, using the elementary inequality $|e^z-1| \leq |z|e^{|z|}$, we conclude that, for any $\lambda \leq c_0\ N^{1/2}$,

$$\left|\mathbb{E}\left[\ e^{i\lambda N^{1/2}\Delta\mathcal{M}^N_n(f)+\frac{\lambda^2}{2}\Delta\mathcal{C}^N_n(f)}\ \middle|\ \mathcal{F}^N_{n-1}\right]-1\right| \leq c\ \frac{\lambda^3}{N^{1/2}}\ e^{c\ \frac{\lambda^3}{N^{1/2}}}.$$

This readily implies that there exists some universal positive constant c_1 such that, for any $\lambda^3 \leq c_1\ N^{1/2}$, we have

$$\left|\mathbb{E}\left[\ e^{i\lambda N^{1/2}\Delta\mathcal{M}^N_n(f)+\frac{\lambda^2}{2}\Delta\mathcal{C}^N_n(f)}\ \middle|\ \mathcal{F}^N_{n-1}\right]-1\right| \leq c\ \frac{\lambda^3}{N^{1/2}},$$

which proves that condition $(H1)$ is met with $a_1(n)=c$ and $c_1(n)=1$.
$\square$

Appendix

Doob's type decompositions

Proposition 3.5 ([5]) *Let $(\overline{Q}_{p,n})_{p\leq n}$ be the semi-group defined in (11). For $f_n \in \mathcal{B}_b(E_n)$ and $p \leq n$ we set $f_{p,n} = \overline{Q}_{p,n}(f_n - \eta_n f_n)$. Then we have the following decomposition,*

$$\eta^N_p(f_{p,n}) = \mathcal{A}^N_{p,n}(f_n) + \mathcal{M}^N_{p,n}(f_n),$$

with the predictable and martingale sequences $\mathcal{A}^N_{p,n}(f_n)$ and $\mathcal{M}^N_{p,n}(f_n)$ given by

$$\mathcal{A}^N_{p,n}(f_n) = \sum_{q=1}^{p}\left[1-\eta^N_{q-1}(\overline{Q}_{q-1,q}1)\right]\ \Phi_q(\eta^N_{q-1})(f_{q,n}) \qquad (24)$$

$$\mathcal{M}^N_{p,n}(f_n) = \sum_{q=0}^{p}\ \left[\eta^N_q(f_{q,n}) - \Phi_q(\eta^N_{q-1})f_{q,n}\right], \qquad (25)$$

with the usual convention $\Phi_0(\eta^N_{-1}) = \eta_0$.

Proof: Note that for any $\varphi_n \in \mathcal{B}_b(E_n)$ we have the decomposition

$$\eta^N_p(\overline{Q}_{p,n}\varphi_n) - \eta^N_0(\overline{Q}_{0,n}\varphi_n) = \sum_{q=1}^{p}\bar{\delta}_q,$$

with $\bar{\delta}_q = \eta_q^N(\overline{Q}_{q,n}\varphi_n) - \eta_{q-1}^N(\overline{Q}_{q-1,n}\varphi_n)$. Choose now $\varphi_n = f_n - \eta_n f_n$. For $q \leq p$, we have, by definition of $f_{q,n}$,

$$\bar{\delta}_q = \eta_q^N(f_{q,n}) - \eta_{q-1}^N(\overline{Q}_{q-1,n}\varphi_n) = U_1 + U_2,$$

with

$$\begin{aligned} U_1 &= \eta_q^N(f_{q,n}) - \left[\Phi_q\left(\eta_{q-1}^N\right)\right](f_{q,n}) \\ U_2 &= \left[\Phi_q\left(\eta_{q-1}^N\right)\right](f_{q,n}) - \eta_{q-1}^N(\overline{Q}_{q-1,n}\varphi_n). \end{aligned}$$

In order to show (12), it is thus enough to verify that

$$\eta_{q-1}^N(\overline{Q}_{q-1,n}\varphi_n) = \eta_{q-1}^N(\overline{Q}_{q-1,q}1)\ \left[\Phi_q\left(\eta_{q-1}^N\right)\right](f_{q,n}). \tag{26}$$

However, we have

$$\begin{aligned} \overline{Q}_{q-1,n}\varphi_n &= \frac{\gamma_{q-1}(1)}{\gamma_n(1)} Q_{q-1,n}\varphi_n = \frac{\gamma_{q-1}(1)}{\gamma_n(1)} Q_q\left(Q_{q,n}\varphi_n\right) \\ &= \frac{\gamma_{q-1}(1)}{\gamma_q(1)} Q_q\left(\overline{Q}_{q,n}\varphi_n\right) = \frac{\gamma_{q-1}(1)}{\gamma_q(1)} Q_q\left(f_{q,n}\right), \end{aligned}$$

and hence

$$\begin{aligned} \eta_{q-1}^N(\overline{Q}_{q-1,n}\varphi_n) &= \frac{\gamma_{q-1}(1)}{\gamma_q(1)} \eta_{q-1}^N\left(Q_q\left(f_{q,n}\right)\right) \\ &= \frac{\gamma_{q-1}(1)\eta_{q-1}^N\left(G_{q-1}\right)}{\gamma_q(1)}\ \left[\Phi_q\left(\eta_{q-1}^N\right)\right](f_{q,n}). \end{aligned} \tag{27}$$

On the other hand, for $q \geq 1$ and $x_{q-1} \in E_{q-1}$,

$$[\overline{Q}_{q-1,q}(1)](x_{q-1}) = \frac{\gamma_{q-1}(1)}{\gamma_q(1)}\left[Q_{q-1,q}(1)\right](x_{q-1}),$$

and

$$[Q_{q-1,q}(1)](x_{q-1}) = \int_{E_q} G_{q-1}(x_{q-1})\, M_q(x_{q-1}, dx_q) = G_{q-1}(x_{q-1}),$$

which yields

$$\frac{\gamma_{q-1}(1)G_{q-1}}{\gamma_q(1)} = \overline{Q}_{q-1,q}(1).$$

Plugging this last equality into (27), we get (26), and hence (12). The martingale property of $\mathcal{M}_{p,n}^N(f_n)$ is clear.

Some asymptotic estimates

The next lemma provides a refined version of Bürkholder's type inequalities for independent sequences of random variables.

Lemma 3.6 ([6]) *Let $m(X) = \frac{1}{N}\sum_{i=1}^{N}\delta_{X^i}$ be the N-empirical measure associated to a collection of independent random variables X^i, with respective distributions μ_i on some measurable space $(E,\mathcal{E})$. For any sequence of $\mathcal{E}$-measurable functions h_i such that $\mu_i(h_i) = 0$ and $\sigma^2(h) = \frac{1}{N}\sum_{i=1}^{N}\mathrm{osc}^2(h_i) < \infty$ we have for any integer $p \geq 1$*

$$\sqrt{N}\ \mathbb{E}(|m(X)(h)|^p)^{\frac{1}{p}} \leq d(p)^{\frac{1}{p}}\ \sigma(h) \tag{28}$$

with the sequence of finite constants $(d(n))_{n\geq 0}$ defined for any $n \geq 1$ by the formulae

$$d(2n) = (2n)_n\ 2^{-n} \quad \textit{and} \quad d(2n-1) = \frac{(2n-1)_n}{\sqrt{n-1/2}}\ 2^{-(n-1/2)} \tag{29}$$

The extension of the above lemma to the interacting particle measures η_n^N and the Feynman-Kac flow η_n is the following

Lemma 3.7 *Let $(d(p))_{p\geq 1}$ be the sequence introduced in (29). For any McKean interpretation model and for any $n \geq 0$, $p \geq 1$, $f_n \in \mathrm{Osc}(E_n)$ and $\varepsilon > 0$ we have*

$$\begin{array}{rcl}
\mathbb{E}\left(|[\eta_n^N - \eta_n](f_n)|^p\right)^{\frac{1}{p}} & \leq & d(p)^{1/p}\ \ b(n)/\sqrt{N} \\
\mathbb{E}\left[e^{\varepsilon N^{1/2}|\eta_n^N(f_n)-\eta_n(f_n)|}\right] & \leq & \left(1+\varepsilon 2^{-1/2}\ b(n)\right)\ e^{(\varepsilon b(n))^2/2},
\end{array}$$

for some finite constant $b(n)$ such that $b(n) \leq 2\ \sum_{q=0}^{n}\ r_{q,n}\ \beta(P_{q,n})$.

Proof: The proof if based on the following decomposition

$$\eta_n^N - \eta_n = \sum_{q=0}^{n}\ [\Phi_{q,n}(\eta_q^N) - \Phi_{q,n}(\Phi_q(\eta_{q-1}^N))] \tag{30}$$

We introduce the random potential functions

$$G_{q,n}^N : x_q \in E_q \longrightarrow G_{q,n}^N(x_q) = \frac{G_{q,n}}{\Phi_q(\eta_{q-1}^N)(G_{q,n})} \in (0,\infty)$$

and the random bounded operators $P_{q,n}^N$ from $\mathcal{B}_b(E_n)$ into $\mathcal{B}_b(E_q)$ defined for any $(f_n, x_q) \in (\mathcal{B}_b(E_n) \times E_q)$ by

$$\begin{array}{rcl}
P_{q,n}^N(f_n)(x_q) & = & P_{q,n}\left(f_n - \Phi_{q,n}\left(\Phi_q(\eta_{q-1}^N)\right)(f_n)\right)(x_q) \\
& = & \displaystyle\int\ (P_{q,n}f(x_q) - P_{q,n}f(y_q))\ G_{q,n}^N(y_q)\ \Phi_q(\eta_{q-1}^N)(dy_q)
\end{array}$$

We associate to the pair $(G^N_{q,n}, P^N_{q,n})$ the random bounded and integral operator $Q^N_{q,n}$ from $\mathcal{B}_b(E_n)$ into $\mathcal{B}_b(E_q)$ defined for any $(f_n, x_q) \in (\mathcal{B}_b(E_n) \times E_q)$ by

$$Q^N_{q,n}(f_n)(x_q) = G^N_{q,n}(x_q) \ \times \ P^N_{q,n}(f_n)(x_q)$$

Each "local" term in (30) can be expressed in terms of $Q^N_{q,n}$ as follows. For any $q \leq n$ and $f_n \in \mathcal{B}_b(E_n)$ with $\mathrm{osc}(f_n) \leq 1$ we have

$$\begin{aligned}
&\Phi_{q,n}(\eta^N_q)([f_n - \Phi_{q,n}(\Phi_q(\eta^N_{q-1}))(f_n)]) \\
&= \frac{1}{\eta^N_q(G_{q,n})} \ \eta^N_q \left(G_{q,n} \ P_{q,n}[f_n - \Phi_{q,n}(\Phi_q(\eta^N_{q-1}))(f_n)]\right) \\
&= \frac{\eta^N_q Q^N_{q,n}(f_n)}{\eta^N_q Q^N_{q,n}(1)}
\end{aligned}$$

By construction we also observe that

$$\Phi_q(\eta^N_{q-1}) \left(\ G^N_{q,n}\right) = 1 \quad \text{and} \quad \Phi_q(\eta^N_{q-1}) \left(Q^N_{q,n}(f_n)\right) = 0$$

The above considerations easily yield the decomposition

$$\Phi_{q,n}(\eta^N_q) - \Phi_{q,n}(\Phi_q(\eta^N_{q-1})) = \frac{1}{\eta^N_q(G^N_{q,n})} \ [\eta^N_q - \Phi_q(\eta^N_{q-1})]Q^N_{q,n}$$

Using the properties of the Dobrushin's contraction coefficient, we also have

$$\begin{aligned}
\|P^N_{q,n}(f_n)\| \ &\leq \ \mathrm{osc}(P_{q,n}f) \leq \ \beta(P_{q,n}) \\
\|Q^N_{q,n}(f_n)\| \ &\leq \ \|G^N_{q,n}\| \ \|P^N_{q,n}(f_n)\| \leq \|G^N_{q,n}\| \ \beta(P_{q,n}),
\end{aligned}$$

and from these estimates, we readily prove the inequality

$$\begin{aligned}
&\left|[\Phi_{q,n}(\eta^N_q) - \Phi_{q,n}(\Phi_q(\eta^N_{q-1}))](f_n)\right| \\
&\qquad\qquad \leq \ r_{q,n} \ \beta(P_{q,n}) \left|[\eta^N_q - \Phi_q(\eta^N_{q-1})]\overline{Q}^N_{q,n}(f_n)\right|,
\end{aligned}$$

with $\overline{Q}^N_{q,n}(f_n) = Q^N_{q,n}(f_n)/\|Q^N_{q,n}(f_n)\|$. Now, using lemma 3.6, we check that for any $p \geq 1$ we have

$$\sqrt{N} \ \mathbb{E}(|[\eta^N_q - \Phi_q(\eta^N_{q-1})]\overline{Q}^N_{q,n}(f_n)|^p \mid F^N_{q-1})^{1/p} \leq 2 \ d(p)^{1/p}$$

with the sequence of finite constants $d(p)$ introduced in (29). This ends the proof of the first assertion. To prove the second one we The

$\mathbb{L}_n$-inequalities stated in lemma 3.6 clearly implies that, for any $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{E}(e^{\varepsilon|\eta_n^N(f_n)-\eta_n(f_n)|}) \\
&= \textstyle\sum_{n\geq 0} \frac{\varepsilon^{2n}}{(2n)!}\ \mathbb{E}(|\eta_n^N(f_n) - \eta_n(f_n)|^{2n}) \\
&\qquad + \textstyle\sum_{n\geq 0} \frac{\varepsilon^{2n+1}}{(2n+1)!}\ \mathbb{E}(|\eta_n^N(f_n) - \eta_n(f_n)|^{2n+1}) \\
&\leq \textstyle\sum_{n\geq 0} \frac{1}{n!}\ \left(\frac{\varepsilon^2 b(n)^2}{2N}\right)^n + \textstyle\sum_{n\geq 0} \frac{1}{n!}\ \left(\frac{\varepsilon^2 b(n)^2}{2N}\right)^{n+1/2},
\end{aligned}$$

from which we conclude that

$$\begin{aligned}
\mathbb{E}(e^{\varepsilon|\eta_n^N(f_n)-\eta_n(f_n)|}) &= \left(1 + \frac{\varepsilon b(n)}{\sqrt{2N}}\right) \sum_{n\geq 0} \frac{1}{n!}\ \left(\frac{\varepsilon^2 b(n)^2}{2N}\right)^n \\
&= \left(1 + \frac{\varepsilon b(n)}{\sqrt{2N}}\right)\ e^{\frac{\varepsilon^2}{2N} b(n)^2}.
\end{aligned}$$

We end the proof of the lemma by replacing ε by $\varepsilon\sqrt{N}$.

□

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