Stability and uniform propagation of chaos prop. of Ensemble Kalman-Bucy filters

P. Del Moral

INRIA & UNSW Maths/Stats

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Synthesis $\subset$ joint works, hyperef. (2016-2017):

- AoAP-17 (Unif. EnKBF)+ J. Tugaut.
- SIAM C.& Opt. (Unif. En-EKBF)+ A. Kurtzmann, J. Tugaut.
- Arxiv 1 (Stability KBF)+ A.N. Bishop
- Arxiv 2 (Stability EKBF)+ A. Kurtzmann, J. Tugaut.
- Arxiv 3 (Perturbation KB)+ A.N. Bishop, S. Pathiraja.
- (Working paper CLT/Bias/Taylor)+ A.N. Bishop + A. Niclas

Kalman-Bucy filter

Linear+Gaussian filtering problem

$$\begin{cases} dX_t &= A\, X_t\, dt \;+\; R^{1/2}\, dW_t \\ dY_t &= C\, X_t\, dt \;+\; \Sigma^{1/2}\, dV_t \end{cases} \qquad \rightsquigarrow \mathcal{F}_t := \sigma(Y_s,\; s \leq t).$$

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Linear+Gaussian filtering problem

$$\left\{\begin{array}{lcl} dX_t & = & A\, X_t\, dt \;+\; R^{1/2}\, dW_t \\ dY_t & = & C\, X_t\, dt \;+\; \Sigma^{1/2}\, dV_t \end{array}\right. \qquad \rightsquigarrow \mathcal{F}_t := \sigma(Y_s,\; s \le t).$$

Optimal $\mathbb{L}_2$-filter = Kalman-Bucy filter

$$\widehat{X}_t := \mathbb{E}(X_t \mid \mathcal{F}_t) \quad \text{and} \quad P_t := \mathbb{E}\left((X_t - \mathbb{E}(X_t \mid \mathcal{F}_t))\,(X_t - \mathbb{E}(X_t \mid \mathcal{F}_t))'\right)$$

$$\Downarrow$$

$$d\widehat{X}_t = A\,\widehat{X}_t\; dt + P_t\; C'\Sigma^{-1}\;\left(dY_t - C\widehat{X}_t\; dt\right)$$

with the Riccati equation

$$\partial_t P_t = \mathrm{Ricc}(P_t) := AP_t + P_tA' - P_tSP_t + R \quad \text{with} \quad S := C'\Sigma C$$

Nonlinear Kalman-Bucy diffusion

Reformulation

$$\mathrm{Law}(X_t \mid \mathcal{F}_t) = \mathcal{N}\left[\widehat{X}_t, P_t\right] = \mathrm{Law}(\overline{X}_t \mid \mathcal{F}_t) = \eta_t$$

in terms of the McKean-Vlasov type diffusion

$$d\overline{X}_t = A\,\overline{X}_t dt + R^{1/2}\,d\overline{W}_t + \mathcal{P}_{\eta_t} C'\Sigma^{-1}\left[dY_t - \left(C\overline{X}_t dt + \Sigma^{1/2}\,d\overline{V}_t\right)\right]$$

with the covariance matrices

$$\mathcal{P}_{\eta_t} = \eta_t\left[(e - \eta_t(e))(e - \eta_t(e))'\right] \quad \text{with} \quad e(x) := x.$$

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$$\Downarrow$$

$$\mathbb{E}\left(\overline{X}_t \mid \mathcal{F}_t\right) = \widehat{X}_t \qquad \text{and} \qquad \mathcal{P}_{\eta_t} = P_t.$$

The Ensemble Kalman-Bucy filter

Mean field interpretation $\rightsquigarrow$ $N+1$ interacting diffusions

$$d\xi_t^i = A\,\xi_t^i dt + R^{1/2} d\overline{W}_t^i + p_t C' \Sigma^{-1} \left[dY_t - \left(C\xi_t^i dt + \Sigma^{1/2}\, d\overline{V}_t^i \right) \right]$$

with the rescaled particle covariance matrices

$$p_t := \left(1 + \frac{1}{N}\right)\ \mathcal{P}_{\eta_t^N} = \frac{1}{N} \sum_{1 \le i \le N+1} \left(\xi_t^i - m_t\right)\left(\xi_t^i - m_t\right)'$$

and the empirical measures

$$\eta_t^N := \frac{1}{N+1} \sum_{1 \le i \le N+1} \delta_{\xi_t^i} \quad \text{and the sample mean} \quad m_t := \frac{1}{N+1} \sum_{1 \le i \le N+1} \xi_t^i.$$

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where is the Riccati equation?

Th1: The EnKF equations

The EnKF equation

$$dm_t = A\ m_t dt + p_t\ C'\Sigma^{-1}\ (dY_t - Cm_t\ dt) + \frac{1}{\sqrt{N+1}}\ d\overline{M}_t$$

with an r_1-dimensional martingale $\overline{M}_t = \left(\overline{M}_t(k)\right)_{1\leq k\leq r_1}$ with angle-brackets

$$\partial_t \langle \overline{M} \mid \otimes \mid \overline{M} \rangle_t = R + p_t S p_t.$$

The particle/ensemble Riccati equation

$$dp_t = \mathrm{Ricc}(p_t)\ dt + \frac{1}{\sqrt{N}}\ dM_t$$

Symmetric matrix-valued martingale $M_t = (M_t(k,l))_{1\leq k,l\leq r_1}$

Angle brackets given by the Wick-type formula

$$\partial_t \langle M \mid \otimes \mid M\rangle_t^{\sharp} = p_t \otimes_{sym} (R + p_t S p_t)$$

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Orthogonality property

$$\forall 1 \leq k,l,l' \leq r_1 \qquad \left\langle M(k,l), \overline{M}(l')\right\rangle_t = 0.$$

(H) Observability + Controllability

Observe that

$$d\overline{X}_t = (\mathbf{A} - \mathbf{P_t S})\ \overline{X}_t dt \ + \ R^{1/2}\ d\overline{W}_t + P_t C' \Sigma^{-1}\ \left[dY_t - \Sigma^{1/2}\ d\overline{V}_t \right]$$

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$$\Downarrow$$

Steady state: $\exists! P > 0$ **such that** $\mathrm{Ricc}(P) = 0$ **and**

$$\varsigma(\mathbf{A} - \mathbf{PS}) := \max\left\{\mathrm{Re}(\lambda)\ :\ \lambda \in \mathrm{Spec}(A - PS)\right\} < 0$$

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Steady state diffusion

$$d\overline{X}_t \simeq (\mathbf{A} - \mathbf{PS})\ \overline{X}_t dt\ +\ R^{1/2}\ d\overline{W}_t + PC' \Sigma^{-1}\ \left[dY_t - \Sigma^{1/2}\ d\overline{V}_t\right]$$

STABLE EVEN WHEN A **is unstable.**

(H) Observability + Controllability

Controllability & Observability Gramians

$$\mathcal{C}_t := \int_0^t e^{sA} R\, e^{sA'}\, ds \qquad \mathcal{O}_t := \int_0^t e^{-sA'} S\, e^{-sA} ds$$

$$\mathcal{O}_t(\mathcal{C}) := \mathcal{C}_t^{-1} \left[\int_0^t e^{(t-s)A}\, \mathcal{C}_s\, S\, \mathcal{C}_s\, e^{(t-s)A'}\, ds \right] \mathcal{C}_t^{-1}$$

$$\mathcal{C}_t(\mathcal{O}) := \mathcal{O}_t^{-1} \left[\int_0^t e^{-(t-s)A'}\, \mathcal{O}_s R\, \mathcal{O}_s\, e^{-(t-s)A}\, ds \right] \mathcal{O}_t^{-1}$$

$$\Downarrow$$

$\Rightarrow\ \exists\ \upsilon > 0$ (a.k.a obs-control interval) $\quad \exists \varpi_{\pm}^{o,c}, \varpi_{\pm}^{c}(\mathcal{O}), \varpi_{\pm}^{o}(\mathcal{C}) > 0$ s.t.

$\varpi_-^c\ Id \leq \mathcal{C}_\upsilon \leq \varpi_+^c\ Id$ *and so on, for the other Gramians.*

Bucy's theorems

Notation $\mathcal{E}_{s,t} = \exp\left[\oint_s^t Q_u \; du\right] = \text{state transition matrix}$

$$\partial_s \mathcal{E}_{s,t} = -\mathcal{E}_{s,t} \; Q_s \quad \text{and} \quad \partial_t \mathcal{E}_{s,t} = Q_t \mathcal{E}_{s,t}$$

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Theo 1 [Bucy] $\forall (t, P_0)$

$$\left(\mathcal{O}_v(\mathcal{C}) + \mathcal{C}_v^{-1}\right)^{-1} \leq P_{t+v} \leq \mathcal{O}_v^{-1} + \mathcal{C}_v(\mathcal{O})$$

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Theo 2 [Bucy]

$\exists \alpha, \beta > 0$ that depends on $\varpi_\pm^{o,c}, \varpi_\pm^c(\mathcal{O}), \varpi_\pm^o(\mathcal{C})$ s.t. $\forall t \geq s \geq \upsilon$

$$\| \exp \oint_s^t (A - P_u S) du \|_2^2 \leq \alpha \exp\{-\beta(t-s)\}$$

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Also true for any $t \geq s \geq 0$ with α depending on P_0.

Some consequences

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Theo $\exists \nu > 0$ s.t. for any $t \geq s \geq 0$, $x_1, x_2 \in \mathbb{R}^{r_1}$, $Q_1, Q_2 \geq 0$, $n \geq 2$

$$\mathbb{E}\left(\|\psi_{s,t}(x_1,Q_1) - \psi_{s,t}(x_2,Q_2)\|_2^n \mid X_s\right)^{1/n}$$

$$\leq c_{Q_1,Q_2}\ e^{-\nu(t-s)}\ \left[\|x_1 - x_2\|_2 + \left\{\|x_2 - X_s\|_2\ +\ \sqrt{n}\right\}\ \|Q_1 - Q_2\|_2\right]$$

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$\oplus$ **same estimate for** $\overline{\psi}_{s,t}(x, Q) =$ **stochastic flow Kalman-Bucy (nonlinear) diffusion**

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Any choice

$$\nu \in \{\beta,\ (1-\epsilon)\ \varsigma(A - PS),\ \lambda_{max}((A - PS)_{sym})\}$$

is fine but c_{Q_1,Q_2} maybe larger than you expect.

Back to the EnKF - Some observations/numerical issues

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- $C = 0 \Rightarrow \xi_t^i$ i.i.d. copies of the signal.
- $\mathrm{rank}(p_t) \leq N < r_1 \Rightarrow (r_1 - N)$ state dimensions not driven by Y_t.
- $r_1 = 1 \Longrightarrow p_t$ has an heavy tailed invariant distribution $\propto x^{-(N+3)}$

$$\Longrightarrow \quad \forall \epsilon > 0 \qquad \mathbb{E}(e^{\epsilon q_t}) = \infty \quad \text{and} \quad \forall m \geq N+2 \qquad \mathbb{E}(q_t^m) = \infty$$

$\rightsquigarrow$ **Moment explosions**

Back to the EnKF

We really need a stable signal (for uniform estimates)

$$\Downarrow$$

$$(\varsigma(A) \leq)\, \mu(A) := \inf\{\alpha \;\;:\;\; \forall t \geq 0 \;\; \|\exp(At)\|_2 \leq \exp(\alpha t)\} < 0$$

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Spectral abscissa $\rightsquigarrow$ *constants (conditioning numbers) depending on diagonalisation basis. Complicated analysis when* $A \rightsquigarrow A - p_t S$.

"Technical" problems even when $\mu(A) < 0$

$\overline{A} := A - PS$ **stable matrices** $\not\Rightarrow$ $\overline{a} = A - pS$ **stable matrices**

$$p = P + N^{-1/2}\,\Sigma \quad \Longrightarrow \quad \overline{a} = \overline{A} + N^{-1/2}\,\Sigma S$$

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BUT:

$$H(AS) \qquad S = s\; Id \quad \text{and} \quad \mu(A) < 0$$

$$\Longrightarrow \forall p \quad \mu(A - pS) \leq \mu(A) + s\;\mu(-p) < \mu(A) < 0$$

$$\Longrightarrow p_t \;\; \text{never hits the divergence set } \{\Sigma \;:\; \varsigma\left(\overline{A} + \Sigma S\right) > 0\}$$

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Same conclusion when $S > 0$ up to a change of signal-drift matrix $A \rightsquigarrow S^{1/2} A S^{-1/2}$

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Same conclusion when $S > 0$ **up to a change of signal-drift matrix** $A \rightsquigarrow S^{1/2} A S^{-1/2}$

More generally $(A - pS)$ may be locally ill-conditioned in the sense that

$$\exists Q \;:\; \mu(A - QS) = \lambda_{max}((A - QS)_{sym}) > 0 > \lambda_{min}((A - QS)_{sym})$$

Under $H(AS)$

Theo $\forall n \geq 1 \; \exists N_n \geq 1$

$$\sup_{N \geq N_n} \sup_{t \geq 0} \sqrt{N} \; \mathbb{E}\left[\|p_t - P_t\|_F^n\right] \vee \sup_{N \geq N_n} \sup_{t \geq 0} \sqrt{N} \; \mathbb{E}\left[\|\xi_t^1 - \overline{X}_t\|^n\right] < \infty$$

$$\Downarrow$$

Cor $\forall n \geq 1 \; \exists N_n \geq 1$

$$\sup_{N \geq N_n} \sup_{t \geq 0} \sqrt{N} \; \mathbb{W}_n\left(\mathrm{Law}(\xi_t^1), \eta_t\right) \vee \sup_{N \geq N_n} \sup_{t \geq 0} \sqrt{N} \; \mathbb{E}\left(\left|\eta_t^N(f) - \eta_t(f)\right|^n\right) < \infty$$

Nonlinear models

Extended Kalman-Bucy-filters

$$d\widehat{X}_t = A(\widehat{X}_t)\ dt + P_t C'\ \Sigma^{-1}\ \left[dY_t - C\widehat{X}_t\ dt\right]$$

with the "stochastic" Riccati equation:

$$\partial_t P_t = \partial A(\widehat{X}_t) P_t + P_t\ \partial A(\widehat{X}_t)' + R - P_t S P_t$$

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McKean-Vlasov interpretation

$$\begin{aligned} d\overline{X}_t &= \mathcal{A}\left(\overline{X}_t, \mathbb{E}[\overline{X}_t \mid \mathcal{G}_t]\right)\ dt + R^{1/2}\ d\overline{W}_t \\ &\quad + \mathcal{P}_{\eta_t} C' R_2^{-1}\ \left[dY_t - \left(C\overline{X}_t\ dt + \Sigma^{1/2}\ d\overline{V}_t\right)\right] \end{aligned}$$

with the drift function

$$\mathcal{A}(x, m) := A[m] + \partial A[m]\ (x - m).$$

Extended Ensemble Kalman-Bucy-filters

En-EKF = Mean field particle model

$$
\begin{aligned}
d\xi_t^i &= \mathcal{A}(\xi_t^i, m_t)\ dt + R^{1/2} d\overline{W}_t^i \\
&\qquad + p_t C' \Sigma^{-1} \left[dY_t - \left(C\xi_t^i\ dt + \Sigma^{1/2}\ d\overline{V}_t^i \right) \right]
\end{aligned}
$$

with the sample means m_t and covariance matrices p_t and the drift

$$
\mathcal{A}(\xi_t^i, m_t) := A[m_t] + \underbrace{\partial A[m_t]\ (\xi_t^i - m_t)}_{\text{Repulsion/Attraction w.r.t. } m_t}
$$

Some illustrations

Langevin type signal processes

$$R = \sigma^2 \; Id \quad \text{and} \quad (A, \partial A) = (-\partial \mathcal{V}, -\partial^2 \mathcal{V})$$

Non quadratic potential $(q \in \mathbb{R}^r, \mathcal{Q}_1, \mathcal{Q}_2 \geq 0)$

$$\mathcal{V}(x) = \frac{1}{2} \langle \mathcal{Q}_1 x, x \rangle + \langle q, x \rangle + \frac{1}{3} \langle \mathcal{Q}_2 x, x \rangle^{3/2}$$

Interacting diffusion gradient flows

$$\mathcal{V}(x) = \sum_{1 \leq i \leq r} \mathcal{U}_1(x_i) + \sum_{1 \leq i \neq j \leq r} \mathcal{U}_2(x_i, x_j)$$

for some convex confining potential $\mathcal{U}_i : \mathbb{R}^i \mapsto [0, \infty[$

Regularity conditions

Full observation $S = s\ Id$ and

$$
\begin{aligned}
-\lambda_{\partial A} &:= \sup_{x \in \mathbb{R}^{r_1}} \lambda_{max}(\partial A(x) + \partial A(x)') < 0 \\
\|\partial A(x) - \partial A(y)\| &\leq \kappa_{\partial A}\ \|x - y\|
\end{aligned}
$$

Examples: Langevin signal-diffusion

$$
(\lambda_{\partial A}, \kappa_{\partial A}) = \beta\ \left(2^{-1}\lambda_{min}(\mathcal{Q}_1),\ 2\lambda_{max}^{3/2}(\mathcal{Q}_2)\right).
$$

more generally $\partial^2 \mathcal{V} \geq v\ Id \oplus$ *Lipschitz condition*

Stability theorem

$$(\overline{X}_t, \overline{Z}_t) := \text{McKean-Vlasov starting at} \quad (\overline{X}_0, \overline{Z}_0)$$

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Theo [+Kurtzmann-Tugaut]

When $\lambda_{\partial A}$ is sufficiently large we have

$$\mathbb{W}_2(\mathrm{Law}(\overline{X}_t), \mathrm{Law}(\overline{Z}_t)) \leq c \ \exp[-t\ \lambda] \quad \text{for some} \quad \lambda > 0.$$

$\exists$ *more explicit description in terms of* $(R, S, \kappa_{\partial A})$.

Propagation of chaos

$$\mathbb{P}_t^N := \mathrm{Law}(m_t, p_t) \qquad \mathbb{P}_t := \mathrm{Law}(\widehat{X}_t, P_t)$$

and

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$$\Downarrow$$

Theo [+Kurtzmann-Tugaut]

When $\lambda_{\partial A}$ is sufficiently large, $\exists \beta \in]0, 1/2]$ s.t.

$$\sup_{t \geq 0} \mathbb{W}_2\left(\mathbb{P}_t^N, \mathbb{P}_t\right) \vee \sup_{t \geq 0} \mathbb{W}_2\left(\mathbb{Q}_t^N, \mathbb{Q}_t\right) \leq c\ N^{-\beta}$$

as soon as $\mathrm{tr}(P_0)$ is not too large and N large enough...