New Insights on Particle MCMC algorithms

P. Del Moral

UNSW, School of Mathematics and Statistics

STM Workshop, IMS Tokyo 2014

Some hyper-refs

- PMCMC - Andrieu, Doucet, Holenstein JRSS-10
- On Feynman-Kac and PMCMC models, with R. Kohn and F. Patras (ArXiv-2014).
- On parallel implementation of Sequential Monte Carlo methods: the island particle model, with C. Vergé, C. Dubarry, and E. Moulines. (Statistics and Computing-2013).
- A Backward Particle Interpretation of Feynman-Kac Formulae, with A. Doucet and S. Singh (Arxiv-2009/M2AN-2010)
- Feynman-Kac formulae, Springer (2004) [+ Refs]
- Mean field simulation for Monte Carlo integration. Chapman - Hall (2013) [+ Refs]

Bayes/Conditioning/Feynman-Kac measures

Origins/Equivalent particle algorithms

Ex.: MCMC with product target measures

Particle measures $\oplus$ 2 key formulae

Conditioning and duality formulae

Taylor expansions for PMCMC transitions

Bayes/Conditioning/Feynman-Kac measures

Origins/Equivalent particle algorithms

Ex.: MCMC with product target measures

Particle measures ⊕ 2 key formulae

Conditioning and duality formulae

Taylor expansions for PMCMC transitions

Conditioning formulae (not°: $\mathbf{z_n} = (z_0, \ldots, z_n) = z_{0:n}$)

- Bayes rule/Filtering

$$p(\mathbf{x_n}|\mathbf{y_n}) \propto p(\mathbf{y_n}|\mathbf{x_n}) \times p(\mathbf{x_n})$$

with *product* likelihood functions

$$p(\mathbf{y_n}|\mathbf{x_n}) \propto \prod_{0 \leq k \leq n} p(y_k|x_k)$$

Conditioning formulae (not°: $\mathbf{z_n} = (z_0, \ldots, z_n) = z_{0:n}$)

- Bayes rule/Filtering

$$p(\mathbf{x_n}|\mathbf{y_n}) \propto p(\mathbf{y_n}|\mathbf{x_n}) \times p(\mathbf{x_n})$$

with *product* likelihood functions

$$p(\mathbf{y_n}|\mathbf{x_n}) \propto \prod_{0 \le k \le n} p(y_k|x_k)$$

- Markov path restricted to a tube $\mathbf{A_n} = (A_0 \times \ldots \times A_n)$

$$p(\mathbf{x_n}|\mathbf{x_n} \in \mathbf{A_n}) \propto p(\mathbf{x_n} \in \mathbf{A_n}|\mathbf{x_n}) \times p(\mathbf{x_n})$$

with *product* indicator functions

$$p(\mathbf{x_n} \in \mathbf{A_n}|\mathbf{x_n}) \propto \prod_{0 \le k \le n} 1_A(x_k)$$

- **(Sequencial) Importance Sampling $\mathbf{x_n} \rightsquigarrow \mathbf{x_{n+1}} = (\mathbf{x_n}, x_{n+1})$**

$$\pi_{n+1}(\mathbf{x_{n+1}}) \overset{hyp.}{=} \pi_n(\mathbf{x_n})\ q(\mathbf{x_{n+1}}|\mathbf{x_n}) \tag{1}$$

- **(Sequencial) Importance Sampling $\mathbf{x_n} \rightsquigarrow \mathbf{x_{n+1}} = (\mathbf{x_n}, x_{n+1})$**

$$\pi_{n+1}(\mathbf{x_{n+1}}) \overset{hyp.}{=} \pi_n(\mathbf{x_n})\ q(\mathbf{x_{n+1}}|\mathbf{x_n}) \tag{1}$$

$$\overset{IS\,/\,Twisted}{=} \underbrace{\left(\frac{\pi_n(\mathbf{x_n})\ \mathbf{q}(\mathbf{x_{n+1}}|\mathbf{x_n})}{\pi_n(\mathbf{x_n})\ \mathbf{p}(\mathbf{x_{n+1}}|\mathbf{x_n})}\right)} \mathbf{p}(\mathbf{x_{n+1}}|\mathbf{x_n})\ \pi_n(\mathbf{x_n})$$

$$\overset{"hyp"}{\propto} \mathbf{G_{n+1}}(\mathbf{x_{n+1}}) \quad \mathbf{p}(\mathbf{x_{n+1}}|\mathbf{x_n})\ \pi_n(\mathbf{x_n})$$

$$\ldots$$

- **(Sequencial) Importance Sampling $\mathbf{x_n} \rightsquigarrow \mathbf{x_{n+1}} = (\mathbf{x_n}, x_{n+1})$**

$$\pi_{n+1}(\mathbf{x_{n+1}}) \overset{hyp.}{=} \pi_n(\mathbf{x_n})\ q(\mathbf{x_{n+1}}|\mathbf{x_n}) \tag{1}$$

$$\overset{IS\,/\,Twisted}{=} \underbrace{\left(\frac{\pi_n(\mathbf{x_n})\ \mathbf{q}(\mathbf{x_{n+1}}|\mathbf{x_n})}{\pi_n(\mathbf{x_n})\ \mathbf{p}(\mathbf{x_{n+1}}|\mathbf{x_n})}\right)} \ \mathbf{p}(\mathbf{x_{n+1}}|\mathbf{x_n})\ \pi_n(\mathbf{x_n})$$

$$\overset{"\,hyp"}{\propto} \mathbf{G_{n+1}}(\mathbf{x_{n+1}}) \qquad \mathbf{p}(\mathbf{x_{n+1}}|\mathbf{x_n})\ \pi_n(\mathbf{x_n})$$

$$\cdots$$

$$\propto \left\{\prod_{0\leq k\leq(n+1)} \mathbf{G_k}(\mathbf{x_k})\right\} \underbrace{\prod_{0\leq k\leq(n+1)} \mathbf{p}(\mathbf{x_k}|\mathbf{x_{k-1}})}_{=p(x_0,\ldots,x_n)}$$

▸ (Sequencial) Importance Sampling $\mathbf{x_n} \rightsquigarrow \mathbf{x_{n+1}} = (\mathbf{x_n}, x_{n+1})$

$$\pi_{n+1}(\mathbf{x_{n+1}}) \overset{hyp.}{=} \pi_n(\mathbf{x_n})\; q(\mathbf{x_{n+1}}|\mathbf{x_n}) \tag{1}$$

$$\overset{IS\,/\,Twisted}{=} \underbrace{\left(\frac{\pi_n(\mathbf{x_n})\; \mathbf{q}(\mathbf{x_{n+1}}|\mathbf{x_n})}{\pi_n(\mathbf{x_n})\; \mathbf{p}(\mathbf{x_{n+1}}|\mathbf{x_n})}\right)} \mathbf{p}(\mathbf{x_{n+1}}|\mathbf{x_n})\; \pi_n(\mathbf{x_n})$$

$$\overset{"hyp"}{\propto} \mathbf{G_{n+1}}(\mathbf{x_{n+1}}) \quad \mathbf{p}(\mathbf{x_{n+1}}|\mathbf{x_n})\; \pi_n(\mathbf{x_n})$$

$$\ldots$$

$$\propto \left\{\prod_{0\leq k\leq (n+1)} \mathbf{G_k}(\mathbf{x_k})\right\} \underbrace{\prod_{0\leq k\leq (n+1)} \mathbf{p}(\mathbf{x_k}|\mathbf{x_{k-1}})}_{=p(x_0,\ldots,x_n)}$$

. . . *or more directly from the initial product formula (1)*

$$q(\mathbf{x_{n+1}}|\mathbf{x_n}) = \mathbf{p}(\mathbf{x_{n+1}}|\mathbf{x_n}) \underbrace{\frac{\mathbf{q}(\mathbf{x_{n+1}}|\mathbf{x_n})}{\mathbf{p}(\mathbf{x_{n+1}}|\mathbf{x_n})}}_{=\mathbf{G_{n+1}}(\mathbf{x_{n+1}})}$$

All ex. are strictly $\subset$ Feynman-Kac models ($\exists\neg$**!**)

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) = \frac{1}{\mathcal{Z}_n}\left\{\prod_{0\leq p<n} G_p(x_p)\right\}\ \mathbb{P}_n(d(x_0,\ldots,x_n))$$

with

$$\mathbb{P}_n := \mathrm{Law}\,(X_0,\ldots,X_n) \quad \text{with} \quad X_n \text{ Markov in } E_n$$

All ex. are strictly $\subset$ Feynman-Kac models ($\exists\neg$!)

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) = \frac{1}{\mathcal{Z}_n}\left\{\prod_{0\leq p<n} G_p(x_p)\right\}\ \mathbb{P}_n(d(x_0,\ldots,x_n))$$

with

$$\mathbb{P}_n := \mathrm{Law}\,(X_0,\ldots,X_n) \quad \text{with} \quad X_n \text{ Markov in } E_n$$

Notation : η_n n-th marginal distribution

All ex. are strictly $\subset$ Feynman-Kac models ($\exists\neg$!)

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) = \frac{1}{\mathcal{Z}_n}\left\{\prod_{0\leq p<n} G_p(x_p)\right\}\ \mathbb{P}_n(d(x_0,\ldots,x_n))$$

with

$$\mathbb{P}_n := \mathrm{Law}\,(X_0,\ldots,X_n) \quad \text{with} \quad X_n \text{ Markov in } E_n$$

Notation : η_n n-th marginal distribution

$$\Downarrow$$

Key obs.: $\mathbf{X_n} = (X_0,\ldots,X_n)$ and $\mathbf{G_k}(\mathbf{X_k}) = G_k(X_k)$

$$\boldsymbol{\eta_n}(\mathbf{f}) = \mathbb{E}\left(\mathbf{f}(\mathbf{X_n}) \prod_{0\leq k<n} \mathbf{G_k}(\mathbf{X_k})\right) = \mathbb{Q}_n(\mathbf{f})$$

$\Leftrightarrow$ **Algo.**	$\mathbf{X_{n-1} \rightsquigarrow X_n}$	$\mathbf{G_n}$
Sequential Monte Carlo	Sampling	Resampling
Particle Filters	Prediction	Updating
Genetic Algorithms	Mutation	Selection
Evolutionary Population	Exploration	Branching-selection
Diffusion Monte Carlo	Free evolutions	Absorption
Quantum Monte Carlo	Walkers motions	Reconfiguration
Sampling Algorithms	Transition proposals	Accept-reject-recycle

$\Leftrightarrow$ **Algo.**	$X_{n-1} \rightsquigarrow X_n$	G_n
Sequential Monte Carlo	Sampling	Resampling
Particle Filters	Prediction	Updating
Genetic Algorithms	Mutation	Selection
Evolutionary Population	Exploration	Branching-selection
Diffusion Monte Carlo	Free evolutions	Absorption
Quantum Monte Carlo	Walkers motions	Reconfiguration
Sampling Algorithms	Transition proposals	Accept-reject-recycle

More botanical names:
bootstrapping, spawning, cloning, pruning, replenish, multi-level splitting, enrichment, go with the winner, quantum teleportation,. . .

$\Leftrightarrow$ **Algo.**	$X_{n-1} \rightsquigarrow X_n$	G_n
Sequential Monte Carlo	Sampling	Resampling
Particle Filters	Prediction	Updating
Genetic Algorithms	Mutation	Selection
Evolutionary Population	Exploration	Branching-selection
Diffusion Monte Carlo	Free evolutions	Absorption
Quantum Monte Carlo	Walkers motions	Reconfiguration
Sampling Algorithms	Transition proposals	Accept-reject-recycle

More botanical names:
bootstrapping, spawning, cloning, pruning, replenish, multi-level splitting, enrichment, go with the winner, quantum teleportation,...

1947/1948 *(Fermi/Kahn-Harris)* $\leq$ Meta-Heuristic style stochastic algo. $\leq$ **1996**

$\Leftrightarrow$ **Algo.**	$\mathbf{X_{n-1} \rightsquigarrow X_n}$	$\mathbf{G_n}$
Sequential Monte Carlo	Sampling	Resampling
Particle Filters	Prediction	Updating
Genetic Algorithms	Mutation	Selection
Evolutionary Population	Exploration	Branching-selection
Diffusion Monte Carlo	Free evolutions	Absorption
Quantum Monte Carlo	Walkers motions	Reconfiguration
Sampling Algorithms	Transition proposals	Accept-reject-recycle

More botanical names:
bootstrapping, spawning, cloning, pruning, replenish, multi-level splitting, enrichment, go with the winner, quantum teleportation,...

1947/1948 *(Fermi/Kahn-Harris)* $\leq$ Meta-Heuristic style stochastic algo. $\leq$ **1996**

Convergence/Performance analysis : CLT, LDP, $\mathbb{L}_p$-estimates, Empirical processes, Moderate deviations, propagations of chaos, unif cv w.r.t. time, **new stochastic models** $\supset$ **PMCMC (2009-2010)**....

$\forall$ Interpolating path measures $\pi_0 \rightsquigarrow \ldots \rightsquigarrow \pi_T$

$$\pi_n(d\theta) \propto \left\{ \prod_{1 \leq k \leq n} h_k(\theta) \right\} \lambda(d\theta)$$

$\forall$ Interpolating path measures $\pi_0 \rightsquigarrow \ldots \rightsquigarrow \pi_T$

$$\pi_n(d\theta) \propto \left\{ \prod_{1 \leq k \leq n} h_k(\theta) \right\} \lambda(d\theta)$$

$\prod =$ *Tuning parameter:* w.r.t. dimensions

$\forall$ **Interpolating path measures** $\pi_0 \rightsquigarrow \ldots \rightsquigarrow \pi_T$

$$\pi_n(d\theta) \propto \left\{ \prod_{1 \leq k \leq n} h_k(\theta) \right\} \lambda(d\theta)$$

$\prod =$ *Tuning parameter:* w.r.t. dimensions parameter $\epsilon \downarrow$

$\forall$ **Interpolating path measures** $\pi_0 \rightsquigarrow \ldots \rightsquigarrow \pi_T$

$$\pi_n(d\theta) \propto \left\{ \prod_{1 \leq k \leq n} h_k(\theta) \right\} \lambda(d\theta)$$

$\prod =$ *Tuning parameter:* w.r.t. dimensions parameter $\epsilon \downarrow$ or Nb obs., ...

$\forall$ **Interpolating path measures** $\pi_0 \rightsquigarrow \ldots \rightsquigarrow \pi_T$

$$\pi_n(d\theta) \propto \left\{ \prod_{1 \leq k \leq n} h_k(\theta) \right\} \lambda(d\theta)$$

$\prod =$ *Tuning parameter:* w.r.t. dimensions parameter $\epsilon \downarrow$ or Nb obs., ...

As the filtering equation:

$$\pi_{n-1} \xrightarrow{\text{Correction/Updating}} d\pi_n \propto h_n \; d\pi_{n-1} \xrightarrow{\pi_n\text{-MCMC/Prediction } M_n} \pi_n$$

$\forall$ **Interpolating path measures** $\pi_0 \rightsquigarrow \ldots \rightsquigarrow \pi_T$

$$\pi_n(d\theta) \propto \left\{ \prod_{1 \leq k \leq n} h_k(\theta) \right\} \lambda(d\theta)$$

$\prod =$ *Tuning parameter:* w.r.t. dimensions parameter $\epsilon \downarrow$ or Nb obs., ...

As the filtering equation:

$$\pi_{n-1} \xrightarrow{\text{Correction/Updating}} d\pi_n \propto h_n \; d\pi_{n-1} \xrightarrow{\pi_n\text{-MCMC/Prediction } M_n} \pi_n$$

$\mathcal{X}_{n-1} = (\mathcal{X}^i_{n-1})_{1 \leq i \leq N}$

$\forall$ **Interpolating path measures** $\pi_0 \rightsquigarrow \ldots \rightsquigarrow \pi_T$

$$\pi_n(d\theta) \propto \left\{ \prod_{1\leq k\leq n} h_k(\theta) \right\} \lambda(d\theta)$$

$\prod =$ *Tuning parameter:* w.r.t. dimensions parameter $\epsilon \downarrow$ or Nb obs., ...

As the filtering equation:

$$\pi_{n-1} \xrightarrow{\text{Correction/Updating}} d\pi_n \propto h_n \, d\pi_{n-1} \xrightarrow{\pi_n\text{-MCMC/Prediction } M_n} \pi_n$$

$$\mathcal{X}_{n-1} = (\mathcal{X}^i_{n-1})_{1\leq i\leq N} \xrightarrow{h_n\text{-Selection}} \widehat{\mathcal{X}}_{n-1} = (\widehat{\mathcal{X}}^i_{n-1})_{1\leq i\leq N}$$

$\forall$ **Interpolating path measures** $\pi_0 \rightsquigarrow \ldots \rightsquigarrow \pi_T$

$$\pi_n(d\theta) \propto \left\{ \prod_{1 \leq k \leq n} h_k(\theta) \right\} \lambda(d\theta)$$

$\prod =$ *Tuning parameter:* w.r.t. dimensions parameter $\epsilon \downarrow$ or Nb obs., ...

As the filtering equation:

$$\begin{array}{ccccc} \pi_{n-1} & \xrightarrow{\text{Correction/Updating}} & d\pi_n \propto h_n \, d\pi_{n-1} & \xrightarrow{\pi_n\text{-MCMC/Prediction } M_n} & \pi_n \\ \chi_{n-1} = (\chi^i_{n-1})_{1 \leq i \leq N} & \xrightarrow{h_n\text{-Selection}} & \widehat{\chi}_{n-1} = (\widehat{\chi}^i_{n-1})_{1 \leq i \leq N} & \xrightarrow{M_n\text{-Mutation}} & \chi_n \end{array}$$

$\forall$ Interpolating path measures $\pi_0 \rightsquigarrow \ldots \rightsquigarrow \pi_T$

$$\pi_n(d\theta) \propto \left\{ \prod_{1\leq k\leq n} h_k(\theta) \right\} \lambda(d\theta)$$

$\prod =$ *Tuning parameter:* w.r.t. dimensions parameter $\epsilon \downarrow$ or Nb obs., ...

As the filtering equation:

$$\pi_{n-1} \xrightarrow{\text{Correction/Updating}} d\pi_n \propto h_n \; d\pi_{n-1} \xrightarrow{\pi_n\text{-MCMC/Prediction } M_n} \pi_n$$

$$\mathcal{X}_{n-1} = (\mathcal{X}^i_{n-1})_{1\leq i\leq N} \xrightarrow{h_n\text{-Selection}} \widehat{\mathcal{X}}_{n-1} = (\widehat{\mathcal{X}}^i_{n-1})_{1\leq i\leq N} \xrightarrow{M_n\text{-Mutation}} \mathcal{X}_n$$

$$\Downarrow$$

$\subset$ n-th marginals $\pi_n = \eta_n$ of a Feynman-Kac model $\mathbb{Q}_n$

$\forall$ Interpolating path measures $\pi_0 \rightsquigarrow \ldots \rightsquigarrow \pi_T$

$$\pi_n(d\theta) \propto \left\{ \prod_{1\leq k\leq n} h_k(\theta) \right\} \lambda(d\theta)$$

$\prod =$ *Tuning parameter:* w.r.t. dimensions parameter $\epsilon \downarrow$ or Nb obs., ...

As the filtering equation:

$$\pi_{n-1} \xrightarrow{\text{Correction/Updating}} d\pi_n \propto h_n \; d\pi_{n-1} \xrightarrow{\pi_n\text{-MCMC/Prediction } M_n} \pi_n$$

$$\mathcal{X}_{n-1} = (\mathcal{X}^i_{n-1})_{1\leq i\leq N} \xrightarrow{h_n\text{-Selection}} \widehat{\mathcal{X}}_{n-1} = (\widehat{\mathcal{X}}^i_{n-1})_{1\leq i\leq N} \xrightarrow{M_n\text{-Mutation}} \mathcal{X}_n$$

$$\Downarrow$$

$\subset$ **n-th marginals $\pi_n = \eta_n$ of a Feynman-Kac model $\mathbb{Q}_n$**

- *Physics $\rightsquigarrow$ Crook/Jarzinsky formula;*
- *Rare event $\oplus$ Black-box $\rightsquigarrow$ subset sampling/multi-level splitting;*
- *Operation Research $\rightsquigarrow$ Interacting simulated annealing ...*

$$\eta_n \simeq \eta_n^N := \frac{1}{N} \sum_{1 \le i \le N} \delta_{\left(\mathcal{X}^i_{0,n}, \mathcal{X}^i_{1,n}, \ldots, \mathcal{X}^i_{n,n}\right) = \text{i-th ancestral line}}$$

$\rightsquigarrow \mathbb{X}_n^a :=$ uniform ancestral line

$$\eta_n \simeq \eta_n^N := \frac{1}{N}\sum_{1\le i\le N} \delta_{\left(X^i_{0,n}, X^i_{1,n}, \ldots, X^i_{n,n}\right) = \text{i-th ancestral line}}$$

$\rightsquigarrow \mathbb{X}^a_n :=$ uniform ancestral line

1) Product formulae/Particle approximation

$$\mathcal{Z}_n = \prod_{0\le k<n} \eta_k(G_k) \overset{\text{unbias}}{\simeq} \prod_{0\le k<n} \eta_k^N(G_k) := \mathcal{Z}_n^N = \prod_{0\le k<n} \mathcal{G}_k(\mathcal{X}_k)$$

with the empirical potential function

$$\mathcal{G}_k(\mathcal{X}_k) = \eta_k^N(G_k) = \frac{1}{N}\sum_{1\le i\le N} G_k(\mathcal{X}^i_k)$$

$\rightsquigarrow$ Many-body FK on path space

FK model with $(\mathbf{X_n}, \mathbf{G_n}) \rightsquigarrow (\mathcal{X}_n, \mathcal{G}_n)$ = **Many-body** FK

$\rightsquigarrow$ Many-body FK on path space

FK model with $(\mathbf{X_n}, \mathbf{G_n}) \rightsquigarrow (\mathcal{X}_\mathbf{n}, \mathcal{G}_\mathbf{n})$ = **Many-body** FK

For any empirical function $\mathbf{F}(\mathcal{X}_\mathbf{n}) = \frac{1}{N} \sum_{1 \leq i \leq N} \mathbf{f}(\mathcal{X}_\mathbf{n}^\mathbf{i})$

Theo [**MPRF-1996**]

$$\mathbb{E}\left(\mathbf{f}(\mathbf{X_n}) \prod_{0 \leq k < n} \mathbf{G_k}(\mathbf{X_k})\right) = \mathbb{E}\left(\mathbf{F}(\mathcal{X}_\mathbf{n}) \prod_{0 \leq k < n} \mathcal{G}_\mathbf{k}(\mathcal{X}_\mathbf{k})\right)$$

$\rightsquigarrow$ Many-body FK on path space

FK model with $(\mathbf{X_n}, \mathbf{G_n}) \rightsquigarrow (\mathcal{X}_n, \mathcal{G}_n)$ = **Many-body** FK

For any empirical function $\mathbf{F}(\mathcal{X}_n) = \frac{1}{N}\sum_{1\leq i\leq N} \mathbf{f}(\mathcal{X}_n^i)$

Theo [**MPRF-1996**]

$$\mathbb{E}\left(\mathbf{f}(\mathbf{X_n}) \prod_{0\leq k<n} \mathbf{G_k}(\mathbf{X_k})\right) = \mathbb{E}\left(\mathbf{F}(\mathcal{X}_n) \prod_{0\leq k<n} \mathcal{G}_k(\mathcal{X}_k)\right)$$

$\rightsquigarrow$ **SC-13 (Island models/Parallel particle models)**

$\rightsquigarrow$ **FTML-02/Arxiv-2011 (independent Metropolis-Hastings/SMC^2)**

The 2nd key

Hypothesis

$$M_{k+1}(x_k, dx_{k+1}) = H_{k+1}(x_k, x_{k+1})\ \lambda(dx_{k+1}) \overset{ex.}{\propto}\ e^{-\frac{1}{2}(x_{k+1}-a(x_k))^2}\ dx_{k+1}$$

The 2nd key

Hypothesis

$$M_{k+1}(x_k, dx_{k+1}) = H_{k+1}(x_k, x_{k+1})\ \lambda(dx_{k+1}) \overset{ex.}{\propto}\ e^{-\frac{1}{2}(x_{k+1}-a(x_k))^2}\ dx_{k+1}$$

$$\Downarrow$$

2) Backward formulae/Backward Particle chain

$$\mathbb{Q}_n(d(x_0, \ldots, x_n)) = \eta_n(dx_n)\ \mathbb{K}_{n,\eta_{n-1}}(x_n, dx_{n-1})\ \ldots\ \mathbb{K}_{1,\eta_0}(x_1, dx_0)$$

with

$$\mathbb{K}_{k+1,\eta_k}(x_{k+1}, dx_k)$$

$$= \frac{\eta_k(dx_k)\ G_k(x_k)H(x_k, x_{k+1})}{\int\ \eta_k(dx'_k)\ G_k(x'_k)H(x'_k, x_{k+1})}$$

The 2nd key

Hypothesis

$$M_{k+1}(x_k, dx_{k+1}) = H_{k+1}(x_k, x_{k+1})\ \lambda(dx_{k+1}) \overset{ex.}{\propto}\ e^{-\frac{1}{2}(x_{k+1}-a(x_k))^2}\ dx_{k+1}$$

$$\Downarrow$$

2) Backward formulae/Backward Particle chain

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) = \eta_n(dx_n)\ \mathbb{K}_{n,\eta_{n-1}}(x_n, dx_{n-1})\ \ldots\ \mathbb{K}_{1,\eta_0}(x_1, dx_0)$$

with

$$\mathbb{K}_{k+1,\eta_k}(x_{k+1}, dx_k)$$

$$= \frac{\eta_k(dx_k)\ G_k(x_k)H(x_k, x_{k+1})}{\int\ \eta_k(dx'_k)\ G_k(x'_k)H(x'_k, x_{k+1})} \simeq \mathbb{K}_{k+1,\eta_k^{\mathbf{N}}}(x_{k+1}, dx_k)$$

$\rightsquigarrow$ backward random path $\mathbb{X}_{\mathbf{n}}^{\mathbf{b}}$

Unbias Many-body FK (cf. MPRF-1996/M2AN-2010)

$$\begin{aligned} \mathbb{X}_n^a &= \text{Uniform ancestral line} \\ \mathbb{X}_{\mathbf{n}}^{\mathbf{b}} &= \text{Backward particle path} \end{aligned}$$

Unbias Many-body FK (cf. MPRF-1996/M2AN-2010)

$$\begin{aligned} \mathbb{X}_n^a &= \text{Uniform ancestral line} \\ \mathbb{X}_{\mathbf{n}}^{\mathbf{b}} &= \text{Backward particle path} \end{aligned}$$

$$\mathbb{E}\left(\mathbf{f}(\mathbf{X_n}) \prod_{0 \leq k < n} \mathbf{G_k}(\mathbf{X_k})\right) = \mathbb{E}\left(\mathbf{f}(\mathbb{X}_n^a \text{ or } \mathbb{X}_{\mathbf{n}}^{\mathbf{b}}) \prod_{0 \leq k < n} \mathcal{G}_{\mathbf{k}}(\mathcal{X}_{\mathbf{k}})\right)$$

Unbias Many-body FK (cf. MPRF-1996/M2AN-2010)

$$\begin{aligned} \mathbb{X}_n^a &= \text{Uniform ancestral line} \\ \mathbb{X}_{\mathbf{n}}^{\mathbf{b}} &= \text{Backward particle path} \end{aligned}$$

$$\mathbb{E}\left(\mathbf{f}(\mathbf{X_n}) \prod_{0\leq k<n} \mathbf{G_k}(\mathbf{X_k})\right) = \mathbb{E}\left(\mathbf{f}(\mathbb{X}_n^a \text{ or } \mathbb{X}_{\mathbf{n}}^{\mathbf{b}}) \prod_{0\leq k<n} \mathcal{G}_{\mathbf{k}}(\mathcal{X}_{\mathbf{k}})\right)$$

$$\oplus$$

Theo 1 [Arxiv-2014]

Law (**Ancestral line** | **Complete tree**)

=

Law of the (**backward particle model**)

Theo 2 [Duality formula for Many-body FK (Arxiv-2014)]

$$\mathrm{Law}\left(\mathcal{X}_{\mathbf{n}}^{\star} \mid \mathbf{X}_{\mathbf{n}} = \mathbf{x}\right) := \mathrm{Law}_{\mathbf{many-body}}\left(\mathcal{X}_{\mathbf{n}} \mid \mathbb{X}_{\mathbf{n}}^{\mathbf{b}} = \mathbf{x}\right)$$

Theo 2 [Duality formula for Many-body FK (Arxiv-2014)]

$$\mathrm{Law}\left(\mathcal{X}_{\mathbf{n}}^{\star} \mid \mathbf{X_n} = \mathbf{x}\right) := \mathrm{Law}_{\mathbf{many-body}}\left(\mathcal{X}_{\mathbf{n}} \mid \mathbb{X}_{\mathbf{n}}^{\mathbf{b}} = \mathbf{x}\right)$$

$$\overset{\textit{sampling} \Leftrightarrow \textit{def.}}{=} \mathrm{Law}\ N \text{ particles with frozen path } \mathbf{X_n} = \mathbf{x}$$

Theo 2 [Duality formula for Many-body FK (Arxiv-2014)]

$$\text{Law}\left(\mathcal{X}_n^\star \mid \mathbf{X_n} = \mathbf{x}\right) := \text{Law}_{\mathbf{many-body}}\left(\mathcal{X}_n \mid \mathbb{X}_n^b = \mathbf{x}\right)$$

$$\overset{\textit{sampling} \Leftrightarrow \textit{def.}}{=} \text{Law } N \textbf{ particles with frozen path } \mathbf{X_n} = \mathbf{x}$$

$$\Downarrow$$

$$\mathbb{E}\left(\mathbf{F}(\mathbb{X}_n^b, \mathcal{X}_n) \prod_{0 \le k < n} \mathcal{G}_k(\mathcal{X}_k)\right) = \mathbb{E}\left(\mathbf{F}(\mathbf{X_n}, \mathcal{X}_n^\star) \prod_{0 \le k < n} \mathbf{G_k}(\mathbf{X_k})\right)$$

Proof ingredient $\chi = (\chi^i)$ i.i.d. $\sim \eta = \text{Law}(X)$

$$\mathcal{X}^\star \mid X \;=\; \text{i.i.d.} \;\sim\; \frac{1}{N}\,\delta_X + \left(1 - \frac{1}{N}\right)\,\frac{1}{N-1}\sum_{2\leq i\leq N}\delta_{\chi^i}$$

and

$$\mathbb{X} \mid \chi \;\sim\; \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\chi^i}$$

$$\Downarrow$$

$$\mathbb{E}\left(F(\mathbb{X},\chi)\right) = \mathbb{E}\left(F(X,\mathcal{X}^\star)\right)$$

Direct consequence

Duality $\Rightarrow$ Andrieu-Doucet-Holenstein Particle Gibbs model

$$\Updownarrow$$

2 reversible transitions $\mathbb{K}_n(\mathbf{x}, d\mathbf{x}')$ w.r.t. $\mathbb{Q}_n$:

$$\mathbf{x} \rightsquigarrow \mathcal{X}_{\mathbf{n}}^{\star} \text{ with frozen path } \mathbf{x} \rightsquigarrow \begin{cases} \text{Random backward path } \mathbf{x}' \\ \text{or} \\ \text{Random ancestral line } \mathbf{x}' \end{cases}$$

$\oplus$

Direct (but too crude) minorization condition $\mathbb{K}_n(x, .) \geq \epsilon_n \; \eta_{\mathbf{n}}$

Using the key observation

$$\mathbb{X}_{\mathbf{n}}^{\mathbf{b}} \perp \overline{\mathbb{X}}_{\mathbf{n}}^{\mathbf{b}} \quad \text{Independent copies}$$

$$\Downarrow$$

$$\mathbb{E}\left(\mathbf{f_1}(\mathbb{X}_{\mathbf{n}}^{\mathbf{b}})\ \mathbf{f_2}(\overline{\mathbb{X}}_{\mathbf{n}}^{\mathbf{b}}) \prod_{0\leq p<n} \mathcal{G}_{\mathbf{p}}(\mathcal{X}_{\mathbf{p}})\right) \stackrel{\text{duality}}{=} \mathbb{E}\left(\mathbf{f_1}(\mathbf{X_n})\ \mathbb{K}_{\mathbf{n}}(\mathbf{f_2})(\mathbf{X_n}) \prod_{0\leq p<n} \mathbf{G_p}(\mathbf{X_p})\right)$$

$\oplus$ Symmetry argument $\mathbf{f_1} \leftrightarrow \mathbf{f_2}$

Bayes/Conditioning/Feynman-Kac measures

Origins/Equivalent particle algorithms

Ex.: MCMC with product target measures

Particle measures $\oplus$ 2 key formulae

Conditioning and duality formulae

Taylor expansions for PMCMC transitions

Theo 3 [Taylor exp. - ancestral-lines PMCMC (Arxiv-2014)]

$$\mathbb{K}_n(x,\,.\,) = \boldsymbol{\eta}_\mathbf{n} + \sum_{1\leq k\leq l} \frac{1}{N^k}\; d^{(k)}\mathbb{K}_n(x,\,.\,) + \mathrm{O}\left(\frac{1}{N^{l+1}}\right)$$

at any order l, with explicit operators $d^{(k)}\mathbb{K}_n$ in terms of coalescent trees.

$\oplus$ **Taylor exp. for the law of q particles in the scheme with frozen x**

Some direct corollaries

- Bias and variance of PMCMC empirical centered function $\mathbf{f_n}$:

$$\mathrm{Var} = \frac{1}{N}\ \eta_{\mathbf{n}}(\mathbf{f}_{\mathbf{n}}^2) + \mathrm{O}\left(\frac{1}{N^2}\right)$$

Some direct corollaries

- Bias and variance of PMCMC empirical centered function $\mathbf{f_n}$:

$$\mathrm{Var} = \frac{1}{N}\ \boldsymbol{\eta}_{\mathbf{n}}(\mathbf{f}_{\mathbf{n}}^{2}) + \mathrm{O}\left(\frac{1}{N^2}\right)$$

- Dobrushin contract. coef.

$$\beta(\mathbb{K}_n) = \frac{1}{N}\ \overbrace{\beta(d^{(1)}\mathbb{K}_n)}^{=\mathrm{O}(\mathrm{n})} + \mathrm{O}\left(\frac{1}{N^2}\right)$$

Some direct corollaries

- Bias and variance of PMCMC empirical centered function $\mathbf{f_n}$:

$$\mathrm{Var} = \frac{1}{N}\ \boldsymbol{\eta}_{\mathbf{n}}(\mathbf{f}_{\mathbf{n}}^{2}) + \mathrm{O}\left(\frac{1}{N^2}\right)$$

- Dobrushin contract. coef.

$$\beta(\mathbb{K}_n) = \frac{1}{N}\ \overbrace{\beta(d^{(1)}\mathbb{K}_n)}^{=\mathrm{O}(\mathrm{n})} + \mathrm{O}\left(\frac{1}{N^2}\right)$$

- $m-$Iterates expansions

$$\mathbb{K}_n^{m}(x,\,.) = \boldsymbol{\eta}_{\mathbf{n}} + \frac{1}{N^{m}}\left[\sum_{0\leq k\leq l}\frac{1}{N^k}\ d^{(m+k)}\mathbb{K}_n^{m}(x,\,.) + \mathrm{O}\left(\frac{1}{N^{l+1}}\right)\right]$$