INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Interacting path systems for credit portfolios risk analysis

Pierre Del Moral — Frédéric Patras

N° 7196

February 2010

Thème NUM

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--7196--FR+ENG

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE
INRIA
centre de recherche BORDEAUX - SUD OUEST

Interacting path systems for credit portfolios risk analysis

Pierre Del Moral*, Frédéric Patras†

Thème NUM — Systèmes numériques
Équipes-Projets ALEA

Rapport de recherche n° 7196 — February 2010 — 6 pages

Abstract: This Note introduces an algorithm (referred to as interacting path systems algorithm, IPaS) based on the first Author multilevel splitting technique [5] and suited to the analysis of multiple defaults in credit portfolios. A full development of this Note incorporating technical details and a survey of the use of Interacting Particle Systems in the field of credit risk, *Interacting path systems for credit risk,* is submitted for publication in *Recent Advancements in the Theory and Practice of Credit Derivatives*, Eds T. Bielecki, D. Brigo, F. Patras, Bloomberg Press (2011). The reader is referred to this article for further details.

Key-words: Credit portfolios risk analysis, rare event simulation, interacting particle systems, stochastic particle methods, genetic algorithms.

* Centre INRIA Bordeaux et Sud-Ouest & Institut de Mathématiques de Bordeaux , Université de Bordeaux I, 351 cours de la Libération 33405 Talence cedex, France, Pierre.Del-Moral@inria.fr
† CNRS UMR 6621, Université de Nice, Laboratoire de Mathématiques J.-A. Dieudonné, Parc Valrose, 06108 Nice Cedex 2, France

Centre de recherche INRIA Bordeaux – Sud Ouest
Domaine Universitaire - 351, cours de la Libération 33405 Talence Cedex
Téléphone : +33 5 40 00 69 00

Systèmes de trajectoires en interaction pour l'analyse de risque de portefeuilles de crédit

Résumé : Cette Note présente un algorithme (que nous nommerons systèmes de trajectoires en interaction, en abrégé IPaS) fondé sur des techniques de branchement multi-niveaux développées par le premier auteur [5], et qui s'appliquent de façon naturelle à l'analyse de défauts multiples de portefeuilles de crédit. La version complète de cette Note contenant les détails techniques ainsi qu'un survey sur l'utilisation des algorithmes fondés sur des systèmes de particules en interaction dans le domaine du risque de crédit est soumise pour publication dans l'ouvrage *Recent Advancements in the Theory and Practice of Credit Derivatives*, Eds T. Bielecki, D. Brigo, F. Patras, Bloomberg Press (2011). Nous renvoyons le lecteur à cet article pour une étude plus approfondie.

Mots-clés : Portefeuilles de crédits, analyse de risque, simulation d'événements rares, systèmes de particules en interaction, méthodes particulaires stochastiques, algorithmes génétiques.

1 Notation

We recall briefly the formalism underlying Interacting Particle Systems (IPS) and the corresponding notations. The reader is referred to [5] for details. For a given sequence of state spaces E_n, we start with a sequence of Markov transitions $M_{n+1}(x, dy)$ from E_n to E_{n+1}. The integer N will stand for the number of particles (the sample size) of the IPS simulations, whereas n should be understood as the index of the time step.

The key idea of the application of IPS methods to credit risk is to add to the risk neutral dynamics of the assets in a portfolio (described by the Markov transitions M_n) selection steps that increase the number of risky trajectories. The selection/mutation pattern of the corresponding scheme is denoted by:

$$\xi_n = (\xi_n^i)_{1\leq i\leq N} \xrightarrow{\text{selection}} \widehat{\xi}_n = (\widehat{\xi}_n^i)_{1\leq i\leq N} \xrightarrow{\text{mutation}} \xi_{n+1} = (\xi_{n+1}^i)_{1\leq i\leq N}$$

To model the selection $\xi_n \rightsquigarrow \widehat{\xi}_n$, a $[0,1]$-valued a sequence $G_n,\ n = 1, ..., T$ of potential functions on the E_n is chosen. For each $i = 1, \ldots, N$, we set then:

$$\widehat{\xi}_n^i = \begin{cases} \xi_n^i & \text{with probability} \quad G_n(\xi_n^i) \\ \tilde{\xi}_n^i & \text{with probability} \quad 1 - G_n(\xi_n^i) \end{cases}$$

where $\tilde{\xi}_n^i$ stands for a random variable with law $\sum_{j=1}^N \frac{G_n(\xi_n^j)}{\sum_{k=1}^N G_n(\xi_n^k)}\ \delta_{\xi_n^j}$. During the mutation phase, the particles explore the state space independently (the interactions between the various particles being created by the selection steps) according to the Markov transitions $M_{n+1}(x, dy)$.

Let us assume from now on that the initial values $\xi_0 = (\xi_0^i)_{1\leq i\leq N} \in E_0^N$ form a sequence of identically distributed and independent (i.i.d.) random variables with common law η_0. One can then prove that, at each time-step n, the random variables $(\xi_n^1, \ldots, \xi_n^N)$ are still approximately independent and follow a common law η_n. A law of large numbers holds:

$$\eta_n^N = \frac{1}{N}\sum_{i=1}^N \delta_{\xi_n^i} \longrightarrow \eta_n \tag{1.1}$$

in the sense that

$$\eta_n^N(f) := \frac{1}{N}\sum_{i=1}^N f(\xi_n^i) \longrightarrow \eta_n(f) := \int f(x)\ \eta_n(dx)$$

for any regular function f, when the sample size $N \uparrow \infty$, and for all time steps $n \in \mathbb{N}^*$. The first rigorous proof of convergence was obtained in [4]. For futher mathematical details, we refer to the series of volumes [5]. The associated unnormalized signed measures γ_n are defined by:

$$\gamma_n(f) = \int f(x)\ \gamma_n(dx) = \mathbb{E}\left(f(X_n)\ \prod_{p=0}^{n-1} G_p(X_p)\right),$$

where f runs over bounded functions on the state space E_n and X_n stands for a Markov chain with initial law η_0, state spaces E_n and Markov kernels M_n. We get unbiaised estimators γ_n^N of γ_n from the formula:

$$\gamma_n^N(f) = \eta_n^N(f) \times \prod_{p=0}^{n-1} \eta_p^N(G_p)$$

with the expected convergence property: $\gamma_n^N \longrightarrow \gamma_n$ when the sample size $N \uparrow \infty$, and for all time steps $n \in \mathbb{N}^*$.

2 IPaS for multiname credit risk

The use of IPS in the field of portfolio credit risk was pioneered recently in seminal articles by Carmona-Fouque-Vestal [3] and Carmona-Crepey [2]. We refer to their articles and to

the long version of this Note for insights on the use of classical Interacting Particle Systems for portfolio credit risk analysis.

We focus here on a new approach to multiname credit risk, that we term "interacting path systems" (IPaS). The method has several advantages when dealing with multiple defaults. Indeed, the method is not deviced specifically for a given loss level (so that, for example, it may be expected to behave correctly simultaneously on all the tranches of a CDO). Besides, due to the choice of the potential functions (that are 0/1- valued), the implementation is particularly simple and the algorithms run faster than usual IPS.

2.1 IPaS

We start with a general description of the algorithm -also applicable to other situations than multiname credit portfolios. Let $X_n, T \geq n \geq 0$ be as usual a Markov chain on a sequence of state spaces F_n and Y_n (resp. $E_n = F_0 \times ... \times F_n$) the corresponding path-space Markov chain (resp. sequence of state spaces). We assume that a sequence of subsets $U_1, ..., U_T$, $U_i \subset E_i$ is fixed and are typically interested in the probability $1 - \mathbb{P}(Y_1 \notin U_1, ..., Y_T \notin U_T)$ that the trajectory does enter at least one of these subsets.

It is not our purpose here to enter the mathematical analysis of the algorithm in full generality, but we should mention that (quite similarly to IPS), technical conditions are required to insure that the IPaS algorithm converges properly.

The key idea of the multilevel splitting approach [5, Chap. 12] is to introduce a series of intermediate events interpolating between the series of the full state spaces $E_1, ..., E_T$ of the path-space and the target rare event series $U_1, ..., U_T$. Let us assume therefore that such a (finite) series is given:

$$\forall i \leq T,\ U_i = U_i^{(k)} \subset U_i^{(k-1)} \subset ... \subset U_i^{(1)} \subset U_i^{(0)} = E_i.$$

The next step consists in the introduction of a new Markov chain (with constant state space $\mathbf{S} := E_0 \coprod ... \coprod E_T \coprod \{c\}$, where $\{c\}$ stands for a "cemetery" space state). Roughly, the cemetery state is associated to the particles that do not reach some of the intermediate rare event sets; these trajectories will be killed at some point of the recursion of the IPaS algorithm, the precise meaning of these rather vague explanations will become clear as soon as the IPaS algorithm is explicited. We write $\mathbf{S}^{(j)}$ for $U_1^{(j)} \coprod ... \coprod U_T^{(j)}$.

Let us define the series of stopping times:

$$\tau_i := (T+1) \wedge \inf\{j, Y_j \in U_j^{(i)}\}.$$

We set: $Y_{T+1} := c$. Then, from the strong Markov properties of discrete Markov chains (see e.g. [5, Sect. 2.2.3]), $Z_0 := Y_0, Z_1 := Y_{\tau_1}, ..., Z_k := Y_{\tau_k}$ is a Markov chain on $\mathbf{S}$.

We introduce now the potential functions:

$$G_i(x) = 1 - 1_{\{c\}}(x)$$

Assuming that the IPS scheme convergence properties hold for the Markov chain Z_i and the potential functions G_i (e.g. if condition $\mathcal{A}$ in [5] is satisfied), we get:

Proposition 2.1. *(Convergence of the IPaS scheme) The IPS algorithm run with the Markov chain $Z_0, ..., Z_k$ and the potential functions $G_1, ..., G_k$ is such that, for an arbitrary $0 < j \leq k$:*

$$\eta_j(1_{\mathbf{S}^{(j)}}) = \mathbb{P}(Z_j \in \mathbf{S}^{(j)} | \forall m \leq j-1,\ Z_m \in \mathbf{S}^{(m)}),$$

$$\gamma_j(1_{\mathbf{S}^{(j)}}) = \mathbb{P}(\forall m \leq j,\ Z_m \in \mathbf{S}^{(m)}).$$

In other terms, the IPaS algorithm computes, at each step $j \leq T$, the conditional and unconditional probabilities that the Markov chain $Z_i,\ i \leq T$ has reached all the intermediate subsets.

2.2 IPaS for credit portfolios

Let us consider now a portfolio of k CDS. We consider a structural model, that is we assume that the default of the i-th CDS is triggered as soon as $V_i(j) \leq v_i(j)$, where $V_i(j),\ j \leq T$ stands for a Markov chain describing the value of the assets of the i-th firm and the $v_i(j)$ are thresholds (the value of which is calibrated on the CDS spread curves, see e.g. [1]).

We wish to compute the probabilities that at least $l \leq k$ entities in the portfolio experienced a credit event before the end of the CDS contract. Notice that we have, with the notations of the previous section, $F_i = \mathbb{R}^k$ and $E_i = (\mathbb{R}^k)^{i+1}$. Since, for each entity, $m = 1, ..., k$, a default has occured between the time step 0 and the time-step $n \leq T$ if and only if $\exists j \leq n,\ V_m(j) \leq v_m(j)$, we introduce accordingly a series of subspaces of the state spaces E_n:

Definition 2.2. *The i-default subset $U_n^{(i)}$ of E_n is defined by:*

$$U_n^{(i)} = \{(\mathbf{x}_0, ..., \mathbf{x}_n) = ((x_1^0, ..., x_k^0), ..., (x_1^n, ..., x_k^n)) \in (\mathbb{R}^k)^{n+1} |$$

$$\exists 0 < a_1 < ... < a_i \leq k, \forall j \leq i, \exists l \leq n,\ x_{a_j}^l \leq v_{a_j}(l)\}.$$

The sequence $U_n := U_n^{(k)} \subset U_n^{(k-1)} \subset ... \subset U_n^{(0)} = E_n$ satisfies the conditions required for the application of the IPaS algorithm. We write as usual Z_i and G_i for the corresponding Markov chain and potential functions. Assume further that the conditions required for the application of the IPaS algorithm are satisfied by the pair (Z_i, G_i), we get:

Proposition 2.3. *When the IPS algorithm is run with the Markov chain $Z_0, ..., Z_k$ and the potential functions $G_1, ..., G_k$ associated to a credit portfolio of k names to compute $\eta_j(1_{\mathbf{S}^{(j)}})$ and $\gamma_j(1_{\mathbf{S}^{(j)}})$ for an arbitrary $j \leq k$, it computes respectively the probability that at least j defaults have occured before termination of the contract, conditionnally to the occurence of $j-1$ defaults, and the unconditional probability that at least j defaults have occured in the portfolio.*

2.3 Example

In this last section, we compare the performances of the IPaS algorithm with a naive Monte Carlo algoritm in a typical credit risk computation, that is for a n-to default with a underlying CDS portfolio of 6 names with flat spreads at 100bp (on a quarterly basis) and a recovery rate at 40%. The maturity is 3 years.

The model we choose for the dynamics of the underlyings is a stylized Black-Cox model. Namely, we assume that the Markov chains $V_i,\ i = 1...6$ are defined by: $V_i(0) = a_0$ (initial firm's value), $V_i(j+1) := V_i(j) + \sigma B_i(j),\ j < T$, where the $B_i(j)$ are standard normal variables with $cor(B_i(j), B_l(k)) = 0$ if $i \neq l$ and $= \rho$ if $i = l, j \neq k$. The thresholds are calibrated as usual by a bootstrop algorithm from flat CDS spreads. We assume from now on that $\sigma = 0.4$ and $\rho = 0.4$

In the table, a pair (i,DP) stands for the probability that (exactly) i entities suffer a credit event, whereas a pair (i,MC/ IP) stands for the empirical standard deviation of the algorithm normalized by the corresponding probability. "MC1;5" means that we compute the result of a naive MC scheme with a sample size 10^5, "IP5;3" means that we run the IPaS algorithm with a sample size $5 \cdot 10^3$, and so on. The empirical variance is computed by running 100 times each algorithm. Values above 300% are reported as non significative (NS).

Default probabilities (n-to default on 6 underlyings)							
	0	1	2	3	4	5	6
DP	$7.6 \cdot 10^{-1}$	$1.95 \cdot 10^{-1}$	$3.7 \cdot 10^{-2}$	$6.1 \cdot 10^{-3}$	$8.2 \cdot 10^{-4}$	$8.2 \cdot 10^{-5}$	$4.2 \cdot 10^{-6}$
MC1;3	1.7%	6.1%	17%	38%	104%	300%	NS
IP1;3	1.6%	5.3%	10%	20%	37%	65%	134%
MC5;3	0.73%	2.6%	7%	20%	47%	170%	NS
IP5;3	0.8%	2.7%	4.6%	8%	16%	30%	60%
MC1;4	0.6%	2%	5%	12%	32%	106%	NS
IP1;4	0.6%	1.9%	3.4%	5.5%	12%	21%	54%

The example shows that IPaS provides a meaningful variance reduction for rare events (roughly at a 99.9% confidence level and above). Whereas it is liquely that IPaS is not the final algorithmic solution to the computation of rare events in portfolio credit risk, we feel that it could certainly become an important element in the MonteCarlo toolbox of the domain. Allowing more general path-dependent potential functions fine-tuned to a particular credit derivative or to a given family of time-dependent credit events could for example most probably improve the efficiency of the algorithm. These questions are open and left for further research.

References

[1] T. Bielecki and M. Rutkowski. *Credit risk: modeling, valuation and hedging.*

[2] R. Carmona and S. Crepey. Importance sampling and interacting particle systems for the estimation of markovian credit portfolio loss distribution. 2009. Preprint.

[3] R. Carmona, J.-P. Fouque, and D. Vestal. Interacting particle systems for the computation of rare credit portfolio losses. *Finance and Stochastics*, 13(4):613–633, 2009.

[4] P. Del Moral. Non linear filtering: Interacting particle solution. *Markov Processes and Related Fields*, 2(4):555–580, 1996.

[5] P. Del Moral. *Feynman-Kac formulae.* Probability and its Applications (New York). Springer-Verlag, New York, 2004. Genealogical and interacting particle systems with applications.

Centre de recherche INRIA Bordeaux – Sud Ouest
Domaine Universitaire - 351, cours de la Libération - 33405 Talence Cedex (France)

Centre de recherche INRIA Grenoble – Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier
Centre de recherche INRIA Lille – Nord Europe : Parc Scientifique de la Haute Borne - 40, avenue Halley - 59650 Villeneuve d'Ascq
Centre de recherche INRIA Nancy – Grand Est : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex
Centre de recherche INRIA Paris – Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex
Centre de recherche INRIA Rennes – Bretagne Atlantique : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex
Centre de recherche INRIA Saclay – Île-de-France : Parc Orsay Université - ZAC des Vignes : 4, rue Jacques Monod - 91893 Orsay Cedex
Centre de recherche INRIA Sophia Antipolis – Méditerranée : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399