A study about the existence of the leverage effect in Stochastic Volatility models

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Abstract

The empirical relationship between the return of an asset and the volatility of the asset has been well documented in the financial literature. Named the *leverage effect* or sometimes *risk-premium* effect, it is observed in real data that, when the return of the asset decreases, the volatility increases and vice-versa.

Consequently, it is important to demonstrate that *any* formulated model for the asset price is capable to generate this effect observed in practice. Furthermore, we need to understand the conditions on the parameters present in the model that guarantee the apparition of the leverage effect.

In this paper we analyze two general specifications of stochastic volatility models and their capability of generating the perceived leverage effect. We derive conditions for the apparition of leverage effect in both of these stochastic volatility models. We exemplify using stochastic volatility models used in practice and we explicitly state the conditions for the existence of the leverage effect in these examples.

Key words: Stochastic volatility, Modeling, Leverage effect, Itô's lemma.
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1 Introduction

We have two major objectives in this paper. We wish to demonstrate that leverage effect can be present in stochastic volatility models even if the two

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Brownian motions driving the processes are uncorrelated. We also wish to show that even if the two motions are correlated the leverage effect is determined by more than just the sign of the correlation.

1.1 What is the leverage and what is the leverage effect?

In economic literature, when a simplified approach is used (i.e., neglecting before/after tax issues), *the leverage of a firm* is defined using the formula: $D_t/(D_t + E_t)$ where D_t is the amount of the debt that the firm has and E_t is the market value of the company stock (or equity value) both at some time t. As explained by [1] when the price of equity falls, its leverage rises, increasing the volatility of returns to equity holders. By the same token, news of increasing volatility reduce the demand for a stock because of risk-aversion of the potential buyers. The consequent decline in stock value is followed by increase in volatility as predicted by the news. This is just one possible explanation for the apparition of the leverage effect. The exact cause of the emergence of the leverage effect in practice is still an open problem. [2] conduct a controlled laboratory experiment where the leverage of the firm is kept constant (the debt/equity ratio) and yet they find significant association between the volatility of the return and the return process. The *economical reason* for appearance of the effect is *not* the subject of the current work.

Fisher Black ([3]) is the first researcher who observed the relationship between equity and its volatility. He reported that implied volatility and historical volatility of individual stocks go up when the stock prices go down. The study was conducted over a fairly large time interval and found that the effect can be quite significant. He names this observed relationship in his paper the *leverage effect*. Since that time, the negative correlation between the return of a stock and the volatility of the return has been documented over and over in literature, to mention only some work: [4], [5], [6], [7], [8].

The last article [8] is worth extra mentioning since the authors extend the study to the relationship between past returns and future volatility. They discover that the two quantities are also negatively correlated and that the correlation decays exponentially as the time lag between return and volatility increases. The peak is obtained as the time delay is small and in the limit it can be taken as the instantaneous correlation between the return and volatility.

All the research about the correlation between return and volatility of an asset suggests that *any mathematical model approximating the evolution of asset price should be able to generate the leverage effect* (i.e. a negative correlation between the return and the volatility). [1] stress this point and present other properties that a good stochastic volatility model should exhibit. Therefore, *it is imperative to show that any formulated stochastic volatility (SV) model for the asset price is capable of generating the leverage effect.*

It is important to note that the leverage effect and the leverage of a corporation as defined above and in any economic textbook are different entities. Indeed, while the leverage in the traditional sense depends on the debt issued by the firm, the leverage effect can be calculated (or estimated) using only the equity (asset) value as represented by the shares in the company. The two notions are obviously related if the debt D_t stays constant or is deterministic, however in reality the debt value depends on the equity [1] and in the context of stochastic volatility model this relationship is not easy to study [2]. For this reason, a study relating the two notions (as conducted for example by [9] in the context of a simple lognormal model for the asset price) is beyond the purpose of our work.

1.2 The two Stochastic Volatility Models of the Equity.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ be a probability space endowed with the usual hypotheses [10, page 3]. We assume that all the processes defined hereafter are adapted to the filtration $\mathcal{F}_t$.

The first model in this study presents the equity S_t as a continuous time stochastic volatility model where the volatility term is driven by a general Itô process Y_t. Specifically:

$$\begin{cases} dS_t &= \mu S_t dt + \sigma\left(Y_t\right) S_t dW_t \\ dY_t &= \alpha\left(Y_t\right) dt + \beta\left(Y_t\right) dZ_t, \end{cases} \tag{1}$$

where the functional form of $\sigma(\cdot)$, $\alpha(\cdot)$ and $\beta(\cdot)$ is known, they are well behaved functions which obey local Lipschitz and local growth conditions so that the system admits a unique solution in the strong sense [11, page 367]. In addition we will require that $\sigma(\cdot)$ is twice differentiable with continuous second derivative on any closed interval included in $(0, \infty)$. The processes W_t and Z_t are standard Brownian motions adapted to the filtration $\mathcal{F}_t$, which in general may be correlated with correlation parameter ρ.

We mention that varying the specification of the functions $\sigma(\cdot)$, $\alpha(\cdot)$ and

[1] The rates of any loan taken by the company are dependent primarily on the outside perception of the health of the firm, whose best indicator is the company's equity value.

[2] Debt can be viewed as options on the asset value in this context. However, options on the asset modeled using a stochastic volatility process have no exact formulas (with a number of notable exceptions).

$\beta(\cdot)$ covers **all** the classical continuous time, continuous state space stochastic volatility models present in literature.

The second model considered specifies directly the return R_t on the equity using:

$$\begin{cases} dR_t &= \mu dt + \sigma\left(Y_t\right) dW_t \\ dY_t &= \alpha\left(Y_t\right) dt + \beta\left(Y_t\right) dZ_t. \end{cases} \tag{2}$$

The functions $\alpha(\cdot)$, $\beta(\cdot)$, $\sigma(\cdot)$ and the Brownian processes W and Z parallel the specification in (1) above.

Many current empirical studies chose to forego the specification (1) of the asset's price in favor of directly modeling the return process as in (2). For example, to quote only few recent works: [12], [13], [14]. This particular specification of the return is motivated by the fact that it greatly simplifies the task of estimating coefficients present in the model, a problem which can be quite challenging[3,4].

We note that the model forms presented in (1) and (2) are just notations for the corresponding integral equations (see for example [16, Chapter 5, page 61]). We use both notations interchangeably in this work.

1.3 The stated goals of the current work.

Recent work calls *the correlation coefficient* between the driving Brownian motions in a SV model (W and Z processes in our notation) as the factor that captures "the leverage effect" ([17], [18], page 41, [19]). One empirical study regarding the specification of SV models [20], page 228 defines "the instantaneous correlation between returns and changes in variance", and a formula (formula (4) in the cited paper) for this correlation is provided, with the sole purpose to connect it with the leverage effect described above. In the present work we show that looking only at the correlation between the driving Brownian motions or at the "instantaneous correlation" is not enough.

The *main goals* we are trying to accomplish in the current work are:

- Mathematically formulate a quantity that would convey the presence of the leverage effect in any model.

[3] This is one of the motivations of the current work (see Subsection 1.3).

[4] We also mention that the weak limit of linear or multiplicative GARCH type processes is not of the type in (2), rather closer to the type specified in (1) (see [15]) so the motivation of using models of the type (2) is not because they arise as limits of time series models.

- Calculate this quantity for the two general Stochastic Volatility models presented above in terms of the parameters existing in the model and the functional form of the coefficients.
- Show that the correlation between the sources of randomness present in the model is not the only determining factor for the apparition of the leverage effect.
- Find conditions on the coefficients present in the model that would guarantee the presence of the leverage effect.

The initial motivation of the current work arises from the last point above. Estimating the parameters in continuous time Stochastic Volatility models from discrete observations of the asset price is a hard problem. The main problem is that the Likelihood function for the parameters given only the discrete asset observations is unknown (i.e., there is a lack of meaningful equations). For this reason alternative approaches are constructed (e.g., [21], [22], [23]), all of which assume the existence of a extraneous time series data, either a derivative (call or put) on the asset or the volatility driving process Y_t itself.

In one of the fundamental papers on estimating the volatility process Y_t using a particle filtering method, [24] the authors give rates of convergence under the assumption that the two driving Brownian motions in (1) are uncorrelated. The authors mention that the methods could be easily extended to correlated Brownian Motions but in our experience the extension is not trivial. Therefore, if we analyze the model (1) and we show that it is capable of producing the leverage effect even when the two driving Brownian motions are uncorrelated we show that a simpler model can accomplish what only a more complex model was thought to be capable off [5], and furthermore we are capable of using the methodology developed in [24].

1.4 The structure of the paper.

In the next section (Section 2) we define the quantity used to indicate the presence of the leverage effect.

Section 3 presents an analysis of the leverage effect when the return is specified directly as it is in the model (2). In Section 4 we calculate the leverage effect when the return dynamics are deduced from a specification of the asset price of the type (1). Both sections 3 and 4 contain examples where calculations for specific form of models are performed.

Finally, in Section 5 we summarize the key contributions of this work.

[5] this essentially means that we eliminate the need to estimate the correlation parameter – one of the hardest parameters to estimate

Some technical lemmas are relegated to the Appendix.

2 Leverage Effect.

Definition 1 *We define the **leverage quantity** $L(t)$ over the interval $[0,t]$ as the correlation between the continuously compounded return of the asset and the volatility process (defined as the time derivative of the square root of the quadratic variance of the asset price process) over the interval $[0,t]$.*

*We say that the **leverage effect** is observed for such a process if the leverage quantity defined above is always negative at any moment t.*

*In addition, if the limit of the leverage quantity $L(t)$ as $t \to \infty$, exists and is finite we say that the **long term leverage effect** is present.*

We note that, unlike the usual interpretation of the leverage as the correlation coefficient between the Brownian motions W_t and Z_t, Definition 1 allows the leverage quantity to vary in time. In fact, the change in time agrees to the empirical observations of the leverage effect. The effect of negative news seem to influence both the return and the volatility, however sometimes the effect is stronger than other times. The simplified interpretation of leverage effect as the correlation between the two Brownian motions present is not able to capture this time varying effect.

Furthermore, the presence of a long term leverage effect is a desirable property of a model. If the leverage quantity is estimated over a long period (e.g., a year, 10 years) we would desire that the estimate over an extended period (e.g. 13 months, 11 years) to be approximately the same as the previous estimate.

3 Directly modeling the asset's return process.

In this section we study this model specification (2) and its potential to generate a leverage effect. Rewriting the model (2) in its integral form we obtain:

$$\begin{cases} R_t - R_0 & = \mu t + \int_0^t \sigma\left(Y_s\right) dW_s \\ Y_t - Y_0 & = \int_0^t \alpha\left(Y_s\right) ds + \int_0^t \beta\left(Y_s\right) dZ_s, \end{cases} \tag{3}$$

where W_t and Z_t are correlated Wiener processes with parameter ρ.

The calculation of the leverage quantity for the general specification in (2) is a hard task due to the correlation between the two Brownian motions. While the calculation can be performed and indeed we present the analysis in Lemma 16 on page 21 in the Appendix, the resulting formula is not very useful since it does not provide simple results when particularized to examples.

Instead we present a calculation on a simpler case that nonetheless covers many of the specific stochastic volatility models encountered in practice.

Lemma 2 *Let $\{\mathcal{F}_t\}_t$ denote the standard filtration generated by both Brownian motions W_t and Z_t. Assume that the volatility process $\sigma(Y_t)$ obeys an Itô process of the form:*

$$\sigma(Y_t) = \sigma(Y_0) + \int_0^t \gamma(s)ds + \int_0^t \delta(t, s, Y_s)dZ_s, \tag{4}$$

with $\gamma(s)$ a deterministic function or a $\mathcal{F}_0$-adapted random variable and Z_s the Brownian motion in the Y_t equation. In these conditions the leverage quantity can be expressed as:

$$L(t) = \rho \frac{\int_0^t \mathbf{E}\left[\sigma\left(Y_s\right)\delta\left(t,s,Y_s\right)\right]ds}{\sqrt{\int_0^t \mathbf{E}\left[\sigma^2\left(Y_s\right)\right]ds \int_0^t \mathbf{E}\left[\delta^2\left(t,s,Y_s\right)\right]ds}} \tag{5}$$

Proof. Given the specifications in the hypothesis we can easily calculate the covariance function. We have:

$$\begin{aligned}
&\mathbf{E}\left[(R_t - R_0)(\sigma(Y_t) - \sigma(Y_0))\right] \\
&\quad = \mathbf{E}\left[\left(\mu t + \int_0^t \sigma\left(Y_s\right)dW_s\right)\left(\int_0^t \gamma(u)du + \int_0^t \delta(t,u,Y_u)dZ_u\right)\right] \\
&\quad = \mu t \int_0^t \gamma(u)du + \mu t \mathbf{E}\left[\int_0^t \delta(t,u,Y_u)dZ_u\right] + \mathbf{E}\left[\int_0^t \gamma(u)du \int_0^t \sigma\left(Y_s\right)dW_s\right] \\
&\quad + \mathbf{E}\left[\int_0^t \sigma\left(Y_s\right)dW_s \int_0^t \delta(t,u,Y_u)dZ_u\right].
\end{aligned}$$

The first integral in the above expression is the product of the two expectations, the middle integrals are zero (expectations of zero mean martingales) and using Itô's isometry with the last integral (recall that W and Z are correlated with coefficient ρ), we obtain:

$$Cov\left(R_t - R_0, \sigma(Y_t) - \sigma(Y_0)\right) = \rho \int_0^t \mathbf{E}\left[\sigma\left(Y_s\right) \delta\left(t, s, Y_s\right)\right] ds,$$

where we have applied Fubini's lemma when appropriate. Calculating the variances of the two processes immediately yields the stated answer. ■

3.2 Remarks.

Remark 3 *It is evident from formula* (5) *that if the two driving Brownian motions W_t and Z_t are uncorrelated, then the leverage effect does not exist.*

Remark 4 *From the same formula is also evident that, taking a negative correlation ρ between the two Brownian motions does not automatically guarantee a leverage effect. We could easily determine conditions on the functions $\sigma()$ and $\delta()$ that will actually prevent the leverage effect when ρ is negative.*

Remark 5 *The condition on the form of the process $\sigma(Y_t)$ is needed if we are to avoid a recursion argument (see Lemma 16 in the Appendix). The condition is satisfied for many realistic models as the examples bellow are showing.*

3.3 Leverage effect in affine volatility models: $\sigma(y) = Ay + B$)

Example 6 (SABR model) *In this example we use a modified*[6] *SABR model [25] where we specify the return equation as:*

$$\begin{cases} R_t - R_0 & = \mu t + \int_0^t Y_s dW_s \\ Y_t - Y_0 & = \int_0^t \beta Y_t dZ_s, \end{cases} \tag{6}$$

The two Brownian motions are correlated with correlation ρ.

For this model the leverage quantity is constant for any t:

$$L(t) = \rho\, sgn(\beta)$$

and $sgn(\cdot)$ is the signum function.

The assertion is obtained directly applying the formula (5) *with $\delta(t, s, Y_s) =$*

[6] The difference is that the classical model specifies the stock not the return and therefore in the drift term of R_t there exist an extra term as in Section 4.

βY_s since the conditions of the lemma 2 are clearly satisfied. We have:

$$L(t) = \rho \frac{\int_0^t \mathbf{E}(Y_s \beta Y_s) ds}{\sqrt{\int_0^t \mathbf{E}(Y_s^2) ds \int_0^t \mathbf{E}(\beta^2 Y_s^2) ds}} = \rho \frac{\beta}{|\beta|} = \rho\, sgn(\beta)$$

It is evident that even in this simplest of the examples the leverage effect depends on more than just the correlation between the Brownian motions.

Example 7 (Stein model) *Consider a specification of the model (3) similar with [26] model:*

$$\begin{cases} R_t - R_0 & = \mu t + \int_0^t Y_s dW_s \\ Y_t - Y_0 & = \int_0^t \alpha\,(m - Y_s)\,ds + \int_0^t \beta dZ_s, \end{cases} \tag{7}$$

with $\alpha > 0$, m and β constants, and W_t and Z_t correlated with parameter ρ. The leverage quantity is given by:

$$L(t) = \frac{\rho\beta\left((\mathbf{E}(Y_0) - m)\sqrt{t}e^{-\alpha t} + \frac{m}{\alpha\sqrt{t}}\left(1 - e^{-\alpha t}\right)\right)}{\sqrt{\frac{\beta^2}{2\alpha}\left(1 - e^{-2\alpha t}\right)\left(m^2 + \frac{\beta^2}{2\alpha} + \left(\mathbf{E}[(Y_0 - m)^2] - \frac{\beta^2}{2\alpha}\right)\frac{1 - e^{-2\alpha t}}{2\alpha t} + 2m\mathbf{E}[Y_0 - m]\frac{1 - e^{-\alpha t}}{\alpha t}\right)}}.$$

From this expression we see that we have two distinct cases.

(1) If the distribution of Y_0 is such $E[Y_0] \neq m$ then the leverage effect is present in the model if and only if the sign of $\rho\beta\,(\mathbf{E}(Y_0) - m)$ is negative.
(2) If the mean of Y_0 is $E[Y_0] = m$ then the leverage effect is present if and only if the sign of $\rho\beta m$ is negative[7].

In either case the long term leverage quantity is:

$$\lim_{t\to\infty} L(t) = 0,$$

To prove the formula in the example we verify that the hypothesis of the lemma 2 is satisfied. Then we apply the formula (5). Using equation (19) in the Appendix we see that we can write:

$$Y_t - Y_0 = (m - Y_0)\left(1 - e^{-\alpha t}\right) + \int_0^t \beta e^{-\alpha(t-s)} dZ_s$$

Thus $\sigma(Y_t) = Y_t$ has a representation of the form (4) in the hypothesis of the Lemma 2 with $\gamma(t)$ $\mathcal{F}_0$-adapted and $\delta(t, s, Y_s) = \beta e^{-\alpha(t-s)}$. Thus we can compute the leverage quantity directly using formula (5):

[7] recall that $\alpha > 0$.

$$L(t) = \rho \frac{\int_0^t \beta e^{-\alpha(t-s)} \mathbf{E}\left[Y_s\right] ds}{\sqrt{\int_0^t \beta^2 e^{-2\alpha(t-s)} ds \cdot \int_0^t \mathbf{E}\left[Y_s^2\right] ds}}$$

Now using the formulas (20) and (24) in the Appendix after regular integrations we obtain the provided formula for the leverage quantity $L(t)$.

The two comments about the sign of the leverage quantity and the limiting behavior as $t \to \infty$ are also straightforward from a simple analysis of the dominating term in the numerator of the expression for $L(t)$.

3.4 Leverage effect in root type volatility models: $\sigma(y) = \sqrt{y}$

Example 8 (Hull-White model) *In this example we use return dynamics of the type found in [27]:*

$$\begin{cases} R_t - R_0 & = \mu t + \int_0^t \sqrt{Y_s} dW_s \\ Y_t - Y_0 & = \int_0^t \alpha Y_s ds + \int_0^t \beta Y_s dZ_s, \end{cases} \tag{8}$$

The leverage quantity is in this case

$$L(t) = \rho \, sgn(\beta) \frac{\mathbf{E}\left[\sqrt{Y_0}\right] + \frac{2}{\alpha t}\left(e^{\frac{\alpha t}{2}} - 1\right)}{\sqrt{\mathbf{E}\left[Y_0\right] + \frac{1}{\alpha t}\left(e^{\alpha t} - 1\right)}}$$

From this expression we see that the parameter α is important for the long term behavior of the process. The limiting value as $t \to \infty$ is always $\rho \, sgn(\beta) \frac{\mathbf{E}\left[\sqrt{Y_0}\right]}{\sqrt{\mathbf{E}[Y_0]}}$, but the behavior of the limit is different depending on the sign of α

(1) If $\alpha = 0$ we obtain a constant leverage quantity for any t much as in the case of the SABR model.
(2) If $\alpha < 0$ the convergence is very fast to the limiting value (exponential order)
(3) If $\alpha > 0$ the convergence is very slow to the limiting value (of order $\sqrt{t}$)

In all these cases the leverage effect is present if and only if $sgn(\rho\beta) < 0$.

We need to express the process $\sigma(Y_t) = \sqrt{Y_t}$ in the form of the lemma 2. To this end let $V_t = \exp(-\frac{\beta^2}{4} t + \frac{\beta}{2} Z_t)$. The Novikov condition is satisfied for this process and V_t is an exponential martingale with equation $dV_t = \frac{\beta}{2} V_t dZ_t$.

Applying Itô's lemma to the process$\sqrt{Y_t}$ we obtain:

$$
\begin{aligned}
d\sqrt{Y_t} &= \frac{1}{2\sqrt{Y_t}}dY_t + \frac{1}{2}\left(-\frac{1}{4Y_t\sqrt{Y_t}}\right) d < Y, Y >_t \\
&= \frac{1}{2}\left(\alpha - \frac{\beta^2}{4}\right)\sqrt{Y_t}dt + \frac{\beta}{2}\sqrt{Y_t}dZ_t
\end{aligned}
$$

We can solve this resulting equation very easily to obtain the solution:

$$
\sqrt{Y_t} - \sqrt{Y_0} = e^{\left(\frac{\alpha}{2} - \frac{\beta^2}{8} - \frac{\beta^2}{8}\right)t + \frac{\beta}{2}Z_t} = e^{\frac{\alpha t}{2}}V_t
$$

Since we know the solution for V_t we can write:

$$
\sqrt{Y_t} = \sqrt{Y_0} + e^{\frac{\alpha t}{2}}\left(1 + \int_0^t \frac{\beta}{2}V_s dZ_s\right) = \sqrt{Y_0} + e^{\frac{\alpha t}{2}} + \int_0^t e^{\frac{\alpha t}{2}}\frac{\beta}{2}V_s dZ_s
$$

Thus the process $\sigma(Y_t)$ could be put in the form (5) *with $\delta(t, s, Y_s) = \frac{\beta}{2}e^{\frac{\alpha t}{2}}V_s = \frac{\beta}{2}e^{\frac{\alpha t}{2}}e^{-\frac{\beta^2}{4}s + \frac{\beta}{2}Z_s}$. Now note that both V_t and $V_t^2 = \exp(-\frac{\beta^2}{2}t + \beta Z_t)$ are martingales with respect to the filtration $\mathscr{F}_t$ with means $\mathbf{E}[V_0] = \mathbf{E}[V_0^2] = 1 = V_0$.*

We can calculate the numerator after this observation:

$$
\begin{aligned}
\int_0^t \mathbf{E}\left[\sigma\left(Y_s\right)\delta\left(t, s, Y_s\right)\right]ds = \int_0^t \mathbf{E}[\sqrt{Y_s}\frac{\beta}{2}e^{\frac{\alpha t}{2}}V_s]ds &= \frac{\beta}{2}e^{\frac{\alpha t}{2}}\int_0^t \mathbf{E}\left[\left(\sqrt{Y_0} + e^{\frac{\alpha s}{2}}V_s\right)V_s\right]ds \\
&= \frac{\beta}{2}e^{\frac{\alpha t}{2}}\int_0^t \left(\mathbf{E}\left[\sqrt{Y_0}\mathbf{E}[V_s|\mathscr{F}_0]\right] + e^{\frac{\alpha s}{2}}\mathbf{E}\left[V_s^2\right]\right)ds \\
&= \frac{\beta}{2}e^{\frac{\alpha t}{2}}\int_0^t \left(\mathbf{E}\left[\sqrt{Y_0}\right] + e^{\frac{\alpha s}{2}}\right)ds = \\
&= \frac{\beta}{2}e^{\frac{\alpha t}{2}}\left(\mathbf{E}\left[\sqrt{Y_0}\right]t + \frac{2}{\alpha}\left(e^{\frac{\alpha t}{2}} - 1\right)\right)
\end{aligned}
$$

We can then calculate the terms in the denominator as:

$$
\begin{aligned}
\int_0^t \mathbf{E}[Y_s]ds &= \mathbf{E}\left[Y_0\right]t + \frac{1}{\alpha}\left(e^{\alpha t} - 1\right) \\
\int_0^t \mathbf{E}[\frac{\beta^2}{4}e^{\alpha t}V_s^2]ds &= \frac{\beta^2}{4}e^{\alpha t}t
\end{aligned}
$$

Using the formula (5) after simplifications we reach the leverage equation given.

3.5 Leverage Effect in an exponential type volatility model: $\sigma(y) = e^y$

The only popular continuous time volatility models that contains an exponential type volatility are the [28] and later [29] models. Unfortunately, putting the volatility process $\sigma(Y_t)$ in the form (5) for those particular models is not that easy and we have to use a slight modification for the volatility process to analyze the model.

Example 9 *Consider a specification of the model* (3) *with $\sigma(y) = e^y$ and Y_t a special case of the mean-reverting process of [30] type. Mathematically:*

$$\begin{cases} R_t - R_0 &= \mu t + \int_0^t e^{Y_s} dW_s \\ Y_t - Y_0 &= \alpha t - \int_0^t \frac{\beta^2}{2} Y_s ds + \int_0^t \beta \sqrt{Y_s} dZ_s, \end{cases} \tag{9}$$

with $\alpha > 0$, β constants, and W_t and Z_t correlated with parameter ρ.

Then the leverage effect is present if and only if the product $\beta\rho < 0$.

We note that the Y_t process above is a particular case of a Heston driving volatility with the rate of mean return proportional to the parameter β. We mention that using the Heston model directly produces a square root type volatility and the approach proceeds exactly as in the Example 8. We have:

$$e^{Y_t} = e^{Y_0 + \alpha t - \int_0^t \frac{\beta^2}{2} Y_s ds + \int_0^t \beta \sqrt{Y_s} dZ_s} = \exp\left(Y_0 + \alpha t\right) V_t$$

where $V_t = \exp\left(- \int_0^t \frac{\beta^2}{2} Y_s ds + \int_0^t \beta \sqrt{Y_s} dZ_s\right)$. Again, V_t is an exponential martingale that solves $dV_t = \beta \sqrt{Y_t} V_t dZ_t$. Therefore we can write:

$$e^{Y_t} = e^{Y_0 + \alpha t} + e^{Y_0 + \alpha t} \int_0^t \beta \sqrt{Y_s} V_s dZ_s$$

We can now apply Lemma 2 with $\delta(t, s, V_s) = \beta e^{Y_0 + \alpha t} \sqrt{Y_s} V_s$. The leverage in not easy to calculate exactly but we can assess its sign by looking at the numerator of $L(t)$. We see that:

$$\mathbf{E}\left[\sigma\left(Y_s\right) \delta\left(t, s, Y_s\right)\right] = \beta \mathbf{E}\left[e^{Y_s} e^{Y_0 + \alpha t} \sqrt{Y_s}\, e^{- \int_0^t \frac{\beta^2}{2} Y_s ds + \int_0^t \beta \sqrt{Y_s} dZ_s}\right]$$

We see immediately that the the terms in the expectation are always positive and that the sign of the leverage quantity is given by the sign of the product $\rho\beta$.

Finally, we mention an example of practical use of the formula (5). [14] compares a discrete time version of the model in the Example 9 (denoted there

ASV1), and the discrete time stochastic volatility model of [13] (dubbed in the paper ASV2). The author obtains estimates for ρ the correlation coefficient between the Brownian motions in both models, the numbers published are $\rho_1 = -0.3179$ in the ASV1 model and $\rho_2 = -0.2599$ in the ASV2 model. He then concludes that one specification is better than the other partly due to the fact that the leverage effect is underestimated in the ASV2 model. In the light of this section, it is entirely possible that the correlations in the two models be completely different and yet, the leverage quantities as defined in Section 2 to be similar.

4 Using a directly specified asset process and deducing the return dynamics.

In this section we deduce the formulation of the return process $R_t = \log S_t$ directly from the dynamics of the asset process as specified in (1). We show that the behavior of the leverage quantity (while more difficult to calculate) is essentially different than the analysis presented in the previous section.

If we apply the Itô formula in (1), for the function $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (\log x, y)$, we obtain:

$$\begin{cases} dR_t &= \left(\mu - \frac{\sigma^2(Y_t)}{2}\right) dt + \sigma\left(Y_t\right) dW_t \\ dY_t &= \alpha\left(Y_t\right) dt + \beta\left(Y_t\right) dZ_t, \end{cases} \tag{10}$$

regardless of the two Brownian motions being correlated or not. We rewrite the system in an integral form:

$$\begin{cases} R_t - R_0 &= \int_0^t \left(\mu - \frac{\sigma^2(Y_s)}{2}\right) ds + \int_0^t \sigma\left(Y_s\right) dW_s \\ Y_t - Y_0 &= \int_0^t \alpha\left(Y_s\right) ds + \int_0^t \beta\left(Y_s\right) dZ_s. \end{cases} \tag{11}$$

In this model *we keep the two Brownian motions W_t and Z_t uncorrelated*. We would love to to calculate the leverage quantity in the more general case of correlated Brownian motions but the reality is that such calculation is very complicated.

Nevertheless, we are able to show that even in this case the leverage effect can still be present.

Lemma 10 (The leverage condition) *Assume that an asset process has dynamics specified by* (1) *and correspondingly its return dynamics are as in the system* (10). *Assume that the two driving Brownian motions W_t and Z_t are uncorrelated.*

The following is a sufficient *condition for the leverage effect to be present in the model at any time t.*

$$\mathbf{E}\left[\sigma^2(Y_s)\sigma'(Y_u)\alpha\left(Y_u\right)+\frac{1}{2}\sigma^2(Y_s)\sigma''(Y_u)\beta^2\left(Y_u\right)\right]$$
$$\geq \mathbf{E}\left[\sigma^2(Y_s)\right]\mathbf{E}\left[\sigma'(Y_u)\alpha\left(Y_u\right)+\frac{1}{2}\sigma''(Y_u)\beta^2\left(Y_u\right)\right], \qquad (12)$$

for every $u, s \in [0,t]$, with strict inequality on a set $A \subseteq [0,t]\times[0,t]$ of non-zero Lebesque measure .

Remark 11 *Before we prove the lemma let us make two simple observations.*

First, the fact that the leverage effect can be present even if the two Brownian motions are uncorrelated is very good news for the estimation of coefficients. Indeed, having uncorrelated Brownian motions means one less parameter to be estimated. And in fact this correlation is one of the hardest parameters to estimate.

Second, the inequality presented represents a sufficient condition *for the leverage effect. In fact, since the numerator of the leverage quantity in definition 1 is the integrated expression above with respect to s and u on the interval $[0,t]\times[0,t]$ it is possible that the condition stated is violated and still the integrated quantity is negative at every t. We could also state the leverage quantity $L(t)$ by integrating the expression in* (12) *and dividing by the square root of the product of variances, but the resulting expression is very long and the reader can calculate it very easily once the underlying principle is understood. Furthermore, to actually calculate the integral terms in the resulting expression is a chore in all but the simplest models.*

Proof of the Lemma 10. Recall that $\sigma(\cdot) \in \mathcal{C}^2(\mathbb{R})$. Itô's lemma gives:

$$\begin{aligned}\sigma(Y_t)-\sigma(Y_0) &= \int_0^t \sigma'(Y_s)dY_s + \frac{1}{2}\int_0^t \sigma''(Y_s)d<Y,Y>_s \\ &= \int_0^t \left(\sigma'(Y_s)\alpha\left(Y_s\right)+\frac{1}{2}\sigma''(Y_s)\beta^2\left(Y_s\right)\right)ds + \int_0^t \sigma'(Y_s)\beta\left(Y_s\right)dZ_s \\ &= \int_0^t \sigma_1(Y_s)ds + \int_0^t \sigma'(Y_s)\beta\left(Y_s\right)dZ_s, \qquad (13)\end{aligned}$$

where we have introduced the notation:

$$\sigma_1(y) = \sigma'(y)\alpha\left(y\right)+\frac{1}{2}\sigma''(y)\beta^2\left(y\right). \qquad (14)$$

We calculate:

$$
\begin{aligned}
&\mathbf{E}\left[(R_t - R_0)(\sigma(Y_t) - \sigma(Y_0))\right] \\
&\quad = \mathbf{E}\left[\left(\int_0^t \left(\mu - \frac{\sigma^2(Y_s)}{2}\right) ds + \int_0^t \sigma\left(Y_s\right) dW_s\right)\left(\int_0^t \sigma_1(Y_u) du + \int_0^t \sigma'(Y_s)\beta\left(Y_s\right) dZ_u\right)\right] \\
&\quad = \mathbf{E}\left[\int_0^t \left(\mu - \frac{\sigma^2(Y_s)}{2}\right) ds \int_0^t \sigma_1(Y_u) du\right] \\
&\quad = \int_0^t \int_0^t \mathbf{E}\left[\left(\mu - \frac{\sigma^2(Y_s)}{2}\right)\sigma_1(Y_u)\right] ds\, du,
\end{aligned}
$$

using Fubini's lemma, the fact that two terms are stochastic integrals with expectation zero and that in the model (11) the two Brownian motions W_t and Z_t are independent.

From the equations in (11) and (13) we obtain the expectations for $R_t - R_0$ and $\sigma(Y_t) - \sigma(Y_0)$ and then we can calculate the covariance as:

$$
\begin{aligned}
Cov\left(R_t - R_0, \sigma(Y_t) - \sigma(Y_0)\right) &= \int_0^t \int_0^t \mathbf{E}\left[\left(\mu - \frac{\sigma^2(Y_s)}{2}\right)\sigma_1(Y_u)\right] ds\, du \\
&\quad - \int_0^t \int_0^t \mathbf{E}\left[\mu - \frac{\sigma^2(Y_s)}{2}\right]\mathbf{E}\left[\sigma_1(Y_u)\right] ds\, du \\
&= \int_0^t \int_0^t \mathbf{E}\left[\left(\mu - \frac{\sigma^2(Y_s)}{2}\right)\sigma_1(Y_u)\right] - \mathbf{E}\left[\mu - \frac{\sigma^2(Y_s)}{2}\right]\mathbf{E}\left[\sigma_1(Y_u)\right] ds\, du \\
&= -\int_0^t \int_0^t \left(\mathbf{E}\left[\frac{\sigma^2(Y_s)}{2}\sigma_1(Y_u)\right] - \mathbf{E}\left[\frac{\sigma^2(Y_s)}{2}\right]\mathbf{E}\left[\sigma_1(Y_u)\right]\right) ds\, du \qquad (15)
\end{aligned}
$$

Noting that the sign of the correlation is determined by the sign of the covariance and replacing $\sigma_1(\cdot)$ with its formula in (14), after simple algebra we obtain the result stated. ■

4.1 Leverage effect for affine volatility models. The return process is deduced from the stock price specifications.

Let us remark that if one chooses a specific volatility model with fixed functional form of the functions $\sigma(\cdot)$, $\alpha(\cdot)$, $\beta(\cdot)$, the condition in Lemma 10 is not easy to verify. However, to make sure that the condition is valid and not purely academic, we give an example where the condition is verified. The next example uses the model of [26].

Example 12 (Stein&Stein) *Assume that the asset price follows the follow-*

ing model:

$$\begin{cases} dS_t &= \mu S_t dt + Y_t S_t dW_t \\ dY_t &= \alpha(m - Y_t)dt + \beta dZ_t, \end{cases} \tag{16}$$

where the Brownian motions W_t and Z_t are uncorrelated, μ, β are any constants, and the process Y_t starts from the variable $Y_0 \in L^3(\Omega)$.

We assume that the parameter α is strictly positive (so that the distribution of Y_t is stationary for any t). Then the following conditions on parameters are sufficient for the presence of leverage effect at any moment $t > 0$:

(1) $m < 0$
(2) $\mathbf{E}[Y_0^3] - \mathbf{E}[Y_0^2]\mathbf{E}[Y_0] < 0$
(3) $\mathbf{E}[Y_0] \leq -2m$

Remark 13 *We note that the condition appearing in the exercise is a* sufficient condition *arising from our calculations. The leverage effect may be present even under weaker conditions than the ones presented. Note that m is negative and thus $\mathbf{E}[Y_0] \leq -2m$ implies an upper positive bound for the expectation.*

To verify whether the estimated coefficients can generate the leverage effect one has to check the exact condition in equation (17) *bellow.*

Proof. We apply the Lemma 10 to directly verify that the leverage effect is present. We have that $\sigma'(y) = 1$, $\sigma''(y) = 0$, and $\alpha(y) = \alpha(m - y)$. Therefore, the condition (12) simplifies to:

$$\begin{aligned} \mathbf{E}\left[Y_t^2 \alpha(m - Y_s)\right] &\geq \mathbf{E}\left[Y_t^2\right] \mathbf{E}\left[\alpha(m - Y_s)\right] \\ -\alpha \mathbf{E}\left[Y_t^2 Y_s\right] &\geq -\alpha \mathbf{E}\left[Y_t^2\right] \mathbf{E}\left[Y_s\right] \\ \alpha\left(\mathbf{E}\left[Y_t^2 Y_s\right] - \mathbf{E}\left[Y_t^2\right] \mathbf{E}\left[Y_s\right]\right) &\leq 0, \end{aligned}$$

where we have used t and s for the times replacing s and u in the lemma to simplify the notation a little bit. In the Appendix (Section 6 formula (25) on page 21) we calculate this difference and we substitute it here to obtain the condition:

$$\begin{aligned} \alpha\left(\mathbf{E}[Y_0^3] - \mathbf{E}[Y_0^2]\mathbf{E}[Y_0]\right) + 2m\alpha V(Y_0)\left(e^{\alpha t} - 1\right) + \beta^2 \mathbf{E}(Y_0)\left(e^{2\alpha t \wedge s} - 1\right) \\ + 2m\beta^2 \left(\frac{e^{3\alpha t \wedge s}}{3} - \frac{e^{2\alpha t \wedge s}}{2}\right) \leq 0, \end{aligned} \tag{17}$$

with strict equality on a set of non-zero Lebesque measure.

The formula above gives us a clear criterion for the choice of parameters in the Stochastic Volatility models. If the condition is true, then the leverage effect is present in the model. Furthermore, under the conditions specified in the hypothesis for a time t large enough all the terms in the expression are negative therefore the leverage effect is going to be present in the model. ■

5 Contributions of the paper

In conclusion, we would like to summarize what we have accomplished in this work.

We have defined in Section 2 a leverage quantity that indicates the presence or absence of the leverage effect at any moment in time.

We have analyzed two general stochastic volatility models and their capability of generating the leverage effect.

First, we study a directly specified return model (2) in Section 3 which is very popular in practice. We show that in this case the leverage quantity is related with the correlation between the driving Brownian motions but it is not entirely determined by it. If the correlation is zero the leverage effect is absent, however if the correlation is positive it is still possible for the leverage effect to be present in the model. We calculate the exact expression in several examples.

Second, we analyze an alternative specification for the return, directly deduced from dynamics of the the model (1) in Section 4. In this case, even if the driving Brownian motions are uncorrelated we show that the leverage effect can still be present in the model and that this fact is determined entirely by the parameters present in the model.

In fact, the formulae that we gave are useful from the perspective of the estimation of the coefficients present in the model. [18] mention that the asset data is not sufficient for the estimation of all the parameters present in the model. The usual estimation techniques are performed using asset data *and* some other extraneous derivative data (such as call options etc.). However, this essentially prevents applying the stochastic volatility model to any other data but finance. For example, earthquake modeling, signal processing and even utilities data have no derivatives whose value or strength is determined outside the original signal model. Therefore, a method that does not use these derivatives needs to be devised.

The formulae we give here could serve to limit the space of possible parameters to only such values that allows the model to exhibit features similar to the observations from the real world.

6 Appendix.

6.1 Formulae for the mean reverting Ornstein-Uhlenbeck process

In the appendix, we present all the calculations needed *for the examples presented earlier*. They are more appropriate to be presented in a separate section to avoid distracting from the logical flow of the paper. We mention also that although apparently simple, with the exception of the expected value of the mean-reverting Ornstein-Uhlenbeck process, we could not find references to the complete formulae we give bellow.

Let Y_t denote a mean-reverting Ornstein-Uhlenbeck process formulated as:

$$dY_t = \alpha(m - Y_t)dt + \beta dZ_t, \tag{18}$$

with α, m, and β real constants and Z_t a standard Brownian motion adapted to $\mathcal{F}_t$.

Applying the Ito's rule to the function $f(t, y) = ye^{\alpha t}$ we can obtain the explicit solution:

$$Y_t = e^{-\alpha t}Y_0 + m\left(1 - e^{-\alpha t}\right) + e^{-\alpha t}\int_0^t \beta e^{\alpha s} dZ_s. \tag{19}$$

Clearly:

$$\mathbf{E}[Y_t] = \mathbf{E}[Y_0]\, e^{-\alpha t} + m(1 - e^{-\alpha t}) \tag{20}$$

The next formulae are hard to find in literature, and we choose to state them as separate results.

Lemma 14 *With the process Y_t a mean-reverting OU process specified as in* (18) *we have the covariance function of the process given by:*

$$Cov(Y_t, Y_s) = e^{-\alpha(s+t)}\left[V(Y_0) + \frac{\beta^2}{2\alpha}\left(e^{2\alpha\, t\wedge s} - 1\right)\right], \tag{21}$$

where $V(Y_0)$ denotes the variance of the initial variable Y_0 and $t \wedge s$ denotes the minimum of the two numbers t and s.

Proof. Using (19) and (20) we can write:

$$Y_t - \mathbf{E}[Y_t] = (Y_0 - \mathbf{E}[Y_0])\, e^{-\alpha t} + e^{-\alpha t}\int_0^t \beta e^{\alpha u} dZ_u,$$

and we can calculate:

$$
\begin{aligned}
Cov(Y_t, Y_s) &= \mathbf{E}\left[(Y_t - \mathbf{E}[Y_t])(Y_s - \mathbf{E}[Y_s])\right] \\
&= e^{-\alpha(t+s)}\mathbf{E}\left[\left(Y_0 - \mathbf{E}[Y_0] + \int_0^t \beta e^{\alpha u} dZ_u\right)\left(Y_0 - \mathbf{E}[Y_0] + \int_0^s \beta e^{\alpha v} dZ_v\right)\right] \\
&= e^{-\alpha(t+s)}\left(V(Y_0) + \mathbf{E}\left[(Y_0 - \mathbf{E}[Y_0])\int_0^s \beta e^{\alpha v} dZ_v\right]\right. \\
&\quad \left. + \mathbf{E}\left[(Y_0 - \mathbf{E}[Y_0])\int_0^t \beta e^{\alpha u} dZ_u\right] + \beta^2 \mathbf{E}\left[\int_0^t e^{\alpha u} dZ_u \int_0^s e^{\alpha v} dZ_v\right]\right) \\
&= e^{-\alpha(t+s)}\left(V(Y_0) + \beta^2 \int_0^{t\wedge s} e^{2\alpha u} du\right).
\end{aligned}
$$

To obtain the last equality, we use the Itô isometry in the last integral, and for the two middle integrals the fact that a stochastic integral is a martingale with zero expectation. Finally, computing the sole remaining integral we obtain the result stated in the lemma. ■

Next we calculate a stochastic differential equation for Y_t^2 which allow us to verify the condition appearing in Example 12. We apply Itô's lemma to the function $f(t, y) = y^2 e^{2\alpha t}$ and the process Y_t.

$$
\begin{aligned}
d(Y_t^2 e^{2\alpha t}) &= 2\alpha e^{2\alpha t} Y_t^2 dt + 2Y_t e^{2\alpha t} dY_t + \frac{1}{2}(2e^{2\alpha t}) d < Y, Y >_t \\
&= e^{2\alpha t}\left(2\alpha m Y_t + \beta^2\right) dt + 2\beta Y_t e^{2\alpha t} dZ_t.
\end{aligned}
$$

Therefore, we can express the equation in integral form as:

$$
Y_t^2 = Y_0^2 e^{-2\alpha t} + 2\alpha m e^{-2\alpha t}\int_0^t e^{2\alpha u} Y_u du + \frac{\beta^2}{2\alpha}\left(1 - e^{-2\alpha t}\right) + 2\beta e^{-2\alpha t}\int_0^t e^{2\alpha u} Y_u dZ_u \tag{22}
$$

Lemma 15 *The second moment of a mean-reverting OU process Y_t specified as in* (18) *is given by:*

$$
\mathbf{E}[Y_t^2] = e^{-2\alpha t}\left[\mathbf{E}[Y_0^2] + \frac{\beta^2}{2\alpha}\left(e^{2\alpha t} - 1\right) + 2m\mathbf{E}[Y_0]\left(e^{\alpha t} - 1\right) + m^2\left(e^{\alpha t} - 1\right)^2\right]. \tag{23}
$$

Furthermore, the integrated second moment is given by:

$$
\int_0^t \mathbf{E}[Y_s^2] dt = t\left(m^2 + \frac{\beta^2}{2\alpha}\right) + \left(\mathbf{E}[(Y_0 - m)^2] - \frac{\beta^2}{2\alpha}\right)\frac{1 - e^{-2\alpha t}}{2\alpha} + 2m\mathbf{E}[Y_0 - m]\frac{1 - e^{-\alpha t}}{\alpha}. \tag{24}
$$

Proof. The proof is an exercise in simple calculus, all we have to do is apply expectations in both sides of (22), substitute $\mathbf{E}[Y_u]$ with the expression (20), and finally integrate with respect to u. The expected value in (23) is integrated to yield after another series of calculations the expression in (24). ■

We need to calculate an expression of the form

$$\mathbf{E}\left[Y_t^2 Y_s\right] - \mathbf{E}\left[Y_t^2\right]\mathbf{E}\left[Y_s\right]$$

To this end we first calculate:

$$\begin{aligned}\mathbf{E}\left[Y_t^2 Y_s\right] = e^{-2\alpha t}e^{-\alpha s}\mathbf{E}\Bigg[\Bigg(&Y_0^2 + 2\alpha m\int_0^t e^{2\alpha u}Y_u du + \frac{\beta^2}{2\alpha}\left(e^{2\alpha t}-1\right) + \\ &2\beta\int_0^t e^{2\alpha u}Y_u dZ_u\Bigg)\left(Y_0 + m\left(e^{\alpha s}-1\right) + \int_0^s \beta e^{\alpha v}dZ_v\right)\Bigg] \\ &= e^{-\alpha(2t+s)}\left(I + II + III + IV\right),\end{aligned}$$

where:

$$\begin{aligned}I &= \mathbf{E}\left[\left(Y_0^2 + 2\alpha m\int_0^t e^{2\alpha u}Y_u du + \frac{\beta^2}{2\alpha}\left(e^{2\alpha t}-1\right)\right)\left(Y_0 + m\left(e^{\alpha s}-1\right)\right)\right] \\ &= \mathbf{E}(Y_0^3) + 2\alpha m\int_0^t e^{2\alpha u}\mathbf{E}[Y_uY_0]du + \frac{\beta^2}{2\alpha}\left(e^{2\alpha t}-1\right)\mathbf{E}(Y_0) + m\left(e^{\alpha s}-1\right)\mathbf{E}(Y_0^2) \\ &\quad + 2\alpha m^2\left(e^{\alpha s}-1\right)\int_0^t e^{2\alpha u}\mathbf{E}[Y_u]du + \frac{\beta^2 m}{2\alpha}\left(e^{2\alpha t}-1\right)\left(e^{\alpha s}-1\right)\end{aligned}$$

II and III are expectations of the cross-product terms which are both zero and for economy of space we neglect to write them, and

$$\begin{aligned}IV &= 2\beta^2\mathbf{E}\left[\int_0^t e^{2\alpha u}Y_u dZ_u\int_0^s e^{\alpha v}dZ_v\right] \\ &= 2\beta^2\int_0^{t\wedge s} e^{3\alpha u}\mathbf{E}(Y_u)du\end{aligned}$$

From this expression we need to subtract $\mathbf{E}\left[Y_t^2\right]\mathbf{E}\left[Y_s\right]$ which is calculated using (20) and (22) as:

$$\begin{aligned}\mathbf{E}\left[Y_t^2\right]\mathbf{E}\left[Y_s\right] &= e^{-\alpha(2t+s)}\left(\mathbf{E}(Y_0^2) + 2\alpha m\int_0^t e^{2\alpha u}\mathbf{E}(Y_u)du + \frac{\beta^2}{2\alpha}\left(e^{2\alpha t}-1\right)\right) \\ &\qquad(\mathbf{E}[Y_0] + m(e^{\alpha s}-1)) \\ &= e^{-\alpha(2t+s)}\Bigg(\mathbf{E}(Y_0^2)\mathbf{E}[Y_0] + 2\alpha m\int_0^t e^{2\alpha u}\mathbf{E}(Y_u)\mathbf{E}[Y_0]du + \frac{\beta^2}{2\alpha}\left(e^{2\alpha t}-1\right)\mathbf{E}[Y_0] \\ &\quad + m(e^{\alpha s}-1)\mathbf{E}(Y_0^2) + 2\alpha m^2(e^{\alpha s}-1)\int_0^t e^{2\alpha u}\mathbf{E}(Y_u)du + \frac{\beta^2 m}{2\alpha}\left(e^{2\alpha t}-1\right)\left(e^{\alpha s}-1\right)\Bigg)\end{aligned}$$

We are finally in position to calculate the $\mathbf{E}\left[Y_t^2 Y_s\right] - \mathbf{E}\left[Y_t^2\right]\mathbf{E}\left[Y_s\right]$. Taking the difference of the expressions calculated above, after simplifications we obtain:

$$e^{-\alpha(2t+s)}\Big(\mathbf{E}(Y_0^3)-\mathbf{E}(Y_0^2)\mathbf{E}[Y_0]+2\alpha m\int_0^t e^{2\alpha u}(\mathbf{E}[Y_uY_0]-\mathbf{E}(Y_u)\mathbf{E}[Y_0])du$$
$$+2\beta^2\int_0^{t\wedge s}e^{3\alpha u}\mathbf{E}(Y_u)du\Big)$$

Noting that in the first integral we have the term $Cov(Y_u, Y_0)$ which we know from (21) is equal to $e^{-\alpha u}V(Y_0)$, and that in the second integral we have the expectation of the mean revering OU process which we calculated in (20), we can substitute these terms, and integrate them to finally obtain the following expression:

$$\mathbf{E}\left[Y_t^2Y_s\right]-\mathbf{E}\left[Y_t^2\right]\mathbf{E}\left[Y_s\right]=e^{-\alpha(2t+s)}\Big(\mathbf{E}(Y_0^3)-\mathbf{E}(Y_0^2)\mathbf{E}[Y_0] \tag{25}$$
$$+\,2mV(Y_0)\left(e^{\alpha t}-1\right)+\frac{\beta^2}{\alpha}\mathbf{E}(Y_0)\left(e^{2\alpha t\wedge s}-1\right)+\frac{2m\beta^2}{\alpha}\left(\frac{e^{3\alpha t\wedge s}}{3}-\frac{e^{2\alpha t\wedge s}}{2}\right)\Big)$$

6.2 A more general version of Lemma 2

To obtain a general formula for the leverage having the usual conditions for existence and uniqueness of the solution of stochastic differential equations do not suffice anymore. We require that the functions $\alpha(\cdot)$, $\beta(\cdot)$ and $\sigma(\cdot)$ are in $\mathscr{C}^\infty(0,\infty)$. We note that all the stochastic volatility models used in practice have this property.

Lemma 16 *Given the asset process with return dynamics specified as in the system* (3)*, assuming in addition that the functions α, β and σ are of class $\mathcal{C}^\infty$, the leverage quantity $L(t)$ is:*

$$L(t)=\rho\frac{A(t)}{\sqrt{B(t)C(t)}} \tag{26}$$

where the quantities in the expression are:

$$A(t)=\sum_{i=0}^{\infty}\int_0^{t_0}\int_0^{t_1}\dots\int_0^{t_i}\mathbf{E}\left[\sigma_i'(Y_{t_{i+1}})\beta\left(Y_{t_{i+1}}\right)\sigma\left(Y_{t_{i+1}}\right)\right]dt_{i+1}dt_i\dots dt_1$$

$$B(t) = \int_0^t \mathbf{E}\left[\sigma^2\left(Y_s\right)\right] ds$$
$$C(t) = V\left(\int_0^t \sigma_1(Y_u)du\right) + \sum_{i=0}^{\infty} a_i \int_0^{t_0}\int_0^{t_1}\dots\int_0^{t_i} \mathbf{E}\left[\sigma_i'(Y_{t_{i+1}})\beta^2\left(Y_{t_{i+1}}\right)\sigma'\left(Y_{t_{i+1}}\right)\right] dt_{i+1}dt_i\dots dt_1,$$

where $t = t_0 > t_1 > t_2 > \dots$, and the functions $\sigma_i(y)$ are calculated using the recursive expression:

$$\begin{cases} \sigma_i(y) & = \sigma_{i-1}'(y)\alpha\left(y\right) + \frac{1}{2}\sigma_{i-1}''(y)\beta^2\left(y\right) \\ \sigma_0(y) & = \sigma(y) \end{cases} \tag{27}$$

Consequently, while providing explicit expressions this lemma is not useful from the applications perspective. Note that the presence of the equations (27) *makes its applicability limited even in the simplest of cases.*

Proof. We have already calculated the $\sigma(Y_t)$ dynamics in (13).

To obtain the covariance between these two processes let us calculate:

$$\begin{aligned} &\mathbf{E}\left[(R_t - R_0)(\sigma(Y_t) - \sigma(Y_0))\right] \\ &\quad = \mathbf{E}\left[\left(\mu t + \int_0^t \sigma\left(Y_s\right)dW_s\right)\left(\int_0^t \sigma_1(Y_u)du + \int_0^t \sigma'(Y_u)\beta\left(Y_u\right)dZ_u\right)\right] \\ &\quad = \mu t\int_0^t \mathbf{E}[\sigma_1(Y_u)]du + \mu t\mathbf{E}\left[\int_0^t \sigma'(Y_u)\beta\left(Y_u\right)dZ_u\right] + \mathbf{E}\left[\int_0^t \sigma\left(Y_s\right)dW_s\int_0^t \sigma_1(Y_u)du\right] \\ &\quad + \mathbf{E}\left[\int_0^t\int_0^t \sigma'(Y_u)\beta\left(Y_u\right)\sigma\left(Y_s\right)dW_s dZ_u\right]. \end{aligned}$$

The first integral in the above expression is the product of the two expectations, the second integral is expectation of a zero mean martingale and using Itô's isometry with the last integral (recall that W and Z are correlated with coefficient ρ), we obtain:

$$Cov\left(R_t - R_0, \sigma(Y_t) - \sigma(Y_0)\right) = \rho\int_0^t \mathbf{E}\left[\sigma'(Y_s)\beta\left(Y_s\right)\sigma\left(Y_s\right)\right]ds + \int_0^t \mathbf{E}\left[\sigma_1(Y_u)\int_0^t \sigma\left(Y_s\right)dW_s\right]du,$$

where we have applied Fubini's lemma when appropriate. The second integral in the covariance expression above is nonzero since the the two Brownian motions are correlated. Calculating this term requires repeated applications of the Itô rule. Using the fact that the functions α, β and σ are in $\mathscr{C}^\infty(0,\infty)$ we obtain similarly with (13):

$$\sigma_1(Y_t) = \sigma_1(Y_0) + \int_0^t \left(\sigma_1'(Y_s)\alpha\left(Y_s\right) + \frac{1}{2}\sigma_1''(Y_s)\beta^2\left(Y_s\right)\right)ds + \int_0^t \sigma_1'(Y_s)\beta\left(Y_s\right)dZ_s$$
$$= \sigma_1(Y_0) + \int_0^t \sigma_2(Y_s)ds + \int_0^t \sigma_1'(Y_s)\beta\left(Y_s\right)dZ_s, \qquad (28)$$

where similarly with (14) we have used the notation

$$\sigma_2(y) = \sigma_1'(y)\alpha\left(y\right) + \frac{1}{2}\sigma_1''(y)\beta^2\left(y\right).$$

We substitute (28) in the second integral in the covariance expression and we obtain three new terms:

$$\int_0^t \mathbf{E}\left[\sigma_1(Y_0)\int_0^t \sigma\left(Y_s\right)dW_s\right]du = \int_0^t \mathbf{E}\left[\int_0^t \sigma_1(Y_0)\sigma\left(Y_s\right)dW_s\right]du = 0$$
$$\int_0^t \mathbf{E}\left[\int_0^u \sigma_1'(Y_v)\beta\left(Y_v\right)dZ_v\int_0^t \sigma\left(Y_s\right)dW_s\right]du = \int_0^t\int_0^u \mathbf{E}\left[\sigma_1'(Y_v)\beta\left(Y_v\right)\sigma\left(Y_v\right)\right]\rho dvdu$$
$$\int_0^t \mathbf{E}\left[\int_0^u \sigma_2(Y_v)dv\int_0^t \sigma\left(Y_s\right)dW_s\right]du = \int_0^t\int_0^u \mathbf{E}\left[\sigma_2(Y_v)\int_0^t \sigma\left(Y_s\right)dW_s\right]dvdu$$

The first term above is due to Y_0 being adapted to the entire common filtration $\mathscr{F}_t$, the second is a simple application of conditional expectation and the third is similar with the integral we started with. A simple induction argument provides the following expression:

$$Cov\left(R_t - R_0, \sigma(Y_t) - \sigma(Y_0)\right) = \rho\left(\int_0^t \mathbf{E}\left[\sigma'(Y_s)\beta\left(Y_s\right)\sigma\left(Y_s\right)\right]ds\right.$$
$$\left.+\sum_{i=1}^{\infty}\int_0^t\int_0^{t_1}\cdots\int_0^{t_i}\mathbf{E}\left[\sigma_i'(Y_{t_{i+1}})\beta\left(Y_{t_{i+1}}\right)\sigma\left(Y_{t_{i+1}}\right)\right]dt_{i+1}dt_i\ldots dt_1\right),$$

with $\sigma_i(y)$ as in (27).

Repeating the argument above to calculate the variance of the process $\sigma(Y_t) - \sigma(Y_0$ produces a similar result:

$$Var\left(\sigma(Y_t) - \sigma(Y_0)\right) = V\left(\int_0^t \sigma_1(Y_u)du\right) + \int_0^t \mathbf{E}\left[\beta^2\left(Y_s\right)\left(\sigma'(Y_s)\right)^2\right]ds$$
$$+ 2\sum_{i=1}^{\infty}\int_0^t\int_0^{t_1}\cdots\int_0^{t_i}\mathbf{E}\left[\sigma_i'(Y_{t_{i+1}})\beta^2\left(Y_{t_{i+1}}\right)\sigma'\left(Y_{t_{i+1}}\right)\right]dt_{i+1}dt_i\ldots dt_1,$$

using the same notation as above. ■

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