Mean field simulation for Monte Carlo integration

Part II : Feynman-Kac models

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Some hyper-refs

- **Mean field simulation for Monte Carlo integration**. Chapman & Hall - Maths & Stats [600p.] (May 2013).
- **Feynman-Kac formulae**, Genealogical & Interacting Particle Systems with appl., Springer [573p.] (2004)
- **Particle approximations of Lyapunov exponents connected to Schrödinger operators and Feynman-Kac semigroups. ESAIM-P&S** (2003) (joint work with L. Miclo).
- **Coalescent tree based functional representations for some Feynman-Kac particle models**. Annals of Applied Probability (2009) (joint work with F. Patras, S. Rubenthaler).
- **On the concentration of interacting processes**. Foundations & Trends in Machine Learning [170p.] (2012). (joint work with P. Hu & L.M. Wu)
- **More references on the websitehttp://www.math.u-bordeaux1.fr/∼delmoral/index.html** [+ Links]

Contents

Introduction

Mean field simulation

=

Universal adaptive & interacting sampling technique

Part I $\rightsquigarrow$ 2 types of stochastic interacting particle models:

- Diffusive particle models with mean field drifts
  [McKean-Vlasov style]
- Interacting jump particle models
  [Boltzmann & Feynman-Kac style]

Part II $\subset$ Interacting jumps models

- **Interacting jumps = Recycling transitions =** 

- **Discrete generation models ($\Leftrightarrow$ geometric jump times)**

Equivalent particle algorithms

Sequential Monte Carlo	Sampling	Resampling
Particle Filters	Prediction	Updating
Genetic Algorithms	Mutation	Selection
Evolutionary Population	Exploration	Branching-selection
Diffusion Monte Carlo	Free evolutions	Absorption
Quantum Monte Carlo	Walkers motions	Reconfiguration
Sampling Algorithms	Transition proposals	Accept-reject-recycle

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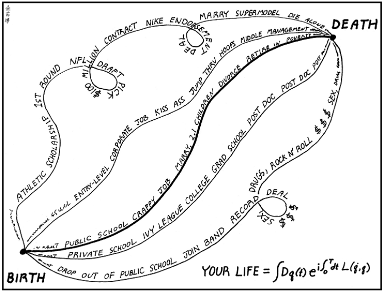
More lively buzzwords:
Bootstrapping, spawning, cloning, pruning, replenish, cloning, splitting, condensation, resampled Monte Carlo, enrichment, go with the winner, subset simulation, rejection and weighting, look-a-head sampling, pilot exploration,..

A single stochastic model

Particle interpretation of Feynman-Kac path integrals



The Path Integral Formulation of Your Life

Feynman-Kac models

FK models $=$ **Markov chain** $X_n \in E_n \oplus$ **functions** $G_n : E_n \to [0,\infty[$

$$d\mathbb{Q}_n := \frac{1}{\mathcal{Z}_n} \left\{ \prod_{0 \leq p < n} G_p(X_p) \right\} d\mathbb{P}_n \quad \text{with} \quad \mathbb{P}_n = \mathrm{Law}(X_0, \ldots, X_n)$$

Flow of n-marginals

$$\eta_n(f) = \gamma_n(f)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f) := \mathbb{E}\left(f(X_n) \prod_{0 \leq p < n} G_p(X_p) \right)$$

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Evolution equations :
with M_n Markov trans. of X_n and $Q_{n+1}(x, dy) = G_n(x) M_{n+1}(x, dy)$

$$\gamma_{n+1} = \gamma_n Q_{n+1} \quad \text{and} \quad \eta_{n+1} = \Psi_{G_n}(\eta_n) M_{n+1}$$

The 3 keys formulae

- Time marginal measures = Path space measures:

$$\gamma_n(f_n) = \mathbb{E}\left(f_n(\mathbf{X_n}) \prod_{0 \leq p < n} \mathbf{G_p}(\mathbf{X_p}) \right)$$

$$[\ \mathbf{X_n} := (X_0, \ldots, X_n) \ \ \& \ \ \mathbf{G_n}(\mathbf{X_n}) = G_n(X_n)] \ \Longrightarrow \ \eta_n = \mathbb{Q}_n$$

- Normalizing constants (= Free energy models):

$$\mathcal{Z}_n = \mathbb{E}\left(\prod_{0 \leq p < n} G_p(X_p) \right) = \prod_{0 \leq p < n} \eta_p(G_p)$$

The last key

- Backward Markov models

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) \propto \eta_0(dx_0)Q_1(x_0,dx_1)\ldots Q_n(x_{n-1},dx_n)$$

with

$$\begin{array}{rcl} Q_n(x_{n-1},dx_n) & := & G_{n-1}(x_{n-1})M_n(x_{n-1},dx_n) \\ & \stackrel{hyp}{=} & H_n(x_{n-1},x_n)\ \nu_n(dx_n) \\ & \Rightarrow & \eta_{n+1}(dx) = \dfrac{1}{\eta_n(G_n)}\ \eta_n\left(H_{n+1}(\centerdot,x)\right)\ \nu_{n+1}(dx) \end{array}$$

If we set

$$\mathbb{M}_{n+1,\eta_n}(x_{n+1},dx_n) = \frac{\eta_n(dx_n)\ H_{n+1}(x_n,x_{n+1})}{\eta_n\left(H_{n+1}(\centerdot,x_{n+1})\right)}$$

then we find the backward equation

$$\eta_{n+1}(dx_{n+1})\ \mathbb{M}_{n+1,\eta_n}(x_{n+1},dx_n) = \frac{1}{\eta_n(G_n)}\ \eta_n(dx_n)\ Q_{n+1}(x_n,dx_{n+1})$$

The last key (continued)

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) \propto \eta_0(dx_0)Q_1(x_0,dx_1)\ldots Q_n(x_{n-1},dx_n)$$

$\bigoplus$

$$\eta_{n+1}(dx_{n+1})\ \mathbb{M}_{n+1,\eta_n}(x_{n+1},dx_n) \propto \eta_n(dx_n)\ Q_{n+1}(x_n,dx_{n+1})$$

$$\Downarrow$$

Backward Markov chain model :

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) = \eta_n(dx_n)\ \mathbb{M}_{n,\eta_{n-1}}(x_n,dx_{n-1})\ldots\mathbb{M}_{1,\eta_0}(x_1,dx_0)$$

with the dual/backward <u>Markov</u> transitions

$$\mathbb{M}_{n+1,\eta_n}(x_{n+1},dx_n) \propto \eta_n(dx_n)\ H_{n+1}(x_n,x_{n+1})$$

Stability properties

- Transition/Excursions/Path spaces

$$X_n = (X'_n, X'_{n+1}) \qquad X_n = X'_{[T_n, T_{n+1}]} \qquad X_n = (X'_0, \ldots, X'_n)$$

- $\supset$ Continuous time models $\supset$ Langevin diffusions

$$X_n = X'_{[t_n, t_{n+1}]} \quad \& \quad G_n(X_n) = \exp \int_{t_n}^{t_{n+1}} V_t(X'_t) dt$$

OR Euler schemes (Langevin diff. $\rightsquigarrow$ Metropolis-Hasting moves)

OR Fully continuous time particle models $\rightsquigarrow$ Schrödinger operators

$$\frac{d}{dt}\gamma_t(f) = \gamma_t(L_t^V(f)) \quad \text{with} \quad L_t^V = L'_t + V_t$$

Important observation:

$$\gamma_t(1) = \mathbb{E}\left(\exp \int_0^t V_s(X'_s) ds\right) = \exp \int_0^t \eta_s(V_s) ds \quad \text{with} \quad \eta_t = \gamma_t/\gamma_t(1)$$

Stability properties

- **Change of probability measures-Importance sampling (IS) - Sequential Monte Carlo methods (SMC) :**

  For any target probability measures of the form

$$\begin{aligned}\mathbb{Q}_{n+1}(d(x_0,\ldots,x_{n+1})) &\propto \mathbb{Q}_n(d(x_0,\ldots,x_n)) \times Q_{n+1}(x_n,dx_{n+1}) \\ &\propto \eta_0(dx_0)Q_1(x_0,dx_1)\ldots Q_{n+1}(x_n,dx_{n+1})\end{aligned}$$

  and any Markov transition M'_{n+1} s.t. $Q_{n+1}(x_n,.) \ll M'_{n+1}(x_n,.)$

$$\begin{aligned}G_n(x_n,x_{n+1}) &= \frac{\text{Target at time } (n+1)}{\text{Target at time } (n) \times \text{Twisted transition}} \\ &= \frac{dQ_{n+1}(x_n,.)}{dM'_{n+1}(x_n,.)}(x_{n+1})\end{aligned}$$

Stability properties

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$$\Downarrow$$

  Feynman-Kac model with $X_n = (X'_n, X'_{n+1})$

$$\mathbb{Q}_n = \frac{1}{\mathcal{Z}_n}\left\{\prod_{0\leq p<n} G_p(X_p)\right\} d\mathbb{P}_n \quad \text{with} \quad \mathbb{P}_n = \mathrm{Law}(X_0,\ldots,X_n)$$

Markov restrictions

- **Confinements:** X_n random walk $\in \mathbb{Z}^d \supset A \quad \& \quad G_n := 1_A$

$$\mathbb{Q}_n = \mathrm{Law}\left((X_0, \ldots, X_n) \mid X_p \in A, \ \forall 0 \leq p < n\right)$$

$$\mathcal{Z}_n = \mathrm{Proba}\left(X_p \in A, \ \forall 0 \leq p < n\right)$$

- **SAW :** $X_n = (X'_p)_{0 \leq p \leq n} \quad \& \quad G_n(X_n) = 1_{X'_n \notin \{X'_0, \ldots, X'_{n-1}\}}$

$$\mathbb{Q}_n = \mathrm{Law}\left((X'_0, \ldots, X'_n) \mid X'_p \neq X'_q, \ \forall 0 \leq p < q < n\right)$$

$$\mathcal{Z}_n = \mathrm{Proba}\left(X'_p \neq X'_q, \ \forall 0 \leq p < q < n\right)$$

Multilevel splitting

Decreasing level sets $A_n \downarrow$, with B non critical recurrent subset.

$$T_n := \inf \{ t \geq T_{n-1} \ : \ X'_t \in (A_n \cup B) \}$$

Excursion valued Feynman-Kac model:

$$X_n = (X'_t)_{t \in [T_n, T_{n+1}]} \quad \& \quad G_n(X_n) = 1_{A_{n+1}}(X'_{T_{n+1}})$$

$$\Downarrow$$

$$\mathbb{Q}_n = \text{Law}\left(X'_{[T_0, T_n]} \mid X' \text{ hits } A_{n-1} \text{ before } B \right)$$

$$\mathcal{Z}_n = \mathbb{P}(X' \text{ hits } A_{n-1} \text{ before } B)$$

Absorbing Markov chains

$$X_n^c \in E_n^c \xrightarrow{\textit{absorption} \sim (1-G_n)} \widehat{X}_n^c \xrightarrow{\textit{exploration} \sim M_{n+1}} X_{n+1}^c$$

$$\mathbb{Q}_n = \mathrm{Law}((X_0^c, \ldots, X_n^c) \mid T^{abs.} \geq n) \quad \& \quad \mathcal{Z}_n = \mathrm{Proba}\left(T^{abs.} \geq n\right)$$

Quasi-invariant measures : $(G_n, M_n) = (G, M)$ **&** M μ**-reversible**

$$\frac{1}{n} \log \mathbb{P}\left(T^{abs.} \geq n\right) \simeq_{n\uparrow\infty} \lambda \ = \text{top spect. of} \ \ Q(x, dy) = G(x)M(x, dy)$$

[Frobenius theo] $Q(h) = \lambda h = \lambda \times$ eigenfunction (ground state)

$$\mathbb{P}(X_n^c \in dx \mid T^{abs.} > n) \simeq_{n\uparrow\infty} \frac{1}{\mu(h)} \ h(x) \ \mu(dx)$$

Doob h-processes X^h

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) \propto \mathbb{P}((X_0^h,\ldots,X_n^h) \in d(x_0,\ldots,x_n))\ h^{-1}(x_n)$$

with

$$M^h(x,dy) = \frac{1}{\lambda} h^{-1}(x) Q(x,dy) h(y) = \frac{M(x,dy)h(y)}{M(h)(x)}$$

- **Invariant measure $\mu_h = \mu_h M^h$ & Additive functionals**

$$\overline{F}_n(x_0,\ldots,x_n) = \frac{1}{n+1} \sum_{0\leq p\leq n} f(x_p) \Longrightarrow \mathbb{Q}_n(\overline{F}_n) \simeq_n \mu_h(f)$$

- **If $G = G^\theta$ depends on some $\theta \in \mathbb{R} \rightsquigarrow f := \frac{\partial}{\partial\theta} \log G^\theta$**

$$\frac{\partial}{\partial\theta} \log \lambda^\theta \simeq_n \frac{1}{n+1} \frac{\partial}{\partial\theta} \log \mathcal{Z}_{n+1}^\theta = \mathbb{Q}_n(\overline{F}_n)$$

Gradient of Markov semigroups

$$X_{n+1}(x) = \mathcal{F}_n(X_n(x), W_n) \ \left(X_0(x) = x \in \mathbb{R}^d\right) \ \rightsquigarrow \ P_n(f)(x) := \mathbb{E}\left(f(X_n(x))\right)$$

First variational equation

$$\frac{\partial X_{n+1}}{\partial x}(x) = A_n(x, W_n)\ \frac{\partial X_n}{\partial x}(x) \quad \text{with} \quad A_n^{(i,j)}(x,w) = \frac{\partial \mathcal{F}_n^i(\,\cdot\,, w)}{\partial x^j}(x)$$

Random process on the sphere $U_0 = u_0 \in \mathbb{S}^{d-1}$

$$U_{n+1} = A_n(X_n, W_n)U_n / \|A_n(X_n, W_n)U_n\| = \frac{\frac{\partial X_n}{\partial x}(x)\ u_0}{\left\|\frac{\partial X_n}{\partial x}(x)\ u_0\right\|}$$

Feynman-Kac model $\mathcal{X}_n = (X_n, U_n, W_n)$ **&** $\mathcal{G}_n\,(x,u,w) = \|\mathcal{A}_n(x,w)\ u\|$

$$\nabla P_{n+1}(f)(x)\ u_0 = \mathbb{E}\left(\underbrace{F(\mathcal{X}_{n+1})}_{\nabla f(X_{n+1})\ U_{n+1}}\ \underbrace{\prod_{0\le p\le n} \mathcal{G}_p\,(\mathcal{X}_p)}_{\left\|\frac{\partial X_n}{\partial x}(x)\ u_0\right\|}\right)$$

Bad tempting ideas

I.i.d. weighted samples X_n^i

$$\mathcal{Z}_n := \mathbb{E}\left(\prod_{0\leq p<n} G_p(X_p)\right) \simeq \mathcal{Z}_n^N := \frac{1}{N}\sum_{i=1}^{N}\prod_{0\leq p<n} G_p(X_p^i)$$

or in terms of killing-absorption models

$$\mathcal{Z}_n = \mathbb{P}\left(T \geq n\right) \simeq \mathcal{Z}_n^N := \frac{1}{N}\sum_{1\leq i\leq N} 1_{T^i\geq n}$$

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Example : X_n simple RW $\in \mathbb{Z}^d$, $G_n = 1_{[-10,10]}$ (killed at the boundary)

$$\Downarrow$$

$$\exists n = n(\omega) \ : \ \mathcal{Z}_n^N = 0$$

Bad tempting ideas

I.i.d. weighted samples X_n^i

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$$\Downarrow$$

$$\exists n = n(\omega) \ : \ \mathcal{Z}_n^N = 0$$

and

$$N\,\mathbb{E}\left(\left[\frac{\mathcal{Z}_n^N}{\mathcal{Z}_n} - 1\right]^2\right) = \frac{1-\mathcal{Z}_n}{\mathcal{Z}_n}$$

$$\simeq \mathrm{Proba}(X_p \in A,\ \forall 0 \leq p < n)^{-1} = \mathbb{P}\left(T \geq n\right)^{-1}$$

Our objective

Find an unbiased estimate $\mathcal{Z}_n^N$ s.t.

$$N \; \mathbb{E}\left(\left[\frac{\mathcal{Z}_n^N}{\mathcal{Z}_n} - 1\right]^2\right) \leq c \times n$$

using the multiplicative formula

$$\mathbb{E}\left(\prod_{0 \leq p < n} G_p(X_p)\right) = \prod_{0 \leq p < n} \eta_p(G_p)$$

And estimating/Learning each (larger) terms in the product

$$\eta_p(G_p) \simeq \eta_p^N(G_p) \quad \text{with} \quad \eta_p^N = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i}$$

Flow of n-marginals [X_n Markov with transitions M_n]

$$\eta_n(f) = \gamma_n(f)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f) := \mathbb{E}\left(f(X_n) \prod_{0 \leq p < n} G_p(X_p)\right)$$

$$\Updownarrow \; (\gamma_n(1) = \mathcal{Z}_n)$$

Nonlinear evolution equation :

$$\begin{aligned} \eta_{n+1} &= \Psi_{G_n}(\eta_n) M_{n+1} \\ \mathcal{Z}_{n+1} &= \eta_n(G_n) \times \mathcal{Z}_n \end{aligned}$$

$$\downarrow$$

Nonlinear m.v.p. = Law of a Markov $\overline{X}_n$ (perfect sampler)

$$\begin{aligned} \eta_{n+1} &= \Phi_{n+1}(\eta_n) \\ &= \eta_n \left(S_{n,\eta_n} M_{n+1}\right) = \eta_n \mathbf{K}_{\mathbf{n+1},\eta_\mathbf{n}} = \mathrm{Law}(\overline{X}_{n+1}) \end{aligned}$$

Examples related to product models

$$\eta_n(dx) := \frac{1}{\mathcal{Z}_n} \left\{ \prod_{p=0}^{n} h_p(x) \right\} \lambda(dx) \quad \text{with} \quad h_p \geq 0$$

2 illustrations:

$$h_p(x) = e^{-(\beta_{p+1}-\beta_p)V(x)} \quad \beta_p \uparrow \quad \Longrightarrow \quad \eta_n(dx) = \frac{1}{\mathcal{Z}_n} \, e^{-\beta_n V(x)} \, \lambda(dx)$$

$$h_p(x) = 1_{A_{p+1}}(x) \quad A_p \downarrow \quad \Longrightarrow \quad \eta_n(dx) = \frac{1}{\mathcal{Z}_n} \, 1_{A_n}(x) \, \lambda(dx)$$

For any MCMC transitions M_n with target η_n, we have

$$\eta_{n+1} = \eta_{n+1} M_{n+1} = \Psi_{h_{n+1}}(\eta_n) M_{n+1} \subset \textbf{Feynman-Kac model}$$

Mean field particle model

McKean Markov chain model

$$\eta_{n+1} = \eta_n K_{n+1,\eta_n} = \mathrm{Law}(\overline{X}_n)$$

$$\Downarrow$$

Markov chain $\xi_n = (\xi_n^i)_{1\leq i\leq N} \in E_n^N$

$$\xi_n^i \quad \rightsquigarrow \quad \xi_{n+1}^i \quad \sim \quad K_{n+1,\eta_n^N}(\xi_n^i, dx) \quad \text{with} \quad \eta_n^N = \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi_n^i}$$

and the (unbiased) particle normalizing constants

$$\mathcal{Z}_{n+1}^N = \eta_n^N(G_n) \times \mathcal{Z}_n^N = \prod_{0\leq p\leq n} \eta_p^N(G_p)$$

Mean field particle model

Mean field FK simulation $\xi_n^i \rightsquigarrow \xi_{n+1}^i \sim K_{n+1,\eta_n^N} = S_{n,\eta_n^N} M_{n+1}$

$$\Updownarrow$$

$\rightsquigarrow$ *Sequential particle simulation technique* (**SMC**)

G_n-acceptance-rejection with recycling $\oplus$ M_{n+1}-propositions

Mean field particle model

Mean field FK simulation $\xi_n^i \rightsquigarrow \xi_{n+1}^i \sim K_{n+1,\eta_n^N} = S_{n,\eta_n^N} M_{n+1}$

$$\Updownarrow$$

$\rightsquigarrow$ *Sequential particle simulation technique* (**SMC**)

G_n-acceptance-rejection with recycling $\oplus$ M_{n+1}-propositions

$\rightsquigarrow$ *Genetic type branching particle algorithm* (**GA**)

$$\xi_n \xrightarrow{G_n-selection} \widehat{\xi}_n \xrightarrow{M_n-mutation} \xi_{n+1}$$

Mean field particle model

Mean field FK simulation $\xi_n^i \rightsquigarrow \xi_{n+1}^i \sim K_{n+1,\eta_n^N} = S_{n,\eta_n^N} M_{n+1}$

$$\Updownarrow$$

$\rightsquigarrow$ *Sequential particle simulation technique* (**SMC**)

G_n-acceptance-rejection with recycling $\oplus$ M_{n+1}-propositions

$\rightsquigarrow$ *Genetic type branching particle algorithm* (**GA**)

$$\xi_n \xrightarrow{G_n-selection} \widehat{\xi}_n \xrightarrow{M_n-mutation} \xi_{n+1}$$

$\rightsquigarrow$ *Reconfiguration Monte Carlo (particles $\rightsquigarrow$ walkers)* (**QMC**)

$$(\text{Selection}, \text{Mutation}) = (\text{Reconfiguration}, \text{exploration})$$

Continuous time Feynman-Kac particle models

Master equation

$$\eta_t(\centerdot) = \frac{\gamma_t(\centerdot)}{\gamma_t(1)} = \mathrm{Law}(\overline{X}_t) \quad \Rightarrow \quad \frac{d}{dt}\eta_t(f) = \eta_t(L_{t,\eta_t}(f))$$

Continuous time Feynman-Kac particle models

Master equation

$$\eta_t(\centerdot) = \frac{\gamma_t(\centerdot)}{\gamma_t(1)} = \mathrm{Law}(\overline{X}_t) \quad \Rightarrow \quad \frac{d}{dt}\eta_t(f) = \eta_t(L_{t,\eta_t}(f))$$

(ex. : $V_t = -U_t \leq 0$)

$$L_{t,\eta_t}(f)(x) = \underbrace{L'_t(f)(x)}_{\text{free exploration}} + \underbrace{U_t(x)}_{\text{acceptance rate}} \int (f(y) - f(x)) \underbrace{\eta_t(dy)}_{\text{interacting jump law}}$$

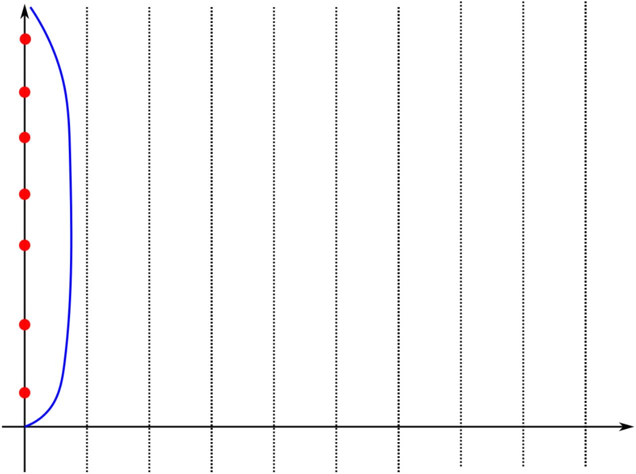
$$\Updownarrow$$

$$L_{t,\eta_t}(f) = \underbrace{L'_t(f) - U_t\ f}_{\text{Schrödinger's op.}} + \underbrace{U_t\ \eta_t(f)}_{\text{normalizing-stabilizing term}}$$

$$\Downarrow$$

Particle model: Survival-acceptance rates $\oplus$ Recycling jumps

Graphical illustration : $\eta_n \simeq \eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i}$



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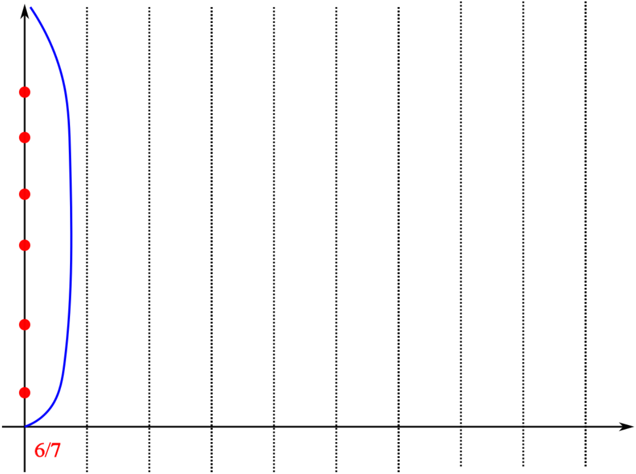

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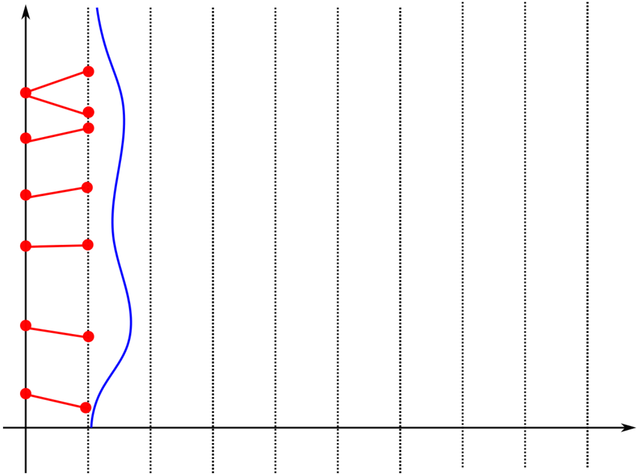

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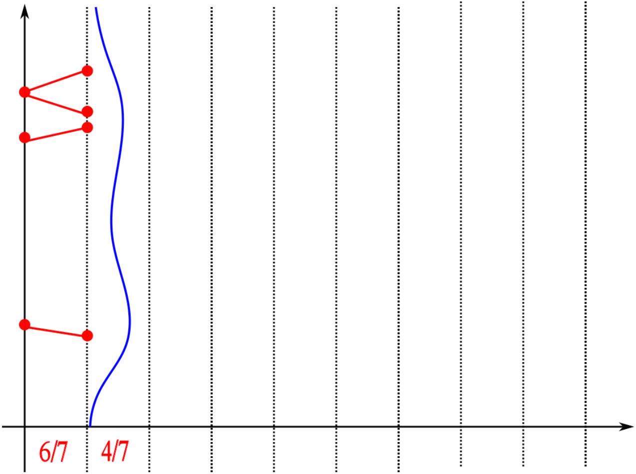

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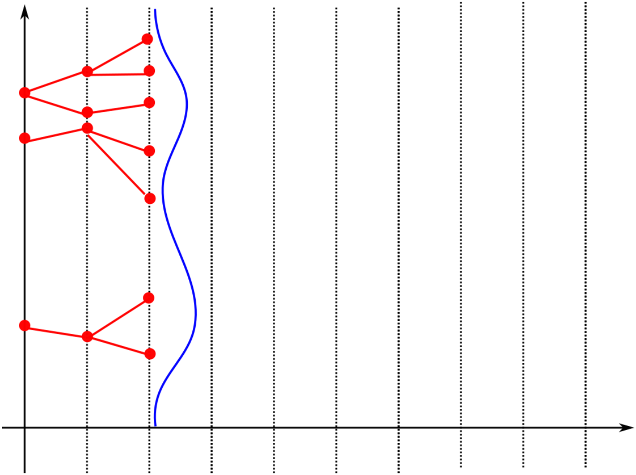

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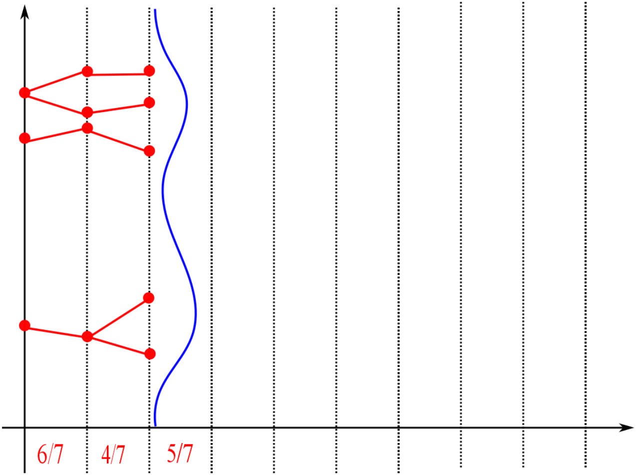

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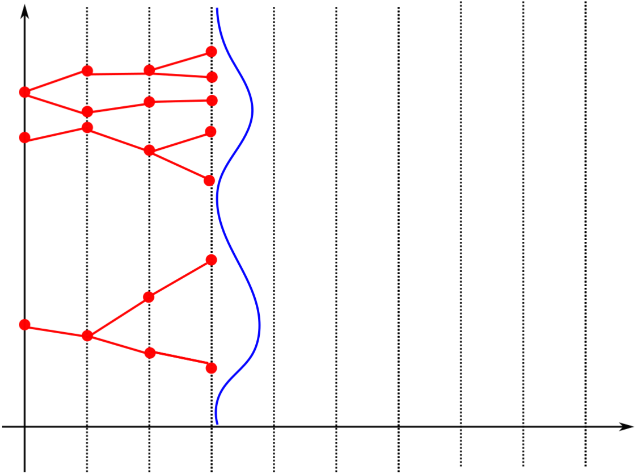

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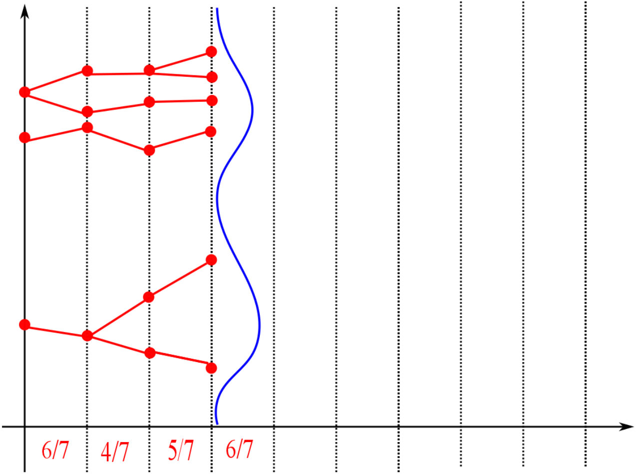

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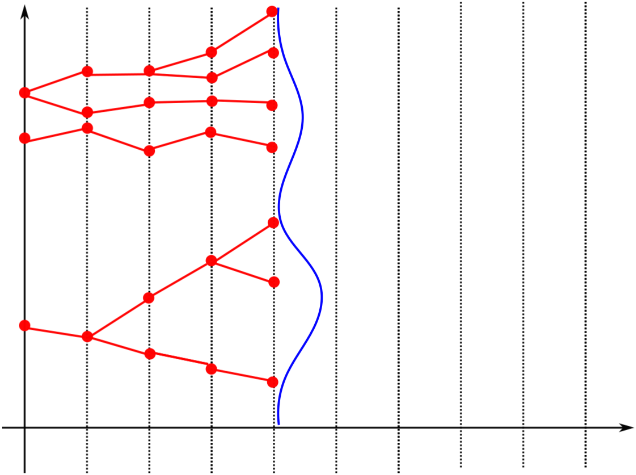

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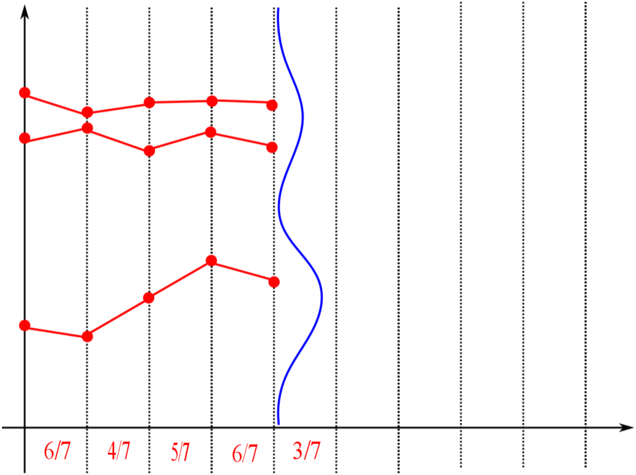

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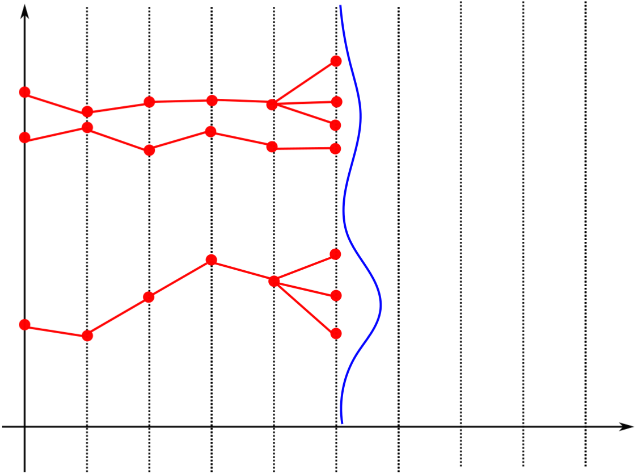

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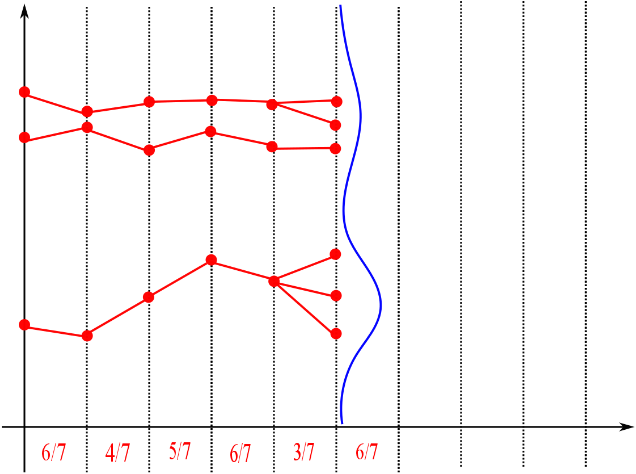

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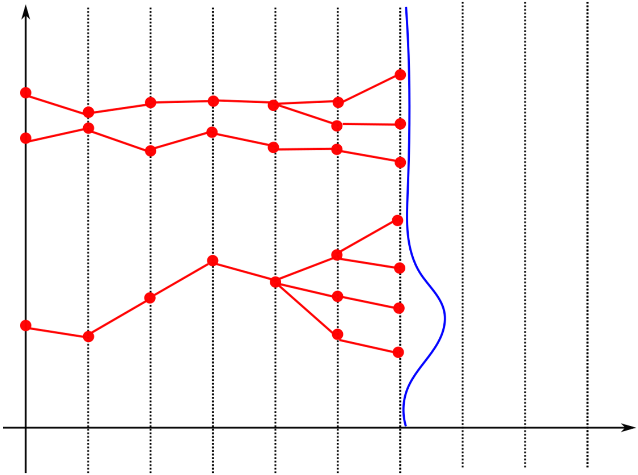

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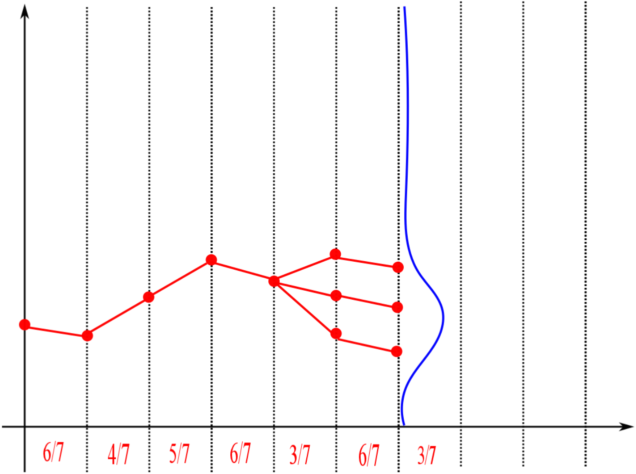

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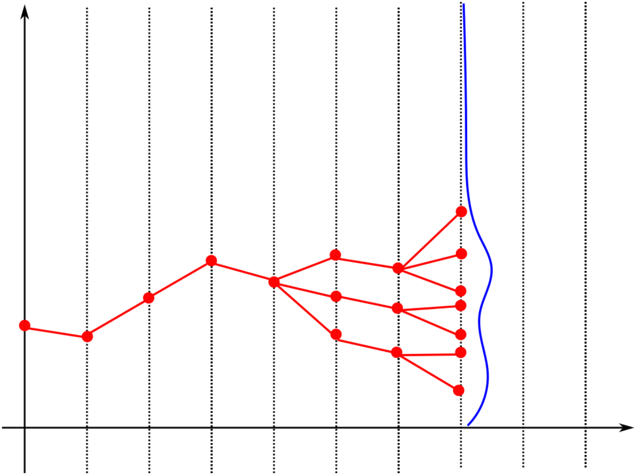

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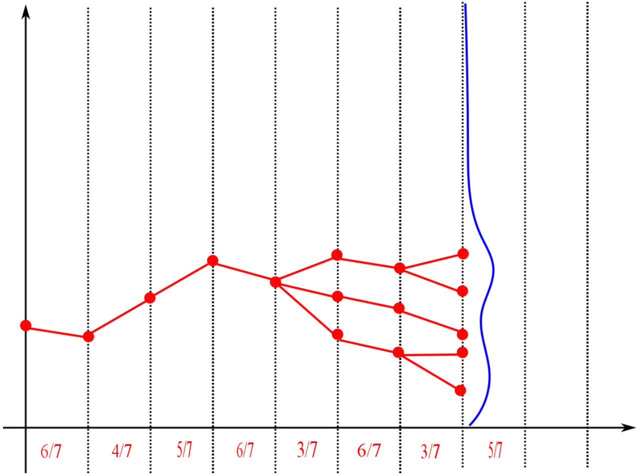

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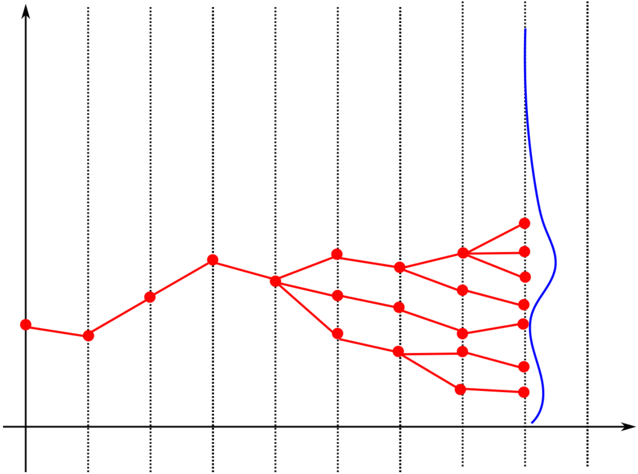

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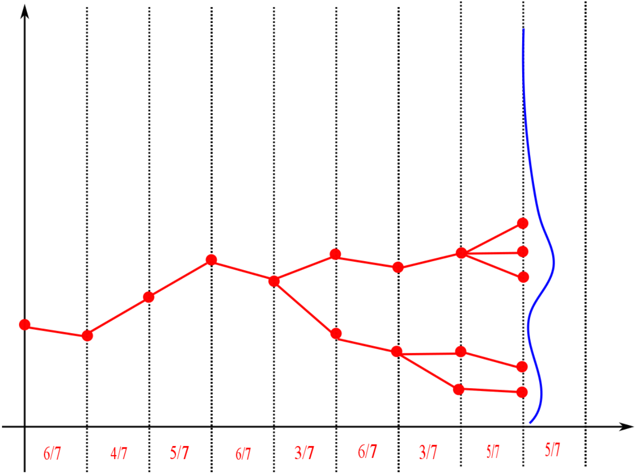

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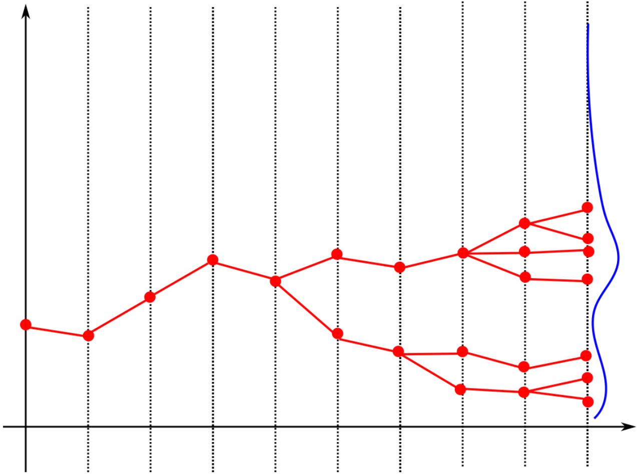

How to use the full ancestral tree model ?

$$G_{n-1}(x_{n-1})M_n(x_{n-1}, dx_n) \stackrel{hyp}{=} H_n(x_{n-1}, x_n)\ \nu_n(dx_n)$$

$$\Rightarrow \mathbb{Q}_n(d(x_0, \ldots, x_n)) = \eta_n(dx_n) \quad \underbrace{\mathbb{M}_{n,\eta_{n-1}}(x_n, dx_{n-1})}_{\propto\ \eta_{n-1}(dx_{n-1})\ H_n(x_{n-1}, x_n)} \ldots \mathbb{M}_{1,\eta_0}(x_1, dx_0)$$

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Particle approximation = Random stochastic matrices

$$\mathbb{Q}_n^N(d(x_0,\ldots,x_n)) = \eta_n^N(dx_n)\ \mathbb{M}_{n,\eta_{n-1}^N}(x_n,dx_{n-1})\ldots\mathbb{M}_{1,\eta_0^N}(x_1,dx_0)$$

How to use the full ancestral tree model ?

$$G_{n-1}(x_{n-1})M_n(x_{n-1}, dx_n) \overset{hyp}{=} H_n(x_{n-1}, x_n)\ \nu_n(dx_n)$$

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Particle approximation = Random stochastic matrices

$$\mathbb{Q}_n^N(d(x_0, \ldots, x_n)) = \eta_n^N(dx_n)\ \mathbb{M}_{n,\eta_{n-1}^N}(x_n, dx_{n-1}) \ldots \mathbb{M}_{1,\eta_0^N}(x_1, dx_0)$$

Ex.: Additive functionals $\mathbf{f_n}(x_0, \ldots, x_n) = \frac{1}{n+1} \sum_{0 \le p \le n} f_p(x_p)$

$$\mathbb{Q}_n^N(\mathbf{f_n}) := \frac{1}{n+1} \sum_{0 \le p \le n} \eta_n^N \underbrace{\mathbb{M}_{n,\eta_{n-1}^N} \ldots \mathbb{M}_{p+1,\eta_p^N}(f_p)}_{\text{matrix operations}}$$

4 particle estimates

- Individuals ξ_n^i "almost" iid with law

$$\eta_n \simeq \eta_n^N = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i}$$

- Path space models $\rightsquigarrow$ Ancestral lines "almost" iid with law

$$\mathbb{Q}_n \simeq \eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\text{Ancestral line}_n(i)}$$

- Backward particle model

$$\mathbb{Q}_n^N(d(x_0, \ldots, x_n)) = \eta_n^N(dx_n)\ \mathbb{M}_{n,\eta_{n-1}^N}(x_n, dx_{n-1}) \ldots \mathbb{M}_{1,\eta_0^N}(x_1, dx_0)$$

- Normalizing constants

$$\mathcal{Z}_{n+1} = \prod_{0 \leq p \leq n} \eta_p(G_p) \simeq_{N\uparrow\infty} \mathcal{Z}_{n+1}^N = \prod_{0 \leq p \leq n} \eta_p^N(G_p) \quad (\text{Unbiased})$$

Island models

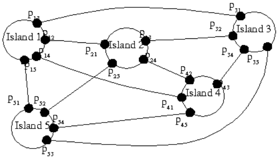

Fig 3.4 Schematic of a genetic algorithm using island migration

Reminder : the unbiased property

$$\mathbb{E}\left(\mathbf{f_n}(\mathbf{X_n}) \prod_{0\leq p<n} \mathbf{G_p}(\mathbf{X_p})\right) = \mathbb{E}\left(\eta_n^N(\mathbf{f_n}) \prod_{0\leq p<n} \eta_p^N(\mathbf{G_p})\right)$$

$$= \mathbb{E}\left(\mathbf{F_n}(\mathcal{X}_n) \prod_{0\leq p<n} \mathcal{G}_p(\mathcal{X}_p)\right)$$

with the Island evolution Markov chain model

$$\mathcal{X}_n := \eta_n^N \qquad \text{and} \quad \mathcal{G}_n(\mathcal{X}_n) = \eta_n^N(\mathbf{G_n}) = \mathcal{X}_n(\mathbf{G_n})$$

$\Rightarrow$ particle model with $(\mathcal{X}_n, \mathcal{G}_n(\mathcal{X}_n))$ = Interacting Island particle model

Some key advantages

- Mean field models=Stochastic linearization/perturbation technique

$$\eta_n^N = \eta_{n-1}^N K_{n,\eta_{n-1}^N} + \frac{1}{\sqrt{N}}\ V_n^N$$

  with $V_n^N \simeq V_n$ independent centered Gaussian fields .

- $\eta_n = \eta_{n-1} K_{n,\eta_{n-1}}$ stable $\Rightarrow$ Non propagation of local sampling errors

  $\Longrightarrow$ Uniform control w.r.t. the time horizon

- "No burning, no need to study the stability of MCMC models".
- Stochastic adaptive grid approximation
- Nonlinear system $\rightsquigarrow$ positive - beneficial interactions.
- Simple and natural sampling algorithm.
- Local conditional iid samples $\oplus$ Stability of nonlinear sg

  $\rightsquigarrow$ New concentration and empirical process theory

Current population models

Constants $(\mathbf{c_1}, \mathbf{c_2})$ related to (**bias**,**variance**), c finite constant
Test functions/observables $\|f_n\| \leq 1$, $\forall\ (x \geq 0, n \geq 0, N \geq 1)$.

When $E_n = \mathbb{R}^d$:

$$F_n(y) := \eta_n\left(1_{(-\infty,y]}\right) \quad \text{and} \quad F_n^N(y) := \eta_n^N\left(1_{(-\infty,y]}\right) \text{ with } y \in \mathbb{R}^d$$

The probability of any of the following events is greater than $1 - e^{-x}$.

$$\left|\eta_n^N - \eta_n\right|(f_n) \leq \frac{c_1}{N}\ \left(1 + x + \sqrt{x}\right) + \frac{c_2}{\sqrt{N}}\ \sqrt{x}$$

$$\sup_{0 \leq p \leq n} \left|\left[\eta_p^N - \eta_p\right](f_p)\right| \leq c\ \sqrt{x \log(n+e)/N}$$

$$\left\|F_n^N - F_n\right\| \leq c\ \sqrt{d\ (x+1)/N}$$

Particle free energy/Genealogical tree models

Constants $(\mathbf{c_1}, \mathbf{c_2})$ related to (**bias**,**variance**), c finite constant $\forall\ (x \geq 0, n \geq 0, N \geq 1)$.

The probability of any of the following events is greater than $1 - e^{-x}$

$$\left|\frac{1}{n}\log \mathcal{Z}_n^N - \frac{1}{n}\log \mathcal{Z}_n\right| \leq \frac{c_1}{N}\ \left(1 + x + \sqrt{x}\right) + \frac{c_2}{\sqrt{N}}\ \sqrt{x}$$

$$\left|\left[\eta_{\mathbf{n}}^{\mathbf{N}} - \mathbb{Q}_n\right](f_n)\right| \leq c_1\ \frac{(n+1)}{N}\ \left(1 + x + \sqrt{x}\right) + c_2\ \sqrt{\frac{(n+1)}{N}}\ \sqrt{x}$$

with $\eta_{\mathbf{n}}^{\mathbf{N}} =$ Genealogical tree models $:= \eta_n^N$ (in path space)

Backward particle models

Constants $(\mathbf{c_1}, \mathbf{c_2})$ related to (**bias**,**variance**), c finite constant.
For any normalized additive functional $\mathbf{f_n}$ with $\|f_p\| \leq 1$, $\forall$ $(x \geq 0, n \geq 0, N \geq 1)$

The probability of the following event is greater than $1 - e^{-x}$

$$\left|\left[\mathbb{Q}_n^N - \mathbb{Q}_n\right](\mathbf{f}_n)\right| \leq c_1 \, \frac{1}{N} \, (1 + (x + \sqrt{x})) + c_2 \, \sqrt{\frac{x}{N(n+1)}}$$