Mean field simulation for Monte Carlo integration

Part I : Intro. nonlinear Markov models (+links integro-diff eq.)

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Some hyper-refs

- **Mean field simulation for Monte Carlo integration**. Chapman & Hall - Maths & Stats [600p.] (May 2013).
- **Feynman-Kac formulae**, Genealogical & Interacting Particle Systems with appl., Springer [573p.] (2004)
- **Concentration Inequalities for Mean Field Particle Models**. `Ann. Appl. Probab` (2011). *(+ Rio)*.
- **Coalescent tree based functional representations for some Feynman-Kac particle models** Annals of Applied Probability (2009) (+ Patras, Rubenthaler)
- **Particle approximations of a class of branching distribution flows arising in multi-target tracking**. `SIAM Control. & Opt.` (2011). *(joint work with Caron, Doucet, Pace)*
- **On the stability & the approximation of branching distribution flows, with applications to nonlinear multiple target filtering**. `Stoch. Analysis and Appl.` (2011). *(joint work with Caron, Pace, Vo)*.
- **More references on the websitehttp://www.math.u-bordeaux1.fr/∼delmoral/index.html** [+ Links]

Contents

Lebesgue integral on a measurable state space E

$(\mu, f) =$(measure, function)$\in (\mathcal{M}(E) \times \mathcal{B}_b(E))$

$$\mu(f) = \int \mu(dx)\ f(x)$$

Delta-Dirac Measure at $a \in E$

$$\mu = \delta_a \quad \Rightarrow \quad \delta_a(f) = \int f(x)\ \delta_a(dx) = f(a)$$

Normalization of a positive measure $\mu \in \mathcal{M}_+(E)$ (when $\mu(1) \neq 0$)

$$\overline{\mu}(dx) := \mu(dx)/\mu(1) = \text{probability} \in \mathcal{P}(E)$$

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Example

$$\mu = 10 \times \text{Law}(X)$$

$$\Downarrow$$

$$\mu(1) = 10 \ \ \& \ \ \overline{\mu} = \text{Law}(X) \ \ \& \ \ \overline{\mu}(f) = \mathbb{E}(f(X))$$

Examples

- E={1,...,d}$\rightsquigarrow$ Matrix-Vector notation

$$\mu := [\mu(1), \ldots, \mu(d)] \quad \text{and} \quad f := \begin{bmatrix} f(1) \\ \vdots \\ f(d) \end{bmatrix} \Rightarrow \mu(f) := \sum_{i=1}^{d} \mu(i)\ f(i)$$

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- $\mathcal{X} = \sum_{1 \leq i \leq N} \delta_{X^i}$ spatial Poisson point process with intensity μ

$$\Updownarrow$$

$$N = \text{Poisson random variable with } \mathbb{E}(N) = \mu(1) \quad \text{and} \quad X^i \;\; iid \quad \sim \overline{\mu}$$

$$\Downarrow$$

$$\mathbb{E}(\mathcal{X}(f)) = \mathbb{E}(\mathbb{E}(\mathcal{X}(f) \mid N)) = \mathbb{E}(N\, \overline{\mu}(f)) = \mu(1)\, \overline{\mu}(f) = \mu(f)$$

$Q(x,dy)$ integral operator from E into E'

Two operator actions :

$$f \in \mathcal{B}_b(E') \mapsto Q(f) \in \mathcal{B}_b(E) \quad \text{and} \quad \mu \in \mathcal{M}(E) \mapsto \mu Q \in \mathcal{M}(E')$$

with

$$\begin{aligned} Q(f)(x) &= \int Q(x,dx')f(x') \\ [\mu Q](dx') &= \int \mu(dx)Q(x,dx') \qquad (\Longleftrightarrow [\mu Q](f) := \mu[Q(f)]\ ) \end{aligned}$$

and the composition

$$(Q_1Q_2)(x,dx'') = \int Q_1(x,dx')Q_2(x',dx'')$$

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Notation (used in variance descriptions)

$$Q([f - Q(f)]^2)(x) := \int \ Q(x, dy) \ [f(y) - Q(f)(x)]^2$$

Q **Markov operator** $\Leftrightarrow \forall x \quad Q(x, dy) \in \mathcal{P}(E)$

$\rightsquigarrow$ notation : M or K (Markov transitions-kernel/Stochastic matrices)

Finite state spaces $E = \{1, \ldots, d\}$ and $E' = \{1, \ldots, d'\}$:

Action on the right

$$f := \left[\begin{array}{c} f(1) \\ \vdots \\ f(d') \end{array}\right] \in \mathcal{B}_b(E') \mapsto Q(f) = \left[\begin{array}{c} Q(f)(1) \\ \vdots \\ Q(f)(d) \end{array}\right] \in \mathcal{B}_b(E)$$

Matrix operation

$$Q(f) = \left[\begin{array}{c} Q(f)(1) \\ \vdots \\ Q(f)(d) \end{array}\right] = \left[\begin{array}{ccc} Q(1,1) & \ldots & Q(1,d') \\ \vdots & \ldots & \vdots \\ Q(d,1) & \ldots & Q(d,d') \end{array}\right] \left[\begin{array}{c} f(1) \\ \vdots \\ f(d') \end{array}\right]$$

Finite state spaces $E = \{1, \ldots, d\}$ and $E' = \{1, \ldots, d'\}$:

Action on the left

$$\mu = [\mu(1), \ldots, \mu(d)] \in \mathcal{M}(E) \mapsto \mu Q = [(\mu Q)(1), \ldots, (\mu Q)(d')] \in \mathcal{M}(E')$$

Matrix operation

$$\mu Q = [\mu(1), \ldots, \mu(d)] \begin{bmatrix} Q(1,1) & \ldots & Q(1,d') \\ \vdots & \ldots & \vdots \\ Q(d,1) & \ldots & Q(d,d') \end{bmatrix}$$

Boltzmann-Gibbs transformation : $G \geq 0$ s.t. $\mu(G) > 0$

$$\Psi_G(\mu)(dx) = \frac{1}{\mu(G)} \; G(x) \; \mu(dx)$$

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Bayes' rule (with fixed observation y)

$$\mu(dx) = p(x)dx \quad \text{and} \quad G(x) = p(y|x)$$

$$\Downarrow$$

$$\Psi_G(\mu)(dx) = \frac{1}{p(y)}\ p(y|x)\ p(x)dx = p(x|y)\ dx$$

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Restriction

$$\mu(dx) = \mathbb{P}(X \in dx) = p(x)dx \quad \text{and} \quad G(x) = 1_A(x)$$

$$\Downarrow$$

$$\Psi_G(\mu)(dx) = \frac{1}{\mathbb{P}(X \in A)}\ 1_A(x)\ p(x)dx = \mathbb{P}(X \in dx \mid X \in A)$$

Boltzmann-Gibbs transformation : $G \geq 0$ s.t. $\mu(G) > 0$

$$\Psi_G(\mu)(dx) = \frac{1}{\mu(G)}\ G(x)\ \mu(dx)$$

$$\Downarrow$$

$\exists$ **Markov transport equation**

$$\Psi_G(\mu)(dy) = \int \mu(dx) S_\mu(x, dy) \Longleftrightarrow \Psi_G(\mu) = \mu S_\mu$$

Example 1 : $(G \leq 1) \rightsquigarrow$ **accept/reject/recycling/interacting jumps**

$$S_\mu(x, dy) = G(x)\delta_x(dy) + (1 - G(x))\ \Psi_G(\mu)(dy)$$

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Note :

$$S_{\frac{1}{N}\sum_{1\leq j\leq N}\delta_{X^j}}(X^i, dy)$$

$$= G(X^i)\delta_{X^i}(dy) + (1 - G(X^i))\ \sum_{1\leq j\leq N} \frac{G(X^j)}{\sum_{1\leq k\leq N} G(X^k)}\ \delta_{X^j}(dy)$$

Other examples of transport equations $\Psi_G(\mu) = \mu S_\mu$

Example 2 : $\forall \epsilon_\mu \quad \text{s.t.} \quad \epsilon_\mu G \leq 1 \quad \mu - \text{a.e.}$

$$S_\mu(x, dy) = \epsilon_\mu \; G(x) \; \delta_x(dy) + (1 - \epsilon_\mu G(x)) \; \Psi_G(\mu)(dy)$$

Example 3 : $\forall a$ **s.t.** $G > a$

$$S_\mu(x, dy) = \frac{a}{\mu(G)} \; \delta_x(dy) + \left(1 - \frac{a}{\mu(G)}\right) \; \Psi_{G-a}(\mu)(dy)$$

Example 4 : $\forall G$

$$S_\mu(x, dy) = \alpha(x) \; \delta_x(dy) + (1 - \alpha(x)) \; \Psi_{[G - G(x)]_+}(\mu)(dy)$$

with the acceptance rate

$$\alpha(x) \quad = \quad \mu[G \wedge G(x)]/\mu(G)$$

Example: Markov chain $X_{n+1} = F_n(X_n, W_n)$

$$\begin{aligned} \eta_0 &= \text{Law}(X_0) \quad \text{and} \quad M_n(x_{n-1}, dx_n) = \mathbb{P}\left(X_n \in dx_n \mid X_{n-1} = x_{n-1}\right) \\ \eta_0 M_1 &= \text{Law}(X_1) \quad \text{and} \quad M_n(f)(x) = \mathbb{E}(f(X_n) \mid X_{n-1} = x) \end{aligned}$$

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as well as

$$\begin{aligned} (M_1 \ldots M_n)(x_0, dx_n) &= \mathbb{P}(X_n \in dx_n \mid X_0 = x_0) \\ \eta_0 M_1 \ldots M_n &= \mathrm{Law}(X_n) \end{aligned}$$

E$=\{1,\ldots,d\} \rightsquigarrow$ Matrix-Vector notation

Markov chain models $X_{n+1} = F_n(X_n, W_n)$

Linear evolution equations $\mathbf{Law(X_n)} := \eta_n$

$$\eta_{n+1} = \eta_n M_{n+1}$$

$$\Downarrow$$

$E=\{1,\ldots,d\} \rightsquigarrow$ simple matrix computations

$$\eta_{n+1} = [\eta_n(1), \ldots, \eta_n(d)] \begin{bmatrix} M_{n+1}(1,1) & \ldots & M_{n+1}(1,d) \\ \vdots & \ldots & \vdots \\ M_{n+1}(d,1) & \ldots & M_{n+1}(d,d) \end{bmatrix}$$

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Note :
Same equations for $X_n \in E_n = \{1, \ldots, d_n\}$ **with** $M_{n+1} \in \mathbb{R}_+^{(d_n \times d_{n+1})}$

(Crude) Monte Carlo methods : Markov chain $X_{n+1} = F_n(X_n, W_n) \in E_n$

Linear evolution equations $\mathbf{Law(X_n) := \eta_n}$

$$\eta_{n+1} = \eta_n M_{n+1}$$

$$\Downarrow$$

N iid copies/samples $\mathbf{X^i_{n+1} = F_n(X^i_n, W^i_n),\ 1 \leq i \leq N}$

$$\Downarrow$$

$$\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{X_n^i} \simeq_{N\uparrow\infty} \eta_n$$

$$\Updownarrow$$

$$\eta_n^N(f) = \frac{1}{N} \sum_{1 \leq i \leq N} f(X_n^i) \simeq_{N\uparrow\infty} \eta_n(f) = \mathbb{E}(f(X_n))$$

(Crude) Monte Carlo methods : Markov chain $X_n \in E_n$ with transitions M_n

Path space models

$$\mathbb{P}_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{(X_0^i, \ldots, X_n^i)} \simeq_{N\uparrow\infty} \mathbb{P}_n = \mathrm{Law}(X_0, \ldots, X_n)$$

Historical process

$$\mathbf{X_n} = (X_0, \ldots, X_n) \in \mathbf{E_{n+1}} = \prod_{0 \leq p \leq n} E_p \quad \text{with Markov transition} \quad \mathbf{M_n}$$

$$\Downarrow$$

Same model as before

$$\mathbb{P}_{n+1} = \boldsymbol{\eta}_{\mathbf{n+1}} = \boldsymbol{\eta}_{\mathbf{n}} \mathbf{M_{n+1}}$$

and

$$\boldsymbol{\eta}_{\mathbf{n}}^{\mathbf{N}} := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\mathbf{X_n^i}} = \mathbb{P}_n^N \quad \simeq_{N\uparrow\infty} \quad \mathbb{P}_n = \boldsymbol{\eta}_{\mathbf{n}}$$

with N iid copies of the historical process

$$\mathbf{X_n^i} = (X_0^i, \ldots, X_n^i)$$

Diffusion type & discrete generation models (d=1)

Markov chain with Gaussian perturbations

$$\begin{aligned} X_{n+1} &= F_n(X_n, W_n) \\ &= X_n + a_n(X_n)\Delta + b_n(X_n)\ \sqrt{\Delta}\ W_n \end{aligned}$$

with the Gaussian random variable

$$\mathbb{P}(W_n \in dw) = \frac{1}{\sqrt{2\pi}}\ \exp\left(-\frac{w^2}{2}\right)$$

$$\downarrow$$

Diffusion limits when $(t_{n+1} - t_n) = \Delta \downarrow 0$ & $(a_n, b_n, X_n) = \left(a'_{t_n}, b'_{t_n}, X'_{t_n}\right)$

$$\downarrow$$

$$dX'_t = a'_t(X'_t)dt + b'_t(X'_t)dW_t$$

Diffusion type & discrete generation models (d=1)

Diffusion model

$$dX'_t = a'_t(X'_t)dt + b'_t(X'_t)dW_t$$

Taylor expansion (Ito's formula)

$$f(X'_t + dX'_t) - f(X'_t) \quad = \quad \partial f(X'_t)\ dX'_t + \frac{1}{2}\ \partial^2 f(X'_t)\ dX'_t dX'_t + \ldots$$

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with the infinitesimal generator

$$L_t(f) = a'_t\ \partial f + \frac{1}{2}\ (b'_t)^2\ \partial^2 f$$

$$\Downarrow$$

Weak parabolic PDE

$$\eta_t = \mathrm{Law}(X'_t) \quad \Leftrightarrow \quad \frac{d}{dt}\eta_t(f) = \eta_t(L_t(f))$$

Two-steps jump-diffusion type model (d=1)

$$X_n \to X_{n+1/2} = F_n(X_n, W_n) \to X_{n+1} = \epsilon_{n+1} \; X_{n+1/2} + (1 - \epsilon_{n+1}) \;\; Y_{n+1}$$

$$\begin{aligned} \epsilon_{n+1} &\sim e^{-V_n(X_{n+1/2})\ \Delta} \; \delta_1 + \left(1 - e^{-V_n(X_{n+1/2})\ \Delta}\right) \delta_0 \\ Y_{n+1} &\sim P_n(X_{n+1/2}, dy) \end{aligned}$$

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Jump-diffusion limits $(a_n, b_n, V_n, P_n, X_n) = \left(a'_{t_n}, b'_{t_n}, V'_{t_n}, P'_{t_n}, X'_{t_n}\right)$
between jumps = diffusion as before, and jumps times

$$T_{n+1} = \inf\left\{t \geq T_n\ :\ \int_{T_n}^{t} V'_s(X'_s)\ ds \geq e_n\right\} \qquad e_n \text{ iid expo}(\lambda = 1)$$

Two-steps jump-diffusion type model (d=1)

$$X_n \xrightarrow{M_{n+1}} X_{n+1/2} \xrightarrow{J_{n+1}} X_{n+1}$$

with the diffusion part

$$M_{n+1}(f)(x) \simeq f(x) + L_t^{diff}(f)(x)\ \Delta$$

and the jump part

$$\begin{aligned} J_{n+1}(f)(x) &= e^{-V'_{t_n}(x)\ \Delta}\ f(x) + \left(1 - e^{-V'_{t_n}(x)\ \Delta}\right)\ P'_{t_n}(f)(x) \\ &= f(x) + \underbrace{V'_{t_n}(x)\ \left[P'_{t_n}(f)(x) - f(x)\right]}_{=L_t^{jump}(f)(x)}\ \Delta \end{aligned}$$

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$$\left[MJ = (I + L^{diff}\ \Delta)\ (I + L^{jump}\ \Delta) \simeq I + (L^{diff} + L^{jump})\ \Delta\right]$$

Two-steps jump-diffusion type model (d=1)

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$$\left[MJ = (I + L^{diff}\ \Delta)\ (I + L^{jump}\ \Delta) \simeq I + (L^{diff} + L^{jump})\ \Delta\right]$$

$\Longrightarrow$**Weak integro-differential equation**

$$\eta_t = \mathrm{Law}(X'_t) \quad \Leftrightarrow \quad \frac{d}{dt}\eta_t(f) = \eta_t(L_t(f))$$

with the infinitesimal generator

$$L_t(f)(x) = a'_t(x)\ \partial f(x) + \frac{1}{2}\ (b'_t(x))^2\ \partial^2 f(x) + V'_t(x)\ \int (f(y) - f(x))\ P'_t(x, dy)$$

Mean field models : Markov chain $X_{n+1} = F_n(X_n, \eta_n, W_n)$

Nonlinear evolution equations $\mathbf{Law(X_n) := \eta_n}$

$$\eta_{n+1} = \Phi_{n+1}(\eta_n) := \eta_n K_{n+1,\eta_n}$$

with Markov transitions $K_{n,\eta}$ indexed by the symplex $\eta \in \mathbb{S}^{(d-1)} \subset \mathbb{R}_+^d$

$$\Updownarrow$$

$$K_{n+1,\eta_n}(x_n, dx_{n+1}) = \mathbb{P}\left(X_{n+1} \in dx_{n+1} \mid X_n = x_n\right)$$

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When $E = \{1, \ldots, d\} \rightsquigarrow$ simple matrix computations

$$\eta_{n+1} = [\eta_n(1), \ldots, \eta_n(d)] \begin{bmatrix} K_{n+1,\eta_n}(1,1) & \ldots & K_{n+1,\eta_n}(1,d) \\ \vdots & \ldots & \vdots \\ K_{n+1,\eta_n}(d,1) & \ldots & K_{n+1,\eta_n}(d,d) \end{bmatrix}$$

Example 1 : Markov chain $X_{n+1} = F_n(X_n, \eta_n, W_n) \in E := \{-1, 1\}$

McKean's two velocities of gases :

$$F_n(X_n, \eta_n, W_n) = X_n \, 1_{W_n \leq \eta_n(1)} + (-X_n) \, 1_{W_n > \eta_n(1)} \Longleftrightarrow X_{n+1} = \epsilon_n \, X_n$$

with the Bernoulli random variable

$$\mathbb{P}(\epsilon_n = 1) = 1 - \mathbb{P}(\epsilon_n = -1) = \eta_n(1)$$

$$\Downarrow$$

Nonlinear equation on the symplex $\mathbb{S}^{(0)} = [0, 1]$:

$$\eta_{n+1}(1) = \eta_n(1)^2 + \eta_n(-1)^2 \qquad = \qquad \eta_n(1)^2 + (1 - \eta_n(1))^2$$

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Nonlinear equation on the symplex $\mathbb{S}^{(0)} = [0, 1]$:

$$\eta_{n+1}(1) = \eta_n(1)^2 + \eta_n(-1)^2 \quad = \quad \eta_n(1)^2 + (1 - \eta_n(1))^2$$

$$\rightarrow_{n\uparrow\infty} \quad 1/2 \quad \text{if} \quad \eta_0(1) \not\in \{0, 1\}$$

Example 2 : $G_n(i)(\stackrel{ex.}{=} e^{-V_n(i)}) \in]0,1]$ & $(M_{n+1}(i,j))_{i,j=1,\ldots,d}$ Markov transition

$$K_{n+1,\eta_n}(i,j) = G_n(i)\, M_{n+1}(i,j) + (1 - G_n(i)) \sum_k \frac{\eta_n(k) G_n(k)}{\sum_l \eta_n(l) G_n(l)}\, M_{n+1}(k,j)$$

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$$\Downarrow$$

$$\sum_k \eta_n(k)\; K_{n+1,\eta_n}(k,j)$$

$$= \sum_k \eta_n(k) G_n(k)\; M_{n+1}(k,j) + \left[1 - \sum_l \eta_n(l) G_n(l)\right] \times \sum_k \frac{\eta_n(k) G_n(k)}{\sum_l \eta_n(l) G_n(l)}\; M_{n+1}(k,j)$$

$$= \sum_k \frac{\eta_n(k) G_n(k)}{\sum_l \eta_n(l) G_n(l)}\; M_{n+1}(k,j)$$

Example 2 : $G_n(i)(\stackrel{ex.}{=} e^{-V_n(i)}) \in]0,1]$ **&** $(M_{n+1}(i,j))_{i,j=1,\ldots,d}$ **Markov transition**

$$K_{n+1,\eta_n}(i,j) = G_n(i)\, M_{n+1}(i,j) + (1-G_n(i)) \sum_k \frac{\eta_n(k) G_n(k)}{\sum_l \eta_n(l) G_n(l)}\, M_{n+1}(k,j)$$

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$$= \sum_k \frac{\eta_n(k) G_n(k)}{\sum_l \eta_n(l) G_n(l)}\; M_{n+1}(k,j) \propto \sum_k \eta_n(k) G_n(k) M_{n+1}(k,j)$$

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$$\Downarrow$$

$$\eta_{n+1} \propto \eta_n \underbrace{\begin{bmatrix} G_n(1) & 0 & \ldots & 0 & 0 \\ \vdots & & \ldots & & \vdots \\ 0 & 0 & \ldots & 0 & G_n(d) \end{bmatrix}}_{\text{Multiplication}} \underbrace{\begin{bmatrix} M_{n+1}(1,1) & \ldots & M_{n+1}(1,d) \\ \vdots & \ldots & \vdots \\ M_{n+1}(d,1) & \ldots & M_{n+1}(d,d) \end{bmatrix}}_{\text{Markov transport}}$$

Example 2 : $G_n(i)(\stackrel{ex.}{=} e^{-V_n(i)}) \in]0,1]$ & $(M_{n+1}(i,j))_{i,j=1,\ldots,d}$ Markov transition

$$\eta_{n+1} \quad = \quad [\Psi_{G_n}(\eta_n)(1),\ldots,\Psi_{G_n}(\eta_n)(d)] \left[\begin{array}{ccc} M_{n+1}(1,1) & \ldots & M_{n+1}(1,d) \\ \vdots & \ldots & \vdots \\ M_{n+1}(d,1) & \ldots & M_{n+1}(d,d) \end{array}\right]$$

with the Boltzmann-Gibbs transformation

$$\Psi_{G_n} \;:\; \eta \in \mathcal{M}(E) \quad \mapsto \Psi_{G_n}(\eta) \in \mathcal{M}(E)$$

defined by

$$\Psi_{G_n}(\eta)(i) = \frac{1}{\eta_n(G_n)}\; G_n(i)\; \eta(i)$$

$$\Downarrow$$

Nonlinear evolution eq. on the $(d-1)$-symplex $\eta_n \in \mathbb{S}^{(d-1)} \subset \mathbb{R}_+^d$

$$\eta_{n+1} = \Phi_{n+1}(\eta_n) := \Psi_{G_n}(\eta_n) M_{n+1}$$

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Nonlinear evolution eq. on the $(d-1)$-symplex $\eta_n \in \mathbb{S}^{(d-1)} \subset \mathbb{R}_+^d$

$$\eta_{n+1} = \Phi_{n+1}(\eta_n) := \Psi_{G_n}(\eta_n) M_{n+1} \Leftrightarrow \textbf{Feynman-Kac models}$$

Mean field models : Markov $X_{n+1} = F_n(X_n, \eta_n, W_n) \in E = \mathbb{R}^d$

Nonlinear evolution equations $\mathbf{Law(X_n) := \eta_n}$

$$\eta_{n+1} = \Phi_{n+1}(\eta_n) := \eta_n K_{n+1,\eta_n}$$

$$\Downarrow$$

Mean field particle-simulation models:

N almost iid copies/samples $\mathbf{X^i_{n+1} = F_n(X^i_n, \eta^N_n, W^i_n),\ 1 \leq i \leq N}$

with the occupation measure of the system at time n :

$$\eta^N_n := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{X^i_n} \simeq_{N\uparrow\infty} \eta_n$$

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$$\Downarrow \quad \text{[by induction on } n\text{]}$$

$$\eta^N_{n+1} = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{X^i_{n+1}} \quad \text{with} \quad X^i_{n+1} \simeq F_n(X^i_n, \eta_n, W^i_n) \Rightarrow \eta^N_{n+1} \simeq_{N\uparrow\infty} \eta_{n+1}$$

An abstract mathematical model

Nonlinear evolution equation

$$\eta_{n+1} = \Phi_{n+1}(\eta_n) := \eta_n K_{n+1,\eta_n}$$

$$\Downarrow$$

Nonlinear Markov model interpretation (not unique)

$$K_{n+1,\eta_n}(x_n, dx_{n+1}) = \mathbb{P}\left(X_{n+1} \in dx_{n+1} \mid X_n = x_n\right) \quad \text{with Law}(X_n) := \eta_n$$

$$\Downarrow$$

Mean field particle-simulation model

$$X_n^i \rightsquigarrow X_{n+1}^i = \text{r.v. with law } K_{n+1,\eta_n^N}(X_n^i, dx) \quad \text{where} \quad \eta_n^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_n^i}$$

Local sampling errors

Nonlinear evolution equation

$$\eta_{n+1} = \Phi_{n+1}(\eta_n)$$

Mean field particle model

$$V_{n+1}^N := \sqrt{N}\left(\eta_{n+1}^N - \Phi_{n+1}\left(\eta_n^N\right)\right) \quad \Leftrightarrow \quad \eta_{n+1}^N = \Phi_{n+1}\left(\eta_n^N\right) + \frac{1}{\sqrt{N}}\ V_{n+1}^N$$

Note $\forall f \ : \ \mathrm{osc}(f) := \sup_{x,y}(f(x) - f(y)) \leq 1$

$$\begin{aligned} \mathbb{E}\left(\eta_{n+1}^N(f) \mid \eta_n^N\right) &= \eta_n^N K_{n+1,\eta_n^N}(f) = \Phi_{n+1}\left(\eta_n^N\right)(f) \\ N\ \mathbb{E}\left(\left[\eta_{n+1}^N - \Phi_{n+1}\left(\eta_n^N\right)\right](f)^2 \mid \eta_n^N\right) &= \eta_n^N \left[K_{n+1,\eta_n^N}\left[f - K_{n+1,\eta_n^N}(f)\right]^2\right] \leq 1 \end{aligned}$$

McKean measures : Markov chain $X_n \in E_n$ with transitions $K_{n,\eta_{n-1}}$

$$\mathbb{P}_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{(X_0^i, \ldots, X_n^i)} \simeq_{N \uparrow \infty} \mathbb{P}_n = \mathrm{Law}(X_0, \ldots, X_n)$$

with the McKean measures

$$\mathbb{P}_n(d(x_0, \ldots, x_n)) = \eta_0(dx_0) \prod_{1 \leq p \leq n} K_{p,\eta_{p-1}}(x_{p-1}, dx_p)$$

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Historical process

$$\mathbf{X_n} = (X_0, \ldots, X_n) \in \mathbf{E_{n+1}} = \prod_{0 \le p \le n} E_p \quad \text{with Markov transition } \mathbf{K_{n,\eta_{n-1}}}$$

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$$\Downarrow$$

Same mathematical model as before

$$\mathbb{P}_{n+1} = \boldsymbol{\eta}_{\mathbf{n+1}} = \boldsymbol{\eta}_{\mathbf{n}} \mathbf{K}_{\mathbf{n+1},\boldsymbol{\eta}_{\mathbf{n}}}$$

and

$$\boldsymbol{\eta}_{\mathbf{n}}^{\mathbf{N}} := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\mathbf{X}_{\mathbf{n}}^{\mathbf{i}}} = \mathbb{P}_n^N \quad \simeq_{N \uparrow \infty} \quad \mathbb{P}_n = \boldsymbol{\eta}_{\mathbf{n}}$$

with **almost** N iid copies of the historical process $\mathbf{X}_{\mathbf{n}}^{\mathbf{i}} = (X_0^i, \ldots, X_n^i)$.

Example 1 : Markov chain $X_{n+1} = F_n(X_n, \eta_n, W_n) \in E = \mathbb{R}^{d=1}$

McKean-Vlasov diffusion type & discrete generation models

$$X_{n+1} = F_n(X_n, \eta_n, W_n) = a_n(X_n, \eta_n) + b_n(X_n, \eta_n)\ W_n$$

with the Gaussian random variable

$$\mathbb{P}(W_n \in dw) = \frac{1}{\sqrt{2\pi}}\ \exp\left(-\frac{w^2}{2}\right)$$

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Two-steps jump type model

$$X_n \to X_{n+1/2} = F_n(X_n, \eta_n, W_n) \to X_{n+1} = \epsilon_{n+1}\ X_{n+1/2} + (1 - \epsilon_{n+1})\ \ Y_{n+1}$$

with

$$\begin{array}{rcl} \epsilon_{n+1} & \sim & e^{-V_{\eta_{n+1/2}}(X_{n+1/2})}\ \delta_1 + \left(1 - e^{-V_{\eta_{n+1/2}}(X_{n+1/2})}\right) \delta_0 \\ Y_{n+1} & \sim & P_{\eta_{n+1/2}}(X_{n+1/2}, dy) \end{array}$$

Example 1 : Markov chain $X_{n+1} = F_n(X_n, \eta_n, W_n) \in \mathbb{R}^{d=1}$

McKean-Vlasov diffusion type & discrete generation models

$$a_n(X_n, \eta_n) = \int \eta_n(dy)\ a'_n(X_n, y) \quad \& \quad b_n(X_n, \eta_n) = \int \eta_n(dy)\ b'_n(X_n, y)$$

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$$X^i_{n+1} = F_n(X^i_n, \eta^N_n, W^i_n) = \frac{1}{N}\sum_{j=1}^{N} a'_n(X^i_n, X^j_n) + \frac{1}{N}\sum_{j=1}^{N} b'_n(X^i_n, X^j_n)\ W^i_n$$

Example 1 : Markov chain $X_{n+1} = F_n(X_n, \eta_n, W_n) \in \mathbb{R}^{d=1}$

McKean-Vlasov diffusion type & discrete generation models

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$$X^i_{n+1} = F_n(X^i_n, \eta^N_n, W^i_n) = \frac{1}{N}\sum_{j=1}^N a'_n(X^i_n, X^j_n) + \frac{1}{N}\sum_{j=1}^N b'_n(X^i_n, X^j_n)\ W^i_n$$

In the same vein : Two-steps mean field jump type model

$$X^i_n \to X^i_{n+1/2} = F_n(X^i_n, \eta^N_n, W^i_n) \to X^i_{n+1} = \epsilon^i_{n+1}\ X^i_{n+1/2} + \left(1 - \epsilon^i_{n+1}\right)\ Y^i_{n+1}$$

with

$$\begin{aligned} \epsilon^i_{n+1} &\sim e^{-V_{\eta^N_{n+1/2}}(X^i_{n+1/2})}\ \delta_1 + \left(1 - e^{-V_{\eta^N_{n+1/2}}(X^i_{n+1/2})}\right)\delta_0 \\ Y^i_{n+1} &\sim P_{\eta^N_{n+1/2}}(X^i_{n+1/2}, dy) \end{aligned}$$

Example 2 : Back to the multiplication-transport model $\eta_{n+1} = \Psi_{G_n}(\eta_n) M_{n+1}$

Nonlinear transport equation

$$\Psi_{G_n}(\mu) = \mu S_{n,\mu} \quad \text{with} \quad S_{n,\mu}(x, .) = G_n(x)\delta_x + (1 - G_n(x))\ \Psi_{G_n}(\mu)$$

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$$\Downarrow$$

Nonlinear Markov chain model

$$\eta_{n+1} = \Phi_{n+1}(\eta_n) := \eta_n K_{n+1,\eta_n} \quad \text{with} \quad K_{n+1,\eta_n} := S_{n,\eta_n} M_{n+1}$$

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$$\Downarrow$$

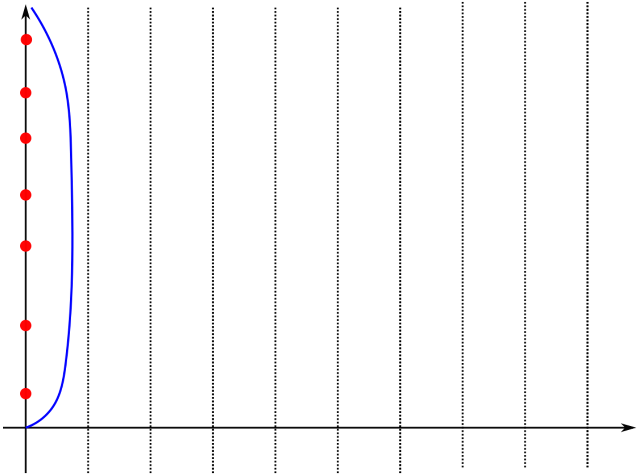
Mean field particle model = Genetic type interacting jump model

$$X_n^i \underbrace{\xrightarrow{\ S_{n,\eta_n^N}\ } \widehat{X}_n^i \xrightarrow{\ M_{n+1}\ }}_{K_{n+1,\eta_n^N}} X_{n+1}^i$$

with

$$S_{n,\eta_n^N}(X_n^i, dy) = G_n(X_n^i)\delta_{X_n^i}(dy) + (1 - G(X_n^i)) \sum_{1 \leq j \leq N} \frac{G_n(X_n^j)}{\sum_{1 \leq k \leq N} G_n(X_n^k)}\ \delta_{X_n^j}(dy)$$

Graphical illustration : $\eta_n \simeq \eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{X_n^i}$



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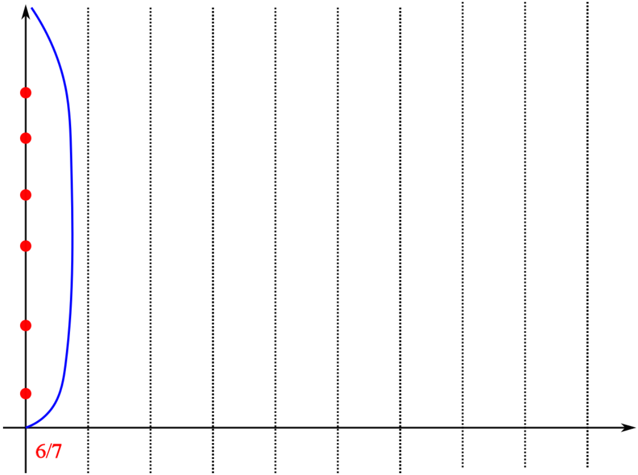

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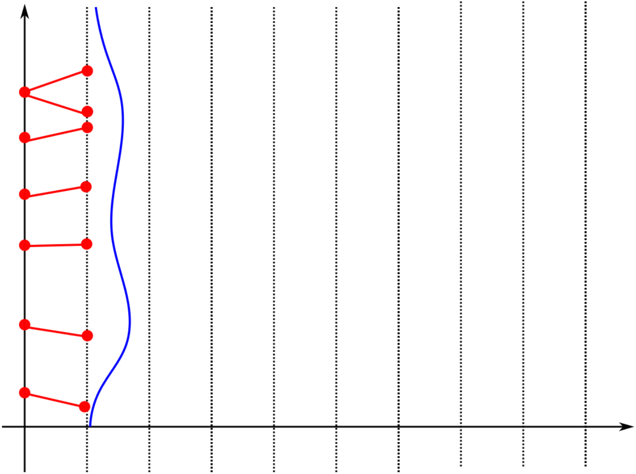

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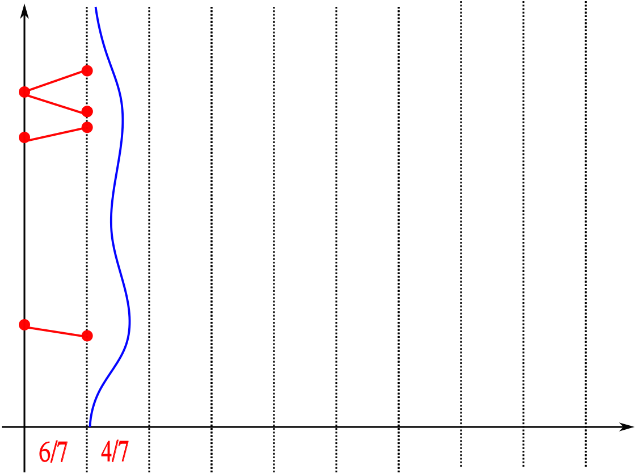

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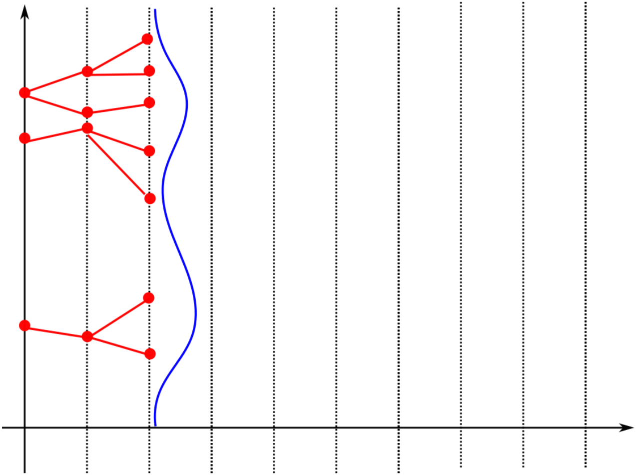

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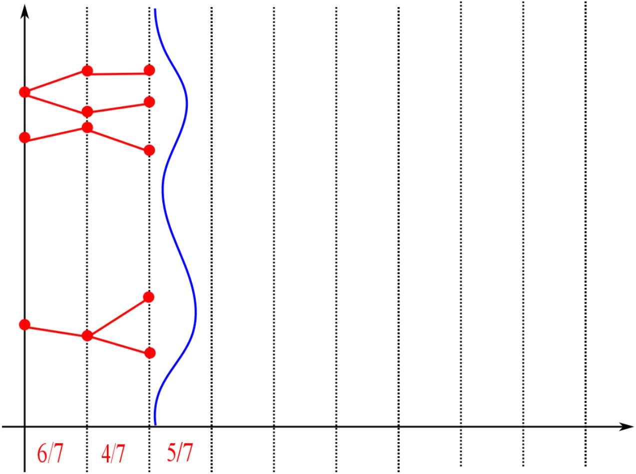

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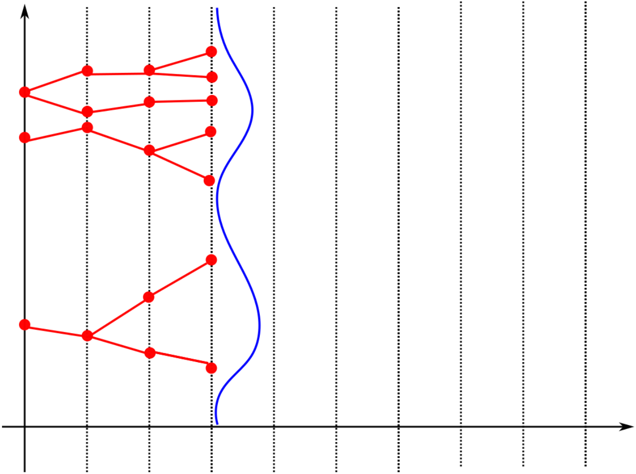

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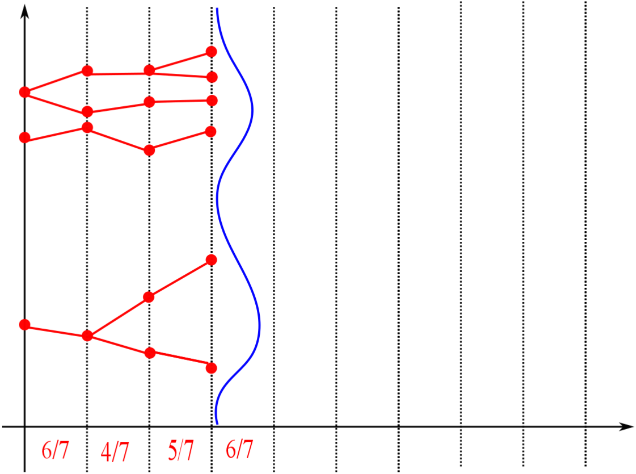

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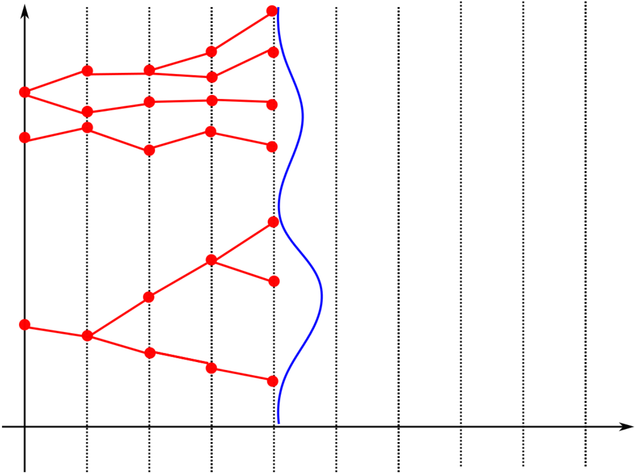

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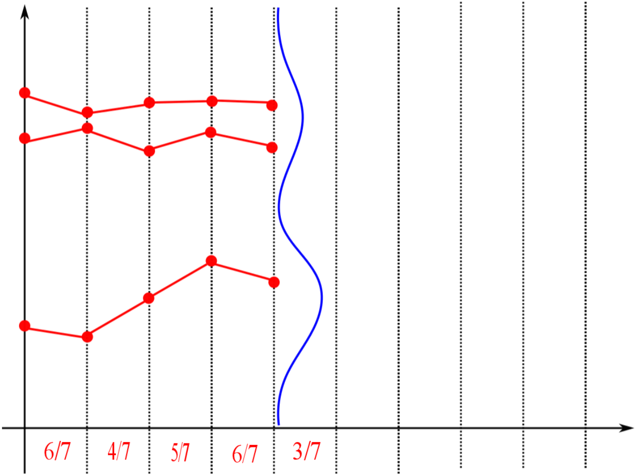

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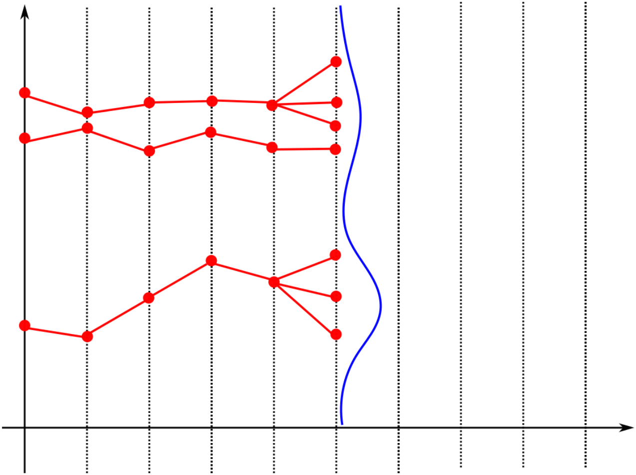
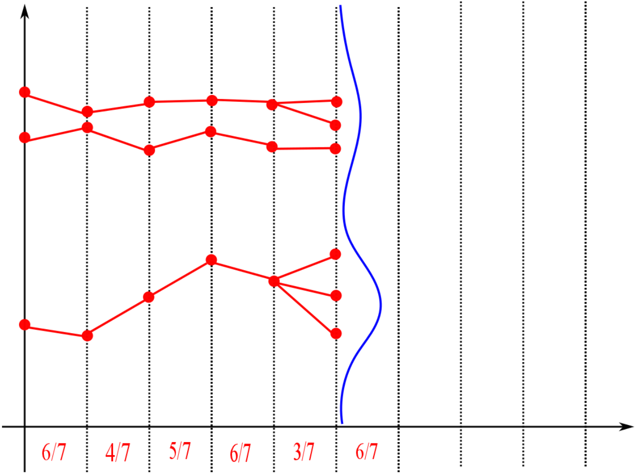


Graphical illustration : $\eta_n \simeq \eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{X_n^i}$



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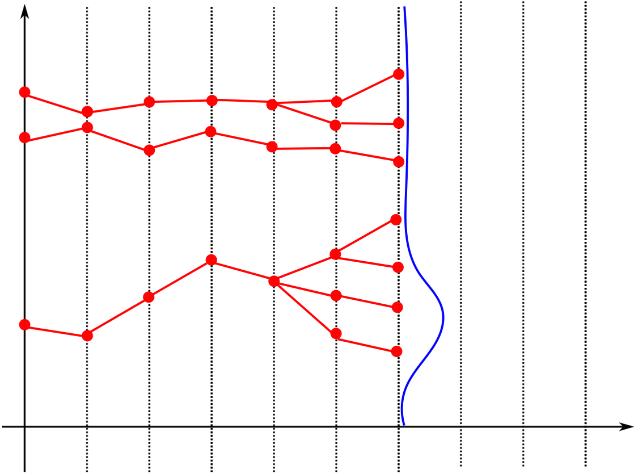

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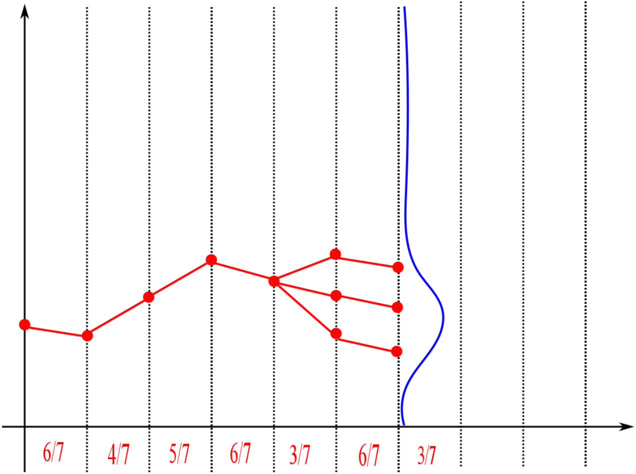

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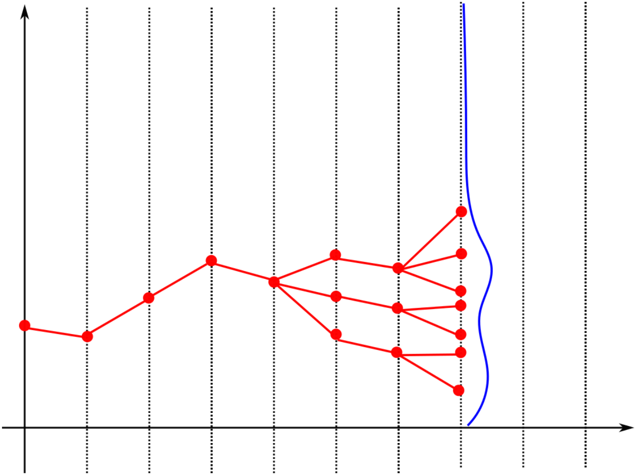

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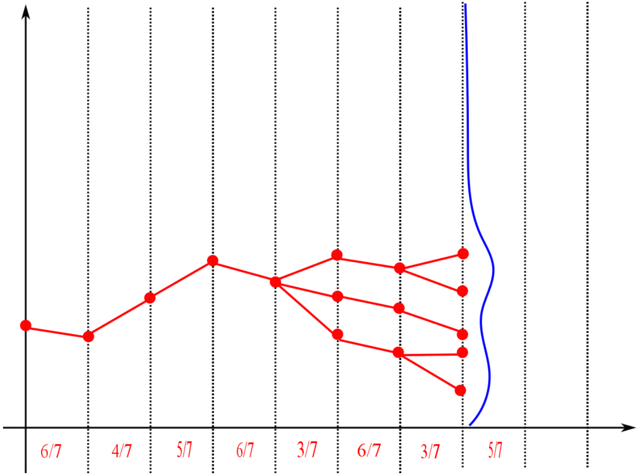

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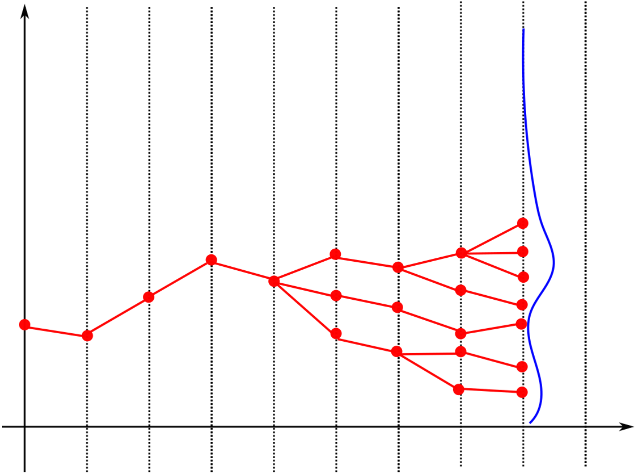

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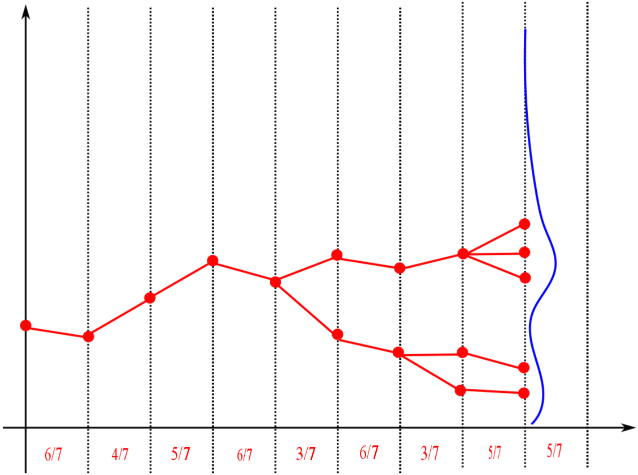

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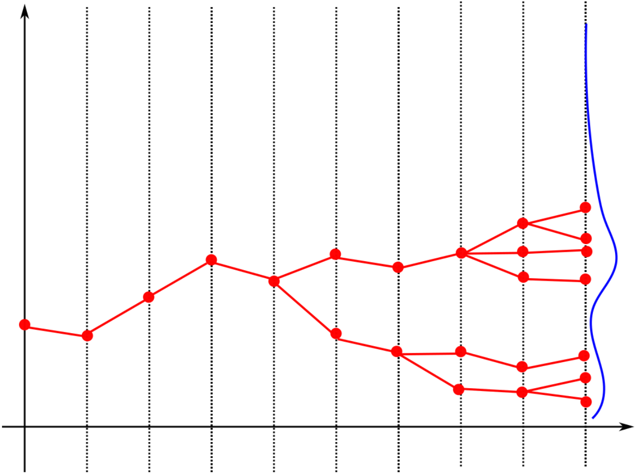

Some questions

- $\text{Law}(X_n^i \mid \text{acceptance}) = \eta_n$?
- Convergence of the occupation measures of the genealogical trees?
- Use of the occupation measures of the complete ancestral tree?
- Convergence of the product of success proportions?

$\rightsquigarrow$ **Feynman-Kac integration models on path spaces**

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- **Link with nonlinear integro-differential equations?**

Nonlinear diffusion type models (d=1)

$$X_{n+1} = F_n(X_n, \eta_n, W_n) = X_n + a_n(X_n, \eta_n)\Delta + b_n(X_n, \eta_n)\ \sqrt{\Delta}\ W_n$$

with the Gaussian random variable

$$\mathbb{P}(W_n \in dw) = \frac{1}{\sqrt{2\pi}}\ \exp\left(-\frac{w^2}{2}\right)$$

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Diffusion limits when

$(t_{n+1} - t_n) = \Delta \downarrow 0\ \&\ (a_n, b_n, X_n, \eta_n) = \left(a'_{t_n}, b'_{t_n}, X'_{t_n}, \eta'_{t_n}\right)$

$$dX'_t = a'_t(X'_t, \eta'_t)dt + b'_t(X'_t, \eta'_t)dW_t$$

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$$\Downarrow$$

Weak parabolic nonlinear PDE

$$\eta'_t = \mathrm{Law}(X'_t) \quad \Leftrightarrow \quad \frac{d}{dt}\eta'_t(f) = \eta'_t(L_{t,\eta'_t}(f))$$

with the infinitesimal generator

$$L_{t,\eta'_t}(f)(x) = a'_t(x, \eta'_t)\ \partial f(x) + \frac{1}{2}\ (b'_t(x, \eta'_t))^2\ \partial^2 f(x)$$

Two-steps jump-diffusion type model (d=1)

$$X_n \to X_{n+1/2} = F_n(X_n, \eta_n, W_n) \to X_{n+1} = \epsilon_{n+1}\ X_{n+1/2} + (1 - \epsilon_{n+1})\ \ Y_{n+1}$$

$$\begin{aligned} \epsilon_{n+1} &\sim e^{-V_{\eta_{n+1/2}}(X_{n+1/2})\ \Delta}\ \delta_1 + \left(1 - e^{-V_{\eta_{n+1/2}}(X_{n+1/2})\ \Delta}\right) \delta_0 \\ Y_{n+1} &\sim P_{\eta_{n+1/2}}(X_{n+1/2}, dy) \end{aligned}$$

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Jump-diffusion limits

$(a_n, b_n, V_n, P_n, X_n, \eta_n) = \left(a'_{t_n}, b'_{t_n}, V'_{\eta_{t_n}}, P'_{\eta_{t_n}}, X'_{t_n}, \eta'_{t_n}\right)$
between jumps = nonlinear diffusion as before, and jumps times

$$T_{n+1} = \inf \left\{ t \geq T_n \;\; : \;\; \int_{T_n}^{t} \; V'_s(X'_s, \eta'_s) \; ds \geq e_n \right\} \qquad e_n \text{ iid expo}(\lambda = 1)$$

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Weak integro-differential equation $\frac{d}{dt}\eta'_t(f) = \eta'_t(L_{t,\eta'_t}(f))$

$$\begin{aligned} &L_{t,\eta'_t}(f)(x) \\ &= a'_t(x,\eta'_t)\ \partial f(x) + \frac{1}{2}\ (b'_t(x,\eta'_t))^2\ \partial^2 f(x) + V'_{\eta'_t}(x)\ \int (f(y)-f(x))\ P'_{\eta'_t}(x,dy) \end{aligned}$$

An abstract mathematical model

Nonlinear evolution equation $\in \mathcal{P}(E)$

$$\frac{d}{dt}\eta_t(f) = \eta_t(L_{t,\eta_t}(f))$$

$$\Downarrow$$

Mean field particle-simulation model= Markov $\in E^N$ **with generator**

$$\mathcal{L}_t(\varphi)(x^1,\ldots,x^N) = \sum_{1\leq i\leq N} L^{(i)}_{t,\frac{1}{N}\sum_{1\leq j\leq N}\delta_{x^j}}(\varphi)(x^1,\ldots,x^i,\ldots,x^N)$$

Local sampling errors

Nonlinear evolution equation

$$d\eta_t(f) = \eta_t(L_{t,\eta_t}(f))\ dt$$

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Mean field particle model (useless in pratice)

$$d\eta_t^N(f) = \eta_t^N(L_{t,\eta_t^N}(f))\ dt + \frac{1}{\sqrt{N}}\ dM_t^N(f)$$

with a sequence of martingales M_t^N with predictable increasing process

$$\langle M^N(f)\rangle_t = \int_0^t \eta_s^N \Gamma_{L_{s,\eta_s^N}}(f,f)\ ds$$

where Γ_L stands for the "carré du champ" operator

$$\Gamma_L(f,f) := L([f - f(x)]^2)(x) = L(f^2)(x) - 2f(x)L(f)(x)$$

How & Why it works?

- (Biology) Individual based models (IBM).
- (Physics) Mean field approximation schemes.
- (Computer Sciences) Stochastic adaptive grid approximation.
- (Stats) Universal acceptance-rejection-recycling sampling schemes.
- (Probab) Stochastic linearization/perturbation technique.

$$\begin{aligned} \eta_n &= \Phi_n\left(\eta_{n-1}\right) \\ \eta_n^N &= \Phi_n\left(\eta_{n-1}^N\right) + \frac{1}{\sqrt{N}}\ V_n^N \end{aligned}$$

Theorem: $(V_n^N)_n \simeq_{N\uparrow\infty} (V_n)_n$ **independent centered Gaussian fields.**

$\Phi_{p,n}(\eta_p) = \eta_n$ stable sg $\Longleftrightarrow$ No propagation of local errors

$\Longrightarrow$ Uniform control w.r.t. the time horizon

$\rightsquigarrow$ **New concentration inequalities for (general) interacting processes**

Stochastic perturbation analysis = Taylor expansion

Gâteaux derivative of $\Phi_n \; : \; \mathcal{P}(E_{n-1}) \to \mathcal{P}(E_n)$

$$\frac{\partial}{\partial \epsilon} \Phi_n(\eta + \epsilon\mu)(f)_{|\epsilon=0} = \mu[d_\eta \Phi_n(f)]$$

with the first order linear/integral operator

$$d_\eta \Phi_n \; : \; f \in \mathcal{B}_b(E_n) \mapsto d_\eta \Phi_n(f) \in \mathcal{B}_b(E_{n-1})$$

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Differential rule

$$d_\eta \left(\Phi_{n+1} \circ \Phi_n \right) = d_\eta \Phi_n \circ d_{\Phi_n(\eta)} \Phi_{n+1}$$

$$\Downarrow$$

Semigroup derivatives

$$\Phi_{p,n}(\mu) - \Phi_{p,n}(\eta) \simeq (\mu - \eta) \, d_\eta \Phi_{p,n}$$

Key idea = First order expansions

Key telescoping decomposition

$$\eta_n^N - \eta_n = \sum_{p=0}^{n} \left[\Phi_{p,n}(\eta_p^N) - \Phi_{p,n}\left(\Phi_p(\eta_{p-1}^N)\right)\right]$$

$\oplus$ First order expansion

$$\sqrt{N}\ \left[\Phi_{p,n}(\eta_p^N) - \Phi_{p,n}\left(\Phi_p(\eta_{p-1}^N)\right)\right]$$

$$= \sqrt{N}\ \left[\Phi_{p,n}\left(\Phi_p(\eta_{p-1}^N) + \frac{1}{\sqrt{N}}\ V_p^N\right) - \Phi_{p,n}\left(\Phi_p(\eta_{p-1}^N)\right)\right]$$

$$\simeq V_p^N D_{p,n} + \frac{1}{\sqrt{N}}\ R_{p,n}^N$$

with $\underbrace{\text{a predictable } D_{p,n} - \text{first order operator}}_{\text{fluctuation term}} \oplus \underbrace{\text{2nd-order measure } R_{p,n}^N}_{\text{bias-term}}$

Stochastic perturbation model

Stochastic perturbation model

$$W_n^{\eta,N} := \sqrt{N}\left[\eta_n^N - \eta_n\right] = \sum_{0\leq p\leq n} V_p^N D_{p,n} + \frac{1}{\sqrt{N}}\, R_n^N$$

Under some mixing condition on the limiting semigroups $\Phi_{p,n}$

$$\operatorname{osc}\left(D_{p,n}(f)\right) \leq Cte\ e^{-(n-p)\alpha}$$

and

$$\mathbb{E}\left(|R_n^N(f)|^m\right) \leq Cte\ 2^{-m}(2m)!/m!$$

$$\Downarrow$$

Uniform concentration estimates w.r.t. the time parameter

Uniform concentration inequalities

Constants $(\mathbf{c_1}, \mathbf{c_2})$ related to (**bias**,**variance**), c finite constant
Test functions/observables $\|f_n\| \leq 1$, $\forall\ (x \geq 0, n \geq 0, N \geq 1)$.

When $E_n = \mathbb{R}^d$:

$$F_n(y) := \eta_n\left(1_{(-\infty,y]}\right) \quad \text{and} \quad F_n^N(y) := \eta_n^N\left(1_{(-\infty,y]}\right) \text{ with } y \in \mathbb{R}^d$$

The probability of any of the following events is greater than $1 - e^{-x}$.

$$\left|\eta_n^N - \eta_n\right|(f_n) \leq \frac{c_1}{N}\ \left(1 + x + \sqrt{x}\right) + \frac{c_2}{\sqrt{N}}\ \sqrt{x}$$

$$\sup_{0 \leq p \leq n} \left|\left[\eta_p^N - \eta_p\right](f_p)\right| \leq c\ \sqrt{x \log(n+e)/N}$$

$$\left\|F_n^N - F_n\right\| \leq c\ \sqrt{d\ (x+1)/N}$$

Coalescent tree based expansions

Weak propagation of chaos Taylor's type expansions:

$$\begin{aligned} \mathbb{P}_n^{(N,q)} &= \text{Law of the first } q \text{ ancestral lines } (q \leq N) \\ &= \mathbb{Q}_n^{\otimes q} + \sum_{1 \leq l \leq m} \frac{1}{N^l}\, d_l \mathbb{P}_n^{(q)} \;+\; \mathrm{O}\left(\frac{1}{N^{m+1}}\right) \end{aligned}$$

with signed measures $d_l \mathbb{P}_n^{(q)}$ expressed in terms of **coalescent trees**.

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$$\Downarrow$$

Romberg-Richardson interpolation: For any order $l \geq 1$

$$\sum_{1\leq m\leq l} \frac{(-1)^{l-m}}{m!}\, \frac{m^l}{(l-m)!}\, \mathbb{P}_n^{(mN,q)} = \mathbb{Q}_n^{\otimes q} + \mathrm{O}(1/N^l)$$