Advanced Monte Carlo Methods

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INRIA Bordeaux

Nelder Fellow Lecture Series, March 2024

Part II - Discrete time Feynman-Kac models

Outline

Feynman-Kac measures

A variety of interpretations/targets/. . .

Feynman-Kac particle recipes/methodologies

Performance/Convergence analysis

Feynman-Kac measures

"Updated" Feynman-Kac measures

Linear evolution semigroups

Nonlinear evolution semigroups

Path-space meas.- Markov triangular arrays

A variety of interpretations/targets/...

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$$\eta_n = \Phi_n(\eta_{n-1}) := \Psi_{G_{n-1}}(\eta_{n-1})P_n$$

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$$\Updownarrow \quad \text{with} \quad \mathbf{P_n(x_{n-1}, dx_n) := \mathbb{P}(X_n \in dx_n \mid X_{n-1} = x_{n-1})}$$

FK solution/semigroup/formulation:

$$\eta_n(f) = \gamma_n(f)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f) = \mathbb{E}\left(f(X_n) \prod_{0 \le p < n} G_p(X_p)\right)$$

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$\rightsquigarrow$ **Path-space measures:**

$$\mathbb{Q}_n(dx) := \frac{1}{\gamma_n(1)} \left\{\prod_{0 \le p < n} G_p(x_p)\right\} \eta_0(dx_0)\ P_1(x_0, dx_1) \ldots P_n(x_{n-1}, dx_n)$$

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Time-uniform expo. concentration [$\|f\| \leq 1$] ($\Rightarrow \mathbb{L}_p$-estimates)

$\rightsquigarrow$ **For any $x \geq 0$ and $n \geq 0$**

$$\mathbb{P}\left(|\eta_n^N(f) - \eta_n(f)| \leq c\ \sqrt{(1+x)/N}\right) \geq 1 - e^{-x}$$

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and

$$\mathbb{P}\left(\sup_{0\leq p\leq n} |\eta_n^N(f) - \eta_n(f)| \leq c\ \sqrt{((1+x)\log(n))/N}\right) \geq 1 - e^{-x}$$

Some refs/more refined:

(AAP2011), (FTML2011), (CRC2013) [compact/minorization conditions]

"Updated" Feynman-Kac measures

$$\widehat{\eta}_n := \Psi_{G_n}(\eta_{\mathbf{n}}) \iff \widehat{\eta}_n(f) \propto \mathbb{E}\left(f(X_n) \prod_{0 \leq p \leq \mathbf{n}} G_p(X_p)\right)$$

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Key observation:

$$\eta_{\mathbf{n}} = \widehat{\eta}_{\mathbf{n}-1}\mathbf{P}_{\mathbf{n}} \Longrightarrow \widehat{\eta}_n(f) = \frac{\widehat{\eta}_{\mathbf{n}-1}\mathbf{P}_{\mathbf{n}}(G_n f)}{\widehat{\eta}_{\mathbf{n}-1}\mathbf{P}_{\mathbf{n}}(G_n)}$$

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with

$$\widehat{G}_{n-1} = P_n(G_n) \quad \text{and} \quad \widehat{P}_n(f) := \frac{P_n(G_n f)}{P_n(G_n)}$$

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But $\neq$ GA Monte Carlo/samplers (mutation,selection) $\sim (\widehat{P}_n, \widehat{G}_n)$

(cf. `Example 3/Sect. 4-2 in (AAP-1998)` & Sect 2-3-2 in (Sp2000)),...

Feynman-Kac semigroups - any (G_n, P_n)

Linear evolution semigroup $(p \leq n)$**:**

$$\gamma_n = \gamma_p Q_{p,n} \quad \text{with} \quad Q_{p,n}(f)(x_p) := \mathbb{E}\left(f(X_n) \prod_{p \leq q < n} G_q(X_q) \mid X_p = x_p \right)$$

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$$\Updownarrow$$

$$Q_{p,n} = Q_{p+1} Q_{p+2} \ldots Q_n \quad \text{with} \quad Q_n(x_{n-1}, dx_n) = G_{n-1}(x_{n-1}) \; P_n(x_{n-1}, dx_n)$$

Conventions: $Q_{n,n} = I$.

Feynman-Kac semigroups - any (G_n, P_n)

Linear evolution semigroup. $(p \leq n)$**:**

$$\gamma_n = \gamma_p Q_{p,n}$$

$\rightsquigarrow$ **Nonlinear evolution semigroup:**

$$\eta_n(f) = \frac{\gamma_n(f)}{\gamma_n(1)} = \frac{\eta_p(G_{p,n}\overline{Q}_{p,n}(f))}{\eta_p(G_{p,n})} = \Psi_{G_{p,n}}(\eta_p)\overline{Q}_{p,n}(f)$$

with the Markov transition

$$\overline{Q}_{p,n}(f)(x_p) := \frac{Q_{p,n}(f)}{Q_{p,n}(1)} \quad \text{and} \quad G_{p,n} = Q_{p,n}(1)$$

Conventions: $\overline{Q}_{n,n} = I$ *and* $G_{n,n} = 1$

Path-space meas. - Markov triangular arrays

$$\mathbb{Q}_n(dx) := \frac{1}{\gamma_n(1)} \left\{ \prod_{0\leq p<n} G_p(x_p) \right\} \eta_0(dx_0)\ P_1(x_0,dx_1)\ldots P_n(x_{n-1},dx_n)$$

Key observation:

$$G_{p,n} = Q_{p,n}(1) = G_p\ P_{p+1}(G_{p+1,n}) \quad \Longleftrightarrow \quad G_p = G_{p,n}/P_{p+1}(G_{p+1,n})$$

$$\Downarrow$$

Triangular array of Markov transitions $\rightsquigarrow$ stability properties

$$\mathbb{Q}_n(dx) = \frac{\eta_0(dx_0)\ G_{0,n}(x_0)}{\eta_0(G_{0,n})} \frac{P_1(x_0,dx_1)G_{1,n}(x_1)}{P_1(G_{1,n})(x_0)} \cdots \frac{P_n(x_{n-1},dx_n)G_{n,n}(x_n)}{P_n(G_{n,n})(x_{n-1})}$$

Triangular arrays/Stability FK/Positive semigroups:

$\rightsquigarrow$ (CRAS1999)/(IHP2021),(SP2000),..., (AAP2023), (SAA2023).

Feynman-Kac measures

A variety of interpretations/targets/...

- Sub-Markov models (hard/soft obstacles)
- Self avoiding walks
- Level crossing excursions
- Filtering problems
- Approximate Bayesian Computation
- Kalman filter/Linear Gaussian
- Nonlinear Kalman/McKean-Vlasov
- The Ensemble Kalman filter
- EnKF vs Particle Filter
- Branching processes
- Quasi-invariant measures
- Path-space meas. - h-process

Feynman-Kac particle recipes/methodologies

Performance/Convergence analysis

$\neq G_n/P_n \rightsquigarrow \neq$ interpretations/targets

Some ex.: $G_n(x_n) = (h_n/h_{n-1})(x_n)$ or $e^{-(U_{n+1}-U_n)(x_n)}$,

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$\supset$ **Continuous time models,...**

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$\oplus$ **Change of measures/Tuning $\rightsquigarrow$ non unique choice of P_n or G_n.**

Sub-Markov models $Q_n(1)(x) := G_{n-1}(x) \in [0,1]$

$$Q_n(x, dy) = G_{n-1}(x)\, P_n(x, dy) \rightsquigarrow$$

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Absorbed chain (extended Markov $P_n(c,c) = 1$ and $f(c) = 0$)

$$X_n^c \in E_n^c \xrightarrow{\textit{absorption} \sim (1-G_n)} \widehat{X}_n^c \xrightarrow{\textit{exploration} \sim P_n} X_{n+1}^c$$

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Conditional probabilities

$$\mathbb{Q}_n = \mathrm{Law}((X_0^c, \ldots, X_n^c) \mid T^{absorption} \geq n)$$

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Normalizing constants

$$\gamma_n(1) = \text{Proba}\left(T^{absorption} \geq n\right) \overset{G_n = 1_{A_n}}{=} \mathbb{P}(X_p \in A_p \,, 0 \leq p < n)$$

Self avoiding walks in $E' := \mathbb{Z}^d \rightsquigarrow$ historical process

$$X_n = (X'_0, \ldots, X'_n) \in E_n := (E' \times \ldots \times E') \quad \& \quad G_n(X_n) = 1_{X'_n \notin \{X'_0, \ldots, X'_{n-1}\}}$$

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$$\gamma_{n+1}(1) = \mathrm{Proba}\left(X'_p \neq X'_q,\ \forall 0 \leq p < q \leq n\right)$$

Self avoiding walks in $E' := \mathbb{Z}^d \rightsquigarrow$ historical process

$$X_n = (X'_0, \ldots, X'_n) \in E_n := (E' \times \ldots \times E') \quad \& \quad G_n(X_n) = 1_{X'_n \notin \{X'_0, \ldots, X'_{n-1}\}}$$

$\rightsquigarrow$ **Conditional distributions**

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Edward's model (a.k.a. Weakly SAW/Domb-Joyce/...)

$$G_n(X_n) = \exp\left(-\frac{1}{\epsilon} \sum_{0 \leq p < n} 1_{X'_p}(X'_n)\right)$$

X'-Excursions - $A_n \downarrow$, with B non critical recurrent

Hitting A_n or B

$$T_n := \inf \{ t \geq T_{n-1} \ : \ X'_t \in (A_n \cup B) \}$$

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$$\mathbb{Q}_n = \mathrm{Law}(n \ excursions \mid hits \ A_n \ before \ B)$$

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$$\gamma_n(1) = \mathbb{P}(X \ hits \ A_n \ before \ B)$$

Filtering: State/Osb $= Z_n = (X_n, Y_n)$ Markov

Markov transitions:

$$\mathbb{P}((X_n, Y_n) \in d(x_n, y_n) \mid Z_{n-1} = z_{n-1}) = P_n(x_{n-1}, dx_n) \underbrace{K_n^Y(x_n, dy_n)}_{g_n(x_n, y_n)\ \nu_n(dy_n)}$$

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BUT ALSO

$$\widehat{\eta}_n = \mathrm{Law}(X_n \mid (Y_0, \ldots, Y_n) = (y_0, \ldots, y_n)), \ldots$$

ABC ($Y_n \in \mathbb{R}^d$) $\rightsquigarrow$ **Intractable likelihoods**

$$\mathbb{P}(Y_n \in y_n \mid X_n = x_n) = \underbrace{K_n^Y(x_n, dy_n)}_{\text{too complex/unkown}}$$

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One key idea: (PTRF-2001)/(Preprint Purdue-1998)]

$$\mathcal{X}_n = (X_n, Y_n) \quad \text{and} \quad \mathcal{Y}_n = Y_n + \epsilon\, \mathcal{N}(0, I)$$

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$\rightsquigarrow$ **ϵ-Approximations**
(or using sufficient statistics $\varsigma(y_n)$/local neighbourhoods/uniform sensor-type noise $\rightsquigarrow 1_{\mathcal{V}_\epsilon(y_n)}(z_n)$,...):

$$\text{Law}(X_n \mid (\mathcal{Y}_0, \ldots, \mathcal{Y}_n) = (y_0, \ldots, y_n)) \simeq_{\epsilon \to 0} \text{Law}(X_n \mid (Y_0, \ldots, Y_n) = (y_0, \ldots, y_n))$$

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Given the observations y $\rightsquigarrow$ FK with

$$\mathcal{X}_n = (X_n, Y_n) \quad \text{and} \quad \mathcal{G}_n(\mathcal{X}_n) = \exp\left(-\frac{1}{2\epsilon}\ \|y_n - Y_n\|^2\right)$$

Kalman filter/Linear-Gaussian models $X_0 \sim \mathcal{N}(m_0, r_0)$

$$X_n = AX_{n-1} + \ \sigma \ W_n \quad \text{and} \quad G_n(x_n) \propto \exp\left(-\frac{1}{2\tau}\ (y_n - Cx_n)^2\right)$$

Updating/Bayes' rule/Regression:

$$\eta_n = \mathcal{N}(m_n, r_n) \Longrightarrow \Psi_{G_n}(\eta_n) = \mathcal{N}(\widehat{m}_n, \widehat{r}_n)$$

with

$$\begin{aligned} \widehat{r}_n &= \frac{r_n}{1+\varsigma r_n} \quad \text{and} \quad \varsigma := C^2/\tau \\ \widehat{m}_n &:= m_n + \text{Gain}(r_n)\ (y_n - Cm_n) \quad \text{and} \quad \text{Gain}(r_n) := \frac{\varsigma r_n}{1+\varsigma r_n} \quad C^{-1} \end{aligned}$$

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Prediction/Mutation/Markov transport:

$$\mathcal{N}(\widehat{m}_n, \widehat{r}_n)P_{n+1} = \mathcal{N}(m_{n+1}, r_{n+1}) \quad \text{with} \quad (m_{n+1}, r_{n+1}) := (A\widehat{m}_n, A^2\widehat{r}_n + \sigma^2)$$

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Linear/Gauss flow (Kalman filter, $y_n = 0 \rightsquigarrow$ quantum harmonic osc.)

$$\eta_0 = \mathcal{N}(m_0, r_0) \Longrightarrow \forall n \geq 0 \qquad \eta_n = \mathcal{N}(m_n, r_n)$$

Stochastic/**Nonlinear** Kalman/McKean-Vlasov

$$\overline{X}_n \sim \mathcal{N}(m_n, \mathbf{r_n}) \quad \text{and} \quad \overline{V}_n \sim \mathcal{N}(0,1)$$

$$\Longrightarrow \quad \check{X}_n := \overline{X}_n + \text{Gain}(\mathbf{r_n})\,(y_n - (C\overline{X}_n + \sqrt{\tau}\,\overline{V}_n) \sim \mathcal{N}(\widehat{m}_n, \widehat{r}_n)$$

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proof: mean ok and

$$\check{X}_n - \widehat{m}_n = (1 - \mathsf{Gain}(r_n)C)\ (\overline{X}_n - m_n) - \mathsf{Gain}(r_n)\sqrt{\tau}\ \overline{V}_n$$

$$\Longrightarrow \text{variance} = \underbrace{(1 - \mathsf{Gain}(r_n)C)^2}_{(\frac{1}{1+\varsigma r_n})^2}\ r_n + \underbrace{(\mathsf{Gain}(r_n)C)^2}_{(\frac{\varsigma r_n}{1+\varsigma r_n})^2}\ \underbrace{(\tau/C^2)}_{1/\varsigma} = \frac{r_n}{1+\varsigma r_n} = \widehat{r}_n$$

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Prediction/Mutation = Markov transition:

$$\check{X}_n \sim \mathcal{N}(\widehat{m}_n, \widehat{r}_n) \quad \text{and} \quad \overline{W}_{n+1} \sim \mathcal{N}(0,1)$$

$$\Longrightarrow \overline{X}_{n+1} := A\check{X}_n + \sigma\,\overline{W}_{n+1} \sim \mathcal{N}(m_{n+1}, r_{n+1})$$

Ensemble Kalman filter - Nonlinear Markov chain

$$\left\{\begin{array}{rcl}\check{X}_n & = & \overline{X}_n + \text{Gain}(r_n)\left(Y_n - (C\,\overline{X} + \sqrt{\tau}\,\overline{V}_n)\right)\\ \overline{X}_{n+1} & = & A\,\check{X}_n + \sigma\,\overline{W}_{n+1}.\end{array}\right.$$

with r_n := conditional variance of the random state $\overline{X}_n$.

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$$\Downarrow$$

EnKF with $i = 1, \ldots, N$ samples:

$$\begin{cases} \widehat{\xi}^i_n &= \xi^i_n + \text{Gain}(r^{\mathbf{N}}_n)\,(y_n - (C\,\xi^i_n + \sqrt{\tau}\,\overline{V}^i_n)) \\ \xi^i_{n+1} &= A\,\widehat{\xi}^i_n + \sigma\,\overline{W}^i_{n+1}. \end{cases}$$

with $r^{\mathbf{N}}_n$:= conditional N-sample variance of the random states ξ^i_n.

EnKF vs Particle filters = GA = SMC = DMC = ...

$$\left(\xi_n^i\right)_{1\leq i\leq N} \in \mathbb{R}^N \xrightarrow{\textbf{Selection}} \left(\widehat{\xi}_{\mathbf{n}}^{\mathbf{j}}\right)_{\mathbf{1\leq j\leq N}} \xrightarrow{\textbf{Mutation}} \left(\xi_{\mathbf{n+1}}^{\mathbf{i}}\right)_{\mathbf{1\leq i\leq N}}$$

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Selection/ **Mutation**:

$$\widehat{\xi}_n^j \sim \sum_{1\leq i\leq N} \frac{e^{-(Y_n - C\xi_n^i)^2/(2\tau)}}{\sum_{1\leq j\leq N} e^{-(Y_n - C\xi_n^j)^2/(2\tau)}} \delta_{\xi_n^i} \quad \textbf{and set} \quad \xi_{n+1}^j := A\,\widehat{\xi}_n^j + \sigma\, W_{n+1}^j$$

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Sample means $\simeq$ Conditional expectations:

$$\forall n \in \mathbb{N} \qquad \widehat{X}_n^{\mathrm{PF}} := \frac{1}{N} \sum_{1\leq i\leq N} \widehat{\xi}_n^i \simeq_{N\to\infty} \widehat{X}_n$$

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CLT variance time-uniformly bounded for any A

(Whiteley (AAP-2013))

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(`Whiteley (AAP-2013)`**) BUT for any $A > 1$**

$$\xi_0^i = x_0^i > \frac{\sigma}{A-1} \sqrt{2 \log N} \Longrightarrow \lim_{n \to \infty} \mathbb{E}\left[\left|\widehat{X}_n^{\mathrm{PF}} - \widehat{X}_n\right|\right] = +\infty$$

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While time-uniform estimates even for any $A > 1$ for the EnKF sample means $\widehat{X}_n^{\mathrm{EnKF}}$ (`(AAP-2023)/(Arxiv-2021)`**)**

$\rightsquigarrow$ continuous time (review `(MC2S-2023)`)

Branching processes - discrete time

- $P_n(x_{n-1}, dx_n)$ Markov $X_{n-1} \in E_{n-1} \rightsquigarrow X_n \in E_n$ (ex. historical)
- $g_n^i(x_n) \in \mathbb{N} =$ i.i.d. branching r.v., indexed by $x_n \in E_n$ and $i \geq 1$ s.t

$$\mathbb{E}(g_n^i(x_n)) \;\propto\; G_n(x_n)$$

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Branching process

$$\mathcal{X}_n := \sum_{1 \leq i \leq N_n} \delta_{\xi_n^i} \xrightarrow{branch.} \widehat{\mathcal{X}}_n := \sum_{1 \leq i \leq \widehat{N}_n} \delta_{\widehat{\xi}_n^i} \xrightarrow{explore} \mathcal{X}_{n+1} := \sum_{1 \leq i \leq N_{n+1} = \widehat{N}_n} \delta_{\xi_{n+1}^i}$$

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Note that

$$\widehat{\mathcal{X}}_n = \sum_{1 \leq i \leq N_n} g_n^i(\xi_n^i) \ \delta_{\xi_n^i}$$

$\rightsquigarrow$ **First moment (a.k.a. many-to-one) = FK**

$$\mathbb{E}(\mathcal{X}_n(f))/N_0 = \gamma_n(f) := \mathbb{E}\left(f(X_n) \prod_{0 \leq p < n} G_p(X_p)\right)$$

(Quasi)-Invariant measures

Any positive semigroup $\rightsquigarrow$ Time homogeneous FK (G, P)

$$Q(x,dy) = G(x)\ P(x,dy) \quad \text{with} \quad G := Q(1) \quad \text{and} \quad P(f) := Q(f)/Q(1)$$

(Quasi)-Invariant measures

Any positive semigroup $\rightsquigarrow$ Time homogeneous FK (G, P)

$$Q(x, dy) = G(x)\ P(x, dy) \quad \text{with} \quad G := Q(1) \quad \text{and} \quad P(f) := Q(f)/Q(1)$$

Hyp. stable semigroup:

$$\eta_n = \Phi(\eta_{n-1}) := \Psi_G(\eta_{n-1})P \longrightarrow_{n\to\infty} \eta_\infty = \Phi(\eta_\infty)$$

Case P is μ-reversible ($\Longrightarrow$ Q is $\Psi_{1/G}(\mu)$-reversible)

$$\Psi_{1/G}(\mu)(f_1 Q(f_2)) \propto \mu(f_1 P(f_2)) = \mu(P(f_1) f_2)$$

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$$\Downarrow$$

Spectral theo. on $\mathbb{L}_2(\Psi_{1/G}(\mu))$: $\exists \lambda_i > 0 \downarrow$ & h_i orthonormal s.t.

$$Q(h_i) = \lambda_i h_i$$

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Spectral theo. on $\mathbb{L}_2(\Psi_{1/G}(\mu))$: $\exists \lambda_i > 0 \downarrow$ & h_i orthonormal s.t.

$$Q(h_i) = \lambda_i h_i$$

and spectral decomposition

$$Q^n(x, dy) = \sum_{i \geq 0} \lambda_i^n \; h_i(x) \; h_i(y) \; \Psi_{1/G}(\mu)(dy)$$

Case P is μ-reversible

$(\lambda_0, h_0) = (\lambda, h) \rightsquigarrow$ **QSD** η_∞ **"more explicit" formulation**

$$\eta_\infty(f) :\propto \mu(P(h)\ f)$$

Case P is μ-reversible

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Also note:

$$\eta_\infty(G) = \frac{\mu(Q(h))}{\mu(h)}$$

Case P is μ-reversible

$(\lambda_0, h_0) = (\lambda, h) \rightsquigarrow$ **QSD** η_∞ **"more explicit" formulation**

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Also note:

$$\eta_\infty(G) = \frac{\mu(Q(h))}{\mu(h)} = \lambda$$

Case P is μ-reversible

$(\lambda_0, h_0) = (\lambda, h) \rightsquigarrow$ **QSD** η_∞ **"more explicit" formulation**

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Also note:

$$\eta_\infty(G) = \frac{\mu(Q(h))}{\mu(h)} = \lambda \Longrightarrow \Psi_G(\eta_\infty)(f) \propto \frac{\mu(Q(h)\ f)}{\lambda}$$

Case P is μ-reversible

$(\lambda_0, h_0) = (\lambda, h) \rightsquigarrow$ **QSD** η_∞ **"more explicit" formulation**

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Also note:

$$\eta_\infty(G) = \frac{\mu(Q(h))}{\mu(h)} = \lambda \Longrightarrow \Psi_G(\eta_\infty)(f) \propto \frac{\mu(Q(h)\ f)}{\lambda} \propto \Psi_h(\mu)(f)$$

Path-space - time homogeneous FK $\rightsquigarrow$ h-process

$$\mathbb{Q}_n(dx) := \frac{1}{\gamma_n(1)} \left\{ \prod_{0 \leq p < n} G(x_p) \right\} \eta_0(dx_0)\ P(x_0, dx_1) \ldots P(x_{n-1}, dx_n)$$

Key observation (including design FK with prescribed h):

$$Q(h) \quad = \quad \lambda\ h \iff G = \lambda h / P(h)$$

Path-space - time homogeneous FK $\rightsquigarrow$ h-process

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$$Q(h) \quad = \quad \lambda\ h \iff G = \lambda h / P(h)$$

$$\Downarrow$$

$$\mathbb{Q}_n(dx) \propto \underbrace{\frac{\eta_0(dx_0)\ h(x_0)}{\eta_0(h)}}_{=\eta_0^h(dx_0)} \left(\prod_{0 \le p < n} \underbrace{\frac{P(x_p, dx_{p+1}) h(x_{p+1})}{P(h)(x_p)}}_{P^h(x_p, dx_{p+1})} \right) \frac{\mathbf{1}}{\mathbf{h(x_n)}}$$

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Caution: even sampling P^h the weight $1/h$ may be very large and

$$\mu(hP(h)P^h(f)) = \mu(hP(hf)) = \mu(P(h)hf)$$

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Caution: even sampling P^h the weight $1/h$ may be very large and

$$\mu(hP(h)P^h(f)) = \mu(hP(hf)) = \mu(P(h)hf) \Longrightarrow \eta^h_\infty(f) \propto \mu(hP(h)f)$$

Feynman-Kac measures

A variety of interpretations/targets/. . .

Feynman-Kac particle recipes/methodologies

- Genealogical/Ancestral trees
- Backward particle models
- Island models
- Many-body FK models
- Particle Gibbs
- Quenched/Annealed models
- (Particle) Metropolis-Hasting
- Conditional Linear-Gaussian models
- Interacting Kalman filters

Performance/Convergence analysis

Historical/Path-space FK/Genealogical/Ancestral trees

$$X_n := (X'_0, \ldots, X'_n) \quad \& \quad G_n(X_n) = G'_n\,(X'_n)$$

$$\Downarrow$$

$$\gamma_n(f) := \mathbb{E}\left(f\,(X'_0, \ldots, X'_n) \prod_{0 \leq p < n} G'_p(X'_p) \right)$$

Historical/Path-space FK/Genealogical/Ancestral trees

$$X_n := (X'_0, \ldots, X'_n) \quad \& \quad G_n(X_n) = G'_n(X'_n)$$

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Back to n**-th time marginals**

$$f(X'_0, \ldots, X'_n) = f'(X'_n) \quad \rightharpoonup \quad \gamma_n(f) = \gamma'_n(f')$$

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Back to *n*-th time marginals

$$f(X'_0, \ldots, X'_n) = f'(X'_n) \quad \rightharpoonup \quad \gamma_n(f) = \gamma'_n(f')$$

$\rightsquigarrow$ **Path-valued GA (historical mutation, path-selection)** $\sim (P_n, G_n)$

$\rightsquigarrow$ **Genealogical trees**, **ancestral lines** $\xi^i_n = \left(\xi^i_{0,n}, \xi^i_{1,n}, \ldots, \xi^i_{n,n}\right)$

Unnormalized particle measures

$$\gamma_n(1) = \mathbb{E}\left(G_{n-1}(X_{n-1}) \prod_{0 \leq p < (n-1)} G_p(X_p) \right) = \frac{\gamma_{n-1}(G_{n-1})}{\gamma_{n-1}(1)} \; \gamma_{n-1}(1)$$

$$\Downarrow$$

$$\gamma_n(f) = \eta_n(f) \prod_{0 \leq p < n} \eta_p(G_p)$$

Unnormalized particle measures

$$\gamma_n(1) = \mathbb{E}\left(G_{n-1}(X_{n-1}) \prod_{0 \leq p < (n-1)} G_p(X_p) \right) = \frac{\gamma_{n-1}(G_{n-1})}{\gamma_{n-1}(1)}\ \gamma_{n-1}(1)$$

$$\Downarrow$$

$$\gamma_n(f) = \eta_n(f) \prod_{0 \leq p < n} \eta_p(G_p)$$

$\rightsquigarrow$ **unbiased unnormalized particle measures**

$$\gamma_n^N(f) := \eta_n^N(f) \prod_{0 \leq p < n} \eta_p^N(G_p)$$

Design unnormalized particle measures + unbiasedness + variance

$\rightsquigarrow$ (MPRF-1996),. . . , (IHP2011),. . .

$\rightsquigarrow$ cf. books+refs: (FK2004), (FTML2012), (CRC2013)

Unbiasedness property/Random ancestral lines

$$\mathbb{E}(\gamma_n^N(f)) = \gamma_n(f)$$

with historical $\quad X_n := (X'_0, \ldots, X'_n) \rightsquigarrow \xi_n^i = \left(\xi_{0,n}^i, \xi_{1,n}^i, \ldots, \xi_{n,n}^i\right)$

$$\Updownarrow$$

$$\mathbb{E}\left(f(\mathbb{X}_n) \prod_{0\le p<n} m(\xi_p)(G_p)\right) = \gamma_n(f)$$

with a randomly chosen ancestral line

$$\mathbb{X}_n \sim m(\xi_n) = \frac{1}{N}\sum_{1\le i\le N} \delta_{\left(\xi_{0,n}^i, \xi_{1,n}^i, \ldots, \xi_{n,n}^i\right)}$$

Density condition $Q_n(x_{n-1}, dx_n) = q_n(x_{n-1}, x_n)\ \nu_n(dx_n)$

$$\mathbb{Q}_n(d(x_0, \ldots, x_n)) \ \propto \ \nu_n(dx_n) q_n(x_{n-1}, x_n) \ldots \nu_1(dx_1)\ q_1(x_0, x_1)\ \eta_0(dx_0)$$

Density condition $Q_n(x_{n-1}, dx_n) = q_n(x_{n-1}, x_n)\ \nu_n(dx_n)$

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Key obs:

$$\eta_n(dx_n) \propto \eta_{n-1}(H_{n-1}(\centerdot, x_n))\ \nu_n(dx_n)$$

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Key obs:

$$\eta_n(dx_n) \propto \eta_{n-1}(H_{n-1}(\,\cdot\,,x_n))\ \nu_n(dx_n)$$

$$\Updownarrow$$

$$\nu_n(dx_n)\ q_n(x_{n-1},x_n)\ \eta_{n-1}(dx_{n-1}) \propto\ \eta_n(dx_n)\ \underbrace{\frac{\eta_{n-1}(dx_{n-1})q_n(x_{n-1},x_n)}{\eta_{n-1}(q_{n-1}(\,\cdot\,,x_n))}}_{:=\mathbb{K}_{n,\eta_{n-1}}(x_n,dx_{n-1})}$$

Density condition $Q_n(x_{n-1}, dx_n) = q_n(x_{n-1}, x_n)\ \nu_n(dx_n)$

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$$\Downarrow$$

Backward FK

$$\mathbb{Q}_n(d(x_0, \ldots, x_n)) = \eta_n(dx_n)\ \mathbb{K}_{n,\eta_{n-1}}(x_n, dx_{n-1}) \ldots \mathbb{K}_{1,\eta_0}(x_1, dx_0)$$

Backward FK and ancestral lines

$$\mathbb{Q}_n(dx) = \eta_n(dx_n)\, \mathbb{K}_{n,\eta_{n-1}}(x_n, dx_{n-1}) \ldots \mathbb{K}_{1,\eta_0}(x_1, dx_0)$$

Backward FK and ancestral lines

$$\mathbb{Q}_n(dx) = \eta_n(dx_n)\, \mathbb{K}_{n,\eta_{n-1}}(x_n, dx_{n-1}) \ldots \mathbb{K}_{1,\eta_0}(x_1, dx_0)$$

Backward particle FK (M2AN10) $=$ **Conditional ancestral line** **(**Arxiv(14)/IHP(16)**)**

$$\mathbb{Q}_n^{\mathbf{N}}(dx) \quad = \quad \eta_n^{\mathbf{N}}(dx_n)\, \mathbb{K}_{n,\eta_{n-1}^{\mathbf{N}}}(x_n, dx_{n-1}) \ldots \mathbb{K}_{1,\eta_0^{\mathbf{N}}}(x_1, dx_0)$$

Backward FK and ancestral lines

$$\mathbb{Q}_n(dx) = \eta_n(dx_n)\, \mathbb{K}_{n,\eta_{n-1}}(x_n, dx_{n-1}) \ldots \mathbb{K}_{1,\eta_0}(x_1, dx_0)$$

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$$\begin{aligned} \mathbb{Q}_n^{\mathbf{N}}(dx) &= \eta_n^{\mathbf{N}}(dx_n)\, \mathbb{K}_{n,\eta_{n-1}^{\mathbf{N}}}(x_n, dx_{n-1}) \ldots \mathbb{K}_{1,\eta_0^{\mathbf{N}}}(x_1, dx_0) \\ &= \mathbb{P}\left(\mathbb{X}_n \in d(x_0, \ldots, x_n) \mid \eta_{\mathbf{p}}^{\mathbf{N}},\ \mathbf{p} \leq \mathbf{n}\right) \end{aligned}$$

Backward FK and ancestral lines

$$\mathbb{Q}_n(dx) = \eta_n(dx_n)\, \mathbb{K}_{n,\eta_{n-1}}(x_n, dx_{n-1}) \ldots \mathbb{K}_{1,\eta_0}(x_1, dx_0)$$

Backward particle FK (M2AN10) = **Conditional ancestral line** (Arxiv(14)/IHP(16))

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$\rightsquigarrow$ **Backward Markov chain on index set** $\{1, \ldots, N\}$

$$\eta_n^{\mathbf{N}} \simeq \left[\frac{1}{N}, \ldots, \frac{1}{N}\right], \quad \mathbb{K}_{n,\eta_{n-1}^{\mathbf{N}}} \simeq \begin{bmatrix} \mathbb{K}_{n,\eta_{n-1}^{\mathbf{N}}}(\xi_n^1, \xi_n^1) & \ldots & \mathbb{K}_{n,\eta_{n-1}^{\mathbf{N}}}(\xi_n^1, \xi_n^N) \\ \vdots & \ldots & \vdots \\ \mathbb{K}_{n,\eta_{n-1}^{\mathbf{N}}}(\xi_n^N, \xi_n^1) & \ldots & \mathbb{K}_{n,\eta_{n-1}^{\mathbf{N}}}(\xi_n^N, \xi_n^N) \end{bmatrix}, \quad f_n \simeq \begin{bmatrix} f(\xi_n^1) \\ \vdots \\ f(\xi_n^N) \end{bmatrix}$$

Unbiasedness property/Island models/(a.k.a. SMC^2) (FTML2011)

$$\mathbb{E}\left(f(X_n)\prod_{0\le p<n} G_p(X_p)\right)$$

$$=\mathbb{E}\left(m(\xi_n)(f)\ \prod_{0\le p<n} m(\xi_p)(G_p)\right)$$

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$$=\mathbb{E}\left(m(\xi_n)(f)\ \prod_{0\leq p<n} m(\xi_p)(G_p)\right)=\mathbb{E}\left(\mathcal{F}(\xi_\mathbf{n})\prod_{0\leq p<n}\mathcal{G}_\mathbf{p}(\xi_\mathbf{p})\right)$$

with

$$\mathcal{F}(\xi_n):=m(\xi_n)(f)\quad\text{and}\quad\mathcal{G}_p(\xi_p):=m(\xi_p)(G_p)$$

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GA = Island model:

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N'-Genetic model with mutation as $\xi_{n-1}\rightsquigarrow\xi_n$

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with

$$\mathcal{F}(\xi_n) := m(\xi_n)(f) \quad \text{and} \quad \mathcal{G}_p(\xi_p) := m(\xi_p)(G_p)$$

GA = **Island model:**

N'-Genetic model with mutation as $\xi_{n-1} \rightsquigarrow \xi_n$

and selection/fitness/potential $\mathcal{G}_n(\xi_n)$.

$\rightsquigarrow$ Many-body-FK $=$ Re-weighted FK-particle

$$\mathcal{Q}_n(F_n) := \frac{1}{\gamma_n(1)}\ \mathbb{E}\left(F_n(\xi_0,\ldots,\xi_n)\prod_{0\leq p<n}\mathcal{G}_p(\xi_p)\right)$$

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Important observation

$$\mathrm{Law}_{\mathcal{Q}_n}(\mathbb{X}_n) = \mathbb{Q}_n$$

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Important observation

$$\mathrm{Law}_{\mathcal{Q}_n}\left(\mathbb{X}_n\right)=\mathbb{Q}_n$$

Easy sampling given an ancestral line (`Arxiv(14)/IHP(16)`)

$$\mathrm{Law}_{\mathcal{Q}_n}\left((\xi_0,\ldots,\xi_n)\mid\mathbb{X}_n\right)$$

$=$ Law N **particles with a given frozen path** $\mathbb{X}_n$

$\rightsquigarrow$ Many-body-FK $=$ Re-weighted FK-particle

$$\mathcal{Q}_n(F_n) := \frac{1}{\gamma_n(1)} \mathbb{E}\left(F_n(\xi_0, \ldots, \xi_n) \prod_{0 \le p < n} \mathcal{G}_p(\xi_p)\right)$$

Important observation

$$\mathrm{Law}_{\mathcal{Q}_n}(\mathbb{X}_n) = \mathbb{Q}_n$$

Easy sampling given an ancestral line (`Arxiv(14)/IHP(16)`)

$$\mathrm{Law}_{\mathcal{Q}_n}((\xi_0, \ldots, \xi_n) \mid \mathbb{X}_n)$$

$$= \mathrm{Law}\ N \text{ particles with a given frozen path } \mathbb{X}_n$$

Same/also work with backward ancestral lines

(Particle) Gibbs sampler

Target/Joint dist.: $\mathrm{Law}_{\mathcal{Q}_n}(\mathbb{X}_n, \mathcal{X}_n)$ **with** $\mathcal{X}_n := (\xi_0, \ldots, \xi_n)$

$$\begin{pmatrix} \mathbb{X}_n \\ \mathcal{X}_n \end{pmatrix} \xrightarrow{\text{pick } \mathbb{X}'_n \text{ in } \mathcal{X}_n} \begin{pmatrix} \mathbb{X}'_n \\ \mathcal{X}_n \end{pmatrix} \xrightarrow{\mathcal{X}'_n \text{ with frozen } \mathbb{X}'_n} \begin{pmatrix} \mathbb{X}'_n \\ \mathcal{X}'_n \end{pmatrix}$$

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Note $\mathbb{X}_n \rightsquigarrow \mathbb{X}'_n$ **Markov with invariant measure**

$$\mathrm{Law}_{\mathcal{Q}_n}(\mathbb{X}_n) = \mathbb{Q}_n$$

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Target/Joint dist.: $\mathrm{Law}_{\mathcal{Q}_n}(\mathbb{X}_n, \mathcal{X}_n)$ **with** $\mathcal{X}_n := (\xi_0, \ldots, \xi_n)$

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Note $\mathbb{X}_n \rightsquigarrow \mathbb{X}'_n$ **Markov with invariant measure**

$$\mathrm{Law}_{\mathcal{Q}_n}(\mathbb{X}_n) = \mathbb{Q}_n$$

Same/also work with backward ancestral lines

Quenched/Annealed FK $\sim$ parameter $\mathrm{Law}(\Theta) = \mu$

$$\theta \mapsto (G_k^{[\theta]}, P_k^{[\theta]}) \rightsquigarrow \gamma_n^{[\theta]}, \eta_n^{[\theta]}, \mathbb{Q}_n^{[\theta]}$$

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$$\theta \mapsto (G_k^{[\theta]}, P_k^{[\theta]}) \rightsquigarrow \gamma_n^{[\theta]}, \eta_n^{[\theta]}, \mathbb{Q}_n^{[\theta]} \quad \text{and} \quad \xi^{[\theta]} := (\xi_0^{[\theta]}, \ldots, \xi_n^{[\theta]})$$

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Note that

$$\gamma_{n+1}^{[\theta]}(1) = \mathcal{Z}_n(\theta) = \prod_{0 \leq p \leq n} \eta_p^{[\theta]}(G_p^{[\theta]})$$

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Some examples:

$$\mathcal{Z}_n(\theta) = p(y_0, \ldots, y_n \mid \theta), \; \mathbb{P}(\text{critical event} \mid \theta), \ldots$$

Objective: Given a time horizon n sample from the annealed

$$\pi_n(d\theta) \; \propto \; \mathcal{Z}_n(\theta) \; \mu(d\theta)$$

Quenched/Annealed FK $\sim$ parameter $\mathrm{Law}(\Theta) = \mu$

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Some examples:

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Some examples:

$$\pi_{n+1}(d\theta) = p(\theta \mid (y_0, \ldots, y_n)), \ \mathbb{P}(\theta \mid \text{critical event}), \ldots$$

Extended target: Given a time horizon n

Design of quenched distributions:

$$\nu = \text{Law}(\theta, \xi^{[\theta]}) \rightsquigarrow \nu(d(\theta, z)) = \mu(d\theta)\, \nu(\theta \rightsquigarrow dz)$$

Extended target: Given a time horizon n

Design of quenched distributions:

$$\nu = \mathrm{Law}(\theta, \xi^{[\theta]}) \;\rightsquigarrow\; \nu(d(\theta, z)) = \mu(d\theta)\; \nu(\theta \rightsquigarrow dz)$$

with $z := (z_0, \ldots, z_n)$; as well as

$$\overline{\mathcal{Z}}_n(\theta, z) := \prod_{0 \leq p \leq n} h_p(\theta, z) \quad \text{and} \quad h_p(\theta, z) := m(z_p)(G_p^{[\theta]})$$

By the unbiasedness property: sample quenched/extended distributions

$$\overline{\pi}_n(d(\theta, z)) \propto \overline{\mathcal{Z}}_n(\theta, z)\; \nu(d(\theta, z))$$

Extended target: Given a time horizon n

Design of quenched distributions:

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By the unbiasedness property: sample quenched/extended distributions

$$\overline{\pi}_n(d(\theta, z)) \propto \overline{\mathcal{Z}}_n(\theta, z)\ \nu(d(\theta, z)) \quad \rightsquigarrow \quad \theta - \text{marginal} = \pi_{\mathbf{n}}$$

(Particle) Metropolis-Hasting on extended space

$$\left(\begin{array}{c}\theta\\ z\end{array}\right) \rightsquigarrow \left(\begin{array}{c}\theta'\\ z'\end{array}\right)$$

Metropolis-Hasting with reversible local θ move: $\theta \rightsquigarrow \theta' \rightsquigarrow z'$

$$\mu(d\theta)\ K(\theta, d\theta') = \mu(d\theta')\ K(\theta', d\theta)$$

Acceptance ratio

$$\frac{\left[\overline{\mathcal{Z}}_n(\theta', z')\ \mu(d\theta')\ \nu(\theta' \rightsquigarrow dz')\right]\ K(\theta', d\theta)\ \nu(\theta \rightsquigarrow dz)}{\left[\overline{\mathcal{Z}}_n(\theta, z)\ \mu(d\theta)\ \nu(\theta \rightsquigarrow dz)\right]\ K(\theta, d\theta')\ \nu(\theta' \rightsquigarrow dz')} = \frac{\overline{\mathcal{Z}}_n(\theta', z')}{\overline{\mathcal{Z}}_n(\theta, z)}$$

Parameter and path $x = (x_0, \ldots, x_n)$

$$\Pi_n(d(x,\theta)) \;\propto\; \mathcal{Z}_n(\theta)\; \mu(d\theta)\; \mathbb{Q}_n^{[\theta]}(dx)$$

Examples:

$$\Pi_{n+1}(d(x,\theta)) = p((x,\theta) \mid (y_0,\ldots,y_n)),\; \mathbb{P}((x,\theta) \mid \text{critical event}),\ldots$$

Gibbs sampler

$$\begin{pmatrix}\theta\\ x\end{pmatrix} \xrightarrow{\text{target } \Pi_n(d\theta \mid x)} \underbrace{\begin{pmatrix}\theta'\\ x\end{pmatrix} \xrightarrow{\text{target } \mathbb{Q}_n^{[\theta']}(dx)} \begin{pmatrix}\theta'\\ x'\end{pmatrix}}_{\text{ex.: particle MH or particle Gibbs with frozen ancestral lines}}$$

Ex. inverse Gamma prior (shape,scale)$= (\alpha, \beta)$

$$\mu(d\theta) \propto \frac{1}{\theta^{1+\alpha}} \; e^{-\beta/\theta} \; 1_{]0,\infty[}(\theta) \; d\theta$$

Only $X_n = b(X_{n-1}) + \sqrt{\theta}\mathcal{N}(0,1) \rightsquigarrow$ inverse Gamma

$$\begin{aligned}
\Longrightarrow \Pi_n(d\theta \mid x) \quad &= \quad p(x \mid \theta) \; \mu(d\theta) \\
&\propto \quad \frac{1}{\theta^{1+\alpha+n/2}} \; \exp\Big(-\frac{1}{\theta} \underbrace{\Big[\beta + \sum_{1\leq k\leq n} (x_k - b(x_{k-1}))^2\Big]}_{:=\beta_n(x)}\Big) \; d\theta
\end{aligned}$$

Conditional Linear-Gaussian models - Θ_n Markov

Linear/Gaussian given Θ:

$$X_n^{[\theta]} = A(\Theta_n) X_{n-1}^{[\theta]} + \sigma(\Theta_n)\, W_n \;\;\&\;\; G_{\mathbf{n}}^{[\theta]}(x) \propto \exp\left(-\frac{1}{2\tau(\Theta_n)}\,(y_n - C(\Theta_n)x)^2\right)$$

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$\rightsquigarrow$ **Gaussian quenched FK** $\rightsquigarrow \eta_n^{[\theta]} = \mathcal{N}(m_n^{[\theta]}, r_n^{[\theta]})$ **AND**

$$\mathbb{E}\left(f(X_n^{[\theta]}, \Theta_n) \prod_{0 \le p < n} G_p^{[\theta]}(X_p^{[\theta]}) \;\mid \boldsymbol{\Theta} = \theta\right) = \eta_n^{[\theta]}(f(\,.\,, \theta_{\mathbf{n}})) \prod_{0 \le p < n} \eta_p^{[\theta]}(G_p^{[\theta]})$$

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Key Obs.:

$$\begin{aligned} \eta_n^{[\theta]}(f(\,.\,, \theta_{\mathbf{n}})) &= \mathcal{F}_n(\theta_n, m_n^{[\theta]}, r_n^{[\theta]}) \\ \eta_p^{[\theta]}(G_p^{[\theta]}) &\propto \mathcal{G}_p(\theta_p, m_p^{[\theta]}, r_p^{[\theta]}) := \exp\left(-\frac{(y_p - C(\Theta_p) m_p^{[\theta]})^2}{2(\tau(\Theta_p) + C(\Theta_p)^2 r_p^{[\theta]})}\right) \end{aligned}$$

Conditional Linear-Gaussian models - Θ_n Markov

Markov chain:

$$\mathcal{X}_n := (\Theta_n, m_n^{[\Theta]}, r_n^{[\Theta]})$$

Annealed FK

$$\mathbb{E}\left(f(X_n^{[\Theta]}, \Theta_n) \prod_{0 \leq p < n} G_p^{[\Theta]}(X_p^{[\Theta]})\right) = \mathbb{E}\left(\mathcal{F}_n(\mathcal{X}_n) \prod_{0 \leq p < n} \mathcal{G}_p(\mathcal{X}_p)\right)$$

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$\leftrightsquigarrow$ **Genetic Monte Carlo samplers (mutation,selection)** $\sim (\mathcal{X}_n, \mathcal{G}_n)$

$=$ **Interacting Kalman filters**

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$\leftrightsquigarrow$ **Same ideas as islands** $\rightsquigarrow$ **Interacting particle filters**

Some application domains

Signal processing/Bayesian inference/machine learning/Physics/Risk-rare events/Maths-bio/...

$\supset$ *Filtering/smoothing, first moment spatial branching, tube conditioning, self-avoiding RW, "quasi"-invariant, rare events, level splitting, interacting MCMC, ground states, leading eigen-triple Schrödinger semigroups, ...*

Real world examples (discrete time):

$$G'_n(x'_n) = p(y_n|x'_n) \qquad G'_n = 1_{A_n} \qquad G_n(x'_0,\ldots,x'_n) = 1_{x'_n \not\in \{x'_0,\ldots,x'_{n-1}\}}$$

$$G_n = e^{-(\beta_{n+1}-\beta_n)V} \quad \& \quad P_n = \ e^{-\beta_n V} - \text{shaker} \qquad \ldots$$

Feynman-Kac measures

A variety of interpretations/targets/...

Feynman-Kac particle recipes/methodologies

Performance/Convergence analysis

- Stochastic perturbation theory
- Variational techniques
- V-norm contraction
- Stability positive semigroups

Sample mean $m_t := \frac{1}{N} \sum_{1 \leq i \leq N} X_t^i$ with **iid** copies X_t^i of X_t

$$\text{Sample mean} \quad m_t := \frac{1}{N} \sum_{1 \le i \le N} X_t^i \quad \text{with } \textbf{iid} \text{ copies } X_t^i \text{ of } X_t$$

⇝ **stochastic perturbation formulation**

$$\mathbf{m_t} := \mathbb{E}(\mathbf{X_t}) + \frac{1}{\sqrt{N}} \mathbb{V}_t^N$$

$$\text{Sample mean} \quad m_t := \frac{1}{N}\sum_{1\leq i\leq N} X_t^i \quad \text{with } \textbf{iid} \text{ copies } X_t^i \text{ of } X_t$$

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$$\mathbf{m_t} := \mathbb{E}(\mathbf{X_t}) + \frac{1}{\sqrt{N}}\ \mathbb{V}_t^N$$

with bias-variance perturbation control

$$\mathbb{E}(\mathbb{V}_t^N) = 0 \quad \& \quad \mathbb{E}((\mathbb{V}_t^N)^2) = \mathbb{E}((X_t - \mathbb{E}(X_t))^2)$$

$$\text{Sample mean} \quad m_t := \frac{1}{N} \sum_{1 \le i \le N} X_t^i \quad \text{with } \textbf{iid} \text{ copies } X_t^i \text{ of } X_t$$

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Key Observation:

$$X_t \text{ stable} \implies \sup_{t \ge 0} \mathbb{E}((m_t - \mathbb{E}(X_t))^2) \le c/N$$

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$$\lim_{t \to \infty} \mathbb{E}((X_t - \mathbb{E}(X_t))^2) = \infty \implies \sup_{t \geq 0} \mathbb{E}((m_t - \mathbb{E}(X_t))^2) = \infty$$

Stochastic perturbation analysis

"Intuitive picture" $\rightsquigarrow$ any nonlinear-type sg = evolution sg. $s \leq t$:

$$\mathcal{X}_t = \Phi_t(\mathcal{X}_{t-1})$$

Stochastic perturbation analysis

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$$\mathcal{X}_t = \Phi_t(\mathcal{X}_{t-1}) = \Phi_{s,t}(\mathcal{X}_s) \Longleftrightarrow \Phi_{s,t} := \Phi_t \circ \ldots \circ \Phi_{s+1}$$

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Perturbation evolution sg :

$$\mathcal{X}_t^\epsilon = \Phi_t^\epsilon(\mathcal{X}_{t-1}^\epsilon)$$

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Perturbation evolution sg :

$$\mathcal{X}_t^\epsilon = \Phi_t^\epsilon(\mathcal{X}_{t-1}^\epsilon) := \Phi_t(\mathcal{X}_{t-1}^\epsilon) + \epsilon\ \mathbb{V}_t^\epsilon$$

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Perturbation evolution sg :

$$\mathcal{X}_t^\epsilon = \Phi_t^\epsilon(\mathcal{X}_{t-1}^\epsilon) := \Phi_t(\mathcal{X}_{t-1}^\epsilon) + \epsilon \, \mathbb{V}_t^\epsilon$$

$$\begin{array}{ccccccccc}
\mathcal{X}_0 & \rightarrow & \mathcal{X}_1 = \Phi_1(\mathcal{X}_0) & \rightarrow & \mathcal{X}_2 = \Phi_{0,2}(\mathcal{X}_0) & \rightarrow & \cdots & \rightarrow & \Phi_{0,t}(\mathcal{X}_0) \\
\Downarrow & & & & & & & & \\
\mathcal{X}_0^\epsilon & \rightarrow & \Phi_1(\mathcal{X}_0^\epsilon) & \rightarrow & \Phi_{0,2}(\mathcal{X}_0^\epsilon) & \rightarrow & \cdots & \rightarrow & \Phi_{0,t}(\mathcal{X}_0^\epsilon) \\
 & & \Downarrow & & & & & & \\
 & & \mathcal{X}_1^\epsilon & \rightarrow & \Phi_2(\mathcal{X}_1^\epsilon) & \rightarrow & \cdots & \rightarrow & \Phi_{1,t}(\mathcal{X}_1^\epsilon) \\
 & & & & \Downarrow & & & & \\
 & & & & \mathcal{X}_2^\epsilon & \rightarrow & \cdots & \rightarrow & \Phi_{2,t}(\mathcal{X}_2^\epsilon) \\
 & & & & & & & & \vdots \\
 & & & & & & \Downarrow & & \\
 & & & & & & \mathcal{X}_{t-1}^\epsilon & \rightarrow & \Phi_t(\mathcal{X}_{t-1}^\epsilon) \\
 & & & & & & & & \Downarrow \\
 & & & & & & & & \mathcal{X}_t^\epsilon
\end{array}$$

Stoch. Alekseev-Gröbner type telescoping formula

$$\mathcal{X}_t^\epsilon - \mathcal{X}_t = \sum_{s=0}^{t} \left[\Phi_{s,t}(\mathcal{X}_s^\epsilon) - \Phi_{s,t}(\Phi_s(\mathcal{X}_{s-1}^\epsilon))\right]$$

Stoch. Alekseev-Gröbner type telescoping formula

$$\begin{aligned} \mathcal{X}_t^\epsilon - \mathcal{X}_t &= \sum_{s=0}^{t} \left[\Phi_{s,t}(\mathcal{X}_s^\epsilon) - \Phi_{s,t}(\Phi_s(\mathcal{X}_{s-1}^\epsilon))\right] \\ &= \sum_{s=0}^{t} \left[\Phi_{s,t}\left(\Phi_s(\mathcal{X}_{s-1}^\epsilon) + \epsilon\, \mathbb{V}_s^\epsilon\right) - \Phi_{s,t}\left(\Phi_s(\mathcal{X}_{s-1}^\epsilon)\right)\right] \end{aligned}$$

Stoch. Alekseev-Gröbner type telescoping formula

$$\begin{aligned}
\mathcal{X}_t^\epsilon - \mathcal{X}_t &= \sum_{s=0}^{t} \left[\Phi_{s,t}(\mathcal{X}_s^\epsilon) - \Phi_{s,t}(\Phi_s(\mathcal{X}_{s-1}^\epsilon))\right] \\
&= \sum_{s=0}^{t} \left[\Phi_{s,t}\left(\Phi_s(\mathcal{X}_{s-1}^\epsilon) + \epsilon\, \mathbb{V}_s^\epsilon\right) - \Phi_{s,t}\left(\Phi_s(\mathcal{X}_{s-1}^\epsilon)\right)\right] \\
&= \textbf{First} \text{ or } \textbf{second order expansions} = \textbf{CLT} \text{ or } \textbf{BIAS}
\end{aligned}$$

$$\downarrow$$

Origins/Bias/CLT/Stability/Time-uniform propagation of chaos/. . .

+Guionnet (LSP No. 03-1998), (CRAS-99)/(IHP-00), +Miclo [Sem.Prob-00], time-uniform exponential concentration (AAP-11)(FTML-11), strong large deviations, books+refs . . .

$\rightsquigarrow$ **stability analysis of linear/nonlinear semigroups.**

Caution: CLT variance uniformly bounded w.r.t. time BUT GA blows up for "unstable" mutations

Variational techniques $X_n(x)$ stoch. flow

$$X_n(x) = \mathcal{F}_n(X_{n-1}(x)) \overset{ex.}{=} F_n(X_{n-1}(x), W_n) \quad \text{with} \quad X_0(x) = x \in \mathbb{R}^d$$

$$\Downarrow$$

First variational (matrix-valued) equation

$$\nabla X_n(x) = \nabla X_{n-1}(x)\ \nabla \mathcal{F}_n(X_{n-1}(x))$$

$$\text{Hyp:} \quad \mathbb{E}(\ \nabla \mathcal{F}_{\mathbf{n}}(\mathbf{y})\ \nabla \mathcal{F}_{\mathbf{n}}(\mathbf{y})') \leq (\mathbf{1} - \delta_{\mathbf{n}})\ \mathbf{I}$$

$$\Longrightarrow \nabla X_n(x) \nabla X_n(x)' \leq (1 - \delta_n)\ \nabla X_{n-1}(x)\ \nabla X_{n-1}(x)' \leq \prod_{1 \leq p \leq n} (1 - \delta_p)\ I$$

Stab. Markov sg. with V-norms ($V \geq 1/2$)

$$\|f\|_V = \|f/V\| \quad \rightsquigarrow \mathcal{B}_V(E) \quad \text{and} \quad |||\mu - \eta|||_V = |\mu - \eta|(V) \quad \rightsquigarrow \mathcal{P}_V(E)$$

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V-Dobrushin contraction coef.:

$$\beta_V(P_{n,n+m}) := \sup_{(\mu,\eta)\in\mathcal{P}_V(E)} \frac{|||(\mu - \eta)P_{n,n+m}|||_V}{|||\mu - \eta|||_V} \leq 1 - \delta_m$$

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$$\Downarrow$$

V-contraction:

$$|||(\mu - \eta)P_{n,n+pm}|||_V \leq (1 - \delta_m)^p \; |||\mu - \eta|||_V$$

Time homogeneous ($P_{k,k+n} = P^n$): $\exists! \eta_\infty = \eta_\infty P \in \mathcal{P}_V(E)$ and $b > 0$ s.t.

$$|||\mu P^n - \eta_\infty|||_V \leq c \; e^{-bn} \; |||\mu - \eta_\infty|||_V$$

V-norm contractions tech.

(+Penev [CRC-Chapman & Hall(2016)],
+Horton-Jasra Arxiv(21)/AAP(22),+Arnaudon-Ouhabaz SAA(23))

$$\beta_V(P_{n,n+m}) \leq 1-\delta_m \Longleftarrow \begin{cases} P_{n,n+\tau}(V) \overset{(1)}{\leq} \epsilon_\tau V + c_\tau \\ \\ \displaystyle\sup_{V(x)\vee V(y)\leq r} \|(\delta_x-\delta_y)P_{n,n+\tau}\|_{tv} \overset{(2)}{\leq} 1-\epsilon_\tau(r) \end{cases}$$

V-norm contractions tech.

(+Penev [CRC-Chapman & Hall(2016)],
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$$\beta_V(P_{n,n+m}) \leq 1-\delta_m \Longleftarrow \left\{ \begin{array}{l} P_{n,n+\tau}(V) \overset{(1)}{\leq} \epsilon_\tau V + c_\tau \\ \\ \sup_{V(x)\vee V(y)\leq r} \|(\delta_x-\delta_y)P_{n,n+\tau}\|_{tv} \overset{(2)}{\leq} 1-\epsilon_\tau(r) \end{array} \right.$$

But also

$$(2)' = \left\{ \begin{array}{c} P_{n,n+\tau}(x,dy) \geq q_{n,n+\tau}(x,y)\ \nu_{n,\tau}(dy) \\ \\ \inf_{V(x)\vee V(y)\leq r} q_{n,n+\tau}(x,y) \geq \iota_r(\tau) > 0 \end{array} \right\} \Longrightarrow (2)$$

Note:
V **bounded** $\Longrightarrow$ Dobrushin coef. $:= \beta(P_{p,n})$ and $(2)'$=Doeblin min cond. $\rightsquigarrow$
+Guionnet (LSP No. 03-1998), (CRAS-99)/(IHP-00), +Miclo [Sem.Prob-00]

Extension to absolutely continuous sg density $q_{s,t}(x,y)$

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Particle estimates for this class of models?

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$(0 < H/V \in \mathcal{B}_0(E)$ & $\eta_0 = \mu \in \mathcal{P}_V(E))$ $\rightsquigarrow$ **Rank one operators**

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and for $H = h$ and $\mu = \eta_\infty$ the above yields

$$\sup_{\|f\|_V \leq 1} \left\| e^{-\rho(t-s)}\ Q_{s,t}(f) - \frac{h}{\eta_\infty(h)}\ \eta_\infty(f) \right\|_V \leq\ c\ e^{-b(t-s)}$$