Feynman-Kac particle models and their applications to stochastic engineering

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Outline

1. Some motivations
   - Stochastic engineering problems
   - Some "heuristic-metaheuristic" like algorithms
   - A pair of "conjectures"
2. Feynman-Kac models
   - Path integral measures
   - Some "wrong" approximation ideas
   - A nonlinear approach
   - Mean field particle methods
3. Some modeling recipes
   - Interacting sampling technique (a brief reminder)
   - A direct conditioning approach
   - Static models and Boltzmann-Gibbs flows
4. Some references

Stochastic engineering $\rightsquigarrow$ Conditional & Boltzmann-Gibbs' measures

- Filtering: Signal-Observation (X_t, Y_t) [Radar, Sonar, GPS, ...]

$$\eta_t = \mathrm{Law}((X_0, \ldots, X_t) \mid (Y_0, \ldots, Y_t))$$

- Rare events: [Overflows, ruin processes, epidemic propagations,...]

$$\eta_t = \mathrm{Law}((X_0, \ldots, X_t) \mid \text{Rare event}) \quad \text{and} \quad \mathbb{P}(X \in \text{Rare event})$$

- Molecular simulation: [ground state energies, directed polymers...]

$$\eta_t := \text{Boltzamnn-Gibbs} \quad \text{or} \quad \eta_t = \mathrm{Law}(X_t \mid \text{non absorption})$$

- Combinatorial counting: [$\not\cap$ random walks, unif. sampling]
- Global optimization, hidden Markov problems

$$\eta_t = \frac{1}{\mathcal{Z}_t}\, e^{-\beta_t V(x)}\, \lambda(dx) \quad \text{or} \quad \eta_t = \frac{1}{\mathcal{Z}_t}\, p_\theta(y_t)\, \lambda(d\theta)$$

Since the 50's $\rightsquigarrow \uparrow$ bio-inspired sampling strategies

- Filtering: Particle, spawning, switching, bootsrap, condensed, ensemble, auxiliary, importance resampling,...filters.
- Rare events: Multi splitting, restart, subset sampling,...
- Molecular simulation: quantum and diffusion Monte-Carlo methods, matrix reconfigurations, reptation, pruning enrichment, go-with-the-winner, ...
- Combinatorial counting: cloning, pruning, branch and cut. ...
- Global optimization, hidden Markov problems: genetic, evolutionary alg., tabu search, survival the fittest, population Monte Carlo,...

$\sim$ **Same natural dynamic heuristic:**

1. *Explore randomly the solution space.*
2. *Duplicate nice proposals and stop-kill the other ones.*

A pair of "conjectures"

1. Previous engineering problems

   $\subset$ A single Feynman-Kac representation model

2. Previous "heuristic-metaheuristics"

   $\subset$ A single mean field particle interpretation model.

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Immediate answer :

1. Both are true
2. Proof: *Partly today ..../....*

   Since $90's$ $\rightsquigarrow$ Series of joints works with: *Arnaud Doucet, Alice Guionnet, Jean Jacod, Laurent Miclo, and many others.*

R.P. Feynman [Princeton Ph.D. in the 40's]

- Feynman-Kac measures: Markov-Potential (X_n, G_n) on E_n

$$d\eta_n = \frac{1}{\mathcal{Z}_n} \left\{ \prod_{0 \leq p < n} G_p(X_p) \right\} d\mathbb{P}_n^X$$

- Weak representation:

  [f_n test funct. on E_n, up to a state enlargement $X_n = (X'_0, \ldots, X'_n)$]

$$\eta_n(f_n) = \frac{\gamma_n(f_n)}{\gamma_n(1)} \quad \text{with} \quad \gamma_n(f_n) = \mathbb{E}\left(f_n(X_n) \prod_{0 \leq p < n} G_p(X_p) \right)$$

- A Key multiplicative formula: ($\rightsquigarrow$ Unbias estimation)

$$\mathcal{Z}_n = \mathbb{E}\left(\prod_{0 \leq p < n} G_p(X_p) \right) = \prod_{0 \leq p < n} \eta_p(G_p)$$

Some "wrong" approximation ideas

- "Pure" weighted Monte Carlo methods : X^i iid copies of X

$$\frac{1}{N}\sum_{i=1}^{N} f_n(X_n^i)\left\{\prod_{0\leq p<n} G_p(X_p^i)\right\} \simeq \mathbb{E}\left(f_n(X_n)\prod_{0\leq p<n} G_p(X_p)\right)$$

$\rightsquigarrow$ bad grids X^i $\oplus$ degenerate weights (running ex $G_n = 1_A$).

- Uncorrelated MCMC for each target measure η_n ($\uparrow$ complex.).
- "Pure" branching $\rightsquigarrow$ critical random population sizes

$$G_n(x) = \mathbb{E}(g_n(x)) \quad \text{with} \quad g_n(x) \text{ r.v. } \in \mathbb{N}$$

- Harmonic/(Gaussian+linearisation) approximations.
- $G.M(H) \propto H \rightsquigarrow G \propto H/M(H) \rightsquigarrow H$-process X^H (unknown).

Nonlinear distribution flows

Evolution equation: [$\eta_n \in \mathcal{P}(E_n)$ probability measures $\uparrow$ complexity].

$$\eta_{n+1} = \Phi_{n+1}(\eta_n) = \Psi_{G_n}(\eta_n) M_{n+1}$$

With only 2 transformations:

- X-Free Markov transport eq. : [$M_n(x_{n-1}, dx_n)$ from E_{n-1} into E_n]

$$(\eta_{n-1} M_n)(dx_n) := \int_{E_{n-1}} \eta_{n-1}(dx_{n-1}) \ M_n(x_{n-1}, dx_n)$$

- Bayes-Boltzmann-Gibbs transformation :

$$\Psi_{G_n}(\eta_n)(dx_n) := \frac{1}{\eta_n(G_n)} \ G_n(x_n) \ \eta_n(dx_n)$$

Nonlinear distribution flows

- **Nonlinear Markov models :** always $\exists K_{n,\eta}(x, dy)$ Markov s.t.

$$\eta_n = \Phi_n(\eta_{n-1}) = \eta_{n-1} K_{n,\eta_{n-1}} = \text{Law}\left(\overline{X}_n\right)$$

i.e. :

$$\mathbb{P}(\overline{X}_n \in dx_n \mid \overline{X}_{n-1}) = K_{n,\eta_{n-1}}(\overline{X}_{n-1}, dx_n)$$

Mean field particle interpretation

- Markov chain $\xi_n = (\xi_n^1, \ldots, \xi_n^N) \in E_n^N$ s.t.

$$\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq_{N \uparrow \infty} \eta_n$$

- Particle approximation transitions $(\forall 1 \leq i \leq N)$

$$\xi_{n-1}^i \rightsquigarrow \xi_n^i \sim K_{n,\eta_{n-1}^N}(\xi_{n-1}^i, dx_n)$$

Discrete generation mean field particle model

Schematic picture : $\xi_n \in E_n^N \rightsquigarrow \xi_{n+1} \in E_{n+1}^N$

$$
\begin{array}{ccc}
\xi_n^1 & \xrightarrow{K_{n+1,\eta_n^N}} & \xi_{n+1}^1 \\
\vdots & & \vdots \\
\xi_n^i & \longrightarrow & \xi_{n+1}^i \\
\vdots & & \vdots \\
\xi_n^N & \longrightarrow & \xi_{n+1}^N
\end{array}
$$

Rationale :

$$
\begin{aligned}
\eta_n^N \simeq_{N\uparrow\infty} \eta_n \quad & \Longrightarrow \quad K_{n+1,\eta_n^N} \simeq_{N\uparrow\infty} K_{n+1,\eta_n} \\
& \Longrightarrow \quad \xi_n^i \text{ almost iid copies of } \overline{X}_n
\end{aligned}
$$

Advantages

- Mean field model=Stoch. linearization/perturbation tech. :

$$\eta_n^N = \Phi_n(\eta_{n-1}^N) + \frac{1}{\sqrt{N}}\; W_n^N$$

  with $W_n^N \simeq W_n$ independent and centered Gauss field.

- $\eta_n = \Phi_n(\eta_{n-1})$ stable $\Rightarrow$ local errors do not propagate

  $\Longrightarrow$ uniform control of errors w.r.t. the time parameter

- "No need" to study the cv of equilibrium of MCMC models.
- Adaptive stochastic grid approximations
- Take advantage of the nonlinearity of the system to define beneficial interactions. Non intrusive methods.
- Natural and easy to implement, etc.

"Intuitive picture" $\rightsquigarrow$ nonlinear sg : $\eta_n = \Phi_n(\eta_{n-1}) = \Phi_{p,n}(\eta_p) = \eta_n$

Local errors

$$W_n^N := \sqrt{N}\ \left[\eta_n^N - \Phi_n\left(\eta_{n-1}^N\right)\right] \simeq W_n \ \perp\ \text{Gaussian field}$$

Local transport formulation :

$$\begin{array}{ccccccccc}
\eta_0 & \rightarrow & \eta_1 = \Phi_1(\eta_0) & \rightarrow & \eta_2 = \Phi_{0,2}(\eta_0) & \rightarrow & \cdots & \rightarrow & \Phi_{0,n}(\eta_0) \\
\Downarrow & & & & & & & & \\
\eta_0^N & \rightarrow & \Phi_1(\eta_0^N) & \rightarrow & \Phi_{0,2}(\eta_0^N) & \rightarrow & \cdots & \rightarrow & \Phi_{0,n}(\eta_0^N) \\
 & & \Downarrow & & & & & & \\
 & & \eta_1^N & \rightarrow & \Phi_2(\eta_1^N) & \rightarrow & \cdots & \rightarrow & \Phi_{1,n}(\eta_1^N) \\
 & & & & \Downarrow & & & & \\
 & & & & \eta_2^N & \rightarrow & \cdots & \rightarrow & \Phi_{2,n}(\eta_2^N) \\
 & & & & & & & & \vdots \\
 & & & & & & \Downarrow & & \\
 & & & & & & \eta_{n-1}^N & \rightarrow & \Phi_n(\eta_{n-1}^N) \\
 & & & & & & & & \Downarrow \\
 & & & & & & & & \eta_n^N
\end{array}$$

$\rightsquigarrow$ **Key decomposition formula** : *[Propagation of local errors]+[works for* **any** *approximating scheme]*

$$\begin{aligned}
\eta_n^N - \eta_n &= \sum_{q=0}^{n} [\Phi_{q,n}(\eta_q^N) - \Phi_{q,n}(\Phi_q(\eta_{q-1}^N))] \\
&\simeq \frac{1}{\sqrt{N}} \sum_{q=0}^{n} W_q^N D_{q,n} \quad \text{first order decomp. } \Phi_{p,n}(\eta) - \Phi_{p,n}(\mu) \simeq (\eta-\mu)D_{p,n} + (\eta-\mu)^{\otimes 2}\ldots
\end{aligned}$$

$\Rightarrow$ Example Unif. estimates + A 2 lines proof CLT : $\sqrt{N}\left[\eta_n^N - \eta_n\right] \simeq \sum_{q=0}^{n} W_q D_{q,n}$

Some Theoretical results : TCL,PGD, PDM,...(n,N) :

- McKean particle measure

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{(\xi_0^i,\ldots,\xi_n^i)} \simeq_N \ \mathrm{Law}(\overline{X}_0,\ldots,\overline{X}_n) \ \& \ \eta_n^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\xi_n^i} \simeq_N \ \eta_n$$

- Empirical Processes : $\sup_{n\geq 0}\sup_{N\geq 1}\sqrt{N}\ \mathbb{E}(\|\eta_n^N-\eta_n\|_{\mathcal{F}_n}^p) < \infty$
- Uniform concentration inequalities :

$$\sup_{n\geq 0}\ \mathbb{P}(|\eta_n^N(f_n)-\eta_n(f_n)| > \epsilon) \leq c\ \exp\left\{-(N\epsilon^2)/(2\ \sigma^2)\right\}$$

- Propagations of chaos : $\mathbb{P}_{n,q}^N := \mathrm{Law}(\xi_n^1,\ldots,\xi_n^q)$

$$\mathbb{P}_{n,q}^N \simeq \eta_n^{\otimes q} + \frac{1}{N}\ \partial^1\mathbb{P}_{n,q} + \ldots + \frac{1}{N^k}\ \partial^k\mathbb{P}_{n,q} + \frac{1}{N^{k+1}}\ \partial^{k+1}\mathbb{P}_{n,q}^N$$

with $\sup_{N\geq 1}\|\partial^{k+1}\mathbb{P}_{n,q}^N\|_{\mathrm{tv}} < \infty$ & $\sup_{n\geq 0}\|\partial^1\mathbb{P}_{n,q}\|_{\mathrm{tv}} \leq c\ q^2$.

Ex.: Feynman-Kac distribution flows

- FK-Nonlinear Markov models :

  $\epsilon_n = \epsilon_n(\eta_n) \geq 0$ s.t. η_n-a.e. $\epsilon_n G_n \in [0,1]$ ($\epsilon_n = 0$ not excluded)

$$K_{n+1,\eta_n}(x,dz) = \int S_{n,\eta_n}(x,dy)\ M_{n+1}(y,dz)$$

$$S_{n,\eta_n}(x,dy) := \epsilon_n G_n(x)\ \delta_x(dy) + (1-\epsilon_n G_n(x))\ \Psi_{G_n}(\eta_n)(dy)$$

- Mean field genetic type particle model :

$$\xi_n^i \in E_n \xrightarrow{\text{accept/reject/selection}} \widehat{\xi}_n^i \in E_n \xrightarrow{\text{proposal/mutation}} \xi_{n+1}^i \in E_{n+1}$$

- *Running ex. : $G_n = 1_A \rightsquigarrow$ killing with uniform replacement.*

Mean field genetic type particle model :

$$
\left.\begin{array}{c}\xi_n^1\\ \vdots\\ \xi_n^i\\ \vdots\\ \xi_n^N\end{array}\right]
\xrightarrow{\;S_{n,\eta_n^N}\;}
\left[\begin{array}{ccc}\widehat{\xi}_n^1 & \xrightarrow{M_{n+1}} & \xi_{n+1}^1\\ \vdots & & \vdots\\ \widehat{\xi}_n^i & \longrightarrow & \xi_{n+1}^i\\ \vdots & & \vdots\\ \widehat{\xi}_n^N & \longrightarrow & \xi_{n+1}^N\end{array}\right]
$$

Accept/Reject/Selection transition :

$$S_{n,\eta_n^N}(\xi_n^i, dx)$$

$$:= \epsilon_n G_n(\xi_n^i)\ \delta_{\xi_n^i}(dx) + \left(1 - \epsilon_n G_n(\xi_n^i)\right)\ \sum_{j=1}^N \frac{G_n(\xi_n^j)}{\sum_{k=1}^N G_n(\xi_n^k)} \delta_{\xi_n^j}(dx)$$

Running Ex. : $G_n = 1_A \rightsquigarrow G_n(\xi_n^i) = 1_A(\xi_n^i)$

Path space models

- $X_n = (X'_0, \ldots, X'_n) \rightsquigarrow$ **genealogical tree/ancestral lines**

$$\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{(\xi_{0,n}^i, \xi_{1,n}^i, \ldots, \xi_{n,n}^i)} \simeq_{N \uparrow \infty} \eta_n$$

- **Unbias particle approximations :**

$$\gamma_n^N(1) = \prod_{0 \leq p < n} \eta_p^N(G_p) \ \simeq_{N \uparrow \infty} \ \gamma_n(1) = \prod_{0 \leq p < n} \eta_p(G_p)$$

- *Running example $G_n = 1_A$:*

$$\Rightarrow \gamma_n^N(1) = \prod_{0 \leq p < n} (\text{success } \% \text{ at p})$$

Feynman-Kac particle sampling recipes

Nonlinear Feynman-Kac type flow $\sim (G_n, M_n)$

$$\eta_n = \Phi_n(\eta_{n-1}) = \Psi_{G_{n-1}}(\eta_{n-1}) M_n$$

- Interacting stochastic algorithm

  accept/reject/select/branch/prune/clone/spawn/enrich $\rightsquigarrow$ G_n

  exploration/proposition/mutation/prediction $\rightsquigarrow$ M_n

- Normalizing constants $\rightsquigarrow$ key multiplicative formula.
- Path space models $\rightsquigarrow$ path-particles=ancestral lines

  Occupation meas. of genealogical trees $\simeq \eta_n \in$ path-space

- **Tuning parameters:** $(G_n, M_n) \sim$ change of ref. measures, path/excursion spaces, selection periods, weights interpretations,...

A direct conditioning approach

- Filtering models: Signal-Observation likelihood functions (X_n, G_n)

$$\eta_n = \mathrm{Law}((X_0,\ldots,X_n) \mid (Y_0,\ldots,Y_n))$$

- Rare events:
  - Confinements potentials: $G_n = 1_{A_n}$

$$\begin{aligned}\eta_n &= \mathrm{Law}((X_0,\ldots,X_n) \mid X_0 \in A_0,\ldots,X_n \in A_n)\\ \mathcal{Z}_n &= \mathbb{P}(X_0 \in A_0,\ldots,X_n \in A_n)\end{aligned}$$

  - Twisted measures $\sim \mathbb{P}(V_n(X_n) \geq a)$?

$$\mathbb{E}(f_n(X_n)\ e^{\lambda V_n(X_n)}) = \mathbb{E}\left(f_n(X_n) \prod_{0\leq p\leq n} e^{\lambda(V_p(X_p)-V_{p-1}(X_{p-1}))}\right)$$

- Hitting B before C:
  - *Multi-level decomposition* $B_0 \supset B_1 \supset \ldots \supset B_m$, $B_0 \cap C = \emptyset$.
  - *Inter-level excursions :*

$$T_n = \inf \{p \geq T_{n-1} \;:\; Y_p \in B_n \cup C\}$$

  - *Level excursions and level detection potentials:*

$$X_n = (Y_p \;;\; T_{n-1} \leq p \leq T_n) \quad \text{and} \quad G_n(X_n) = 1_{B_n}(Y_{T_n})$$

$$\Updownarrow$$

$$\mathbb{P}(Y \text{ hits } B_m \text{ before } C) = \mathbb{E}\left(\prod_{1\leq p\leq m} G_p(X_p)\right)$$

$$\mathbb{E}(f(Y_0,\ldots,Y_{T_m})\, 1_{B_m}(Y_{T_m})) = \mathbb{E}\left(f(X_0,\ldots,X_m) \prod_{1\leq p\leq m} G_p(X_p)\right)$$

- Markov processes with fixed terminal values
  - π "target type" measure$+(K, L)$ pair Markov transitions

$$\text{Metropolis potential} \quad G(x_1, x_2) = \frac{\pi(dx_2)L(x_2, dx_1)}{\pi(dx_1)K(x_1, dx_2)}$$

  - *[A Time reversal formula]* :

$$\mathbb{E}^L_\pi(f_n(X_n, X_{n-1} \ldots, X_0)|X_n = x)$$

$$= \frac{\mathbb{E}^K_x(f_n(X_0, X_1, \ldots, X_n) \ \{\prod_{0 \leq p < n} G(X_p, X_{p+1})\})}{\mathbb{E}^K_x(\{\prod_{0 \leq p < n} G(X_p, X_{p+1})\})}$$

- Non intersecting random walks & connectivity constants:

$$X_n := (X'_0, \ldots, X'_n) \quad \text{and} \quad G_n(X_n) = 1_{\not\in \{X'_p \ , \ p<n\}}(X'_n)$$

$$\eta_n = \text{Law}((X'_0, \ldots, X'_n) \mid \forall p < q < n \quad X'_p \neq X'_q)$$

- Molecular simulation $\sim$ Particle absorption models
  - X_n Markov $\in (E_n, \mathcal{E}_n)$ with transitions M_n, and $G_n : E_n \longrightarrow [0,1]$

$$Q_n(x, dy) = G_{n-1}(x)\ M_n(x, dy) \quad \text{sub-Markov operator}$$

  - $\rightsquigarrow E_n^c = E_n \cup \{c\}$.

$$X_n^c \in E_n^c \xrightarrow{\text{absorption } \sim G_n} \widehat{X}_n^c \xrightarrow{\text{exploration } \sim M_n} X_{n+1}^c$$

With:

- **Absorption:** $\widehat{X}_n^c = X_n^c$, with proba $G(X_n^c)$; otherwise $\widehat{X}_n^c = c$.
- **Exploration:** elementary free explorations $X_n \rightsquigarrow X_{n+1}$

Feynman-Kac integral model

- $T = \inf\{n : \widehat{X}^c_n = c\}$ **absorption time :** $\forall f_n \in \mathcal{B}_b(E_n)$

$$\mathbb{P}(T \geq n) = \gamma_n(1) := \mathbb{E}\left(\prod_{0\leq p<n} G(X_p)\right)$$

$$\mathbb{E}(f_n(X^c_n)\ ;\ (T \geq n)) = \gamma_n(f_n) := \mathbb{E}\left(f_n(X_n) \prod_{0\leq p<n} G_p(X_p)\right)$$

- **Continuous time models :** Δ = time step

$$(M, G) = (Id + \Delta\ L, e^{-V\Delta}) \Longrightarrow Q \rightsquigarrow L^V := L - V$$

$\rightsquigarrow$ L-motions $\oplus$ expo. clocks rate V $\oplus$ Uniform selection.

Spectral radius-Lyapunov exponents

- $Q(x,dy) = G(x)M(x,dy)$ sub-Markov operator on $\mathcal{B}_b(E)$
- Normalized FK-model : $\eta_n(f) = \gamma_n(f)/\gamma_n(1)$.

$$\mathbb{P}(T \geq n) = \mathbb{E}\left(\prod_{0\leq p\leq n} G(X_p)\right) = \prod_{0\leq p\leq n} \eta_p(G) \simeq e^{-\lambda n}$$

with $e^{-\lambda} \stackrel{M\ reg.}{=}$ Q-top eigenvalue or

$$\begin{aligned}
\lambda &= -\mathrm{LogLyap}(Q) = \lim_{n\to\infty} -\frac{1}{n}\log \|Q^n\| \\
&= -\frac{1}{n}\log \mathbb{P}(T \geq n) = -\frac{1}{n}\sum_{0\leq p\leq n} \log \eta_p(G) = -\log \eta_\infty(G)
\end{aligned}$$

Feynman-Kac-Shroedinger ground states energies

$M \quad \mu-\text{reversible}:$

$$\Rightarrow \mathbb{E}(f(X_n^c) \mid T > n) \simeq \frac{\mu(H\ f)}{\mu(H)} \quad \text{with} \quad Q(H) = e^{-\lambda} H$$

Limiting FK-measures

$$\eta_n = \Phi(\eta_{n-1}) \rightarrow_{n\uparrow\infty} \eta_\infty \quad \text{with} \quad \frac{\eta_\infty(G\ f)}{\eta_\infty(G)} = \frac{\mu(H\ f)}{\mu(H)}$$

leadsto Particle approximations :

$$\lambda \ \simeq_{n,N\uparrow} \ \lambda_n^N := \frac{1}{n} \sum_{0\leq p\leq n} \log \eta_p^N(G) \quad \text{and} \quad \eta_\infty \ \simeq_{n,N\uparrow} \ \eta_n^N$$

$$\text{Law}((X_0^c, \ldots, X_n^c) \mid (T \geq n)) \simeq \text{Genealogical tree measures}$$

Boltzmann-Gibbs measures

- X r.v. $\in (E, \mathcal{E})$ with $\mu = \mathrm{Law}(X)$
- Target measures associated with $g_n : E \to \mathbb{R}_+$

$$\eta_n(dx) := \Psi_{g_n}(\mu)(dx) = \frac{1}{\mu(g_n)}\ g_n(x)\ \mu(dx)$$

Running examples :

$$\begin{array}{rcl} g_n = 1_{A_n} & \Rightarrow & \eta_n(dx) \propto 1_{A_n}(x)\ \mu(dx) \\ g_n = e^{-\beta_n V} & \Rightarrow & \eta_n(dx) \propto e^{-\beta_n V(x)}\ \mu(dx) \\ g_n = \prod_{0 \le p \le n} h_p & \Rightarrow & \eta_n(dx) \propto \left\{ \prod_{0 \le p \le n} h_p(x) \right\}\ \mu(dx) \end{array}$$

Applications : global optimization pb., tails distributions, hidden Markov chain models, etc.

Boltzmann-Gibbs distribution flows

- Target distribution flow : $\eta_n(dx) \propto g_n(x)\ \mu(dx)$
- Product hypothesis :

$$g_n = g_{n-1} \times G_{n-1} \Longrightarrow \eta_n = \Psi_{G_{n-1}}(\eta_{n-1})$$

Running Examples:

$$\begin{array}{lcllcl} g_n & = & 1_{A_n} & \text{with } A_n \downarrow & \Rightarrow & G_{n-1} = 1_{A_n} \\ g_n & = & e^{-\beta_n V} & \text{with } \beta_n \uparrow & \Rightarrow & G_{n-1} = e^{-(\beta_n-\beta_{n-1})V} \\ g_n & = & \prod_{0\leq p\leq n} h_p & & \Rightarrow & G_{n-1} = h_n \end{array}$$

- Problem : $\eta_n = \Psi_{G_{n-1}}(\eta_{n-1})$ = unstable equation.

FK-stabilization

- Choose $M_n(x, dy)$ s.t. local fixed point eq. $\rightarrow \eta_n = \eta_n M_n$ (Metropolis, Gibbs,...)
- **Stable equation :**

$$\begin{aligned} g_n = g_{n-1} \times G_{n-1} \quad &\Longrightarrow \quad \eta_n = \Psi_{G_{n-1}}(\eta_{n-1}) \\ &\Longrightarrow \quad \eta_n = \eta_n M_n = \Psi_{G_{n-1}}(\eta_{n-1}) M_n \end{aligned}$$

- **Feynman-Kac "dynamical" formulation (X_n Markov M_n)**

$$\int f(x)\; g_n(x)\; \mu(dx) \propto \mathbb{E}\left(f(X_n) \prod_{0 \leq p < n} G_p(X_p) \right)$$

- $\rightsquigarrow$ Interacting Metropolis/Gibbs/... stochastic algorithms.

Interacting stochastic simulation algorithms

- **Mean field and Feynman-Kac particle models :**
  - `Feynman-Kac formulae. Genealogical and interacting particle systems`, Springer (2004) ⊕ Refs.
  - joint work with L. Miclo. A Moran particle system approximation of Feynman-Kac formulae. *Stochastic Processes and their Applications*, Vol. 86, 193-216 (2000).
  - joint work with L. Miclo. Branching and Interacting Particle Systems Approximations of Feynman-Kac Formulae. *Séminaire de Probabilités XXXIV, Lecture Notes in Mathematics, Springer-Verlag Berlin*, Vol. 1729, 1-145 (2000).
- **Sequential Monte Carlo models :**
  - joint work with Doucet A., Jasra A. `Sequential Monte Carlo Samplers`. JRSS B (2006).
  - joint work with A. Doucet. On a class of genealogical and interacting Metropolis models. Sém. de Proba. 37 (2003).

Particle rare event simulation algorithms

- **Twisted Feynman-Kac measures**
  - joint work with J. Garnier. Genealogical Particle Analysis of Rare events. Annals of Applied Probab., 15-4 (2005).
  - joint work with J. Garnier. Simulations of rare events in fiber optics by interacting particle systems. Optics Communications, Vol. 267 (2006).
- **Multi splitting excursion models**
  - joint work with P. Lezaud. `Branching and interacting particle interpretation of rare event proba..` Stochastic Hybrid Systems : Theory and Safety Critical Applications, eds. H. Blom and J. Lygeros. Springer (2006).
  - joint work with F. Cerou, Le Gland F., Lezaud P. `Genealogical Models in Entrance Times Rare Event Analysis`, *Alea*, Vol. I, (2006).

Particle rare event simulation algorithms

- **Particle absorption models**
  - joint work with L. Miclo. Particle Approximations of Lyapunov Exponents Connected to Schrodinger Operators and Feynman-Kac Semigroups. ESAIM Probability & Statistics, vol. 7, pp. 169-207 (2003).
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A new generation of i-MCMC algorithms

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