Stochastic Processes

MATH5835, P. Del Moral

UNSW, School of Mathematics & Statistics

Lectures Notes, No. 9

Consultations (RC 5112):

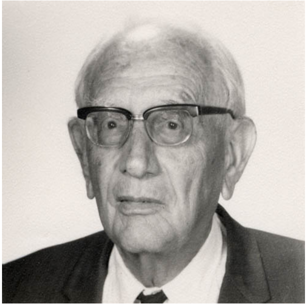
Wednesday 3.30 pm $\rightsquigarrow$ 4.30 pm & Thursday 3.30 pm $\rightsquigarrow$ 4.30 pm

Reminder + Information

References in the slides

- **Material for research projects** $\rightsquigarrow$ Moodle
  (*Stochastic Processes and Applications* $\ni$ variety of applications)
- **Important results**

  $\subset$ **Assessment/Final exam** $=$ LOGO $=$



Mathematics consists in proving the most obvious thing
in the least obvious way. – *George (György) Pólya (1887-1985)*

Plan of the lecture

Purely stochastic techniques

- Coupling distances
  - Total variation distance
  - Wasserstein metric

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- Coupling distances
  - Total variation distance
  - Wasserstein metric
- Stopping times
  - Coupling times of chains
  - Strong stationary times

Three objectives

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- **Quantify rate of convergence to equilibrium**
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- **Choose the right tool** to analyze stability
- **Quantify rate of convergence to equilibrium**
  - Coupling distances
  - Find judicious stopping times
- **develop intuition [without calculations !]**

Coupling

Remember...

$$\mathrm{Law}(X) = \mu_1 \ \& \ \mathrm{Law}(Y) = \mu_2 \ \Rightarrow \ \|\mu_1 - \mu_2\|_{tv} \leq \mathbb{P}(X \neq Y)$$

In fact...

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Theorem

$$\|\mu_1 - \mu_2\|_{tv} = \inf\{\mathbb{P}(X \neq Y) \ : \ (X, Y) \text{ s.t. } \text{Law}(X) = \mu_1 \ \& \ \text{Law}(Y) = \mu_2\}$$

Proof: Maximal coupling

Blackboard

The Wasserstein metric

Probabilities μ_i on (S, d)

$$\mathbb{W}(\mu_1, \mu_2) = \inf \left\{ \mathbb{E}\left(d(X,Y)\right) \ : \ (X,Y) \text{ s.t. Law}(X) = \mu_1 \ \& \ \text{Law}(Y) = \mu_2 \right\}$$

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Example:

$$\begin{aligned} \mathbb{W}(\mathcal{N}(m_1,\sigma_1), \mathcal{N}(m_2,\sigma_2)) &= \mathbb{E}\,(|(m_1 - m_2) + (\sigma_1 - \sigma_2)\,\mathcal{N}(0,1))|) \\ &\le |m_1 - m_2| + |\sigma_1 - \sigma_2| \end{aligned}$$

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Kantorovich-Rubinstein duality theorem:

$$\mathbb{W}(\mu_1, \mu_2) = \sup \{|\mu_1(f) - \mu_2(f)| \; : \; f \in \mathrm{Lip}(S) \text{ s.t. } \mathrm{lip}(f) \leq 1\}$$

Proof $\geq$

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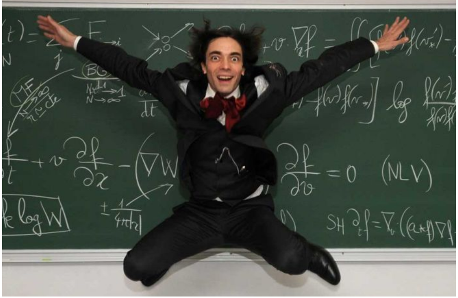
Proof $\geq$ ($\leq$ cf. C. Villani book!) Blackboard



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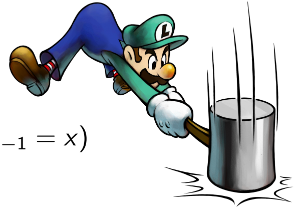
Connexion with optimal transport : cf. C. Villani book!)

Intermediate tool



Prop.:

$$(\delta_x M)(dy) = M(x, dy) = \mathbb{P}(X_n \in dy \mid X_{n-1} = x)$$

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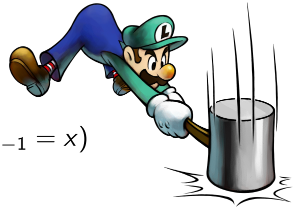

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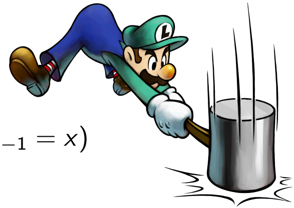

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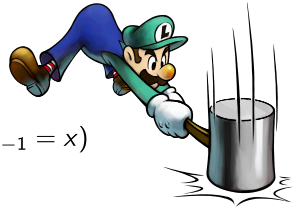

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Example $U \sim \gamma(du)$:

$$X_n = F(X_{n-1}, U_n) \quad \text{with} \quad \int \|F(x,u) - F(y,u)\| \, \gamma(du) \leq a \; \|x - y\|$$

Proof: Blackboard

Stopping times and coupling

Stopping time: $\{T = n\} \sim (X_0, \ldots, X_n)$

(As far as we know, photo is public domain)

Stopping times and coupling ⚠

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Coupling time T of two chains

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Ex.: 2 copies of the same chain

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$$\leq \int \mathbb{P}(X_0 \in dx)\mathbb{P}(Y_0 \in dy)\, \|\mathrm{Law}(X_n \mid X_0 = x) - \mathrm{Law}(Y_n \mid Y_0 = y))\|_{tv}$$

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Examples: when $M(x, dy) \geq \epsilon \, \lambda(dy) \ldots$ (cf. previous lectures)

Strong stationary times

X_n **with invariant measure** π**:**

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Prop.:

$$\|\mathrm{Law}(X_n) - \pi\|_{tv} \leq \mathbb{P}(T > n)$$

Proof: Blackboard