Stochastic Processes

MATH5835, P. Del Moral

UNSW, School of Mathematics & Statistics

Lectures Notes, No. 8

Consultations (RC 5112):

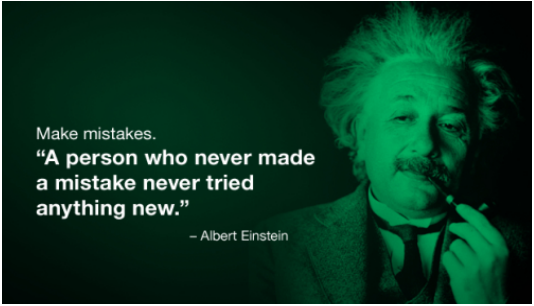
Wednesday 3.30 pm $\rightsquigarrow$ 4.30 pm & Thursday 3.30 pm $\rightsquigarrow$ 4.30 pm

Reminder + Information

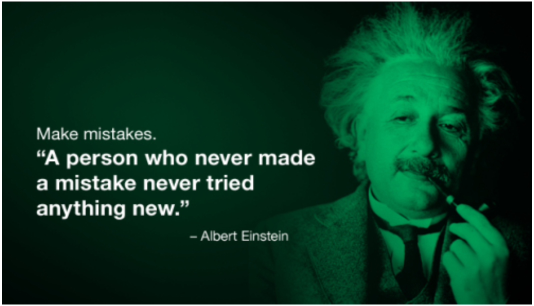
References in the slides

- **Material for research projects** $\rightsquigarrow$ Moodle
  (*Stochastic Processes and Applications* $\ni$ variety of applications)
- **Important results**

  $\subset$ **Assessment/Final exam** = LOGO =



– Albert Einstein (1879-1955)



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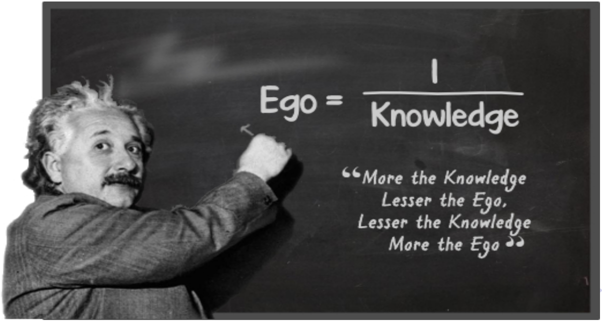

Plan of the lecture

- Markov chain models
  - Elementary transitions
  - Random dynamical systems

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- Stability properties
  - 2 states model
  - Perron Frobenius theorem
  - Spectral analysis
  - Total variation norms

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  - Total variation norms
- Quantitative rates
  - Spectral Gaps
  - Dobrushin contraction/ergodic coef.
- Poisson equation

Three objectives

- **Formalize/Recognize** a Markov chain model

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- **Analyze the stability properties**
  - Analysis on reduced and toy models
  - $\mathbb{L}_2$ techniques and spectral tools
  - Total variation norms and Dobrushin contractions

Three objectives

- **Formalize/Recognize** a Markov chain model
- **Analyze the stability properties**
  - Analysis on reduced and toy models
  - $\mathbb{L}_2$ techniques and spectral tools
  - Total variation norms and Dobrushin contractions
- **Open/Ask questions [$\sim$ continuous/discrete time models?]**

Markov transitions

$$\mathbb{P}(X_n \in dx_n \mid X_0, \ldots, X_{n-2}, X_{n-1}) = \mathbb{P}(X_n \in dx_n \mid X_{n-1})$$

$$\Downarrow$$

$$\mathbb{P}(X_n \in dx_n \mid X_{n-1} = x_{n-1}) = M_n(x_{n-1}, dx_n)$$

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- $S = \mathbb{R}$

$$M_n(x_{n-1}, dx_n) = \frac{1}{2}\ \delta_0(dx_n) + \frac{1}{2}\ \frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}\ (x_n - a(x_{n-1}))^2}\ dx_n$$

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- $S = \{e_1, \ldots, e_d\}$

$$M_n = \begin{pmatrix} M_n(e_1, e_1) & \ldots & M_n(e_1, e_d) \\ \vdots & \vdots & \vdots \\ M_n(e_d, e_1) & \ldots & M_n(e_d, e_d) \end{pmatrix}$$

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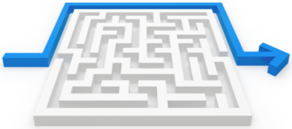
- $\{e_1, \ldots, e_d\} \subset S = \mathbb{R}^d$

$$\forall x_{n-1} = e_i \qquad M_n(x_{n-1}, dx_n) = \sum_{1 \le j \le d} M_n(e_i, e_j)\,\delta_{e_j}(dx_n)$$

Advantages

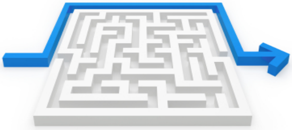


(Chapman-Kolmogorov) Transport equation

$$\underbrace{\mathbb{P}\left(X_n \in dx_n\right)}_{} = \int_{x_{n-1}} \overbrace{\mathbb{P}\left(X_n \in dx_n \mid X_{n-1} = x_{n-1}\right)}^{M_n(x_{n-1}, dx_n)} \overbrace{\mathbb{P}\left(X_{n-1} \in dx_{n-1}\right)}^{=\eta_{n-1}(dx_{n-1})}$$

$$\eta_n(dx_n) = \int_{x_{n-1}} \eta_{n-1}(dx_{n-1})\; M_n(x_{n-1}, dx_n)$$

Advantages



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$$\Updownarrow$$

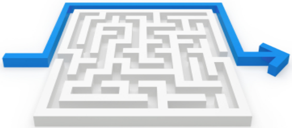
Dynamical system representation

$$\eta_n = \eta_{n-1} M_n = \ldots = \eta_0 M_1 \ldots M_n$$

with

$$\begin{aligned}(M_1 \ldots M_n)(x_0, dx_n) &= \int_{x_1,\ldots,x_{n-1}} M_1(x_0, dx_1) \ldots M_n(x_{n-1}, dx_n) \\ &= \mathbb{P}\left(X_n \in dx_n \mid X_0 = x_0\right)\end{aligned}$$

Advantages ⚠



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Note:

$S = \{e_1, \ldots, e_d\} \simeq \{1, \ldots, d\} \rightsquigarrow$ **matrix/vector operations**

Random dynamical systems ⚠

State space models

$$X_n = F_n(X_{n-1}, W_n) \quad \text{with i.i.d. } W_n \text{ and some initial r.v. } X_0$$

Random dynamical systems

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$$X_n = (A_n \dots A_1)\, X_0 + \sum_{1 \leq p \leq n} (A_n \dots A_{p+1})\, B_p W_p$$

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Dimension 1 :

$[A_n = a_n = a \in [0,1[$ and X'_n a copy of X_n starting at X'_0 (same W_n)]

$$\Downarrow$$

$$X_n - X'_n = a^n\,(X_0 - X'_0) \longrightarrow_{n \uparrow \infty} 0$$

Stability properties

Limit random states

$$X_n = F(X_{n-1}, W_n) \longrightarrow_{n\uparrow\infty} X_\infty \ ??$$

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or

$$\mathrm{Law}(X_n) \longrightarrow_{n\uparrow\infty} \mathrm{Law}(X_\infty) := \eta_\infty \text{ ??} \overset{(\eta_n=\eta_{n-1}M)}{\Longrightarrow} \eta_\infty = \eta_\infty M$$

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Limiting occupation measures

$$\frac{1}{n}\sum_{0\leq p<n} \delta_{X_p} \longrightarrow_{n\uparrow\infty} \mathrm{Law}(X_\infty)\text{??}$$

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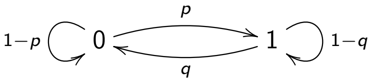
$$\Updownarrow$$

$\forall f \;:\; S \mapsto \mathbb{R}$ **(a.k.a. observable [physics literature])**

$$\int f(x) \left(\frac{1}{n}\sum_{0\leq p<n} \delta_{X_p}\right)(dx) = \frac{1}{n}\sum_{0\leq p<n} f(X_p)$$

$$\longrightarrow_{n\uparrow\infty} \mathbb{E}(f(X_\infty)) = \int f(x)\, \mathbb{P}(X_\infty \in dx)$$

2 states model



$$\Updownarrow$$

$$M = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

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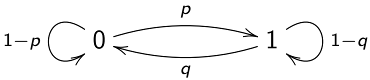

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Invariant measure

$$\pi = \left[\frac{q}{p+q}, \frac{p}{p+q}\right] \implies \pi M \propto [q,p] \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = [q,p] \propto \pi$$

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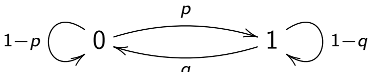

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Some question

$$\frac{1}{n} \sum_{0 \le p < n} \delta_{X_p}(1_0) = \frac{1}{n} \sum_{0 \le p < n} 1_{X_p = 0} \simeq_{n \uparrow \infty} \pi(0) = \frac{q}{p+q} \;??$$

Perron-Frobenius theo

$$M = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

Exercise:

- Find eigenvalues λ_1, λ_2 and eigenvectors $\overline{\varphi}_1$, $\overline{\varphi}_2$.

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- Using the change of variable matrix

$$P := (\overline{\varphi}_1, \overline{\varphi}_2) \rightsquigarrow M = PDP^{-1} \quad \text{with} \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

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$$M^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1} \Rightarrow \ldots \Rightarrow M^n = PD^nP^{-1}$$

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Solution: Blackboard

Perron Frobenius theorem

M on a finite space S s.t. $M^m(x,y) > 0$ for some $m \geq 1$.

$$\Downarrow$$

$\exists! \pi$ on S s.t. $\pi(x) > 0$ and

$$\pi M = \pi \quad \text{with } \forall x, y \in S \quad \lim_{n \to \infty} M^n(x,y) = \pi(y)$$

In addition, 1 is a simple root of the characteristic polynomial of M.

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$$K(x,y) = M^m(x,y) \geq \delta = \overbrace{(\delta \mathrm{Card}(S))}^{:=\epsilon} \overbrace{\mathrm{Card}(S)^{-1}}^{=\nu(x)>0}$$

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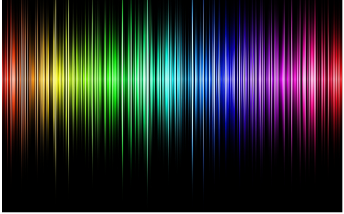
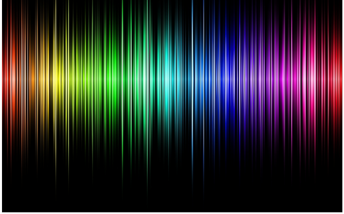
$$\exists!\ \pi = \pi K = \pi M^m \quad \text{and} \quad |K^n(x,y) - \pi(y)| = |M^{mn}(x,y) - \pi(y)| \leq (1-\epsilon)^n$$

Spectral analysis



2 states model

Spectral analysis



2 states model

M reversible w.r.t. some proba. π on some finite S with cardinality d, s.t. $M^m(x,y) > 0$ for some $m \geq 1$.

$$\Downarrow$$

Finite set of real valued eigenvalues $\lambda_1 = 1 \geq \lambda_2 \geq \ldots \geq \lambda_d > -1$
$\oplus$ $\exists$ orthonormal basis of $l_2(\pi)$ made of real valued eigenfunctions $(\psi_i)_{1\leq i\leq d}$ of $(\lambda_i)_{1\leq i\leq d}$, with $\psi_1 = 1$ the unit function.
$\oplus$ Spectral decomposition

$$M^n(x,y) = \pi(y) + \sum_{1<i\leq d} \lambda_i^n \; \psi_i(x) \;\; \pi(y) \; \psi_i(y)$$

The difference $\lambda_2 - \lambda_1 = \lambda_2 - 1$ is called the spectral gap.

Quantitative rates

Exponential decays to equilibrium

$$|M^n(x,y) - \pi(y)| \leq \lambda_\star^n \sqrt{\pi(y)/\pi(x)} \leq e^{-\rho n} \sqrt{\pi(y)/\pi(x)}$$

with the absolute spectral gap

$$\rho = 1 - \lambda_\star \quad \text{with} \quad \lambda_\star := \sup_{1<i\leq d} |\lambda_i|$$

Proof:

Quantitative rates

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Proof: (exercise)

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Proof: (exercise) $\rightsquigarrow$ **Solution** Blackboard

Exercise

Ingredients

- I_n i.i.d. r.v. $\in \mathcal{I} = \{1, \ldots, d\}$ with law μ.
- $\forall i \in \mathcal{I}$, M_i Markov transition on S_i with $\pi_i = \pi_i M_i$.

Product Markov chain with transition M on $S = \prod_{1 \leq i \leq d} S_i$

$$X_{n-1} = (X^1_{n-1}, \ldots, X^d_{n-1}) \rightsquigarrow X_n = (X^1_n, \ldots, X^d_n) \quad \text{s.t. } X^{I_n}_n \sim M_{I_n}(X^{I_n}_{n-1}, dx)$$

1. $$\boldsymbol{\pi(dx)} = \prod_{1 \leq i \leq d} \pi_i(dx^i) \quad \Longrightarrow \quad \boldsymbol{\pi M = \pi}$$
2. $\forall (\lambda_i, \varphi_i)$=*eigen(value,function) system* of M_i

$$\begin{cases} \boldsymbol{\varphi(x)} = \prod_{1 \leq i \leq d} \varphi(x^i) \quad \text{and} \quad \boldsymbol{\lambda} = \sum_{1 \leq i \leq d} \mu(i) \lambda_i \\ \Longrightarrow \quad \boldsymbol{M(\varphi) = \lambda\, \varphi} \end{cases}$$

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Product Markov chain with transition $\boldsymbol{M}$ on $\boldsymbol{S} = \prod_{1 \leq i \leq d} S_i$

$$X_{n-1} = (X^1_{n-1}, \ldots, X^d_{n-1}) \rightsquigarrow X_n = (X^1_n, \ldots, X^d_n) \quad \text{s.t. } X^{I_n}_n \sim M_{I_n}(X^{I_n}_{n-1}, dx)$$

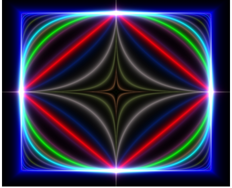
1. $$\boldsymbol{\pi(dx)} = \prod_{1 \leq i \leq d} \pi_i(dx^i) \quad \Longrightarrow \quad \boldsymbol{\pi M = \pi}$$

2. $\forall (\lambda_i, \varphi_i)$=*eigen(value,function) system* of M_i

$$\begin{cases} \boldsymbol{\varphi(x)} = \prod_{1 \leq i \leq d} \varphi(x^i) \qquad \text{and} \qquad \boldsymbol{\lambda} = \sum_{1 \leq i \leq d} \mu(i) \lambda_i \\ \\ \Longrightarrow \quad \boldsymbol{M(\varphi) = \lambda\, \varphi} \end{cases}$$

Solution $\rightsquigarrow$ Blackboard

Total variation norm



Finite spaces

$$\|\mu_1 - \mu_2\|_{tv} = \frac{1}{2}\sum_{x\in S}|\mu_1(x) - \mu_2(x)| = 1 - \sum_{x\in S}\ [\mu_1(x) \wedge \mu_2(x)]$$

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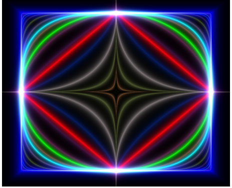

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Absolutely continuous measures

$$\mu_1(dx) = p_1(x)\ \lambda(dx) \quad \text{and} \quad \mu_2(dx) = p_2(x)\ \lambda(dx)$$

$$\Downarrow$$

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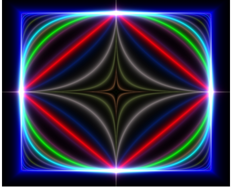

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Proof of =

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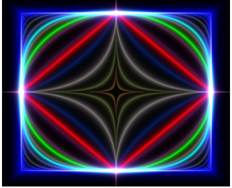

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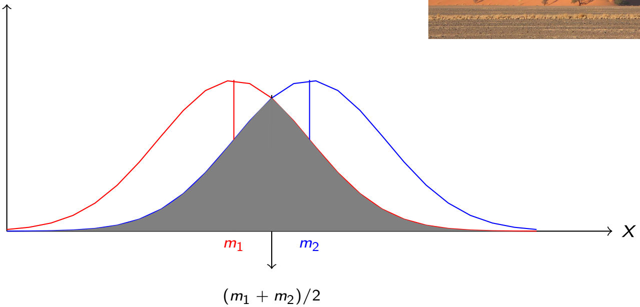
Proof of = $\rightsquigarrow$ Blackboard

An example/Exercise

$p_1 = \mathcal{N}(m_1, 1)$ & $p_2 = \mathcal{N}(m_2, 1)$



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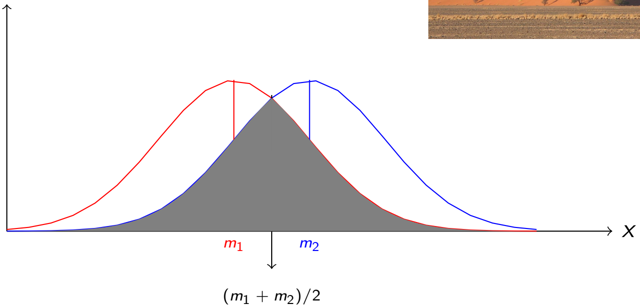

$$\|\mu_1 - \mu_2\|_{tv} = \mathbb{P}\left(|N(0,1)| \leq \frac{m_2 - m_1}{2}\right) \leq \frac{(m_2 - m_1)}{\sqrt{2\pi}}$$

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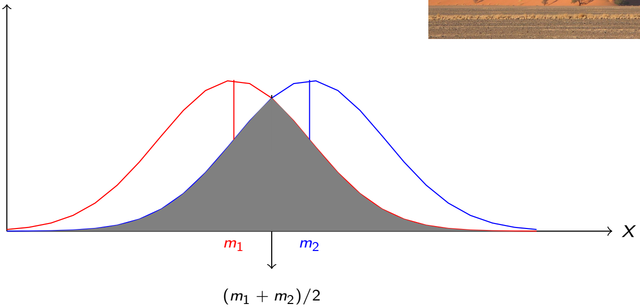

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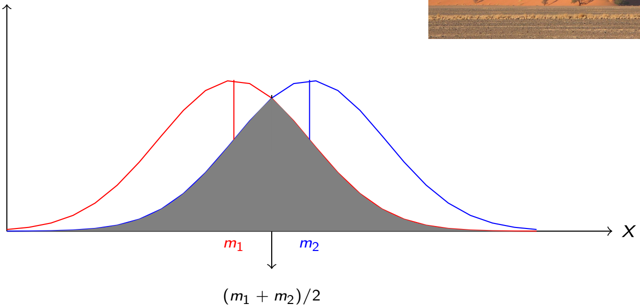

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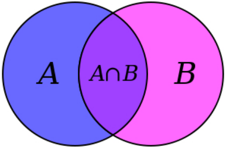
Solution: $\rightsquigarrow$ Blackboard

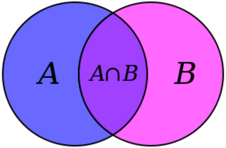
General formulations



General state space S

$$\begin{aligned}
\|\mu_1 - \mu_2\|_{tv} &= \sup\left\{|\mu_1(f) - \mu_2(f)| \;:\; f \text{ s.t. } \operatorname{osc}(f) \leq 1\right\} \\
&= \frac{1}{2}\sup\left\{|\mu_1(f) - \mu_2(f)| \;:\; f \text{ s.t. } \|f\| \leq 1\right\} \\
&= \sup\left\{|\mu_1(A) - \mu_2(A)| \;:\; A \subset S\right\} = 1 - [\mu_1 \wedge \mu_2](S)
\end{aligned}$$

General formulations



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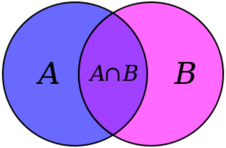
Proof (for finite spaces)

General formulations



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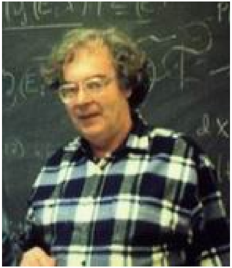
Proof (for finite spaces) $\rightsquigarrow$ Blackboard

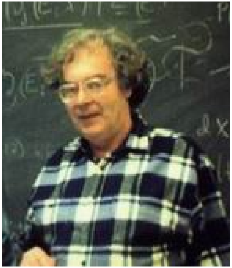
Dobrushin Contraction coef.



M Markov transition on some state space S

$$\beta(M) := \sup_{x,y\in S} \|M(x,.) - M(y,.)\|_{\mathrm{tv}} = \sup_{f\ :\ \mathrm{osc}(f)\leq 1} \mathrm{osc}(M(f))$$

Dobrushin Contraction coef.⚠



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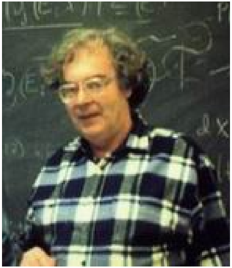

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$$\sup_{f \,:\, \mathrm{osc}(f) \leq 1} \sup_{x,y \in S} \ldots = \sup_{x,y \in S} \sup_{f \,:\, \mathrm{osc}(f) \leq 1} \ldots$$

Contraction - Stability Theorem ⚠

M **Markov transition s.t.** $\beta(M) < 1 \Rightarrow \exists! \pi = \pi M$

$$\mathrm{osc}(M^n(f)) \leq \beta(M)^n \, \mathrm{osc}(f) \rightarrow_{n\rightarrow\infty} 0$$

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$$\|\mu_1 M - \mu_2 M\|_{tv} = \sup_{f\ :\operatorname{osc}(f)\leq 1} \left(\operatorname{osc}(M(f)) \times \left|(\mu_1 - \mu_2)\left[\frac{M(f)}{\operatorname{osc}(M(f))}\right]\right|\right)$$

Poisson equation

M **Markov transition s.t.** $\beta(M^n) \leq a\ e^{-bn}$ $(\Rightarrow \exists! \pi = \pi \boldsymbol{M})$

Poisson equation ⚠

M **Markov transition s.t.** $\beta(M^n) \leq a\, e^{-bn}$ ($\Rightarrow \exists! \pi = \pi M$)

$$\Downarrow$$

$$\forall f \;:\; \mathrm{osc}(f) \leq 1 \quad \text{and} \quad \pi(f) = 0$$

$$g = P(f) = \sum_{n \geq 0} M^n(f) \quad \text{solution of the Poisson eq.} \quad (Id - M)g = f$$