Stochastic Processes

MATH5835, P. Del Moral

UNSW, School of Mathematics & Statistics

Lectures Notes, No. 5

Consultations (RC 5112):

Wednesday 3.30 pm $\rightsquigarrow$ 4.30 pm & Thursday 3.30 pm $\rightsquigarrow$ 4.30 pm

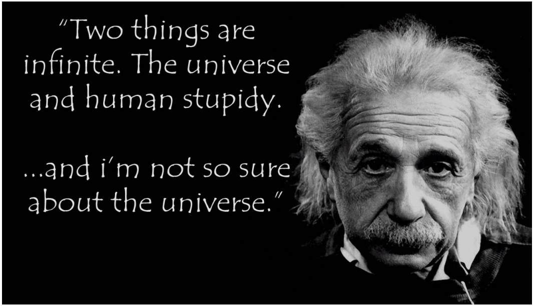
Reminder + Information

References in the slides

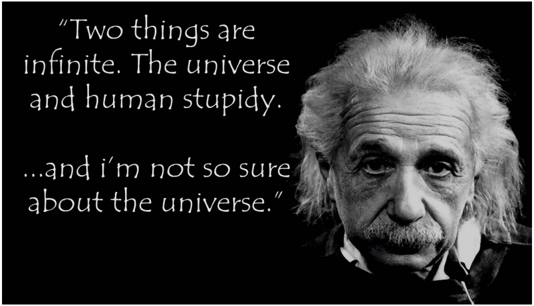
- **Material for research projects** $\rightsquigarrow$ Moodle
  (*Stochastic Processes and Applications* $\ni$ variety of applications)
- **Important results**

  $\subset$ **Assessment/Final exam** = LOGO =

Citations of the day

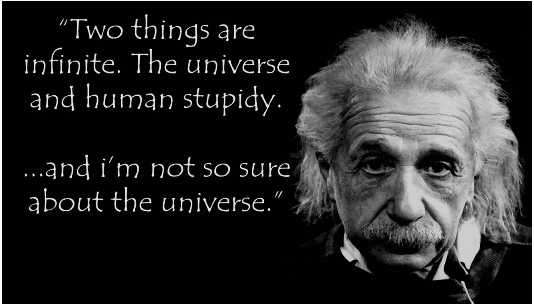


Citations of the day



Since the mathematicians have invaded the theory of relativity, I do not understand it myself anymore.

Citations of the day

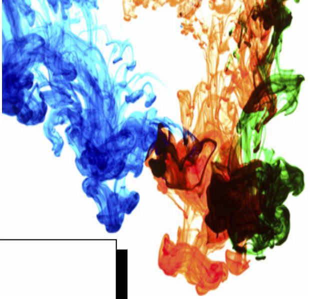


Since the mathematicians have invaded the theory of relativity, I do not understand it myself anymore.

It should be possible to explain the laws of physics to a barmaid.

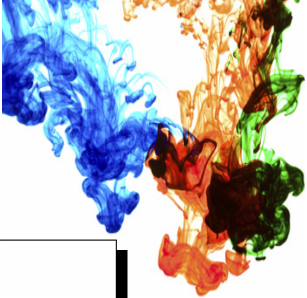
– Albert Einstein (1879-1955)

Mixture of 3 subjects



1. A complement on martingales
2. A brief reminder on dynamical systems
3. Intro to **continuous time stochastic calculus**
   - Brownian motion
   - Ito(-Doeblin) formula
   - The heat equation

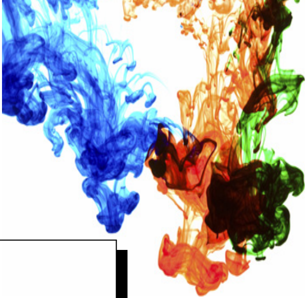
Mixture of 3 subjects



1. A complement on martingales
2. A brief reminder on dynamical systems
3. Intro to **continuous time stochastic calculus**
   - Brownian motion
   - Ito(-Doeblin) formula
   - The heat equation

Central/fundamental subjects in stochastic process theory!!!!

Mixture of 3 subjects

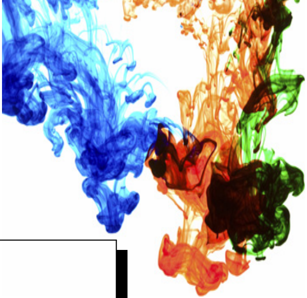


1. A complement on martingales
2. A brief reminder on dynamical systems
3. Intro to **continuous time stochastic calculus**
   - Brownian motion
   - Ito(-Doeblin) formula
   - The heat equation

Central/fundamental subjects in stochastic process theory!!!!

$\uparrow$ **attention**

Mixture of 3 subjects

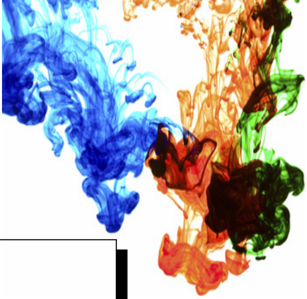


1. A complement on martingales
2. A brief reminder on dynamical systems
3. Intro to **continuous time stochastic calculus**
   - Brownian motion
   - Ito(-Doeblin) formula
   - The heat equation

Central/fundamental subjects in stochastic process theory!!!!

$\uparrow$ attention $\oplus$ $\uparrow$ consultation times

Mixture of 3 subjects

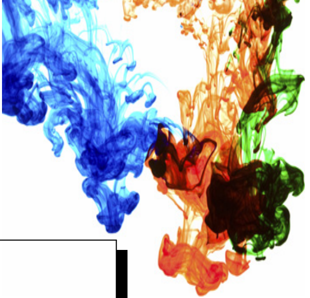


1. A complement on martingales
2. A brief reminder on dynamical systems
3. Intro to **continuous time stochastic calculus**
   - Brownian motion
   - Ito(-Doeblin) formula
   - The heat equation

Central/fundamental subjects in stochastic process theory!!!!

$\uparrow$ attention $\oplus$ $\uparrow$ consultation times $\oplus$ $\uparrow$ questions

Mixture of 3 subjects



1. A complement on martingales
2. A brief reminder on dynamical systems
3. Intro to **continuous time stochastic calculus**
   - Brownian motion
   - Ito(-Doeblin) formula
   - The heat equation

Central/fundamental subjects in stochastic process theory!!!!

$\uparrow$ attention $\oplus$ $\uparrow$ consultation times $\oplus$ $\uparrow$ questions $\Rightarrow$ $\downarrow$ speed

Designing martingales

$$X_n \quad = \quad \varphi_n(\epsilon_0, \ldots, \epsilon_n) \; \in \; S \text{ (colors, tails/heads,} \mathbb{R}^d\text{,...)} \mapsto f(X_n) \in \mathbb{R}^{d=1}$$

The natural filtration of information:

$$\mathcal{F}_n = \sigma(\epsilon_p \;, \quad 0 \leq p \leq n) = \; \uparrow \textit{information} \sim \textit{random process}$$

Designing martingales

$$X_n = \varphi_n(\epsilon_0, \ldots, \epsilon_n) \in S \text{ (colors, tails/heads,}\mathbb{R}^d\text{,\ldots)} \mapsto f(X_n) \in \mathbb{R}^{d=1}$$

The natural filtration of information:

$$\mathcal{F}_n = \sigma(\epsilon_p \ , \quad 0 \le p \le n) = \ \uparrow \textit{information} \sim \textit{random process}$$

Predictable and martingale parts of $\Delta f(X_n) = f(X_n) - f(X_{n-1})$

Designing martingales ⚠

$$X_n \;=\; \varphi_n(\epsilon_0,\ldots,\epsilon_n) \;\in\; S \text{ (colors, tails/heads,}\mathbb{R}^d\text{,\ldots)} \mapsto f(X_n) \in \mathbb{R}^{d=1}$$

The natural filtration of information:

$$\mathcal{F}_n = \sigma(\epsilon_p\;,\;\; 0 \leq p \leq n) = \;\uparrow \textit{information} \sim \textit{random process}$$

Predictable and martingale parts of $\Delta f(X_n) = f(X_n) - f(X_{n-1})$

$$\Delta A_n(f) \;\;:=\;\; \mathbb{E}\left(\Delta f(X_n) \mid \mathcal{F}_{n-1}\right) = \text{predictable increment}$$

Designing martingales

$$X_n \quad = \quad \varphi_n(\epsilon_0, \ldots, \epsilon_n) \; \in \; S \text{ (colors, tails/heads,} \mathbb{R}^d, \ldots) \mapsto f(X_n) \in \mathbb{R}^{d=1}$$

The natural filtration of information:

$$\mathcal{F}_n = \sigma(\epsilon_p \;, \quad 0 \leq p \leq n) = \;\; \uparrow \textit{information} \sim \textit{random process}$$

Predictable and martingale parts of $\Delta f(X_n) = f(X_n) - f(X_{n-1})$

$$\Delta A_n(f) \quad := \quad \mathbb{E}\left(\Delta f(X_n) \mid \mathcal{F}_{n-1}\right) = \text{predictable increment}$$

$$\Delta M_n(f) \quad := \quad \Delta f(X_n) - \mathbb{E}\left(\Delta f(X_n) \mid \mathcal{F}_{n-1}\right) = \text{martingale increment}$$

Designing martingales ⚠

$$X_n = \varphi_n(\epsilon_0, \ldots, \epsilon_n) \in S \text{ (colors, tails/heads,} \mathbb{R}^d, \ldots) \mapsto f(X_n) \in \mathbb{R}^{d=1}$$

The natural filtration of information:

$$\mathcal{F}_n = \sigma(\epsilon_p \ , \ 0 \leq p \leq n) = \ \uparrow \textit{information} \sim \textit{random process}$$

Predictable and martingale parts of $\Delta f(X_n) = f(X_n) - f(X_{n-1})$

$$\Delta A_n(f) := \mathbb{E}\left(\Delta f(X_n) \mid \mathcal{F}_{n-1}\right) = \text{predictable increment}$$

$$\Delta M_n(f) := \Delta f(X_n) - \mathbb{E}\left(\Delta f(X_n) \mid \mathcal{F}_{n-1}\right) = \text{martingale increment}$$

Martingale decomposition

$$f(X_n) = f(X_0) + \underbrace{\sum_{1 \leq p \leq n} \mathbb{E}\left(\Delta f(X_p) \mid \mathcal{F}_{p-1}\right)}_{\text{Predictable part}} + \underbrace{\sum_{1 \leq p \leq n} \left[\Delta f(X_p) - \mathbb{E}\left(\Delta f(X_p) \mid \mathcal{F}_{p-1}\right)\right]}_{\text{Martingale part}}$$

An example = The simple Random walk

$$\Delta X_n := X_n - X_{n-1} = \epsilon_n \quad \text{i.i.d. } \epsilon_n = +1/-1 \text{ proba } 1/2$$

$f(x) = x$ & $\mathcal{F}_n = \sigma(\epsilon_p,\ p \leq n)$ info on the game at time n

$$\Delta A_n(f) \quad := \quad \mathbb{E}(\Delta X_n \mid \mathcal{F}_{n-1}) = 0 =$$ predictable increment

An example = The simple Random walk

$$\Delta X_n := X_n - X_{n-1} = \epsilon_n \quad \text{i.i.d. } \epsilon_n = +1/-1 \text{ proba } 1/2$$

$f(x) = x$ & $\mathcal{F}_n = \sigma(\epsilon_p,\ p \leq n)$ info on the game at time n

$$\begin{aligned} \Delta A_n(f) &:= \mathbb{E}(\Delta X_n \mid \mathcal{F}_{n-1}) = 0 = \text{predictable increment} \\ \Delta M_n(f) &:= \Delta X_n - \mathbb{E}(\Delta X_n \mid \mathcal{F}_{n-1}) = \epsilon_n = \text{martingale increment} \end{aligned}$$

$f(x) = x^3$ (**exo**)

An example = The simple Random walk

$$\Delta X_n := X_n - X_{n-1} = \epsilon_n \quad \text{i.i.d. } \epsilon_n = +1/-1 \text{ proba } 1/2$$

$f(x) = x$ & $\mathcal{F}_n = \sigma(\epsilon_p,\ p \leq n)$ info on the game at time n

$$\begin{aligned} \Delta A_n(f) &:= \mathbb{E}\left(\Delta X_n \mid \mathcal{F}_{n-1}\right) = 0 = \text{predictable increment} \\ \Delta M_n(f) &:= \Delta X_n - \mathbb{E}\left(\Delta X_n \mid \mathcal{F}_{n-1}\right) = \epsilon_n = \text{martingale increment} \end{aligned}$$

$f(x) = x^3$ (**exo**)

$$X_n^3 - X_{n-1}^3 = (X_{n-1} + \epsilon_n)^3 - X_{n-1}^3 = 3\ X_{n-1} + (3X_{n-1}^2 + 1)\ \epsilon_n$$

An example = The simple Random walk

$$\Delta X_n := X_n - X_{n-1} = \epsilon_n \quad \text{i.i.d. } \epsilon_n = +1/-1 \text{ proba } 1/2$$

$f(x) = x$ & $\mathcal{F}_n = \sigma(\epsilon_p,\ p \leq n)$ info on the game at time n

$$\begin{aligned} \Delta A_n(f) &:= \mathbb{E}\left(\Delta X_n \mid \mathcal{F}_{n-1}\right) = 0 = \text{predictable increment} \\ \Delta M_n(f) &:= \Delta X_n - \mathbb{E}\left(\Delta X_n \mid \mathcal{F}_{n-1}\right) = \epsilon_n = \text{martingale increment} \end{aligned}$$

$f(x) = x^3$ **(exo)**

$$X_n^3 - X_{n-1}^3 = (X_{n-1} + \epsilon_n)^3 - X_{n-1}^3 = 3\ X_{n-1} + (3X_{n-1}^2 + 1)\ \epsilon_n$$

$$\Downarrow$$

$$\Delta A_n(f) := \mathbb{E}\left(X_n^3 - X_{n-1}^3 \mid \mathcal{F}_{n-1}\right) = 3\ X_{n-1} = \text{predictable increment}$$

An example = The simple Random walk

$$\Delta X_n := X_n - X_{n-1} = \epsilon_n \quad \text{i.i.d. } \epsilon_n = +1/-1 \text{ proba } 1/2$$

$f(x) = x$ & $\mathcal{F}_n = \sigma(\epsilon_p,\ p \leq n)$ info on the game at time n

$$\begin{aligned} \Delta A_n(f) &:= \mathbb{E}\left(\Delta X_n \mid \mathcal{F}_{n-1}\right) = 0 = \text{predictable increment} \\ \Delta M_n(f) &:= \Delta X_n - \mathbb{E}\left(\Delta X_n \mid \mathcal{F}_{n-1}\right) = \epsilon_n = \text{martingale increment} \end{aligned}$$

$f(x) = x^3$ (**exo**)

$$X_n^3 - X_{n-1}^3 = (X_{n-1} + \epsilon_n)^3 - X_{n-1}^3 = 3\ X_{n-1} + (3X_{n-1}^2 + 1)\ \epsilon_n$$

$$\Downarrow$$

$$\begin{aligned} \Delta A_n(f) &:= \mathbb{E}\left(X_n^3 - X_{n-1}^3 \mid \mathcal{F}_{n-1}\right) = 3\ X_{n-1} = \text{predictable increment} \\ \Delta M_n(f) &:= (3X_{n-1}^2 + 1)\ \epsilon_n = \text{martingale increment} \end{aligned}$$

The martingale^2 [with $M_0 = 0$]

$$M_n^2 = \sum_{1\leq p\leq n} (\Delta M^2)_p \quad \text{with} \quad (\Delta M^2)_p = M_p^2 - M_{p-1}^2$$

The martingale^2 [with $M_0 = 0$] ⚠

$$M_n^2 = \sum_{1\le p\le n} (\Delta M^2)_p \quad \text{with} \quad (\Delta M^2)_p = M_p^2 - M_{p-1}^2$$

$$= \sum_{1\le p\le n} \mathbb{E}\left((\mathbf{\Delta M^2})_\mathbf{p} \mid \mathcal{F}_\mathbf{p-1}\right) + \underbrace{\sum_{1\le p\le n} \left[(\Delta M^2)_p - \mathbb{E}\left((\mathbf{\Delta M^2})_\mathbf{p} \mid \mathcal{F}_\mathbf{p-1}\right)\right]}_{=\ \textbf{martingale (exo 1)}}$$

The martingale^2 [with $M_0 = 0$]

$$M_n^2 = \sum_{1\leq p\leq n} (\Delta M^2)_p \quad \text{with} \quad (\Delta M^2)_p = M_p^2 - M_{p-1}^2$$

$$= \sum_{1\leq p\leq n} \mathbb{E}\left((\boldsymbol{\Delta M^2})_{\mathbf{p}} \mid \mathcal{F}_{\mathbf{p-1}}\right) + \underbrace{\sum_{1\leq p\leq n} \left[(\Delta M^2)_p - \mathbb{E}\left((\boldsymbol{\Delta M^2})_{\mathbf{p}} \mid \mathcal{F}_{\mathbf{p-1}}\right)\right]}_{= \text{ martingale (exo 1)}}$$

$$\mathbb{E}\left((\Delta M^2)_p \mid \mathcal{F}_{p-1}\right) = \mathbb{E}\left(M_p^2 - M_{p-1}^2 \mid \mathcal{F}_{p-1}\right)$$

$$= \mathbb{E}\left((M_p - M_{p-1})^2 \mid \mathcal{F}_{p-1}\right) \quad \textbf{(exo 2)}$$

The martingale^2 [with $M_0 = 0$] ⚠

$$M_n^2 = \sum_{1\le p\le n} (\Delta M^2)_p \quad \text{with} \quad (\Delta M^2)_p = M_p^2 - M_{p-1}^2$$

$$= \sum_{1\le p\le n} \mathbb{E}\left((\Delta M^2)_p \mid \mathcal{F}_{p-1}\right) + \underbrace{\sum_{1\le p\le n} \left[(\Delta M^2)_p - \mathbb{E}\left((\Delta M^2)_p \mid \mathcal{F}_{p-1}\right)\right]}_{=\ \text{martingale (exo 1)}}$$

$$\mathbb{E}\left((\Delta M^2)_p \mid \mathcal{F}_{p-1}\right) = \mathbb{E}\left(M_p^2 - M_{p-1}^2 \mid \mathcal{F}_{p-1}\right)$$

$$= \mathbb{E}\left((M_p - M_{p-1})^2 \mid \mathcal{F}_{p-1}\right) \quad \text{(exo 2)}$$

Predictable quadratic variation = angle bracket

$$M_n^2 = \langle M\rangle_n + \text{Martingale} \quad \text{with} \quad \langle M\rangle_n := \sum_{1\le p\le n} \mathbb{E}\left((\Delta M_p)^2 \mid \mathcal{F}_{p-1}\right)$$

An example = The simple Random walk

$$M_n = M_{n-1} + \epsilon_n \quad \text{i.i.d. } \epsilon_n = +1/-1 \text{ proba } 1/2$$

$$\Downarrow$$

$$\begin{aligned} M_n^2 - M_{n-1}^2 &= (M_{n-1} + \epsilon_n)^2 - M_{n-1}^2 \\ &= 2\ M_{n-1}\ \epsilon_n + \epsilon_n^2 = 2\ M_{n-1}\ \epsilon_n + 1 \end{aligned}$$

$$\Downarrow$$

$$\mathbb{E}\left(M_n^2 - M_{n-1}^2 \mid \mathcal{F}_{n-1}\right) = \mathbb{E}\left((M_n - M_{n-1})^2 \mid \mathcal{F}_{n-1}\right) = 1$$

$$\Downarrow$$

$$M_n^2 = \langle \boldsymbol{M} \rangle_{\boldsymbol{n}} + \textbf{Martingale} \quad \text{with} \quad \langle \boldsymbol{M} \rangle_{\boldsymbol{n}} := \sum_{1 \leq p \leq n} 1 = n$$

A brief reminder on dynamical systems

$$\dot{X}_t = b(X_t)$$

A brief reminder on dynamical systems

$$\dot{X}_t = b(X_t) \iff dX_t = b(X_t)\, dt$$

Key properties:

1. Smooth differentiable trajectories.
2. Fully predictable when we know the initial condition.
3. Well adapted to standard differential calculus.

Leibnitz "long s" $= \int$

$$\dot{X}_t = b(X_t)$$

Leibnitz "long s" $= \int$

$$\dot{X}_t = b(X_t) \quad \Longleftrightarrow \quad dX_t = b(X_t)\, dt$$

Leibnitz "long s" $= \int$

$$\dot{X}_t = b(X_t) \iff dX_t = b(X_t)\, dt$$
$$\iff X_{t+dt} = X_t + b(X_t)\, dt$$

Leibniz "long s" $= \int$

$$\dot{X}_t = b(X_t) \quad \Longleftrightarrow \quad dX_t = b(X_t)\, dt$$

$$\Longleftrightarrow \quad X_{t+dt} = X_t + b(X_t)\, dt$$

Integral interpretation of the increments $dX_t = X_{t+dt} - X_t$

$$X_t = X_0 + \sum_{s \leq t} dX_s$$

Leibnitz "long s" $= \int$

$$\dot{X}_t = b(X_t) \iff dX_t = b(X_t)\, dt$$
$$\iff X_{t+dt} = X_t + b(X_t)\, dt$$

Integral interpretation of the increments $dX_t = X_{t+dt} - X_t$

$$X_t = X_0 + \sum_{s \leq t} dX_s = X_0 + \sum_{s \leq t} b(X_s)\, ds$$

Leibnitz "long s" $= \int$

$$\dot{X}_t = b(X_t) \quad \Longleftrightarrow \quad dX_t = b(X_t)\, dt$$
$$\Longleftrightarrow \quad X_{t+dt} = X_t + b(X_t)\, dt$$

Integral interpretation of the increments $dX_t = X_{t+dt} - X_t$

$$X_t = X_0 + \sum_{s \leq t} dX_s = X_0 + \sum_{s \leq t} b(X_s)\, ds := X_0 + \int_0^t b(X_s)\, ds$$

Leibnitz "long s" $= \int$

$$\dot{X}_t = b(X_t) \iff dX_t = b(X_t)\, dt$$
$$\iff X_{t+dt} = X_t + b(X_t)\, dt$$

Integral interpretation of the increments $dX_t = X_{t+dt} - X_t$

$$X_t = X_0 + \sum_{s \leq t} dX_s = X_0 + \sum_{s \leq t} b(X_s)\, ds := X_0 + \int_0^t b(X_s)\, ds$$

For smooth functions $f \rightsquigarrow f(X_t) = f \circ X_t$ **??**

Leibniz "long s" $= \int$

$$\dot{X}_t = b(X_t) \quad \Longleftrightarrow \quad dX_t = b(X_t)\ dt$$
$$\Longleftrightarrow \quad X_{t+dt} = X_t + b(X_t)\ dt$$

Integral interpretation of the increments $dX_t = X_{t+dt} - X_t$

$$X_t = X_0 + \sum_{s \leq t} dX_s = X_0 + \sum_{s \leq t} b(X_s)\ ds := X_0 + \int_0^t b(X_s)\ ds$$

For smooth functions $f \rightsquigarrow f(X_t) = f \circ X_t$ **??**

$$f(X_t) = f(X_0) + \sum_{s \leq t} df(X_s) = f(X_0) + \int_0^t \ \ldots \ ????$$

with the increment of the function

$$df(X_t) := f(X_{t+dt}) - f(X_t)$$

Brook Taylor's expansions ⚠

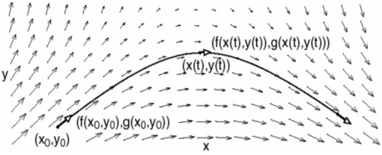
Taylor expansion for smooth functions

$$
\begin{aligned}
f(X_{t+dt}) &= f(X_t + dX_t) \\
&= f(X_t) + \frac{\partial f}{\partial x}(X_t)\ dX_t + \frac{1}{2}\ \frac{\partial^2 f}{\partial x^2}(X_t)\ dX_t dX_t + \ldots \\
&= f(X_t) + \frac{\partial f}{\partial x}(X_t)\ b(X_t)\ dt + \frac{1}{2}\ \frac{\partial^2 f}{\partial x^2}(X_t)\ b^2(X_t)\ (dt)^2 + \ldots
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
f(X_t) &= f(X_0) + \sum_{s \le t} df(X_s) \\
&= f(X_0) + \sum_{s \le t} \frac{\partial f}{\partial x}(X_s)\ b(X_s)\ ds + \mathrm{o}(dt) \\
&= f(X_0) + \int_0^t \frac{\partial f}{\partial x}(X_s)\ b(X_s)\ ds
\end{aligned}
$$

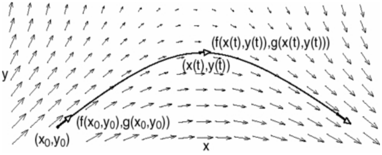
Vector fields (dimension 2 ⇝)



Differential calculus (dimension 1)

$$dX_t = b(X_t)\ dt$$

Vector fields (dimension 2 $\rightsquigarrow$)



Differential calculus (dimension 1)

$$dX_t = b(X_t)\,dt \iff df(X_t) = L(f)(X_t)\,dt$$

with the first order operator/vector field : $f \mapsto L(f)$

$$L(f)(x) := b(x)\,\frac{\partial f}{\partial x}(x)$$

Exercise: $dX_t = b \times X_t\ dt \rightsquigarrow f(X_t) = \log X_t \Rightarrow \ldots ??$

Non homogeneous case $(t, x) \mapsto f(t, x)$

Taylor expansion for smooth functions

$$f(t+dt, X_{t+dt}) = f(t, X_t) + \frac{\partial f}{\partial t}(t, X_t)\ dt + \frac{\partial f}{\partial x}(t, X_t)\ b(X_t)dt + \mathrm{O}((dt)^2)$$

$$\Downarrow$$

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \sum_{s \leq t} df(s, X_s) \\ &= f(X_0) + \int_0^t \left[\frac{\partial f}{\partial s}(s, X_s)ds + \frac{\partial f}{\partial x}(s, X_s)\ b(X_s)\ ds\right] \end{aligned}$$

Non homogeneous functions/models

Differential calculus

$$dX_t \;=\; b_t(X_t)\; dt$$

Non homogeneous functions/models

Differential calculus

$$dX_t \;=\; b_t(X_t)\;dt \quad\Longleftrightarrow\quad df(t,X_t) = \left[\frac{\partial}{\partial t} + L\right](f)(t,X_t)\;dt$$

Exercise: $dX_t = a\;(b - X_t)\;dt \rightsquigarrow f(t,X_t) = e^{at}X_t \Rightarrow ...??$

Evolution semigroups

Flow maps & semigroups: $(s \leq t)$

$$\left\{ \begin{array}{rcl} dX^x_{s,t} & = & b_t(X^x_{s,t})\ dt \\ X^x_s & = & x \end{array} \right. \rightsquigarrow P_{s,t}(f)(x) = f(X^x_{s,t})$$

For any $r \leq s \leq t$ we have $X^x_{r,t} = X^{X^x_{r,s}}_{s,t}$:

$$\Rightarrow \quad P_{r,t}(f)(x) \quad = \quad f(X^{X^x_{r,s}}_{s,t}) = P_{s,t}(f)(X^x_{r,s}) = P_{r,s}(P_{s,t}(f))(x)$$

Exercises

$$\frac{\partial}{\partial t} P_{s,t}(f) = P_{s,t}(L(f)) \quad \text{and} \quad \frac{\partial}{\partial s} P_{s,t}(f) = -L(P_{s,t}(f))$$

Evolution semigroups

Flow maps & semigroups: $(s \leq t)$

$$\left\{ \begin{array}{rcl} dX^x_{s,t} & = & b_t(X^x_{s,t})\ dt \\ X^x_s & = & x \end{array} \right. \rightsquigarrow P_{s,t}(f)(x) = f(X^x_{s,t})$$

For any $r \leq s \leq t$ we have $X^x_{r,t} = X^{X^x_{r,s}}_{s,t}$:

$$\Rightarrow \quad P_{r,t}(f)(x) \quad = \quad f(X^{X^x_{r,s}}_{s,t}) = P_{s,t}(f)(X^x_{r,s}) = P_{r,s}(P_{s,t}(f))(x)$$

Exercises

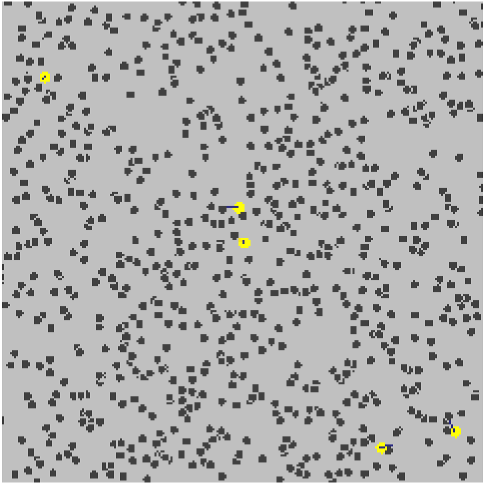
$$\frac{\partial}{\partial t} P_{s,t}(f) = P_{s,t}(L(f)) \quad \text{and} \quad \frac{\partial}{\partial s} P_{s,t}(f) = -L(P_{s,t}(f))$$

and for homogeneous models

$$b_t = b \quad \Rightarrow \quad X^x_{s,t} = X^x_{0,t-s}$$

$$\Rightarrow \quad P_{s,t} = P_{0,t-s} := P_{t-s} \Rightarrow \frac{\partial}{\partial t} P_t(f) = P_t(L(f)) = L(P_t(f))$$

The lost equation - Introduction to Brownian motion

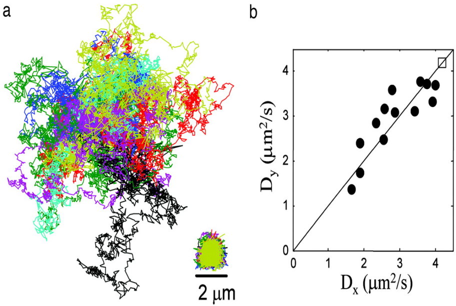


More concrete : Nano particles in water (laser+camera)

A sugar molecule in a cell (simulation)

⊕ pretty nice pedagogical animation

Brownian motion



Key properties:

1. *Continuous but nowhere differentiable* trajectories.
2. *Fully unpredictable/random* even if we know the initial condition and the statistics of perturbations.
3. *Badly & non adapted to standard differential calculus*.

Brownian motion B_t or W_t

Discrete time version : "dt" time steps $\oplus$ fair coin tossing

$$W_t := W_{t-dt} + \begin{cases} +\sqrt{dt} & \text{if } \textbf{Heads} \\ -\sqrt{dt} & \text{if } \textbf{Tails} \end{cases} \quad (1)$$

or

$$W_t := W_{t-dt} + \sqrt{dt} \times \mathrm{N}(0,1)$$

$$\Downarrow$$

$$dt = 10^{-10000000} ??$$

Brownian motion B_t or W_t

Discrete time version : "dt" time steps $\oplus$ fair coin tossing

$$W_t := W_{t-dt} + \begin{cases} +\sqrt{dt} & \text{if } \textbf{Heads} \\ -\sqrt{dt} & \text{if } \textbf{Tails} \end{cases} \quad (1)$$

or

$$W_t := W_{t-dt} + \sqrt{dt} \times \mathrm{N}(0,1)$$

$$\Downarrow$$

$$dt = 10^{-10000000}??$$

- $\simeq_{dt \sim 0}$ **Continuous time model $\oplus$ stochastic calculus**

Brownian motion B_t or W_t

Discrete time version : "dt" time steps $\oplus$ fair coin tossing

$$W_t := W_{t-dt} + \begin{cases} +\sqrt{dt} & \text{if} \quad \textbf{Heads} \\ -\sqrt{dt} & \text{if} \quad \textbf{Tails} \end{cases} \tag{1}$$

or

$$W_t := W_{t-dt} + \sqrt{dt} \times \mathrm{N}(0,1)$$

$$\Downarrow$$

$$dt \;=\; 10^{-10000000}??$$

- $\simeq_{dt\sim 0}$ **Continuous time model $\oplus$ stochastic calculus**
- Wikipedia - Brownian motion
- Section 3.3 (further readings in Section 14.1 in the textbook)

Brownian motion B_t or W_t ⚠

Simple random walk model

$$W_t := W_{t-dt} + \begin{cases} +\sqrt{dt} & \text{if} \quad \textbf{Heads} \\ -\sqrt{dt} & \text{if} \quad \textbf{Tails} \end{cases} \tag{2}$$

Brownian motion B_t or W_t ⚠

Simple random walk model

$$W_t := W_{t-dt} + \begin{cases} +\sqrt{dt} & \text{if } \textbf{Heads} \\ -\sqrt{dt} & \text{if } \textbf{Tails} \end{cases} \qquad (2)$$

⇝ **Only "3 simple ingredients":**

$$(2) \Rightarrow \begin{array}{rcl} dW_t \times dW_t & = & \pm\sqrt{dt} \times \pm\sqrt{dt} = dt \\ dt \times dt & = & 0 \\ dt \times dW_t & = & dt \times \pm\sqrt{dt} = 0 \end{array}$$

⊕ **Randomness encapsulated in** $\mathcal{F}_t = \sigma(W_s \ : \ s \leq t)$ ★.

Taylor $\rightsquigarrow$ Ito-(Doeblin) formula ⚠

THINK:

$$\begin{aligned} df(W_t) &= f(W_{t+dt}) - f(W_t) = f(W_t + dW_t) - f(W_t) \\ &= f'(W_t)\, dW_t + \frac{1}{2}\, f''(W_t)\, \overbrace{dW_t dW_t}^{=dt} + \text{"O}(dt\sqrt{dt})\text{"} \end{aligned}$$

WRITE:

$$df(W_t) = f'(W_t)\, dW_t + L(f)(W_t)\, dt$$

with the "Laplacian" operator = infinitesimal generator

$$f \mapsto L(f) := \frac{1}{2}\, f''$$

Example $f(x) = x^2$ & W_t s.t. $W_0 = 0$

Ito-(Doeblin) formula

$$f'(x) = 2x \quad f''(x) = 2 \;\Rightarrow dW_t^2 = 2W_t dW_t + dt$$

$$\Downarrow$$

$$W_t^2 = 2 \int_0^t W_s \; dW_s + t$$

Compare with Taylor expansions for dynamical systems

$$dX_t = b(X_t) \; dt \quad \text{s.t. } X_0 = 0$$

$$\Downarrow$$

$$dX_t^2 = 2X_t dX_t \implies X_t^2 = \int_0^t 2X_s dX_s$$

Using martingale decompositions

$$W_t = \sum_{s\leq t} (W_{s+ds} - W_s) \quad \text{Martingale w.r.t. } \mathcal{F}_t = \sigma(W_s,\ s\leq t)$$

$$\begin{aligned}
\mathbb{E}\left((W_{s+ds} - W_s) \mid \mathcal{F}_s\right) &= 0 \\
\mathbb{E}\left((W_{s+ds} - W_s)^2 \mid \mathcal{F}_s\right) &= ds
\end{aligned}$$

$$\begin{aligned}
W_{s+ds}^2 - W_s^2 &= (W_s + (W_{s+ds} - W_s))^2 - W_s^2 \\
&= 2W_s(W_{s+ds} - W_s) + (W_{s+ds} - W_s)^2 \\
&= 2W_s(W_{s+ds} - W_s) + ds
\end{aligned}$$

$\Longleftrightarrow$ **Martingale^2 & its angle bracket**

$$W_t^2 = \overbrace{\sum_{s\leq t} 2W_s(W_{s+ds} - W_s)}^{\text{martingale}} + \underbrace{\sum_{s\leq t} ds}_{:=\langle W\rangle_t = t} = \overbrace{\int_0^t 2W_s dW_s}^{\text{martingale}} + t$$

THINK ⚠

$$f(W_{t+dt}) - f(W_t) \quad = \quad L(f)(W_t)\, dt \;+\; \underbrace{f'(W_t)\,(W_{t+dt} - W_t)}_{=[M_{t+dt}(f) - M_t(f)]}$$

with martingale increments

$$\mathbb{E}\left(f'(W_t)\,(W_{t+dt} - W_t) \mid \mathcal{F}_t\right) \quad = \quad f'(W_t)\,\mathbb{E}\left((W_{t+dt} - W_t) \mid W_t\right) = 0$$

THINK ⚠

$$f(W_{t+dt}) - f(W_t) \quad = \quad L(f)(W_t)\, dt \; + \; \underbrace{f'(W_t)\;(W_{t+dt} - W_t)}_{=[M_{t+dt}(f) - M_t(f)]}$$

with martingale increments

$$\mathbb{E}\left(f'(W_t)\;(W_{t+dt} - W_t) \mid \mathcal{F}_t\right) \quad = \quad f'(W_t)\,\mathbb{E}\left((W_{t+dt} - W_t) \mid W_t\right) = 0$$

$$\Downarrow$$

The predictable increment

THINK ⚠

$$f(W_{t+dt}) - f(W_t) \quad = \quad L(f)(W_t)\, dt \; + \; \underbrace{f'(W_t)\, (W_{t+dt} - W_t)}_{=[M_{t+dt}(f) - M_t(f)]}$$

with martingale increments

$$\mathbb{E}\left(f'(W_t)\, (W_{t+dt} - W_t) \mid \mathcal{F}_t\right) \quad = \quad f'(W_t)\, \mathbb{E}\left((W_{t+dt} - W_t) \mid W_t\right) = 0$$

$$\Downarrow$$

The predictable increment

$$\mathbb{E}\left(f(W_{t+dt}) - f(W_t) \;\mid \mathcal{F}_t\right) = L(f)(W_t)\, dt$$

THINK ⚠

$$f(W_{t+dt}) - f(W_t) \quad = \quad L(f)(W_t)\, dt \; + \; \underbrace{f'(W_t)\,(W_{t+dt} - W_t)}_{=[M_{t+dt}(f) - M_t(f)]}$$

with martingale increments

$$\mathbb{E}\left(f'(W_t)\,(W_{t+dt} - W_t) \mid \mathcal{F}_t\right) \quad = \quad f'(W_t)\,\mathbb{E}\left((W_{t+dt} - W_t) \mid W_t\right) = 0$$

$$\Downarrow$$

The predictable increment

$$\mathbb{E}\left(f(W_{t+dt}) - f(W_t) \;\mid \mathcal{F}_t\right) = L(f)(W_t)\; dt$$

$$\Downarrow$$

$$f(W_t) \quad = \quad f(W_0) + \sum_{s \le t} [f(W_{s+ds}) - f(W_s)]$$

THINK ⚠

$$f(W_{t+dt}) - f(W_t) \quad = \quad L(f)(W_t)\, dt \; + \; \underbrace{f'(W_t)\,(W_{t+dt} - W_t)}_{=[M_{t+dt}(f) - M_t(f)]}$$

with martingale increments

$$\mathbb{E}\left(f'(W_t)\,(W_{t+dt} - W_t) \mid \mathcal{F}_t\right) \quad = \quad f'(W_t)\,\mathbb{E}\left((W_{t+dt} - W_t) \mid W_t\right) = 0$$

$$\Downarrow$$

The predictable increment

$$\mathbb{E}\left(f(W_{t+dt}) - f(W_t) \mid \mathcal{F}_t\right) = L(f)(W_t)\, dt$$

$$\Downarrow$$

$$\begin{aligned} f(W_t) &= f(W_0) + \sum_{s \leq t} [f(W_{s+ds}) - f(W_s)] \\ &= f(W_0) + \sum_{s \leq t} L(f)(W_s)\, ds + \underbrace{\sum_{s \leq t} [M_{s+ds}(f) - M_s(f)]}_{\text{martingale} = M_t(f)} \end{aligned}$$

WRITE ⚠

$$df(W_t) \quad = \quad L(f)(W_t)\, dt \; + \; dM_t(f)$$

$$\Updownarrow$$

$$f(W_t) \quad = \quad f(W_0) + \int_0^t df(W_s)$$

WRITE ⚠

$$df(W_t) \quad = \quad L(f)(W_t)\ dt \ + \ dM_t(f)$$

$$\Updownarrow$$

$$\begin{aligned} f(W_t) &= f(W_0) + \int_0^t df(W_s) \\ &= f(W_0) + \int_0^t L(f)(W_s)\ ds + \underbrace{\int_0^t dM_s(f)}_{\text{martingale}=M_t(f)} \end{aligned}$$

THINK ⚠

The martingale remainder term

$$M_t(f) \quad = \quad \sum_{s\leq t} \underbrace{f'(W_s)\,(W_{s+ds} - W_s)}_{=(M_{s+ds}(f)-M_s(f))=dM_s(f)}$$

with

$$\mathbb{E}\left(dM_t(f) \mid \mathcal{F}_t\right) \quad = \quad f'(W_t)\,\mathbb{E}\left((W_{t+dt} - W_t) \mid W_t\right) = 0$$

THINK ⚠

The martingale remainder term

$$M_t(f) = \sum_{s \leq t} \underbrace{f'(W_s)\,(W_{s+ds} - W_s)}_{=(M_{s+ds}(f) - M_s(f)) = dM_s(f)}$$

with

$$\begin{aligned}
\mathbb{E}\left(dM_t(f) \mid \mathcal{F}_t\right) &= f'(W_t)\,\mathbb{E}\left((W_{t+dt} - W_t) \mid W_t\right) = 0 \\
\mathbb{E}\left((dM_t(f))^2 \mid \mathcal{F}_t\right) &= (f'(W_t))^2\,\mathbb{E}\left((W_{t+dt} - W_t)^2 \mid W_t\right) = (f'(W_t))^2\,dt
\end{aligned}$$

THINK ⚠

The martingale remainder term

$$M_t(f) \quad = \quad \sum_{s \le t} \underbrace{f'(W_s)\,(W_{s+ds} - W_s)}_{=(M_{s+ds}(f)-M_s(f))=dM_s(f)}$$

with

$$\begin{aligned}
\mathbb{E}\left(dM_t(f) \mid \mathcal{F}_t\right) &= f'(W_t)\,\mathbb{E}\left((W_{t+dt} - W_t) \mid W_t\right) = 0 \\
\mathbb{E}\left((dM_t(f))^2 \mid \mathcal{F}_t\right) &= (f'(W_t))^2\,\mathbb{E}\left((W_{t+dt} - W_t)^2 \mid W_t\right) = (f'(W_t))^2\,dt
\end{aligned}$$

$$\Downarrow$$

Predictable quadratic variation = angle bracket

$$\langle M(f)\rangle_t := \sum_{s \le t} \mathbb{E}\left((dM_s(f))^2 \mid \mathcal{F}_s\right) = \sum_{s \le t} (f'(W_s))^2\,ds$$

WRITE ⚠

The martingale remainder term

$$M_t(f) \quad = \quad \int_0^t f'(W_s)\, dW_s$$

with the predictable quadratic variation = angle bracket

$$\langle M(f)\rangle_t := \int_0^t (f'(W_s))^2\, ds$$

Ito-(Doeblin) formula

⇒ **Ito-(Doeblin) formula:**

$$df(W_t) \;=\; L(f)(W_t)\,dt + dM_t(f)$$

Ito-(Doeblin) formula ⚠

⇒ **Ito-(Doeblin) formula:**

$$df(W_t) \quad = \quad L(f)(W_t)\ dt \ + \ dM_t(f)$$

with a martingale $M_t(f)$ with angle bracket

$$\langle M(f)\rangle_t := \int_0^t (f'(W_s))^2\ ds$$

Important observation

$$\Gamma_L(f,f)(x) := L((f-f(x))^2)(x) = L(f^2)(x) - 2f(x)L(f)(x) = (f')^2(x)$$

$$\Downarrow$$

$$\langle M(f)\rangle_t := \int_0^t \Gamma_L(f,f)(W_s)\ ds$$

Examples

$$df(W_t) \;=\; \frac{1}{2}\, f''(W_t)\, dt \;+\; dM_t(f) \quad \text{with} \quad \langle M(f)\rangle_t := \int_0^t (f'(W_s))^2\, ds$$

- Powers $\alpha > 0$

$$W_t^\alpha = W_0^\alpha + \frac{\alpha(\alpha-1)}{2} \int_0^t W_s^{\alpha-2}\, ds + M_t$$

with a martingale M_t with angle bracket

$$\langle M(f)\rangle_t := \alpha^2 \int_0^t W_s^{2(\alpha-1)}\, ds$$

- $\exp(\dots)$, $\sin(\dots)$, …

A simple extension to $f(t,x)$

$\Rightarrow$ **Ito-(Doeblin) formula:**

$$df(t, W_t) = \left[\frac{\partial}{\partial t} + L\right](f)(t, W_t)\,dt + dM_t(f)$$

A simple extension to $f(t,x)$ ⚠

⇒ **Ito-(Doeblin) formula:**

$$df(t,W_t) \quad = \quad \left[\frac{\partial}{\partial t} + L\right](f)(t,W_t)\; dt \; + \; dM_t(f)$$

with a martingale $M_t(f)$ with angle bracket

$$\langle M(f)\rangle_t := \int_0^t \left(\frac{\partial f}{\partial x}(s,W_s)\right)^2 \; ds$$

Important observation

$$\begin{aligned}\Gamma_{\frac{\partial}{\partial t}+L}(f,f)(t,x) \quad &:= \quad \left[\frac{\partial}{\partial t} + L\right]((f - f(t,x))^2)(t,x) \\ &= \quad \Gamma_L(f(t,.),f(t,.))(x) = (f'(t,.))^2(x)\end{aligned}$$

Important exercise: show that $Z_t = e^{\alpha W_t - \frac{\alpha^2}{2} t}$ is a martingale!

A general/abstract formula

$\forall$ **(even non-homogeneous) process** X_t **s.t.**

$$\mathbb{E}([f(t+dt, X_{t+dt}) - f(t, X_t)] \mid \mathcal{F}_t) = \left[\frac{\partial}{\partial t} + L_t\right](f)(t, X_t)\ dt$$

we have

$$df(t, X_t) \quad = \quad \left[\frac{\partial}{\partial t} + L_t\right](f)(t, X_t)\ dt \ + \ dM_t(f)$$

A general/abstract formula

$\forall$ (even non-homogeneous) process X_t s.t.

$$\mathbb{E}([f(t+dt, X_{t+dt}) - f(t, X_t)] \mid \mathcal{F}_t) = \left[\frac{\partial}{\partial t} + L_t\right](f)(t, X_t)\ dt$$

we have

$$df(t, X_t) \quad = \quad \left[\frac{\partial}{\partial t} + L_t\right](f)(t, X_t)\ dt \ + \ dM_t(f)$$

with a martingale $M_t(f)$ with angle bracket

$$d\langle M(f)\rangle_t := \Gamma_{L_t}(f(t, \centerdot), f(t, \centerdot))(X_t)\ dt$$

Law(W_t) & The heat equation

$$\mathbb{E}\left(f(W_t)\right) = \int f(x)\ \mathbb{P}(W_t \in dx) = \int_{-\infty}^{+\infty} f(x)\ p_t(x)\ dx$$

Exo: $\forall f$ **twice diff** $\oplus$ **all** $f^{(k)}(+/-\infty) = 0$ **for** $k = 0, 1, 2$ $(\star)$

- **First step:**

$$d\mathbb{E}(f(W_t)) = \ldots = \frac{1}{2}\ \mathbb{E}\left(f''(W_t)\right)\ dt$$

- **Second step:**

$$d\mathbb{E}(f(W_t)) = \ldots = \left[\int f(x)\ \frac{\partial p_t}{\partial t}(x)\ dx\right]\ dt$$

- **Third step:**

$$\mathbb{E}\left(f''(W_t)\right) = \ldots = \int f(x)\ \frac{\partial^2 p_t}{\partial x^2}(x)\ dx$$

- **Conclusion:** ...

Law(W_t) & The heat equation & The Gaussian ⚠

$$\mathbb{P}(W_t \in dx) = p_t(x)\ dx \quad \& \quad \frac{\partial}{\partial t} p_t(x) = \frac{1}{2}\ \frac{\partial^2}{\partial x^2} p_t(x)$$

Exercise slide 28:

$$\mathbb{E}\left(e^{\alpha W_t}\right) = e^{\frac{1}{2}\alpha^2 t}$$

Law(W_t) & The heat equation & The Gaussian ⚠

$$\mathbb{P}(W_t \in dx) = p_t(x)\ dx \quad \& \quad \frac{\partial}{\partial t} p_t(x) = \frac{1}{2}\ \frac{\partial^2}{\partial x^2} p_t(x)$$

Exercise slide 28:

$$\mathbb{E}\left(e^{\alpha W_t}\right) = e^{\frac{1}{2}\alpha^2 t} \Rightarrow \quad W_t \sim N(0, \sigma^2 = t)$$

$$\Downarrow$$

$$p_t(x) = \frac{1}{\sqrt{2\pi t}}\ \exp\left[-\frac{x^2}{2t}\right]\ ??$$

Monte Carlo simulation ⚠

Law of large numbers with i.i.d. copies W_t^i of W_t:

$$\mathbb{E}(f(W_t)) = \int f(x)\,\mathbb{P}(W_t \in dx) = \int_{-\infty}^{+\infty} f(x)\, p_t(x)\, dx$$

$$\simeq \frac{1}{N} \sum_{1 \leq i \leq N} f(W_t^i)$$

Two simulation techniques

$$W_{t+dt} := W_t + \epsilon_t \sqrt{dt} \quad \text{with} \quad \epsilon_t := \pm 1 \;\; \text{Proba} \;\; 1/2$$

$$\underline{\text{or}} \qquad \epsilon_t \sim N(0,1) \tag{3}$$

Note:

$$(3) \Rightarrow W_t = \int_0^s dW_s \simeq \underbrace{\sum_{s \leq t} \epsilon_s \sqrt{ds}}_{t/dt \text{ centered Gaussians}} \sim N(0,t)$$

Brownian fluid flow models

Fluid particle $(X_0 = 0)$**:**

$$dX_t = v \, dt + \sqrt{2D} \, dW_t$$

- Fluid velocity flow v.
- Diffusion coefficient = D

$$\Downarrow$$

$$X_t = \int_0^t dX_s = \int_0^t v \, ds + \int_0^t \sqrt{2D} \, dW_t = v \, t + \sqrt{2D} \, W_t$$

$\Rightarrow \quad X_t = v \, t + \sqrt{2Dt} \, N(0,1)$ Heat equation (exercise)?

$\leadsto$ *Wolfram -[Brownian-Fluid-model-(v,D).cdf] - Mathworld*