Stochastic Processes

MATH5835, P. Del Moral

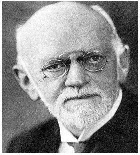
UNSW, School of Mathematics & Statistics

Lectures Notes 3

Consultations (RC 5112):

Wednesday 3.30 pm $\rightsquigarrow$ 4.30 pm & Thursday 3.30 pm $\rightsquigarrow$ 4.30 pm

Citations of the day



– David Hilbert (1862-1943)

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

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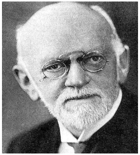 – *David Hilbert (1862-1943)*

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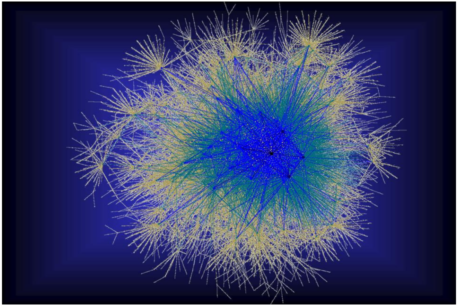
 `- David Hilbert - CSIRO Atherton, QLD`

Google PageRank algorithm



Stanford University patent [`Larry Page` ⊕ `Sergey Brin`] 1996

- Counts the number and quality of page links ⇝ importance index.
- **Hyp.:** Important sites receive more links from others.

Google PageRank - Some information

Using the web-spider bot Googlebot:

- $d \simeq 25 \times 10^9$ Web pages (March 2014).
- d_i outgoing links from each website $i \in \{1, \ldots, d\}$.

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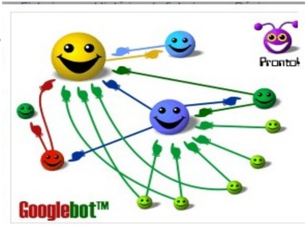
> - How to use this data?
> - Ranking stochastic model?

Google PageRank - Stochastic model 1

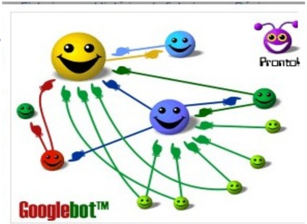


A stochastic (sparse) matrix on $\{1, \ldots, d\}$

$$P(i,j) = \begin{cases} \frac{1}{d_i} & \text{if} \quad j \text{ is one of the } d_i \text{ outgoing links} \\ 0 & \text{if} \quad d_i = 0 \text{ (a.k.a. a dangling node)} \end{cases}$$

Markov chain model ? $\rightsquigarrow$

Google PageRank - Stochastic model 1



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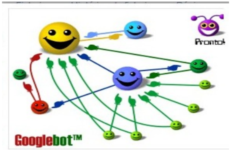
Markov chain model ? $\rightsquigarrow$ Blackboard

Google PageRank - Stochastic model 2/4



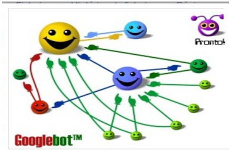
More regular Markov transitions:

$$M(i,j) = \epsilon \; P(i,j) + (1-\epsilon) \; \mu(j)$$

with

- Damping factor $\epsilon \in]0,1[$ (restart rate).
- $\mu(i) = 1/d$ uniform on $\{1, \ldots, d\}$

Google PageRank - Stochastic model 2/4



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WHY?

Google PageRank - Stochastic model 2/4

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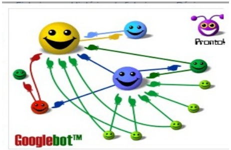
Google PageRank - Stochastic model 2/4



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Consequences for 2 $\perp$ Surfers (X_n, X'_n) (start $\neq$ sites)

$$\overbrace{\mathbb{P}(X_n = i)}^{=p_n(i)} - \overbrace{\mathbb{P}(X'_n = i)}^{=p'_n(i)} \quad = \quad ???$$

$$\mathbb{P}\,(X_n \ \neq \ X'_n) \quad = \quad ??? \quad \rightsquigarrow$$

Google PageRank - Stochastic model 2/4

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$$\mathbb{P}\left(X_n \; \neq \; X'_n\right) \quad = \quad ??? \quad \rightsquigarrow$$ Blackboard $\oplus$ Lecture slides 2!

Google PageRank - Stochastic model 3/4

Surfers X_n starting at $X_0 = i$:

$$p_0(j) = \mathbb{P}\left(X_0 = j\right) = 1_i(j) \Leftrightarrow p_0 := \left[0, \ldots, 0, \overbrace{1}^{i-th}, 0 \ldots, 0\right]$$

$\Downarrow$ **[Forgetting the initial condition]**

$$\mathbb{P}\left(X_n = j\right) = \mathbb{P}\left(X_n = j \mid X_0 = i\right) = p_0 M^n = M^n(i,j) \rightarrow_{n\uparrow\infty} p_\infty(j)$$

Google PageRank - Stochastic model 4/4

More general situations (i.e. $\forall p_0$)

$$p_n = p_0 M^n \Longrightarrow p_n(j) = \sum_k p_0(k)\ M^n(k,j) \longrightarrow_{n\uparrow\infty} p_\infty(j)$$

$$\Downarrow$$

Fixed point equation = invariant/stationary

$$p_n \longrightarrow_{n\uparrow\infty} p_\infty = p_\infty M \rightsquigarrow \text{Wolfram - Mathworld}$$

Google PageRank -Ranking

- Rate of convergence to equilibrium:

$$\|p_n - p_\infty\|_{tv} \overset{admitted}{:=} \frac{1}{2}\sum_{i=1}^{d} |p_n(i) - p_\infty(i)| \leq \ldots ??$$

- How to rank sites using the surfer exploration?

Google PageRank -Ranking

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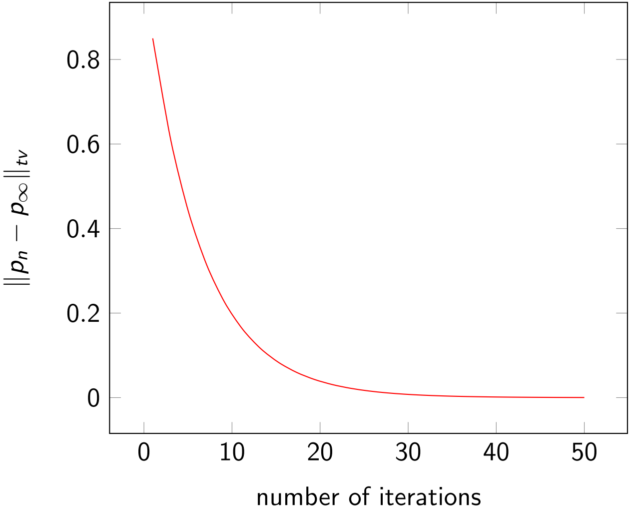
$\rightsquigarrow$ Blackboard $\oplus$ **Lecture notes** $\oplus$ next slide

Google PageRank $\epsilon = .85$

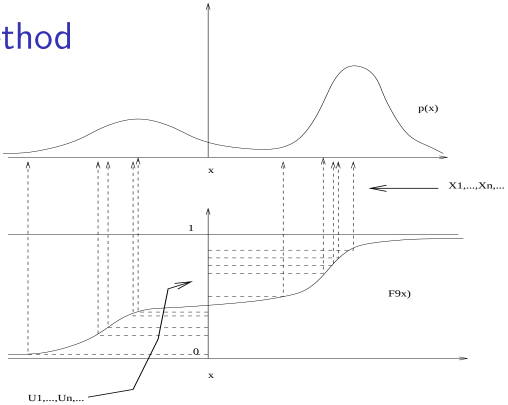


From Monte Carlo to Los Alamos

An introduction to simulation

- 3 simple ways to sample elementary random variables
- The Metropolis-Hasting model
  ($\simeq$ 1960 [Metropolis-Rosenbluth (2)-Teller (2) cf. lect. notes]):
  $\in$ Top-10 algo. in the 20th century.
- In the 21th century ...

The inverse method



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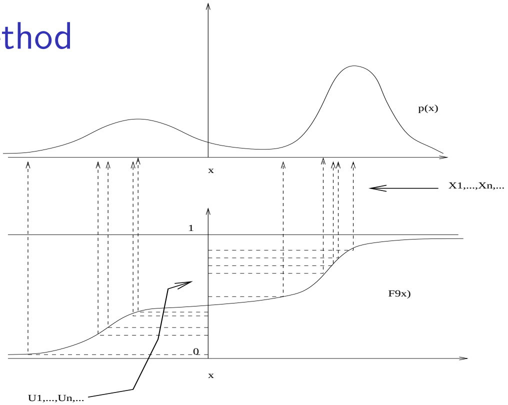
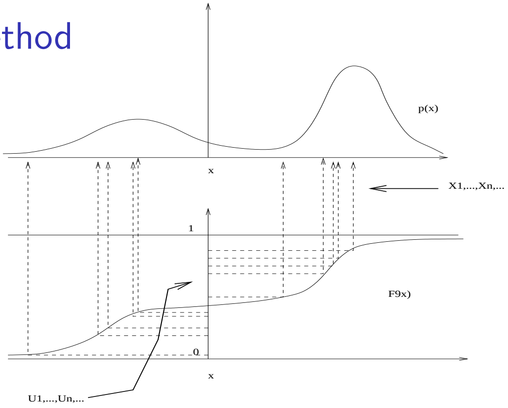


Formula

$$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} \mathbb{P}(X \in dy) \quad \Rightarrow \quad X \stackrel{def}{=} F^{-1}(U)$$

Examples: *Exp(λ), discrete, binomial, multinomial,. . .*

The inverse method



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$\rightsquigarrow$ **Wolfram - Mathworld** $\oplus$ **Section 4.1 pp. 51-53**

The change of variables

$$\int_{\phi(a)}^{\phi(b)} f(x)\,dx = \int_a^b f(\phi(t))\phi'(t)\,dt.$$

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Some formulae (U_i $\perp$ Unif $[0,1]$)

$[a_1, b_1]\times[a_2, b_2] \rightsquigarrow (X_1, X_2) = (a_1 + (b_1 - a_1)U_1, a_2 + (b_2 - a_2)U_2)$??

and

$$\begin{cases} Y_1 := \sqrt{-2\log(U_1)}\ \cos(2\pi U_2) \\ Y_2 := \sqrt{-2\log(U_1)}\ \sin(2\pi U_2) \end{cases} \quad ??\rightsquigarrow$$

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Blackboard

Uniform on the unit circle ?? $\rightsquigarrow$

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Uniform on the unit circle ?? $\rightsquigarrow$ Blackboard

Lecture notes section 4.2 pp. 54-55

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Bb Blackboard

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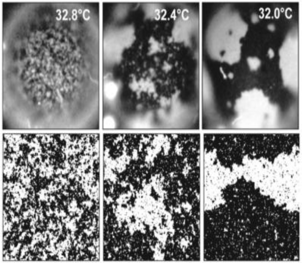
??$\rightsquigarrow$ Blackboard

Uniform on the unit circle ?? $\rightsquigarrow$ Blackboard

$\rightsquigarrow$ **Wolfram - Mathworld** $\oplus$ **Section 4.2 pp. 54-55**

Boltzmann-Gibbs measures

$$\pi(dx) := \frac{1}{\mathcal{Z}_\beta} \, e^{-\beta \; V(x)} \, \lambda(dx)$$



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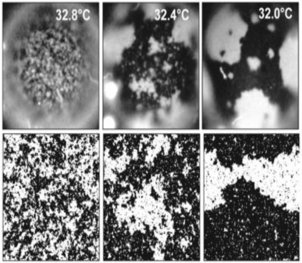

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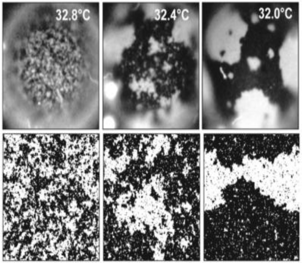
Some examples: (**see also section 6.4**)

- *Ising/Sherrington-Kirkpatrick model*: $x \in \{-1,+1\}^{\{1,\ldots,L\}^2}$ with $\lambda(x) = 2^{-L^2}$

$$V(x) = h \sum_{i \in E} x(i) - J \sum_{i \sim j} \theta_{i,j}\ x(i)\ x(j)$$

Boltzmann-Gibbs measures



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- *Traveling Salesman m cities* e_i: $x \in \mathcal{G}_m$ with $\lambda(x) = \frac{1}{m!}$

$$V(x) = \sum_{p=1}^{m}\ d(e_{x(p)}, e_{x(p+1)})$$

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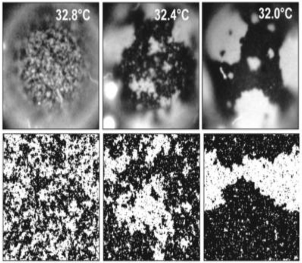

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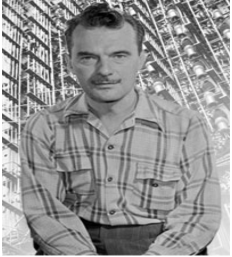
- *Black Box problems:*

$$\text{Inputs} \;=\; X \;\rightarrow\; \boxed{\text{Numerical codes}\quad \mathsf{F}} \;\rightarrow\; \text{Outputs} \;=\; Y = F(X)$$

$$e^{-\beta V(x)} \;\simeq\; 1_{F^{-1}(A)}(x) \;\Rightarrow\; \pi = \mathrm{Law}(X \mid X \in A)$$

The Metropolis Hasting sampler



Markov chain $X_{n-1} \rightsquigarrow X_n$ with 2 steps:

- Propose a transition $X_{n-1} = x \rightsquigarrow y$ with some probability density $P(x, dy) \sim \pi(dy)$
- Accept $X_n = y$ **or** reject $X_n = x$ with acceptance probability

$$a(x,y) = \min\left(1, \frac{\pi(dy)P(y,dx)}{\pi(dx)P(x,dy)}\right)$$

$$\Downarrow$$

$$\pi M = \pi$$

The Metropolis Hasting sampler

The Markov transition:

$$M(x,dy) := P(x,dy) \;\times\; a(x,y) + \left(1 - \int \; P(x,dz)\; a(x,z)\right) \; \delta_x(dy)$$

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$$\Downarrow$$

Master equation $\Leftrightarrow$ π-reversible property of M

$$\pi(dx)M(x, dy) = \pi(dy)M(y, dx) \ \Rightarrow \ \pi M = \pi$$

$\rightsquigarrow$ **Wolfram - Mathworld**

Reversible Proposals

Reversible Proposals:

$$\pi(dx)P(x, dy) = \pi(dy)P(y, dx)$$

$$\Downarrow$$

Maximal acceptance rate

$$a(x, y) = \min\left(1, \frac{\pi(dy)P(y, dx)}{\pi(dx)P(x, dy)}\right) = 1$$

Ex.- Gibbs Sampler on $x = (x_1, x_2) \in S = (S_1 \times S_2)$

$$\pi(d(x_1, x_2)) = \pi_1(dx_1)\ L_{1,2}(x_1, dx_2) = \pi_2(dx_2)\ L_{2,1}(x_2, dx_1)$$

$$\Updownarrow$$

$$(X_1, X_2) \sim \pi \Rightarrow \pi_1 = \mathrm{Law}(X_1) \quad \text{and} \quad L_{1,2}(x_1, dx_2) = \mathbb{P}\left(X_2 \in dx_2 \mid X_1 = x_1\right)$$

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Example:

$$p(x_1, x_2) = \frac{1}{\pi}\ 1_{x_1^2 + x_2^2 \leq 1} = 1_{0 \leq x_1 \leq 1} \ \times\ 1_{|x_2| \leq \sqrt{1 - x_1^2}}$$

Gibbs sampling types of proposals

$$P = K_1 K_2 \quad \text{or} \quad P = K_2 K_1 \quad \text{or} \quad P = \frac{1}{2}\, K_1 + \frac{1}{2}\, K_2$$

with the *"matrix-like"* compositions:

$$(K_1 K_2)(x_1, dx_3) := \int_{x_2} K_1(x_1, dx_2)\; K_2(x_2, dx_3)$$

$\oplus$ **the conditional transitions with a fixed coordinate:**

$$\begin{array}{rcl} K_1((x_1, x_2), d(y_1, y_2)) & := & \delta_{x_1}(dy_1) L_{1,2}(y_1, dy_2) \\ K_2((x_1, x_2), d(y_1, y_2)) & := & \delta_{x_2}(dy_2) L_{2,1}(y_2, dy_1) \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right) \xrightarrow{\quad K_2 \quad} \left(\begin{array}{c} {\color{red} y_1} \\ x_2 \end{array}\right) \xrightarrow{\quad K_1 \quad} \left(\begin{array}{c} {\color{red} y_1} \\ {\color{blue} y_2} \end{array}\right)$$

$$\xrightarrow{\qquad\qquad\qquad K_2 K_1 \qquad\qquad\qquad}$$

The unit disk example!!

Reversibility check - Back to K_i !

Proposition: The "frozen first" coordinate transition

$$K_1((y_1, y_2), d(x_1, x_2)) := \delta_{y_1}(dx_1) L_{1,2}(x_1, dx_2)$$

is π-reversible.

Reversibility check - Back to K_i !

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Proof/Exercise:

Reversibility check - Back to K_i !

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Proof/Exercise:

$$\pi(d(y_1, y_2)) \times K_1((y_1, y_2), d(x_1, x_2))$$

$$= \pi_1(dy_1) L_{1,2}(y_1, dy_2) \times \delta_{y_1}(dx_1) L_{1,2}(x_1, dx_2)$$

$$= \underbrace{\pi_1(dy_1)\delta_{y_1}(dx_1)}_{=\pi_1(dx_1)\delta_{x_1}(dy_1)} \times \underbrace{(L_{1,2}(y_1, dy_2) L_{1,2}(x_1, dx_2))}_{(x,y)-symmetry}$$

$$\Downarrow \ (x = (x_1, x_2) \ \& \ y = (y_1, y_2))$$

Reversibility property

$$\pi(dy) \times K_1(y, dx) = \pi(dx) \times K_1(x, dy)$$

Exercise 1

Exercise/Proposition:
If M_1 and M_2 two π-reversible Markov transitions on S

$$\forall i = 1, 2 \qquad \pi(dx)M_i(x, dy) = \pi(dy)M_i(y, dx)$$

Then

$$\pi(dx_1)M_1(x_1, dx_2)M_2(x_2, dx_3) \;=\; \pi(dx_3)\; M_2(x_3, dx_2)\; M_1(x_2, dx_1)$$

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Proof:

$$\begin{aligned}
\pi(dx_1)M_1(x_1,dx_2)M_2(x_2,dx_3) &:= \pi(dx_2)\ M_1(x_2,dx_1)\ M_2(x_2,dx_3) \\
&= M_1(x_2,dx_1)\ \pi(dx_2)\ \ M_2(x_2,dx_3) \\
&= M_1(x_2,dx_1)\ \pi(dx_3)\ \ M_2(x_3,dx_2) \\
&= \pi(dx_3)\ M_2(x_3,dx_2)\ M_1(x_2,dx_1)
\end{aligned}$$

Exercise 2

Exercise/Proposition: The transition $X = x \rightsquigarrow Y \in dy$

$$Y = \sqrt{1-\epsilon}\, X \; + \; \sqrt{\epsilon}\, W \quad \text{with} \quad W \sim \mathrm{N}(0,1)$$

is $N(0,1)$-reversible for any $\epsilon \in [0,1]$.

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$$M(x,dy) \propto \exp\left[-\frac{1}{2\epsilon}\;\left(y - \sqrt{1-\epsilon}\; x\right)^2\right]\; dy$$

$$\begin{aligned} x^2 + \frac{1}{\epsilon}\left(y - \sqrt{1-\epsilon}\; x\right)^2 &= x^2 + \frac{1}{\epsilon}\left(y^2 - 2\sqrt{1-\epsilon}\; yx + (1-\epsilon)\; x^2\right) \\ &= \frac{1}{\epsilon}\underbrace{\left(y^2 - 2\sqrt{1-\epsilon}\; yx + \; x^2\right)}_{(x,y)-symmetry} \end{aligned}$$

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Consequences? $\rightsquigarrow$ Blackboard