Stochastic Processes

MATH5835, P. Del Moral

UNSW, School of Mathematics & Statistics

Lectures Notes No. 11

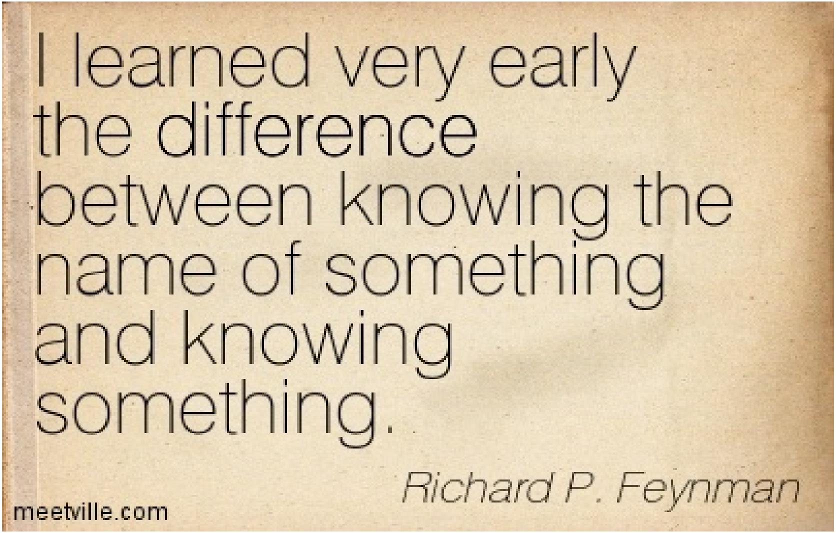
Consultations (RC 5112):

Wednesday 3.30 pm $\rightsquigarrow$ 4.30 pm & Thursday 3.30 pm $\rightsquigarrow$ 4.30 pm

Reminder + Information

References in the slides

- **Material for research projects** $\rightsquigarrow$ Moodle
  (*Stochastic Processes and Applications* $\ni$ variety of applications)



– *Richard P. Feynman (1918-1988)* ⊕ video

Three objectives

Understanding & Solving

- Classical stochastic algorithms
- Some advanced Monte Carlo schemes
- Intro to computational physics/biology

Plan of the lecture

- **Stochastic algorithms**
  - Robbins Monro model
  - Simulated annealing

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- **Some advanced Monte Carlo models**
  - Interacting simulated annealing
  - Rare event sampling
  - Black box and inverse problems

Plan of the lecture

- **Stochastic algorithms**
  - Robbins Monro model
  - Simulated annealing
- **Some advanced Monte Carlo models**
  - Interacting simulated annealing
  - Rare event sampling
  - Black box and inverse problems
- **Computational physics/biology**
  - Molecular dynamics
  - Schödinger ground states
  - Genetic type algorithms

Robbins Monro model

Objectives

$$\text{Given} \quad U \; : \; \mathbb{R}^d \mapsto \mathbb{R}^d \ni a \qquad \text{find} \quad U_a = \{x \in \mathbb{R}^d \; : \; U(x) = a\}$$

Robbins Monro model

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Examples

- *Concentration of products (therapeutic,...):* $U(x) = \mathbb{E}(\mathcal{U}(x, Y))$

  $\mathcal{U}(x, Y) := \mathcal{U}(\text{"drug" dose } x, \text{ "data" patients } Y) = \text{dosage effects}$

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- *Median and quantiles estimation*

$$U(x) = \mathbb{P}(Y \leq x) \quad \rightsquigarrow \quad \text{find } x_a \quad \text{s.t.} \quad \mathbb{P}(Y \leq x_a) = a$$

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- *Optimization problems* (V smooth & convex)

  $$U(x) = \nabla V(x) \quad \rightsquigarrow \quad \text{find } x_0 \quad \text{s.t.} \quad \nabla V(x_0) = 0$$

When U is known

Hypothesis

$$U_a = \{x_a\} \qquad \& \qquad \langle (x - x_a), U(x) - U(x_a) \rangle \ > \ 0$$

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$$\Updownarrow \; d = 1$$

Same sign!

$$U(x) \geq U(x_a) \quad \Rightarrow \quad x \geq x_a$$
$$U(x) \leq U(x_a) \quad \Rightarrow \quad x \leq x_a$$

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Algorithm?

$$x_{n+1} = x_n + \gamma_n \ (U(x_a) - U(x_n))$$

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Algorithm?

$$x_{n+1} = x_n + \gamma_n \, (U(x_a) - U(x_n))$$

with some technical conditions

$$\sum_n \gamma_n = \infty \quad \text{and} \quad \sum_n \gamma_n^2 < \infty$$

When $U(x) = \mathbb{E}(\mathcal{U}(x, Y))$ is **un**known

Examples

- **Quantiles**

$$U(x) = \mathbb{P}(Y \leq x) = \mathbb{E}(\mathcal{U}(x, Y)) \quad \mathcal{U}(x, Y) := 1_{]-\infty, x]}(Y)$$

When $U(x) = \mathbb{E}(\mathcal{U}(x, Y))$ is **unknown**

Examples

- **Quantiles**

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- **Dosage effects** $Y =$ absorption curves of drugs w.r.t. time

$$U(x) = \mathbb{E}(\mathcal{U}(x, Y))$$

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Examples

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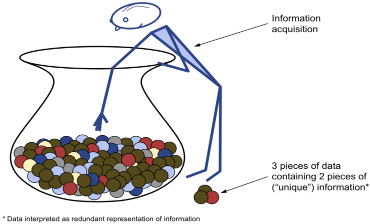
- **Dosage effects** Y = absorption curves of drugs w.r.t. time

$$U(x) = \mathbb{E}(\mathcal{U}(x, Y))$$

- **Noisy measurements**

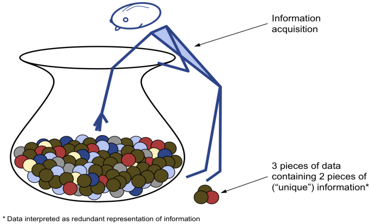
$$x \longrightarrow \boxed{\text{sensor/black box}} \longrightarrow \mathcal{U}(x, Y) := U(x) + Y$$

Unknown ⇝ Sampling



Ideal deterministic algorithm

Unknown ⇝ Sampling



Ideal deterministic algorithm

$$x_{n+1} = x_n + \gamma_n \left(U(x_a) - U(x_n)\right) = x_n + \gamma_n \left(a - U(x_n)\right)$$

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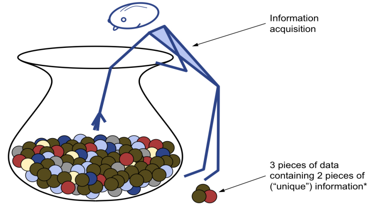

* Data interpreted as redundant representation of information

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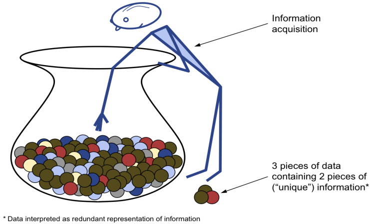

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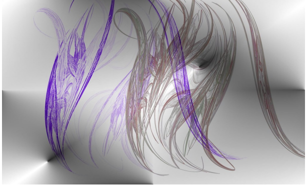
Robbins Monro algorithm

$$X_{n+1} = X_n + \gamma_n \left(a - \mathcal{U}(x_n, Y_n)\right)$$

Stochastic gradient



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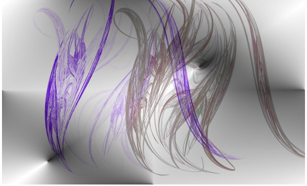

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$$\Downarrow \;\; a = 0 \quad \& \quad \mathcal{U}(x, Y_n) = \nabla \mathcal{V}_x(x, Y_n) \quad (\rightsquigarrow U(x) = \nabla_x \mathbb{E}(\mathcal{V}_x(x, Y)))$$

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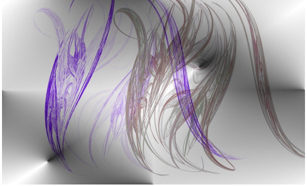

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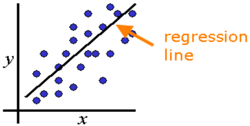
Stochastic gradient

$$X_{n+1} = X_n - \underbrace{\gamma_n}_{\text{learning rate}} \nabla \mathcal{V}_x(X_n, Y_n)$$

Example (linear regression)



N data set $z^i \in \mathbb{R}^d$ $\rightsquigarrow$ observation $y^i \in \mathbb{R}^{d'}$

Best $x \in \mathbb{R}^d$? such that

$$y^i \simeq h_x(z^i) + N(0,1) \quad \text{with} \quad h_x(z) = \sum_{1 \le i \le d} x_i z_i$$

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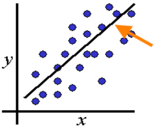

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Averaging criteria

$$\mathcal{U}(x) = \mathbb{E}\left(\mathcal{V}(x,(y^I,z^I))\right) \overset{I \ unif \in \{1,\ldots,N\}}{=\!=} \frac{1}{2N} \sum_{1 \leq i \leq N} \left(h_x(z^i) - y^i\right)^2$$

with

$$\mathcal{V}(x,(y^i,z^i)) = \frac{1}{2}\ \left(h_x(z^i) - y^i\right)^2 \Rightarrow \nabla_x \mathcal{V} = \begin{pmatrix} \left(h_x(z^i) - y^i\right)\ z^i_1 \\ \ldots \\ \left(h_x(z^i) - y^i\right)\ z^i_d \end{pmatrix}$$

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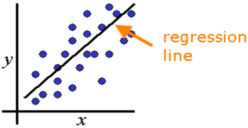

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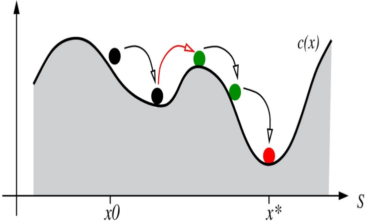
Stochastic gradient process

$$X_{n+1} \quad = \quad X_n - \gamma_n\ \nabla_x \mathcal{V}(X_n, (Y^{I_n}, Z^{I_n}))$$

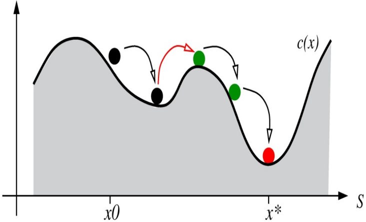
Simulated annealing



Objectives

$$\text{given} \quad V : S \mapsto \mathbb{R} \qquad \text{find} \quad V^\star = \{x \in S : V(x) = \inf_y V(y)\}$$

Simulated annealing



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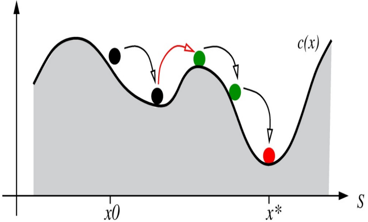
$$\Updownarrow$$

Probabilist viewpoint: $\Leftrightarrow$ *Sampling the Boltzmann-Gibbs distribution*

$$\mu_\beta(dx) := \frac{1}{\mathcal{Z}_\beta} e^{-\beta V(x)} \lambda(dx)$$

for some reference measure λ.

Simulated annealing



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for some reference measure λ. **A couple of examples**

$$\begin{array}{rcl} S &=& \{x_1,\ldots,x_k\} \quad \lambda(\{x_i\}) := \lambda(x_i) = \frac{1}{d} \\ S &=& \mathbb{R}^k \quad \lambda(dx) := \prod_{1\leq i\leq k} dx^i = \text{Lebesgue measure on } \mathbb{R}^k \end{array}$$

Optimization vs. Sampling

Finite state spaces $S = \{x_1, \ldots, x_k\} \ni x_i$

$$\mu_\beta(x_i) := \frac{e^{-\beta V(x_i)}\ \lambda(x_i)}{\sum_{y \in S} e^{-\beta V(y)}\ \lambda(y)} = \frac{e^{-\beta V(x_i)}}{\sum_{1 \le j \le k} e^{-\beta V(x_j)}}$$

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Proposition

$$\mu_\beta(x_i) \quad \longrightarrow_{\beta \uparrow \infty} \quad \mu_\infty(x_i) = \frac{1}{\mathrm{Card}(V^\star)}\ 1_{V^\star}(x_i)$$

Proof:

Metropolis-Hastings transition

Reversible proposition w.r.t. λ (local moves/neighbors)

$$\lambda(x)P(x,y) = \lambda(y)P(y,x)$$

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Acceptance/rejection transition

$$\begin{aligned} M_\beta(x,y) &= P(x,y)\ \min\left(1, \frac{\mu_\beta(y)P(y,x)}{\mu_\beta(x)P(x,y)}\right) + \ldots \delta_x(dy) \\ &= P(x,y)\ e^{-\beta(V(y)-V(x))_+} + \ldots \delta_x(dy) \end{aligned}$$

$$\Downarrow$$

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$$\Downarrow$$

Balance/Reversibility equation

$$\mu_\beta(y)M_\beta(y,x) = \mu_\beta(x)M_\beta(x,y)$$

Simulated Annealing

$$X_0 \xrightarrow{M_{\beta_0}^{n_0}} \underset{\sim\, \mu_{\beta_0}}{X_{n_0}} \xrightarrow{M_{\beta_1}^{n_1}} \underset{\sim\, \mu_{\beta_1}}{X_{n_0+n_1}} \xrightarrow{M_{\beta_2}^{n_2}} \underset{\sim\, \mu_{\beta_2}}{X_{n_0+n_1+n_2}} \quad \cdots / \cdots$$

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You Tube illustrations

- **SA and Travelling Salesman problem**
- **Automatic Label placement**
- **Ising model with SA**
- **Artist view of the SA & the Ising model**

Interacting Simulated Annealing

$$e^{-\beta_1 V} = e^{-(\beta_1-\beta_0)V} \times e^{-\beta_0 V}$$

Interacting Simulated Annealing

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$\Leftrightarrow$ **Bayes' type multiplication rule**

$$\mu_{\beta_1}(dx) \propto e^{-(\beta_1-\beta_0)V(x)} \times \mu_{\beta_0}(dx)$$

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$\forall i = 1, \ldots, N \rightsquigarrow$ Interacting Simulated Annealing

$$\begin{array}{ccc} \widehat{X}_0^i & \xrightarrow{M_{\beta_0}^{n_0}} & X_{n_0}^i \\ \sim \mu_{\beta_0} & & \Downarrow \end{array}$$

$$\mu_{\beta_0}^N = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{X_{n_0}^i} \sim \mu_{\beta_0}$$

Interacting Simulated Annealing

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$\forall i = 1, \ldots, N \rightsquigarrow$ **Interacting Simulated Annealing**

$$\widehat{X}_0^i \sim \mu_{\beta_0} \xrightarrow{M_{\beta_0}^{n_0}} X_{n_0}^i$$

$$\Downarrow$$

$$\mu_{\beta_0}^N = \frac{1}{N}\sum_{1\leq i\leq N} \delta_{X_{n_0}^i} \sim \mu_{\beta_0}$$

$$\Downarrow$$

$$(\star) \sum_{1\leq i\leq N} \frac{e^{-(\beta_1-\beta_0)V(X_{n_0}^i)}}{\sum_{1\leq j\leq N} e^{-(\beta_1-\beta_0)V(X_{n_0}^j)}}\delta_{X_{n_0}^i} \sim \mu_{\beta_1}$$

Interacting Simulated Annealing

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$\forall i = 1, \ldots, N \rightsquigarrow$ **Interacting Simulated Annealing**

$$\underset{\sim \mu_{\beta_0}}{\widehat{X}_0^i} \xrightarrow{M_{\beta_0}^{n_0}} X_{n_0}^i \xrightarrow{N \text{ samples from } (\star)} \underset{\sim \mu_{\beta_1}}{\widehat{X}_{n_1}^i}$$

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$\forall i = 1,\ldots,N \rightsquigarrow$ Interacting Simulated Annealing

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$$(\star) \sum_{1\leq i\leq N} \frac{e^{-(\beta_1-\beta_0)V(X_{n_0}^i)}}{\sum_{1\leq j\leq N} e^{-(\beta_1-\beta_0)V(X_{n_0}^j)}} \delta_{X_{n_0}^i} \sim \mu_{\beta_1}$$

Rare event sampling

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Rare event sampling

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$$\mu_A = \mathrm{Law}(X \mid Y \in B) = \mathrm{Law}(X \mid X \in A)$$

Subset shakers

Reversible proposition w.r.t. λ (local moves/neighbors)

$$\lambda(x)P(x,dy) = \lambda(y)P(y,dx)$$

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Acceptance/rejection transition = A-Shaker

$$M_A(x,dy) \quad = \quad P(x,dy)\, 1_A(y) + (1 - P(x,A))\, \delta_x(dy)$$

$$\Downarrow$$

$$\mu_A(dy) M_A(y,dx) = \mu_A(dx) M_A(x,dy)$$

Interacting Subset Sampling $A_n \downarrow$

$$1_{A_1} = 1_{A_1 \cap A_0} = 1_{A_1} \times 1_{A_0}$$

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Interacting Subset Sampling $A_n \downarrow$

Local approximations

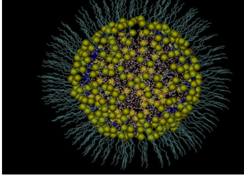
$$\begin{aligned}\mathbb{P}(X \in A_{p+1} \mid X \in A_p) &= \int 1_{A_{p+1}}(x)\ \mu_{A_p}(dx) \\ &\simeq \int 1_{A_{p+1}}(x)\ \mu^N_{A_p}(dx) = \frac{1}{N} \sum_{1 \leq j \leq N} 1_{A_{p+1}}(X^j_{n_0+\ldots+n_p})\end{aligned}$$

$$\Downarrow$$

Unbias estimate

$$\begin{aligned}\mathbb{P}(X \in A_n \mid X \in A_0) &= \mathbb{P}(X \in A_n \mid X \in A_{n-1}) \times \mathbb{P}(X \in A_{n-1} \mid X \in A_0) \\ &= \prod_{0 \leq p < n} \mathbb{P}(X \in A_{p+1} \mid X \in A_p) \\ &\simeq \prod_{0 \leq p \leq n} \frac{1}{N} \sum_{1 \leq j \leq N} 1_{A_{p+1}}(X^j_{n_0+\ldots+n_p})\end{aligned}$$

Molecular dynamics



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\begin{aligned}
q &= (q_i)_{1 \le i \le k} = \text{k atomic particles} \in \mathbb{R}^3 \\
m &= (m_i)_{1 \le i \le k} = \text{k masses} \in \mathbb{R}_+ \\
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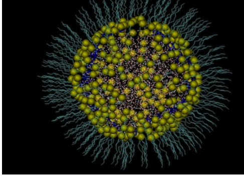
Molecular dynamics



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Hamiltonian energy functional $x = (q, p)$=phase vector

$$
H(q,p) = \sum_{i=1}^{k} \underbrace{\frac{\|p_i\|^2}{2m_i}}_{\text{kinetic energy}} + \underbrace{V(q_1,\ldots,q_k)}_{\text{interparticle potential}}
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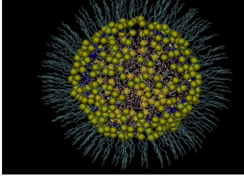

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Example: Lennard Jones potential

$$V(q_1,\ldots,q_k) = \sum_{1\leq i<j\leq k} V_{LJ}(\|q_j - q_i\|)$$

with weak van de Waals bonds energies

$$V_{LJ}(r) = 4\epsilon\left[\left(\frac{\tau}{r}\right)^{12} - \left(\frac{\tau}{r}\right)^{6}\right]$$

Molecular dynamics

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Dynamical gradient flow equations

$$\begin{cases} \frac{dq_i}{dt} = \frac{p_i}{m_i} = \frac{\partial H}{\partial p_i}(q,p) \\ \frac{dp_i}{dt} = -\frac{\partial V}{\partial q_i}(q) = -\frac{\partial H}{\partial q_i}(q,p) \end{cases}$$

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Time discretizations: Beeman, Leapfrog and Verlet schemes

Molecular dynamics

Boltzmann-Gibbs measures

$$H(x) = H(q,p) \rightsquigarrow \mu_\beta(dx) = \frac{1}{\mathcal{Z}_\beta}\; e^{-\beta H(x)}\; dx \quad \text{with} \quad \beta = \frac{1}{\text{temperature}}$$

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=Invariant measures of the Langevin stochastic gradient process

$$\begin{cases} dq_i &= \beta\; \overbrace{\frac{\partial H}{\partial p_i}(q,p)}^{p_i/m_i}\; dt \\ dp_i &= -\beta\; \underbrace{\left[\frac{\partial H}{\partial q_i}(q,p) + \sigma^2 \frac{\partial H}{\partial p_i}(q,p)\right]}_{=\frac{\partial V}{\partial q_i}(q)+\sigma^2\; p_i/m_i} dt + \sigma\sqrt{2}\; \underbrace{dW^i_t}_{\text{iid Brownian}} \end{cases}$$

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(1 trillion simulation steps ($\sim$ O(year)) for 1millisecond...)

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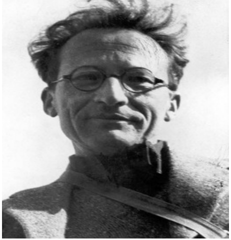
Introduction to MD (a.k.a. the SPC water model).
Supercritical water by MDSimulator (YouTube).
Oil and water separation

Schrödinger ground states



Schrödinger equation
$\simeq$ Quantum type Newton law (De Broglie 1924)

["Physics reasoning"]

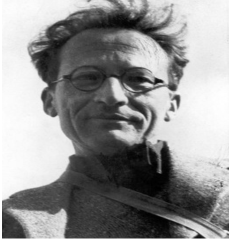
Wave function of a massive particle with:

- Velocity/momentum $p = k\hbar$
- Energy $E_c = p^2/(2m) = \hbar\omega \Rightarrow$ frequency $\omega = E_c/\hbar$

is given by

$$\psi(t,x) = \psi_0 \; e^{i(kx-\omega t)} \overset{\text{Blackboard}}{\Longrightarrow} i\hbar\frac{\partial\psi}{\partial t} = E_c \; \psi = -\frac{\hbar^2}{2m} \; \frac{\partial^2\psi}{\partial x}$$

Schrödinger ground states



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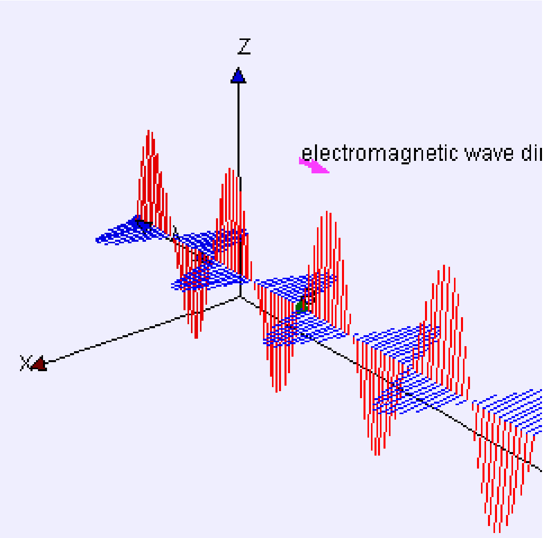
In a potential energy

$$E = E_c + V(x) \Rightarrow i\hbar\frac{\partial \psi}{\partial t} = E_c\psi + V \ \psi = \underbrace{-\frac{\hbar^2}{2m}\ \frac{\partial^2\psi}{\partial x} + V \ \psi}_{-L^V(\psi)}$$

Schrödinger ground states

The wave function is the result of two traveling waves in the x and t directions.



Schrödinger ground states

Schrödinger eq. $\simeq$ Quantum version of Newton law

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$$\Downarrow \quad u(\tau,x) = \psi(-i\tau\hbar,x)$$

Feynman-Kac model/Heat equation

$$\frac{\partial u}{\partial\tau} = L^V(u) := \frac{\hbar^2}{2m}\ \frac{\partial^2 u}{\partial x} - Vu$$

Feynman-Kac/Heat equation

$$\frac{\partial u}{\partial \tau} = L^V(u) := \underbrace{\frac{\hbar^2}{2m}\ \frac{\partial^2 u}{\partial x}}_{:=L^0(u)} - Vu$$

Solution s.t. $u(0,x) = f(x) \rightsquigarrow$ **Feynman-Kac model**

$$\begin{aligned} u(\tau, x) &= Q_\tau(f)(x) \\ &:= \mathbb{E}\left(f(X_\tau)\ e^{-\int_0^\tau V(X_s)ds} \mid X_0 = x \right) \end{aligned}$$

with the diffusion:

$$dX_s = (\hbar/\sqrt{m})\ \underbrace{dW_s}_{\text{Brownian}}$$

Proof: Blackboard

Spectral decomposition of L^V

Reversibility

$$\begin{aligned}\int g(x)\ L^V(f)(x)\ dx &= \int g(x)\ \frac{\hbar^2}{2m}\ \frac{\partial^2 f}{\partial x}\ dx - \int g(x)\ V(x)\ f(x)\ dx \\ &= \int f(x)\ \frac{\hbar^2}{2m}\ \frac{\partial^2 g}{\partial x}\ dx - \int f(x)\ V(x)\ g(x)\ dx \\ &= \int f(x)\ L^V(g)(x)\ dx := \langle f, L^V(g)\rangle\end{aligned}$$

$$\Downarrow$$

Spectral decomposition on $\mathbb{L}_2(\mathbb{R}^d)$ $\exists E_i \uparrow \in [0,\infty[$ and $\exists \psi_i$ orthonormal eigenfunctions s.t.

$$Q_t(f) = \sum_{i\geq 0}\ e^{-tE_i}\ \langle \varphi_i, f\rangle\ \varphi_i$$

Spectral decomposition of L^V

$$Q_t(f) = \sum_{i\geq 0} e^{-tE_i} \, \varphi_i \, \langle \varphi_i, f \rangle$$

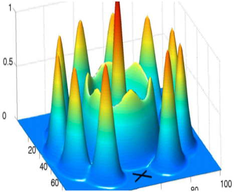
Consequences:

$$\begin{aligned} \frac{dQ_t(f)}{dt} &= -\sum_{i\geq 0} E_i \, e^{-tE_i} \, \langle \varphi_i, f \rangle \, \varphi_i \\ &= \sum_{i\geq 0} e^{-tE_i} \, \langle \varphi_i, f \rangle \, L^V(\varphi_i) \Rightarrow L^V(\varphi_i) = -E_i \, \varphi_i \end{aligned}$$

and for the "top" eigenvalue and its eigenvector φ_0 (ground state)

$$-\frac{1}{t} \log Q_t(1) \longrightarrow_{t\uparrow\infty} E_0 \quad \text{and} \quad \frac{Q_t(f)}{Q_t(1)} \simeq_{t\uparrow\infty} \frac{\langle f, \varphi_0 \rangle}{\langle 1, \varphi_0 \rangle}$$

Quantum/Diffusion Monte Carlo methods



$$\mathbb{E}\left(f(X_\tau)\ e^{-\int_0^\tau V(X_s)ds}\right)$$

$$\simeq \mathbb{E}\left(f(X_{t_n})\ \prod_{0\leq t_k<t_n} e^{-V(X_{t_k})(t_k-t_{k-1})}\right)$$

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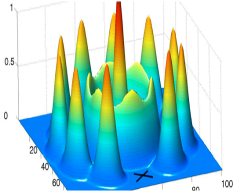

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N interacting walkers/replica evolving in two steps (toy model)

$$(X^i_{t_k})_{1\leq i\leq N} \xrightarrow{V\text{-reconfiguration}} (\widehat{X}^i_{t_k})_{1\leq i\leq N} \xrightarrow{X\text{-exploration}} (X^i_{t_{k+1}})_{1\leq i\leq N}$$

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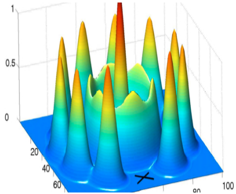

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- **Reconfigurations (selected of low energies)**

$$(\widehat{X}^i_{t_k})_{1\leq i\leq N} \quad \text{iid} \quad \sum_{1\leq j\leq N} \frac{e^{-V(X^i_{t_k})(t_k-t_{k-1})}}{\sum_{1\leq j\leq N} e^{-V(X^j_{t_k})(t_k-t_{k-1})}} \delta_{X^i_{t_k}}$$

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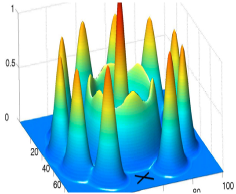

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- **Explorations**

$$X^i_{t_{k+1}} := \widehat{X}^i_{t_k} + (\hbar/\sqrt{m})\ \underbrace{(W^i_{t_{k+1}} - W^i_{t_k})}_{\text{iid Brownian}}$$

Some Monte Carlo estimates

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$$\simeq \left[\frac{1}{N}\sum_{1\leq i\leq N} f(X^i_{t_n})\right]\ \prod_{0\leq t_k<t_n} \underbrace{\frac{1}{N}\sum_{1\leq i\leq N} e^{-V(X^i_{t_k})(t_k-t_{k-1})}}$$

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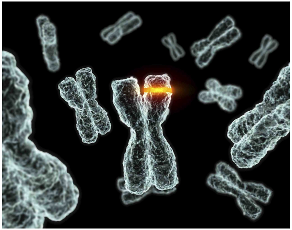
$\rightsquigarrow$ **Log-Lyapunov exponent/top eigenvalue**

$$(f=1) \Rightarrow -\frac{1}{t}\log \mathbb{E}\left(e^{-\int_0^\tau V(X_s)ds}\right) \simeq E_0 \simeq \frac{1}{t_n}\sum_{t_k<t_n}\frac{1}{N}\sum_{1\leq i\leq N} V(X^i_{t_k})(t_k-t_{k-1})$$

and the eigenvector/ground state energy

$$N^{-1}\sum_{1\leq i\leq N}\delta_{X^i_{t_n}} \simeq_{t_n\uparrow\infty}\ \psi_0(x)dx/\langle 1,\psi_0\rangle$$

Genetic type algorithms



Population of N individuals

- Mutation ($\sim$ some given Markov transition M_n)
- Selection w.r.t. some fitness functions $G_n(x)$

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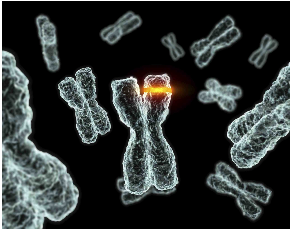

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Synthetic picture

$$(X_0^i)_{1\leq i\leq N} \xrightarrow{\text{selection } \sim\ G_0} (\widehat{X}_0^i)_{1\leq i\leq N} \xrightarrow{\text{mutation } \sim\ M_1} (X_1^i)_{1\leq i\leq N}$$

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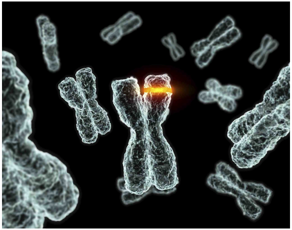

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$$\xrightarrow{\text{selection } \sim G_1} (\widehat{X}_1^i)_{1\leq i\leq N} \xrightarrow{\text{mutation } \sim M_2} (X_2^i)_{1\leq i\leq N} \quad \ldots / \ldots$$

Genetic type algorithms

$G_n = e^{-\beta_n V}$ & $M_n =$ Simulated annealing move $\rightsquigarrow$ interacting SA (optimization)

More generally:

$N \uparrow \infty$ computational power $\Rightarrow \widehat{X}_n^i$ almost iid with Feynman-Kac law

$$\frac{1}{N} \sum_{1 \leq i \leq N} f(X_n^i) \propto_{N \uparrow \infty} \mathbb{E}(f(X_n) \prod_{0 \leq p < n} G_p(X_p))$$

Somme illustrations - Artificial Intelligence

- **Painting Mona Lisa**
- **Darwin - Genetic programming**
- **GA robot controller**
- **Learning how to walk**
- **GA vs Tetris**
- **Evolutionary computation (Danubia 2011)**