Stochastic Processes

MATH5835, P. Del Moral

UNSW, School of Mathematics & Statistics

Lectures Notes, No. 10

Consultations (RC 5112):

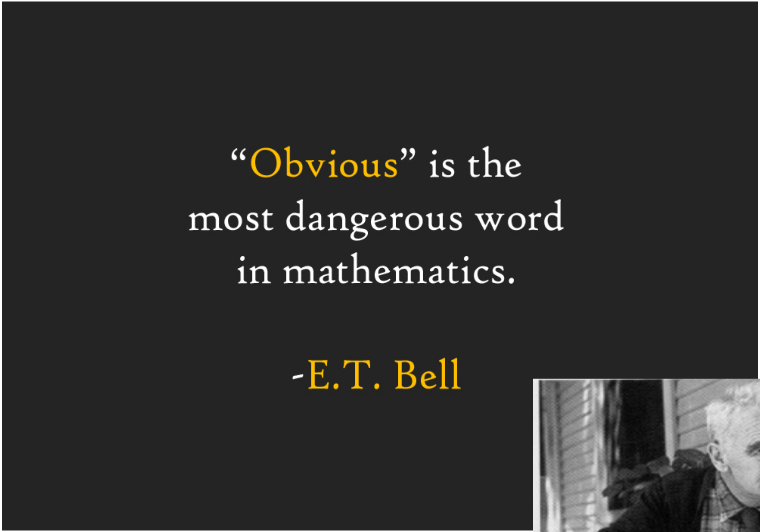
Wednesday 3.30 pm $\rightsquigarrow$ 4.30 pm & Thursday 3.30 pm $\rightsquigarrow$ 4.30 pm

Reminder + Information

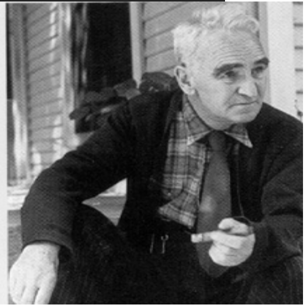
References in the slides

- **Material for research projects** $\rightsquigarrow$ Moodle
  (*Stochastic Processes and Applications* $\ni$ variety of applications)
- **Important results**

  $\subset$ **Assessment/Final exam** = LOGO =



– *Éric Temple Bell (1883-1960)*



Plan of the lecture

An introduction to Martingales

- Stochastic adaptation
  - Filtration of information
  - Projection of processes

Plan of the lecture

An introduction to Martingales

- Stochastic adaptation
  - Filtration of information
  - Projection of processes
- Martingale processes
  - A couple of brackets
  - Applications to Markov chain theory
  - A weak form of the ergodic theorem

Three objectives

- **Decomposition of the information**

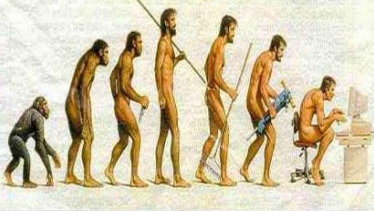
Three objectives

- **Decomposition of the information**
- **Analysis of occupation measures**
    - Bias estimates
    - Variance calculations

Three objectives

- **Decomposition of the information**
- **Analysis of occupation measures**
  - Bias estimates
  - Variance calculations
- **Powerful martingale limit theorems**

Adaptation



Filtration $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$ **w.r.t.** $(X_n)_{n\geq 0}$

$$\mathcal{F}_n = \sigma(X_0, \ldots, X_n) \subset \mathcal{F}_{n+1} = \sigma(X_0, \ldots, X_n, X_{n+1})$$

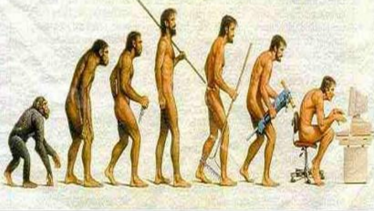
Adapted process Y_n **w.r.t.** $\mathcal{F}$

$$Y_n \in \mathcal{F}_n \iff \exists h_n \; : \; Y_n = h_n(X_0, \ldots, X_n)$$

Projection of a process Z_n **w.r.t.** $\mathcal{F}$

$$n \mapsto \widehat{Z}_n = \mathbb{E}\left(Z_n \mid \mathcal{F}_n\right)$$

Adaptation



Filtration $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$ **w.r.t.** $(X_n)_{n\geq 0}$

$$\mathcal{F}_n = \sigma(X_0, \ldots, X_n) \subset \mathcal{F}_{n+1} = \sigma(X_0, \ldots, X_n, X_{n+1})$$

Adapted process Y_n **w.r.t.** $\mathcal{F}$

$$Y_n \in \mathcal{F}_n \iff \exists h_n \ : \ Y_n = h_n(X_0, \ldots, X_n)$$

Projection of a process Z_n **w.r.t.** $\mathcal{F}$

$$n \mapsto \widehat{Z}_n = \mathbb{E}\left(Z_n \mid \mathcal{F}_n\right) = h_n(X_0, \ldots, X_n)$$

Example : Gambling

$$\begin{aligned} X_n &= \text{outcomes of the game } \in \{-1, +1\} \\ Y_n &= Y_0 + X_1 + \ldots + X_n = \text{gains/debts} \end{aligned}$$

Martingales ⚠

M_n **martingale** $(\in \mathbb{R})$ **w.r.t.** $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$

$$\Delta M_{n+1} = M_{n+1} - M_n \quad \Longrightarrow \quad \mathbb{E}\left(\Delta M_{n+1} \mid \mathcal{F}_n\right) = 0$$

Martingales

M_n **martingale** $(\in \mathbb{R})$ **w.r.t.** $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$

$$\Delta M_{n+1} = M_{n+1} - M_n \quad \Longrightarrow \quad \mathbb{E}\left(\Delta M_{n+1} \mid \mathcal{F}_n\right) = 0$$

Some properties

- Conditioning $\forall p \leq n \qquad \mathbb{E}(M_n \mid \mathcal{F}_p) = M_p$

Martingales ⚠

M_n martingale $(\in \mathbb{R})$ w.r.t. $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$

$$\Delta M_{n+1} = M_{n+1} - M_n \quad \Longrightarrow \quad \mathbb{E}(\Delta M_{n+1} \mid \mathcal{F}_n) = 0$$

Some properties

- Conditioning $\forall p \leq n \qquad \mathbb{E}(M_n \mid \mathcal{F}_p) = M_p$
- Martingale design $M_n = Y_0 + \sum_{0<p\leq n} (\Delta Y_p - \mathbb{E}(\Delta Y_p \mid \mathcal{F}_{p-1}))$

Breaking news

Unfortunately . . . for any $\mathcal{F}$-adapted process H

$$(H \bullet M)_n := \sum_{0<p\leq n} H_{p-1}\,\Delta M_p \quad \mathcal{F}\text{-martingale}$$

Breaking news

Unfortunately . . . for any $\mathcal{F}$-adapted process H

$$(H \bullet M)_n := \sum_{0<p\leq n} H_{p-1}\, \Delta M_p \quad \mathcal{F}\text{-martingale}$$

Example: $\forall \mathcal{F}$-adapted bet sizes H

$$M_n = Y_0 + X_1 + \ldots + X_n \quad \Rightarrow \quad \left\{ \begin{array}{l} (H \bullet M)_n := \sum_{0<p\leq n} H_{p-1}\, X_p \\ \\ \mathcal{F}\text{-martingale} \end{array} \right.$$

The last bad news

Optional stopping theorem

$\forall\ T$ stopping time (i.e. $\{T = n\} \in \mathcal{F}_n$

$$M_n \quad \mathcal{F}-\mathrm{martingale} \implies M_{T\wedge n} \quad \mathcal{F}-\mathrm{martingale}$$

Proof: Blackboard

Squares & Angle brackets

Theorem: M_n martingale $(\in \mathbb{R})$ w.r.t. $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$

$$M_n^2 - [M]_n \quad \text{and} \quad M_n^2 - \langle M\rangle_n \quad \text{martingales}$$

with the quadratic variation and the predictable quadratic variation

$$[M]_n := \sum_{0<k\leq n} (\Delta M_k)^2 \quad \text{and} \quad \langle M\rangle_n := \sum_{0<k\leq n} \mathbb{E}\left((\Delta M_k)^2 \mid \mathcal{F}_{k-1}\right)$$

Proof: Blackboard

Martingales & Markov chains

Markov chain X_n $(\in S)$ with transitions

$$P_n(x_{n-1}, dx_n) = \mathbb{P}(X_n \in dx_n \mid X_{n-1} = x_{n-1})$$

1st key formula $f_n \; : \; x \in S \mapsto \mathbb{R}$ **(observable/test function/...)**

For any time horizon n **(fixed)**

$$\forall k \leq n \qquad M_k = \mathbb{E}(f_n(X_n) \mid \mathcal{F}_k) = \mathbb{E}(f_n(X_n) \mid X_k) = P_{k,n}(f_n)(X_k)$$

is an $\mathcal{F}_k := \sigma(X_0, \ldots, X_k) - \mathrm{martingale}$ with fixed terminal value

$$M_n = f_n(X_n)$$

(Doob)-Martingale decomposition

Doob decomposition

$$\begin{aligned} f(X_n) &:= f(X_0) + \sum_{0<p\leq n} \Delta f(X_p) \quad \text{with} \quad \Delta f(X_p) = f(X_p) - f(X_{p-1}) \\ &= f(X_0) + \sum_{0<p\leq n} \mathbb{E}\left(\Delta f(X_p) \mid \mathcal{F}_{p-1}\right) + M_n(f) \end{aligned}$$

with the martingale

$$\begin{aligned} M_n(f) &:= \sum_{0<p\leq n} \left[\Delta f(X_p) - \mathbb{E}\left(\Delta f(X_p) \mid \mathcal{F}_{p-1}\right)\right] \\ &= \sum_{0<p\leq n} \underbrace{\left[f(X_p) - \mathbb{E}\left(f(X_p) \mid \mathcal{F}_{p-1}\right)\right]}_{=\Delta M_p(f)} \end{aligned}$$

The predictable angle bracket

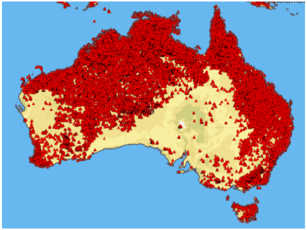
$$\mathbb{E}\left(\left(\Delta M_n(f)\right)^2 \mid \mathcal{F}_{n-1}\right) = \mathbb{E}\left(\left([f(X_n) - \mathbb{E}(f(X_n) \mid \mathcal{F}_{n-1})]\right)^2 \mid \mathcal{F}_{n-1}\right)$$
$$= P_n(f^2)(X_{n-1}) - (P_n(f)(X_{n-1}))^2 \leq \mathrm{osc(f)}^2$$

$$\Downarrow$$

Predictable angle bracket

$$\langle M(f)\rangle_n := \sum_{0<k\leq n} \left[P_k(f^2)(X_{k-1}) - (P_k(f)(X_{k-1}))^2\right] \leq n\ \mathrm{osc(f)}^2$$

Occupation measures



$$\pi^n := \frac{1}{n+1} \sum_{0 \le p \le n} \delta_{X_p}$$

Regularity properties

$$\beta(P) < 1 \quad (\text{or } \exists m \ : \ \beta(P^m) < 1)$$

$$\Longrightarrow \begin{cases} \exists! \pi = \pi P \\ (Id - P)(g) = f - \pi(f) \quad [\text{Poisson eq.}] \end{cases}$$

$$\Downarrow$$

Decomposition

$$\underbrace{\frac{1}{n+1}\left(g(X_{n+1}) - g(X_0)\right)}_{=\mathrm{O}(n^{-1})} = [\pi(f) - \pi^n(f)] + \frac{1}{n+1} \ M_{n+1}(g)$$

with $M_0(g) = 0$ and the martingale increment

$$\Delta M_n(g) := \left(g(X_n) - \mathbb{E}\left(g(X_n) \mid X_{n-1}\right)\right)$$

A weak form of the ergodic theorem

$$\sqrt{n+1}\ [\pi(f) - \pi^n(f)] = -\frac{M_{n+1}(g)}{\sqrt{n+1}} - \left(g(X_{n+1}) - g(X_0)\right)/\sqrt{n+1}$$
$$= -\frac{M_{n+1}(g)}{\sqrt{n+1}} + \mathrm{O}(1/\sqrt{n})$$

A weak form of the ergodic theorem

$$\sqrt{n+1}\ [\pi(f) - \pi^n(f)] = -\frac{M_{n+1}(g)}{\sqrt{n+1}} - (g(X_{n+1}) - g(X_0))\,/\sqrt{n+1}$$

$$= -\frac{M_{n+1}(g)}{\sqrt{n+1}} + \mathrm{O}(1/\sqrt{n})$$

$$\Downarrow$$

The bias term

$$\mathbb{E}\left([\pi(f) - \pi^n(f)]\right) = \mathrm{O}(n^{-1})$$

A weak form of the ergodic theorem

$$\sqrt{n+1}\ [\pi(f) - \pi^n(f)] = -\frac{M_{n+1}(g)}{\sqrt{n+1}} - (g(X_{n+1}) - g(X_0))\,/\sqrt{n+1}$$
$$= -\frac{M_{n+1}(g)}{\sqrt{n+1}} + \mathrm{O}(1/\sqrt{n})$$

$$\Downarrow$$

The bias term

$$\mathbb{E}\left([\pi(f) - \pi^n(f)]\right) = \mathrm{O}(n^{-1})$$

and the variance term

$$(n+1)\ \mathbb{E}\left(\left[[\pi(f) - \pi^n(f)]^2\right]\right) = \frac{1}{n+1}\ \underbrace{\mathbb{E}\left(\langle M(g)\rangle_{n+1}\right)}_{\leq (n+1)\ \mathrm{osc}(g)^2} + \mathrm{o}(1/n)$$

$$\mathbb{E}\left(\left[[\pi(f) - \pi^n(f)]^2\right]\right) = \mathrm{O}(n^{-1}) = \mathbb{E}\left([\pi(f) - \pi^n(f)]\right)$$

The limiting variance

A limiting result

$$n^{-1}\,\mathbb{E}\left(\langle M(g)\rangle_n\right) = \mathbb{E}\left(n^{-1}\sum_{0<k\leq n}\left[P(g^2)-P(g)^2\right](X_{k-1})\right)$$

$$= \mathbb{E}\left(\pi^{n-1}(P(g^2)-P(g)^2)\right) \longrightarrow_{n\to\infty} \pi\left(P(g^2)-P(g)^2\right)$$

The limiting variance

A limiting result

$$
\begin{aligned}
n^{-1}\,\mathbb{E}\left(\langle M(g)\rangle_n\right) &= \mathbb{E}\left(n^{-1}\sum_{0<k\leq n}\left[P(g^2)-P(g)^2\right](X_{k-1})\right)\\
&= \mathbb{E}\left(\pi^{n-1}(P(g^2)-P(g)^2)\right) \longrightarrow_{n\to\infty} \pi\left(P(g^2)-P(g)^2\right)
\end{aligned}
$$

$$\Downarrow$$

$$(n+1)\,\mathbb{E}\left(\left[[\pi(f)-\pi^n(f)]^2\right]\right) \longrightarrow_{n\to\infty} \sigma^2(f) := \pi\left(g^2\right) - \pi\left(P(g)^2\right)$$

The limiting variance

A limiting result

$$n^{-1}\,\mathbb{E}\left(\langle M(g)\rangle_n\right) = \mathbb{E}\left(n^{-1}\sum_{0<k\leq n}\left[P(g^2)-P(g)^2\right](X_{k-1})\right)$$
$$= \mathbb{E}\left(\pi^{n-1}(P(g^2)-P(g)^2)\right) \longrightarrow_{n\to\infty} \pi\left(P(g^2)-P(g)^2\right)$$

$$\Downarrow$$

$$(n+1)\,\mathbb{E}\left(\left[[\pi(f)-\pi^n(f)]^2\right]\right) \longrightarrow_{n\to\infty} \sigma^2(f) := \pi\left(g^2\right)-\pi\left(P(g)^2\right)$$

A more explicit formula

$$g = P(g) + (f-\pi(f))$$

$$\Downarrow$$

$$\sigma^2(f) = \pi((f-\pi(f))^2) + 2\sum_{k\geq 1}\pi([f-\pi(f)]P^k[f-\pi(f)])$$