A note on Laplace-Varadhan's integral lemma

P. Del Moral[*], T. Zajic[†]

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Abstract

In this article we propose a complement to an integral lemma of Laplace-Varadhan arising in the large deviations literature. We examine a situation in which the state space may depend on the rate of deviations. In the final part of this work we use this framework to discuss large deviations principles for an abstract class of discrete generation interacting particle systems as well as for pure jump and McKean-Vlasov diffusions.

1 Introduction

Laplace-Varadhan's integral lemma is a powerful change of reference probability technique which allows to transfer a large deviations principle (LDP) from a sequence of probability measures $\{\overline{\mathbb{Q}}^N\ ;\ N \geq 1\}$ to another $\{\overline{\mathbb{P}}^N\ ;\ N \geq 1\}$. A collection of probability measures $\{\overline{\mathbb{Q}}^N\ ;\ N \geq 1\}$ on some metric space (M, d) is said to satisfy a (LDP) with rate function H if there exists a lower semi-continuous function $H : M \to [0, \infty]$ such that for each open set $\mathcal{A}$ and for each closed subset $\mathcal{B}$

$$-\inf_{m\in\mathcal{A}} H(m) \leq \liminf_{N\to\infty} \log \frac{1}{N}\overline{\mathbb{Q}}^N(\mathcal{A}) \qquad \text{and} \qquad \limsup_{N\to\infty} \log \frac{1}{N}\overline{\mathbb{Q}}^N(\mathcal{B}) \leq -\inf_{m\in\mathcal{B}} H(m). \quad (1)$$

The rate function is good if it has compact level sets, that is for each $h \in [0, \infty)$ the level set $\{m \in M\ ;\ H(m) \leq h\}$ is compact.

The integral connection of Laplace-Varadhan consists in a pair $(\overline{\mathbb{P}}^N, \overline{\mathbb{Q}}^N)$ of absolutely continuous measures on some metric space (M, d) such that

$$\frac{d\overline{\mathbb{P}}^N}{d\overline{\mathbb{Q}}^N}(u) = \exp\left(NC(u)\right) \qquad \overline{\mathbb{Q}}^N - \text{a.e.} \quad (2)$$

for some measurable mapping $C : M \to \mathbb{R}$. When C is bounded continuous the lemma can be stated as follows: if the sequence $\{\overline{\mathbb{Q}}^N\ ;\ N \geq 1\}$ satisfies a LDP with good rate function H then $\{\overline{\mathbb{P}}^N\ ;\ N \geq 1\}$ satisfies a LDP with good rate function $H - C$. The above sequences are frequently defined in terms of the image measures

$$\overline{\mathbb{P}}^N = \mathbb{P}^N \circ \pi_N^{-1} \quad \text{and} \quad \overline{\mathbb{Q}}^N = \mathbb{Q}^N \circ \pi_N^{-1}$$

[*]CNRS-UMR C55830, Univ. P. Sabatier, 31062 Toulouse, France, delmoral@cict.fr

[†]Lockheed Martin, Eagan, MN, U.S.A.; IMA, Univ. of Minnesota, 207 Church St. S.E., Minneapolis, MN 55455, U.S.A., zajic@ima.umn.edu

for some probabilities $\mathbb{P}^N$ and $\mathbb{Q}^N$ on some measurable space Ω^N which may depend on N and for some measurable mapping $\pi_N : \Omega^N \to M$. Now we start by observing that if $\mathbb{P}^N$ and $\mathbb{Q}^N$ are absolutely continuous and for $\mathbb{Q}^N$-almost every $x \in \Omega^N$

$$\frac{d\mathbb{P}^N}{d\mathbb{Q}^N}(x) = \exp\left(NC(\pi_N(x))\right) \tag{3}$$

then the probability images $\overline{\mathbb{P}}^N$ and $\overline{\mathbb{Q}}^N$ are absolutely continuous and their Radon-Nykodim derivative satisfies (2) and Laplace-Varadhan's lemma applies as soon as C is a bounded continuous mapping. In this paper we propose a strategy to relax the analytic representation (3) of the Radon-Nykodim derivative of $\mathbb{P}^N$ with respect to $\mathbb{Q}^N$. Our approach consists in replacing $\mathbb{P}^N$ and $\mathbb{Q}^N$ by a pair of sequences $\mathbb{P}^N_{\alpha,m}$ and $\mathbb{Q}^N_m$ indexed respectively by a parameter pair (α, m) with $\alpha \in \mathbb{R}$ and $m \in M$ and by a parameter $m \in M$. Instead of (3) we suppose that for any index pair $(\alpha, m) \in (\mathbb{R} \times M)$ we have $\mathbb{P}^N_{\alpha,m} \sim \mathbb{Q}^N_m$ and for $\mathbb{Q}^N_m$-almost every $x \in \Omega^N$

$$\frac{d\mathbb{P}^N_{\alpha,m}}{d\mathbb{Q}^N_m}(x) = \exp\left(N[\alpha S_N(x,m) + C_\alpha(\pi_N(x), m)]\right) \tag{4}$$

for some measurable functions $S_N : \Omega^N \times M \to \mathbb{R}$ and $C_\alpha : M \times M \to \mathbb{R}$. We also assume that $\mathbb{P}^N_{1,m}$ is independent of m and we denote the former by $\mathbb{P}^N_1$. For any $(\alpha, m) \in (\mathbb{R} \times M)$ we define the image measures

$$\overline{\mathbb{P}}^N_{\alpha,m} = \mathbb{P}^N_{\alpha,m} \circ \pi_N^{-1} \quad \text{and} \quad \overline{\mathbb{Q}}^N_m = \mathbb{Q}^N_m \circ \pi_N^{-1}.$$

We are now in position to state our main result.

Integral Lemma *Suppose the sequence of probability measures* $\{\overline{\mathbb{Q}}^N_m\ ;\ N \geq 1\}$ *satisfies an LDP with good rate function* $H_m : M \to [0, \infty]$, *for each* $m \in M$. *Also assume that the mappings* $\{C_\alpha(\,.\,, m)\ ;\ \alpha \in \mathbb{R}\}$ *, are continuous at each* m, $C_\alpha(m, m) = 0$ *and the following exponential moment condition holds*

$$\limsup_{N\to\infty} \frac{1}{N} \log \int_{\Omega^N} \exp nN[S_N(x,m) + C_1(\pi_N(x), m)]\ \ d\mathbb{Q}^N_m(x) < \infty \tag{5}$$

for some $(m, n) \in M \times (1, \infty)$. *Then* $\{\overline{\mathbb{P}}^N_1\ ;\ N \geq 1\}$ *satisfies an LDP with good rate function*

$$I : m \in M \to I(m) = H_m(m) \in [0, \infty]$$

We remark that (5) holds when the mappings $C_1(\,.\,, m)$ and $C_n(\,.\,, m)$ are bounded for some $(m, n) \in M \times (1, \infty)$. To see this, note that

$$\begin{aligned}
& \int_{\Omega^N} \exp nN[S_N(x,m) + C_1(\pi_N(x), m)]\ \ d\mathbb{Q}^N_m(x) \\
= & \int_{\Omega^N} \exp N[nC_1(\pi_N(x), m) - C_n((\pi_N(x), m))]\ \ d\mathbb{P}^N_n(x) \\
= & \int_M \exp N[nC_1(u, m) - C_n((u, m))]\ \ d\overline{\mathbb{P}}^N_n(u) \\
\leq & \ \exp\left(\|C_1 - C_n/n\| N\right).
\end{aligned}$$

Notice that when $S_N = 0$ are the null mappings then we have for any $u \in M$ and $N \geq 1$

$$\frac{d\overline{\mathbb{P}}^N_1}{d\overline{\mathbb{Q}}^N_m}(u) = \exp N[C_1(u, m)]$$

If $C_1(\cdot, m)$ is continuous and (5) holds, Varadhan's integral lemma says that the family of probability measures $\overline{\mathbb{P}}_1^N = \mathbb{P}_1^N \circ \pi_N^{-1}$ satisfies the large deviation principle with rate function $I : M \to [0, \infty]$ given for any $u \in M$ by

$$I(u) = H_m(u) - C_1(u, m)$$

In case $C_1(\cdot, m)$ is continuous and (5) holds for all m, since $C_1(u, u) = 0$ we have that

$$I(u) = H_u(u)$$

Note that under our assumptions we only require the continuity of $C_1(\cdot, m)$ at the point m and that (5) holds for one m. Therefore, even in the case that $S_N = 0$, our result does not follow from Laplace-Varadhan's integral lemma.

To illuminate the structure of the Radon-Nykodim derivative (4) we discuss the different roles played by the two parameters $(m, \alpha) \in M \times \mathbb{R}$. One natural and very useful strategy in many applications of large deviations is to find judicious reference probability measures under which the random sequence at hand satisfies an LDP with a good rate function. The next stage consists in transferring this result to the desired sequence of distributions.
The choice of the reference sequence $\mathbb{Q}_m^N$, $m \in M$, is often dictated by the problem at hand. In the context of interacting particle systems (abbreviate IPS) $\mathbb{Q}_m^N$ is often chosen as an N-fold tensor product measure so that the particle are $\mathbb{Q}_m^N$-independent. In this situation Sanov's theorem tells us that an LDP holds with a good rate function. Using the integral lemma the LDP transfer is guaranteed as soon as we can find a collection of distributions $\mathbb{P}_{\alpha,m}^N$ satisfying (4). Intuitively speaking the pair (α, m) can be regarded as a deformation parameter of the sequence of measures $\mathbb{P}_1^N$. In the context of IPS each $\mathbb{P}_{\alpha,m}^N$ is the distribution law of an N-IPS model with an interaction function depending of the parameters $\alpha \in \mathbb{R}$ and $m \in M$. In some sense (α, m) measures the strength of interaction. For instance, in the forthcoming examples, when $\alpha \to 0$ the particles become independent.

2 Proof of the integral lemma

The following technical proposition states the exponential tightness property and two key estimates needed for proving our result.

Proposition 2.1 *Under the assumptions of the above integral lemma the sequence of probability measures $\overline{\mathbb{P}}_1^N$ on M is exponentially tight. For any Borel subset $A \subset M$ and for any $1/n + 1/n' = 1$, $1 < n, n' < \infty$, and $m \in M$ we have*

$$\overline{\mathbb{P}}_1^N(A) \leq \overline{\mathbb{Q}}_m^N(A)^{1/n'} \; \overline{\mathbb{P}}_{m,n}^N(A)^{1/n} \; \exp[N\delta_n(m, A)] \tag{6}$$

$$\overline{\mathbb{Q}}_m^N(A) \leq \overline{\mathbb{P}}_1^N(A)^{1/n} \; \overline{\mathbb{P}}_{\alpha(n),m}^N(A)^{1/n'} \; \exp[N \; \delta_{\alpha(n)}(m, A)/n] \tag{7}$$

with $\alpha(n) = -n'/n$ and for any $\alpha \neq 0$

$$\delta_\alpha(m, A) = \sup_{u \in A} |C_1(u, m) - C_\alpha(u, m)/\alpha|$$

Since the proof of this proposition is rather technical we have chosen to present first how it is used to prove our main result. By the exponential tightness we already know from Puhalskii [11] that there exists a subsequence along which an LDP holds with good rate function.

Proof of the integral lemma: If we take in (6) the closure of the ball of radius ϵ and center $m \in M$, that is

$$A = \overline{B}(m, \epsilon) = \{u \in M \ : \ d(u, m) \leq \epsilon\}$$

we find that for any conjugate integers $1/n + 1/n' = 1$ with $1 < n, n' < \infty$

$$\overline{\mathbb{P}}_1^N(\overline{B}(m, \epsilon)) \leq \ \overline{\mathbb{Q}}_m^N(\overline{B}(m, \epsilon))^{1/n'} \ \exp\left[N\delta_n(m, \overline{B}(m, \epsilon))\right]$$

Recalling that $\{\overline{\mathbb{Q}}_m^N \ ; \ N \geq 1\}$, satisfies the large deviation principle with a good rate function H_m this implies that

$$\limsup_{N\to\infty} \frac{1}{N} \log \overline{\mathbb{P}}_1^N(\overline{B}(m, \epsilon)) \leq -\frac{1}{n'} \inf_{\overline{B}(m,\epsilon)} H_m + \delta_n(m, \overline{B}(m, \epsilon))$$

Since H_m is a good rate function and $\{\overline{B}(m, \epsilon) \ ; \ \epsilon > 0\}$ is a nested family of closed sets, that is

$$\overline{B}(m, \epsilon) \subset \overline{B}(m, \epsilon') \qquad \text{if} \quad \epsilon < \epsilon'$$

from Lemma 4.1.6., p. 104, in [3] we have

$$H_m(m) = \lim_{\epsilon\to 0} \inf_{\overline{B}(m,\epsilon)} H_m$$

Since each mapping $C_n(., m) : M \to \mathbb{R}$ is continuous at the point m and $C_n(m, m) = 0$ by definition of δ_n we also have that

$$\lim_{\epsilon\to 0} \delta_n(m, \overline{B}(m, \epsilon)) = 0$$

Considering the limit $\epsilon \downarrow 0$, one obtains for any $n' > 1$

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \overline{\mathbb{P}}_1^N(\overline{B}(m, \epsilon)) \leq -\frac{1}{n'} \ H_m(m)$$

Letting $n' \to 1$ we get

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \overline{\mathbb{P}}_1^N(\overline{B}(m, \epsilon)) \leq -I(m) \tag{8}$$

Now if we take in (7) the open ball $A = B(m, \epsilon) = \{u \in M \ : \ d(u, m) < \epsilon\}$ we get

$$\overline{\mathbb{Q}}_m^N(B(m, \epsilon)) \ \leq \ \overline{\mathbb{P}}_1^N(B(m, \epsilon))^{1/n} \ \ \exp\left[N \ \delta_{\alpha(n)}(m, B(m, \epsilon))/n\right]$$

Our assumptions on $\{\overline{\mathbb{Q}}_m^N \ ; \ N \geq 1\}$ imply that

$$-H_m(m) \leq -\inf_{B(m,\epsilon)} H_m \leq \liminf_{N\to\infty} \frac{1}{N} \log \overline{\mathbb{Q}}_m^N(B(m, \epsilon))$$

Arguing as above this implies that

$$-I(m) \leq \liminf_{N\to\infty} \frac{1}{N} \log \overline{\mathbb{Q}}_m^N(B(m, \epsilon)) \leq \frac{1}{n}\left[\liminf_{N\to\infty} \frac{1}{N} \log \overline{\mathbb{P}}_1^N(B(m, \epsilon)) + \delta_{\alpha(n)}(m, B(m, \epsilon))\right]$$

Considering the limit $\epsilon \downarrow 0$, one obtains for any $n > 1$

$$-n \ I(m) \leq \lim_{\epsilon\to 0} \liminf_{N\to\infty} \frac{1}{N} \log \overline{\mathbb{P}}_1^N(B(m, \epsilon))$$

Letting $n \to 1$ we get from (8)

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\overline{\mathbb{P}}_1^N(\overline{B}(m,\epsilon)) \le -I(m) \le \lim_{\epsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\overline{\mathbb{P}}_1^N(B(m,\epsilon))$$

Since

$$\lim_{\epsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\overline{\mathbb{P}}_1^N(B(m,\epsilon)) \le \lim_{\epsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\overline{\mathbb{P}}_1^N(\overline{B}(m,\epsilon))$$

it follows that

$$-I(m) = \lim_{\epsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\overline{\mathbb{P}}_1^N(\overline{B}(m,\epsilon)) = \lim_{\epsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\overline{\mathbb{P}}_1^N(B(m,\epsilon))$$

As noticed in Remark 2.2. in [4] this statement implies that $\overline{\mathbb{P}}_1^N$ satisfies the weak large deviation principle (i.e. with upper bounds only for compact sets). Since the sequence $\overline{\mathbb{P}}_1^N$ is exponentially tight the weak large deviation principle is equivalent to the full large deviation (see for instance Lemma 1.2.18, p. 8 in [3]) and the proof of the lemma is now complete. ∎

We now come to the proof of the technical proposition.

Proof of proposition 2.1: Fixing $n > 1$ so that (5) holds and denoting the left hand side of (5) by nc_n, we have, for N large enough,

$$\begin{aligned} \int\left(\frac{d\mathbb{P}_1^N}{d\mathbb{Q}_m^N}\right)^n d\mathbb{Q}_m^N &= \int_{\Omega^N}\exp nN[S_N(x,m)+C_1(\pi_N(x),m)]\ d\mathbb{Q}_m^N(x) \\ &\le \exp(nc_nN) \end{aligned} \tag{9}$$

Since each probability $\overline{\mathbb{Q}}_m^N$ is a tight measure on M and the sequence $\{\overline{\mathbb{Q}}_m^N\ ;\ N\ge 1\}$ satisfies a full large deviation principle one concludes that $\{\overline{\mathbb{Q}}_m^N\ ;\ N\ge 1\}$ is exponentially tight. For any $a<\infty$ there exists a compact set $K(m,a)\subset M$ such that

$$\limsup_{N\to\infty}\frac{1}{N}\log\overline{\mathbb{Q}}_m^N(K^c(m,a)) < -a \qquad \text{with} \quad K^c(m,a) = M - K(m,a)$$

To prove that $\{\overline{\mathbb{P}}_1^N\ ;\ N\ge 1\}$ is exponentially tight we first notice that

$$\overline{\mathbb{P}}_1^N(K_n^c(m,a)) = \mathbb{P}_1^N(1_{K_n^c(m,a)}(\pi_N(x)))$$

with

$$\frac{1}{n}+\frac{1}{n'} = 1\,, \quad \text{and} \quad K_n^c(m,a) = K^c(m, n'(c_n+a))$$

Thus, using Holder's inequality we check that

$$\begin{aligned} \overline{\mathbb{P}}_1^N(K_n^c(m,a)) &= \mathbb{Q}_m^N\left((1_{K_n^c(m,a)}\circ\pi_N)\ \frac{d\mathbb{P}_1^N}{d\mathbb{Q}_m^N}\right) \\ &\le \overline{\mathbb{Q}}_m^N(K_n^c(m,a))^{1/n'}\ \mathbb{Q}_m^N\left(\left(\frac{d\mathbb{P}_1^N}{d\mathbb{Q}_m^N}\right)^n\right)^{\frac{1}{n}} \\ &\le \overline{\mathbb{Q}}_m^N(K_n^c(m,a))^{1/n'}\ \exp(c_nN) \end{aligned}$$

Recalling (9), the above estimate implies that

$$\limsup_{N\to\infty}\frac{1}{N}\log\overline{\mathbb{P}}_1^N(K_n^c(m,a)) < -\frac{1}{n'}\ [n'(c_n+a)] + c_n = -a$$

This clearly ends the proof of the exponential tightness of the sequence $\{\overline{\mathbb{P}}_1^N \ ; \ N \geq 1\}$. In the same way, for any Borel subset $A \subset M$ and for any $1/n + 1/n' = 1$, $1 < n, n' < \infty$, and $m \in M$ we have

$$\begin{aligned}
\overline{\mathbb{P}}_1^N(A) &= \mathbb{Q}_m^N\left((1_A \circ \pi_N) \, \frac{d\mathbb{P}_1^N}{d\mathbb{Q}_m^N}\right) \\
&\leq \overline{\mathbb{Q}}_m^N(A)^{1/n'} \ \mathbb{Q}_m^N\left((1_A \circ \pi_N) \ \left(\frac{d\mathbb{P}_1^N}{d\mathbb{Q}_m^N}\right)^n\right)^{1/n} \\
&\leq \overline{\mathbb{Q}}_m^N(A)^{1/n'} \ \mathbb{Q}_m^N\left((1_A \circ \pi_N) \exp\left(nN[S_N(x,m) + C_1(\pi_N(x),m)]\right)\right)^{1/n}
\end{aligned}$$

Since we have

$$\begin{aligned}
&\mathbb{Q}_m^N\left((1_A \circ \pi_N) \exp\left(nN[S_N(.,m) + C_1(\pi_N(.),m)]\right)\right) \\
&= \mathbb{Q}_m^N\left((1_A \circ \pi_N) \ \exp\left(N[nS_N(.,m) + C_n(\pi_N(.),m)]\right)\right. \\
&\qquad\qquad \left.\times \ \exp\left(-N[C_n(\pi_N(.),m) + nC_1(\pi_N(.),m)]\right)\right) \\
&= \mathbb{P}_{n,m}^N\left((1_A \circ \pi_N) \ \ \exp\left(N[C_n(\pi_N(.),m) - nC_1(\pi_N(.),m)]\right)\right) \\
&\leq \ \overline{\mathbb{P}}_{n,m}^N(A) \ \exp\left(N \sup_{u \in A} |nC_1(u,m) - C_n(u,m)|\right) = \overline{\mathbb{P}}_{n,m}^N(A) \ \exp\left(nN\delta_n(m,A)\right)
\end{aligned}$$

we find that

$$\overline{\mathbb{P}}_1^N(A) \leq \overline{\mathbb{Q}}_m^N(A)^{1/n'} \ \overline{\mathbb{P}}_{n,m}^N(A)^{1/n} \ \exp\left(N\delta_n(m,A)\right)$$

This establishes (6). To prove (7) we first use the decomposition

$$\begin{aligned}
1_A(\pi_N(x)) = &\left[1_A(\pi_N(x)) \ \exp\left(\frac{N}{n}[S_N(x,m) + C_1(\pi_N(x),m)]\right)\right] \\
&\qquad \times \left[1_A(\pi_N(x)) \ \exp\left(-\frac{N}{n}[S_N(x,m) + C_1(\pi_N(x),m)]\right)\right]
\end{aligned}$$

and Holder's inequality to prove that

$$\begin{aligned}
&\overline{\mathbb{Q}}_m^N(A) \\
&\leq \mathbb{Q}_m^N\left(1_A \circ \pi_N \ \exp\left(N[S_N(.,m) + C_1(\pi_N(.),m)]\right)\right)^{1/n} \\
&\qquad\qquad \times \mathbb{Q}_m^N\left(1_A \circ \pi_N \ \exp\left(-N\frac{n'}{n}[S_N(.,m) + C_1(\pi_N(.),m)]\right)\right)^{1/n'} \qquad (10) \\
&= \overline{\mathbb{P}}_1^N(A)^{1/n} \times \mathbb{Q}_m^N\left(1_A \circ \pi_N \ \exp\left(-N\frac{n'}{n}[S_N(.,m) + C_1(\pi_N(.),m)]\right)\right)^{1/n'}
\end{aligned}$$

Now we observe that

$$\begin{aligned}
&\mathbb{Q}_m^N\left(1_A \circ \pi_N \ \exp\left(-N\frac{n'}{n}[S_N(.,m) + C_1(\pi_N(.),m)]\right)\right) \\
&= \mathbb{Q}_m^N\left(1_A \circ \pi_N \ \exp\left(N\alpha(n)[S_N(.,m) + C_1(\pi_N(.),m)]\right)\right) \\
&= \mathbb{Q}_m^N\left(1_A \circ \pi_N \ \exp\left(N[\alpha(n) \ S_N(.,m) + C_{\alpha(n)}(\pi_N(.),m)]\right)\right. \\
&\qquad\qquad \left.\times \ \exp\left(N[\alpha(n)C_1(\pi_N(.),m) - C_{\alpha(n)}(\pi_N(.),m)]\right)\right) \\
&\leq \overline{\mathbb{P}}_{\alpha(n),m}^N(A) \ \times \ \exp\left(N|\alpha(n)| \ \delta_{\alpha(n)}(m,A)\right)
\end{aligned}$$

We find from (10) that

$$\overline{\mathbb{Q}}_m^N(A) \leq \overline{\mathbb{P}}_1^N(A)^{1/n} \ \ \overline{\mathbb{P}}_{\alpha(n),m}^N(A)^{1/n'} \ \times \ \exp\left(N\delta_{\alpha(n)}(m,A)/n\right)$$

This establishes (7) and the proof of the lemma is complete. ■

3 Large deviations for interacting particle systems

We illustrate the impact of the integral lemma in the context of IPS models. Our general and abstract context is ideally suited to treat in the same frame discrete generation IPS as well as pure jump and McKean-Vlasov IPS diffusions.

We let $I = \{0, 1, \ldots, T\} \subset \mathbb{N}$ or $I = \mathbb{R}_+ = [0, T] \subset \mathbb{R}_+ = [0, \infty)$ be a discrete or continuous time index with a finite time horizon $T \in \mathbb{N}$ or $T \in \mathbb{R}_+$. For E a complete separable metric space we denote $\mathcal{P}(E)$ the set of all probability measures on E furnished with the weak topology. By $D(I, E)$ we denote the set of all càdlàg paths from I into E with the Skorohod metric (when $I = \{0, 1, \ldots, T\}$ the set is simply the product space E^{T+1}). We also denote by $\phi(\eta) = (\eta_t)_{t\in I}$ the distribution flow of the marginals with respect to time of a given measure $\eta \in \mathcal{P}(\mathcal{D}(I, E))$. Finally, we slightly abuse notations and denote by $m(x)$ the empirical measure associated to a point in a given product space $x = (x^1, \ldots, x^N) \in E^N$, that is

$$m(x) = \frac{1}{N}\sum_{i=1}^{N} \ \delta_{x^i}$$

where δ_a stands for the Dirac measure at $a \in E$.

- In the discrete time situation we start with a distribution $\eta_0 \in \mathcal{P}(E)$ and a collection of Markov transitions $K = \{K_{n,\eta} \ ; \ n \in I \ , \ \eta \in \mathcal{P}(E)\}$.
  For any distribution flow $\gamma = (\gamma_n)_{n\in I} \in \mathcal{P}(E)^{T+1}$ we denote $\mathbb{Q}_\gamma$ the measure on E^{T+1} defined by

  $$\mathbb{Q}_\gamma(d(x_0, \ldots, x_T)) = \eta_0(dx_0) \ K_{1,\gamma_0}(x_0, dx_1) \ldots K_{T,\gamma_{T-1}}(x_{T-1}, dx_T)$$

  It is important to notice that the McKean distribution defined by

  $$\mathbb{P}(d(x_0, \ldots, x_T)) = \eta_0(dx_0) \ K_{1,\eta_0}(x_0, dx_1) \ldots K_{T,\eta_{T-1}}(x_{T-1}, dx_T)$$

  and associated to the distribution flow

  $$\eta_n = \eta_{n-1}K_{n,\eta_{n-1}} \quad (= \int \eta_{n-1}(dx)K_{n,\eta_{n-1}}(x, .) \ )$$

  is a fixed point of the mapping $\eta \in \mathcal{P}(D(I, E)) \to \mathbb{Q}_{\phi(\eta)} \in \mathcal{P}(D(I, E))$. As usual, $\mathbb{P}$ can be interpretated as the law of a time inmogeneous Markov process X_n with elementary transitions $K_{n,\eta_{n-1}}$ with $\eta_{n-1} = \mathbb{P} \circ X_{n-1}^{-1}$. More precisely $\mathbb{P}$ is the solution of the following discrete time and time-inhomogeneous martingale problem defined on the canonical space $(\Omega, F, \mathcal{X}) = (D(I, E), (F_t)_{t\geq 0}, (X_t)_{t\in I})$:

  - $\mathbb{P} \circ X_0^{-1} = \eta_0$.
  - For any bounded measurable test function f on E the sequence

    $$M_n(f) = f(X_n) - f(X_0) - \sum_{p=1}^{n} \left[f(X_p) - K_{p,\eta_{p-1}}(f)(X_{p-1})\right]$$

    is a F-martingale under $\mathbb{P}$ and the distribution flow $\{\eta_n \ ; \ n \in I\}$ coincides with the set of time marginals of $\mathbb{P}$, that is $\mathbb{P} \circ X_n^{-1} = \eta_n$.

The N-IPS associated to the collection K and the distribution $\eta_0 \in \mathcal{P}(E)$ is an E^N-valued Markov chain $\xi_n = (\xi_n^1, \ldots, \xi_n^N)$, $n \in I$, with initial distribution $\eta_0^{\otimes N}$ and elementary transitions

$$\mathrm{Prob}(\xi_n \in d(x^1, \ldots, x^N) \mid \xi_{n-1}) = \prod_{i=1}^N K_{n,m(\xi_{n-1})}(\xi_{n-1}^i, dx^i)$$

- In the continuous time situation we start with a distribution $\eta_0 \in \mathcal{P}(E)$ and a collection of generators $L = \{L_{t,\mu} \; ; \; t \in I \; , \; \mu \in \mathcal{P}(E)\}$ defined on some dense domain $D(L)$ in the space of bounded continuous functions. For any distribution flow $\gamma = (\gamma_t)_{t\in I}$ with $\gamma_t \in \mathcal{P}(E)$, $t \in I$, we suppose there exist a solution $\mathbb{Q}_\gamma \in \mathcal{P}(\mathcal{D}(I,E))$ of the non-homogeneous martingale problem associated to $\{L_{t,\gamma_t} \; ; \; t \in I\}$ and starting at η_0. In this framework the McKean measure $\mathbb{P}$ can again be characterized as the fixed point of the mapping $\eta \in \mathcal{P}(D(I,E)) \to \mathbb{Q}_{\phi(\eta)} \in \mathcal{P}(D(I,E))$. More generally $\mathbb{P}$ is defined as the solution of the following time-inhomogeneous martingale problem defined on the canonical space $(\Omega, F, \mathcal{X}) = (D(I,E), (F_t)_{t\geq 0}, (X_t)_{t\geq 0})$:
  - $\mathbb{P} \circ X_0^{-1} = \eta_0$.
  - For any $f \in D(L)$ the process

    $$M_t(f) = f(X_t) - f(X_0) - \int_0^t L_{s,\eta_s}(f)(X_s) \; ds$$

    is a F-martingale under $\mathbb{P}$ and the distribution flow $\{\eta_t \; ; \; t \geq 0\}$ coincides with the set of time marginals of $\mathbb{P}$, that is $\mathbb{P} \circ X_t^{-1} = \eta_t$.

  In this case the marginal distribution flow $(\eta_t)_{t\in I}$ solves the following non linear evolution equation defined for any $f \in D(L)$ by

  $$\frac{d}{dt}\eta_t(f) = \eta_t(L_{t,\eta_t}(f))$$

  The N-interacting particle system associated to L and $\eta_0 \in \mathcal{P}(E)$ is an E^N-valued Markov process $\xi_t = (\xi_t^1, \ldots, \xi_t^N)$ having the initial distribution $\eta_0^{\otimes N}$ with generator

  $$\mathcal{L}_t(f)(x^1, \ldots, x^N) = \sum_{i=1}^N L_{t,m(x)}^{(i)}(f)(x^1, \ldots, x^N)$$

  where $L_{t,\mu}^{(i)}$ is used instead of $L_{t,\mu}$ when it acts on the i-th variable of $f(x^1, \ldots, x^N)$.

In the discrete or continuous time situation it is also known that the empirical distribution $\frac{1}{N}\sum_{i=1}^N \delta_{(\xi_t^i)_{t\in I}}$ on path space $D(I,E)$ converges as $N \to \infty$ to the McKean measure $\mathbb{P}$ in the sense that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N F((\xi_t^i)_{t\in I}) = \int_\Omega F(\omega) \; d\mathbb{P}(\omega)$$

in probability and for any bounded continuous function F on $D(I,E)$.

In the remainder of this section we use the following notation:

$$\Omega = D(I,E) \qquad M = \mathcal{P}(\Omega)$$

Let $\mathbb{P}^N$ be the probability measure induced by the N-IPS process $(\xi_t)_{t\in I}$ on the product path space $\Omega^N = \Omega \times \ldots \times \Omega$. By $\overline{\mathbb{P}}^N = \mathbb{P}^N \circ \pi_N^{-1}$ we denote the image probability measure of the empirical measure on path space with

$$\pi_N : \omega \in \Omega^N \to \pi_N(\omega) = \frac{1}{N}\sum_{i=1}^N \delta_{\omega^i} \in M$$

For each $\eta \in M$ we also denote by $\mathbb{Q}_\eta^N = (\mathbb{Q}_{\phi(\eta)})^{\otimes N}$ the N-fold tensor product of the measure $\mathbb{Q}_{\phi(\eta)}$ and by $\overline{\mathbb{Q}}_\eta^N = \mathbb{Q}_\eta^N \circ \pi_N^{-1}$ the corresponding image measure. For each $\eta \in M$ Sanov's theorem tells us that the sequence $\overline{\mathbb{Q}}_{\phi(\eta)}^N$, $N \geq 1$, satisfies a LDP with good rate function

$$\eta' \in M \to H_\eta(\eta') = \mathrm{Ent}(\eta' \, |\mathbb{Q}_{\phi(\eta)} \,) \in [0,\infty]$$

Here, $\mathrm{Ent}(\eta' \, |\mathbb{Q}_{\phi(\eta)} \,)$ denotes the relative entropy of η' with respect to $\phi(\eta)$. To transfer this LDP to the sequence $\overline{\mathbb{P}}^N$, $N \geq 1$, using the above integral lemma we need to find a collection of measures $\mathbb{P}_{\alpha,m}^N$ of the form (4) with $\overline{\mathbb{P}}^N = \overline{\mathbb{P}}_1^N$. Next we indicate a simple way to construct these distributions. The construction will be notationally complicated but it is a straighforward application of Girsanov's theorem.

There have been many works concerning LDP for interacting particle systems on path space. To motivate our work let us briefly connect our strategy with existing results in the literature on the subject. In the case of jump processes, Feng [5] proved that the collection of measures $\overline{\mathbb{P}}^N$ satisfies the LDP with a good rate function

$$\mathcal{I} : \eta \in M \longrightarrow \mathcal{I}(\eta) = \mathrm{Ent}(\eta|\mathbb{Q}_{\phi(\eta)}) \tag{11}$$

while Feng [6] and Léonard [9] obtain the LDP for the empirical process $(m(\xi_t))_{t\in I}$. In the diffusion case Dawson and Gartner [1] also obtained the LDP for the empirical process. In the discrete time situation Guionnet and one of the authors [2] obtain the LDP on path space when the Markov transitions $K_{n,\mu}$ are such that $K_{n,\mu}(x,.) = K_{n,\mu}(y,.)$ for any $x,y \in E$. Here we present a straightforward approach to obtaining the LDP in path space for discrete, pure jump and diffusion IPS based on a direct application of the integral lemma stated in Section 1. In all situations the good rate function is given by (11).

Discrete time IPS:

Suppose that for each pair of measures $(\mu,\eta) \in \mathcal{P}(E)^2$ and for each $n \in I$, $x \in E$ and $\alpha \in \mathbb{R}$ we have that

$$K_{n,\mu}(x,.) \sim K_{n,\eta}(x,.) \quad \text{and} \quad Z_n^{(\alpha)}(\mu,\eta)(x) = \int [\frac{dK_{n,\mu}(x,.)}{dK_{n,\eta}(x,.)}(y)]^\alpha \; K_{n,\eta}(x,dy) \in (0,\infty)$$

We also suppose that the mappings

$$\mu \in \mathcal{P}(E) \to \int \; \mu(dx) \;\; \log Z_n^{(\alpha)}(\mu,\eta)(x) \tag{12}$$

are bounded and continuous at each $\mu = \eta$. Let η be a fixed distribution on path space Ω $(= E^{T+1})$ and let $\mathbb{P}_{\alpha,\eta}^N$, $\alpha \in \mathbb{R}$, denote the collection of distributions on $\Omega^N = (E^{T+1})^N$ defined by

$$\frac{d\mathbb{P}_{\alpha,\eta}^N}{d\mathbb{Q}_\eta^N}(\omega) = \exp\left(N[\alpha S_N(\omega,\eta) + C_\alpha(\pi_N(\omega),\eta)]\right)$$

for $\mathbb{Q}_\eta^N$-almost every $\omega \in \Omega^N$ with

$$\begin{aligned}
S_N(\omega,\eta) &= \sum_{p=1}^T \int \; m(\omega_{p-1},\omega_p)(d(u,v)) \;\, \log\,[\frac{dK_{p,m(\omega_{p-1})}(u,.)}{dK_{p,\eta_{p-1}}(u,.)}(v)] \\
C_\alpha(\mu,\eta) &= -\sum_{p=1}^T \int \; \mu_{p-1}(du) \;\; \log\,[\int \; \left(\frac{dK_{p,\mu_{p-1}}(u,.)}{dK_{p,\eta_{p-1}}(u,.)}(v)\right)^\alpha \;\; K_{p,\eta_{p-1}}(u,dv)\;]
\end{aligned}$$

Under $\mathbb{P}^N_{\alpha,\eta}$ the N-IPS model $(\xi_n)_{n\in I}$ is the N-IPS model associated to the collection of Markov transitions $K^{(\alpha)} = \{K^{(\alpha)}_{n,\mu}\ ;\ n\in I\ ,\ \mu\in\mathcal{P}(E)\}$ on E defined by

$$K^{(\alpha)}_{n,\mu}(x,dy) = \frac{1}{Z^\alpha_n(\mu,\eta_{n-1})(x)}\ \ [\frac{dK_{n,\mu}(x,\,.)}{dK_{n,\eta_{n-1}}(x,\,.)}(y)]^\alpha\ \ K_{n,\eta_{n-1}}(x,dy)$$

As mentionned above the parameter α clearly measures the degree of interaction in the system. For instance, for $\alpha=0$ we have $\mathbb{P}^N_{0,\eta} = \mathbb{Q}^N_\eta = (\mathbb{Q}_{\phi(\eta)})^{\otimes N}$ and under $\mathbb{P}^N_{0,\eta}$ the N-IPS consists in N independent particles with elementary transitions $K_{n,\eta_{n-1}}$. To illustrate this observation we examine the case of Gaussian mean field transitions on $E=\mathbb{R}$ defined by

$$K_{n,\mu}(x,dy) = \frac{1}{\sqrt{2\pi}}\ \exp -\frac{1}{2}\,(y-b[x,\mu])^2\ \ dy$$

for some measurable drift function $b: E\times\mathcal{P}(E)\rightarrow E$. In this special case we can check that for any $(x,\mu)\in E\times\mathcal{P}(E)$

$$K^{(\alpha)}_{n,\mu}(x,dy) = \frac{1}{\sqrt{2\pi}}\ \exp[-\frac{1}{2}\,(y-(b[x,\eta_{n-1}]\ +\ \alpha\ (b[x,\mu]-b[x,\eta_{n-1}])\ )^2]\ \ dy$$

In addition we can also check that for any $\mu,\eta\in M\ (=\mathcal{P}(E^{T+1}))$

$$C_\alpha(\mu,\eta) = \frac{\alpha(1-\alpha)}{2}\sum_{p=0}^{n-1}\int \mu_p(dx)\ (b[x,\eta_p]-b[x,\mu_p])^2$$

We note that the technical assumption (12) is satisfied if and only if the drift function b is chosen so that the mappings

$$\mu\in\mathcal{P}(E)\rightarrow\int \mu(dx)\ (b[x,\eta]-b[x,\mu])^2$$

are bounded and continuous at each $\mu=\eta$. To connect this example with the McKean-Vlasov diffusion model examined hereafter let us suppose the drift function $b[x,\mu]$ takes the form

$$b[x,\mu] = \int \mu(dy)\ b(x,y)$$

for some measurable function $b(x,y)$ on the product space E^2. In this situation we can check that

$$\int \mu(dx)\ (b[x,\eta]-b[x,\mu])^2 = \mu\otimes[(\mu\otimes\mu)-2(\mu\otimes\eta)+(\eta\otimes\eta)](B)$$

with $B(x,y,z)=b(x,y)b(x,z)$. The desired regularity property is therefore met as soon as $b(x,y)$ is bounded continuous.

Continuous time IPS:

As announced we consider two traditional situations.

1. *Pure jump generator*:
   In the first situation we suppose that $L=\{L_\mu\ ;\ \mu\in\mathcal{P}(E)\}$ is a collection of bounded generators and $D(L)=\mathcal{B}_b(E)$ is the set of bounded measurable functions. If $L^+_\mu(x,\,.)$ and $L^-_\mu(x,\,.)$ is a Jordan's decomposition of the signed kernel $L_\mu(x,\,.)$ then we have $L^+_\mu(x,E)=L^-_\mu(x,E)$ and

$$L_\mu(x,dy) = \lambda_\mu(x)\ [Q_\mu(x,dy)-\delta_x(dy)]$$

$$\text{with}\qquad \lambda_\mu(x)=L^+_\mu(x,E)\quad\text{and}\quad Q_\mu(x,dy)=L^+_\mu(x,dy)/L^+_\mu(x,E).$$

In this situation we assume that for any μ, ν and u we have $L^+_\mu(u,.) \sim L^+_\nu(u,.)$ with $dL^+_\mu(u,.)/dL^+_\nu(u,.)(v) \geq \epsilon > 0$. We also assume there exists some countable subset $\mathcal{H} \subset \mathcal{C}_b(E)$ with

$$\left|1 - dL^+_\nu(u,.)/dL^+_\mu(u,.)(v)\right| \leq \sum_{h\in\mathcal{H}} |\mu(h) - \nu(h)| \qquad \text{and} \qquad |\mathcal{H}| = \sum_{h\in\mathcal{H}} \|h\| < \infty$$

2. *Diffusion generator*:
   In this situation we suppose that $E = \mathbb{R}^d$, $d \geq 1$. If $b : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded continuous function then for each $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\varphi \in D(L) = \mathcal{C}^2_b(\mathbb{R}^d)$ we set

$$L_\mu(\varphi)(x) = \frac{1}{2}\sum_{i=1}^d \frac{\partial^2\varphi}{\partial x_i^2}(x) + \sum_{i=1}^d b_i[x,\mu]\,\frac{\partial\varphi}{\partial x_i}(x) \qquad \text{with} \quad b_i[x,\mu] = \int_{\mathbb{R}^d} \mu(dy)\; b_i(x,y)$$

In both situations it is known that there exists a unique McKean measure $\mathbb{P}$ (see for instance [7] in the jump case and [12] for diffusions).

1. In the first situation, with $\eta \in M$ fixed, Girsanov's theorem implies that $\mathbb{P}^N \sim \mathbb{Q}^N_\eta$. In addition for $\mathbb{Q}^N_\eta$-almost every $\omega \in \Omega^N$ we have that

$$\frac{d\mathbb{P}^N}{d\mathbb{Q}^N_\eta}(\omega) \quad = \quad \exp N[S_N(\omega,\eta) + C_1(\pi_N(\omega),\eta)]$$

with the mappings $S_N : \Omega^N \times M \to \mathbb{R}$ and $C_1 : M \times M \to \mathbb{R}$ defined by

$$\begin{aligned} S_N(\omega,\eta) &= \sum_{s\leq T} \int_{E\times E} m(\omega_{s-},\omega_s)(d(u,v))\; \log \frac{dL^+_{m(\omega_{s-})}(u,.)}{dL^+_{\eta_{s-}}(u,.)}(v)\; 1_{u\neq v} \\ C_1(\mu,\eta) &= -\int_0^T \mu_{s-}\left(L^+_{\mu_{s-}} - L^+_{\eta_{s-}}\right)(1)\; ds \end{aligned}$$

In the same way, for any bounded Borel function $f \in \mathcal{B}_b(I \times \Omega^N \times \Omega^N)$ the process defined for any $\omega \in \Omega^N$ by

$$\mathcal{E}_t(f)(\omega) = \exp\left[\sum_{\substack{s\leq t \\ \omega_s\neq\omega_{s-}}} f(s,\omega_{s-},\omega_s) - \int_0^t \int_{E^N} \left(e^{f(s,\omega_{s-},x)} - 1\right)\; \mathcal{L}^+_{\phi(\eta)}(\omega_{s-},dx)\; ds\right]$$

with, for any $\varphi \in \mathcal{B}_b(\Omega^N)$,

$$\mathcal{L}^+_{\phi(\eta)}(\varphi)(\omega_{s-}) = \sum_{i=1}^N \int_E L^+_{\eta_{s-}}(x^i,du)\; \varphi(\theta^i(\omega_{s-},u))\;,$$

is a local $\mathbb{Q}_{\phi(\eta)}$-martingale. Here, for $1 \leq i \leq N$, $u \in E$ and $x = (x_1,\cdots,x_N) \in E^N$, $\theta^i(x,u) = (\theta^i_j(x,u))_{1\leq j\leq N}$ is the element of E^N given by

$$\theta^i_j(x,u) \quad = \quad \begin{cases} x_j & ,\ \text{if } j \neq i \\ u & ,\ \text{if } j = i \end{cases}$$

Applying this result to the collection of functions

$$f(s,x,y) = \alpha \sum_{i=1}^N 1_{y^i\neq x^i}\; \log \frac{dL^+_{m(x)}(x^i,.)}{dL^+_{\eta_{s-}}(x^i,.)}(y^i)\;, \qquad \alpha \in \mathbb{R}$$

we define a family of probability measures $\mathbb{P}^N_{\alpha,\eta}$, with $\alpha \in \mathbb{R}$, by setting

$$\frac{d\mathbb{P}^N_{\alpha,\eta}}{d\mathbb{Q}^N_\eta}(\omega) = \exp N[\alpha S_N(\omega,\eta) + C_\alpha(\pi_N(\omega),\eta)]$$

with the mappings $C_\alpha : M \times M \to \mathbb{R}$ defined by

$$C_\alpha(\mu,\eta) = -\int_0^T \int_{E\times E} \mu_{s-}(du) L^+_{\eta_{s-}}(u,dv) \left(\left[\frac{dL^+_{\mu_{s-}}(u,.)}{dL^+_{\eta_{s-}}(u,.)}(v) \right]^\alpha - 1 \right) ds$$

Note that under $\mathbb{P}^N_{\alpha,\eta}$ the N-IPS model $(\xi_t)_{t\in I}$ becomes the N-IPS model associated to a signed kernel $L^{(\alpha)}_{t,\mu}$ with Jordan decomposition positive part

$$L^{(\alpha)\,+}_{t,\mu}(x,dy) = \left(\frac{dL^+_\mu(x,.)}{dL^+_{\eta_{t-}}(x,.)}(y) \right)^\alpha \; L^+_{\eta_{t-}}(x,dy)$$

In view of the integral lemma and Sanov's theorem [3, Theorem 6.2.10], it remains to prove that for each $m \in M$ and $\alpha \in \mathbb{R}$ the mapping $C_\alpha(.,m)$ is bounded and continuous on M. First we note that for any $\epsilon \leq x \leq 1/\epsilon$ and $\alpha \in \mathbb{R}$

$$|1 - x^\alpha| = |1 - \exp(\alpha \log x)| \leq |\alpha| \; |\log x| \; |1 + x^\alpha|$$

and therefore

$$|1 - x^\alpha| \leq \theta(\alpha,\epsilon) \; |1 - x| \qquad \text{with} \quad \theta(\alpha,\epsilon) = |\alpha| \; \left(\epsilon^{-1} + \epsilon^{-(1+|\alpha|)}\right) (\leq 2|\alpha|/\epsilon^{1+|\alpha|})$$

Hence, under our assumptions we obtain

$$\begin{aligned}
|C_\alpha(\mu,\eta)| &\leq \theta(\alpha,\epsilon) \int_0^T \int_{E\times E} \mu_{s-}(du) L^+_{\eta_{s-}}(u,dv) \left| \frac{dL^+_{\mu_{s-}}(u,.)}{dL^+_{\eta_{s-}}(u,.)}(v) - 1 \right| ds \\
&\leq \theta(\alpha,\epsilon) \sum_{h\in\mathcal{H}} \int_0^T [\mu_{s-} L^+_{\eta_{s-}}(1)] \; |\mu_{s-}(h) - \eta_{s-}(h)| \; ds
\end{aligned}$$

and finally

$$|C_\alpha(\mu,\eta)| \leq \theta_\eta(\alpha,\epsilon) \sum_{h\in\mathcal{H}} \int_0^T |\mu_t(h) - \eta_t(h)| \; ds$$

with $\theta_\eta(\alpha,\epsilon) = \theta(\alpha,\epsilon)\; [\sup_{0\leq t\leq T} \|\lambda_{\eta_t}\|] < \infty$. On the other hand noting that

$$\int_0^T |\mu_t(h) - \eta_t(h)|^2 \; dt = \int_0^T \mu_t(h)^2 \; dt + \int_0^T \eta_t(h)^2 \; dt - 2\int_0^T \mu_t(h)\eta_t(h) \; dt$$

and for any $h \in \mathcal{C}_b(E)$ and $\mu, \eta \in M$

$$\int_0^T \mu_t(h)\eta_t(h) \; dt \;=\; \int_0^T \int_{\Omega_T\times\Omega_T} h(\omega_t) \; h(\omega'_t) \; \mu(d\omega)\eta(d\omega') \; dt = (\mu\otimes\eta)(H_h)$$

with

$$H_h(\omega,\omega') = \int_0^T (h\otimes h)(\omega_t,\omega'_t) \; dt = \int_0^T h(\omega_t)h(\omega'_t) \; dt$$

we have

$$C^2_\alpha(\mu,\eta) \leq \; [\theta^2_\eta(\alpha,\epsilon) \; |\mathcal{H}|] \sum_{h\in\mathcal{H}} \left((\mu\otimes\mu) + (\eta\otimes\eta) - 2(\mu\otimes\eta)\right)(H_h)$$

The space $\Omega = D([0,T],E)$ with the Skorohod metric is topologically complete and separable. Thus, by Lemma 1.1 in [10] if μ^n and η^n weakly converge to μ and η then $(\mu^n \otimes \eta^n)$ weakly converge to $(\mu \otimes \eta)$. For each continuous mapping h the integral function

$$\omega \in \Omega \to \int_0^T h(\omega_t)\ dt$$

is continuous for the Skorohod metric, see for instance Example 8.2, p. 32 in [8]. Since $\mathcal{H} \subset \mathcal{C}_b(E)$ we conclude that $H_h \in \mathcal{C}_b(\Omega_T)$ for any $h \in \mathcal{H}$ and

$$\mu^n \text{ weakly converge to } \mu \Longrightarrow \lim_{n\to\infty} C_\alpha(\mu^n,\eta) = C_\alpha(\mu,\eta).$$

2. Now we examine mean field diffusions. We first notice that for any $\eta \in M$ and $\omega \in \Omega$ the stochastic process $(B_t^\eta)_{t\in I}$ defined by

$$t \in I \to B_t^\eta(\omega) = \omega_t - \omega_0 - \int_0^t b[\omega_t,\eta_t]\ ds$$

is an E-valued Brownian motion with respect to $\mathbb{Q}_{\phi(\eta)}$. By Girsanov's theorem we have $\mathbb{P}^N \sim \mathbb{Q}_\eta^N$ and for $\mathbb{Q}_\eta^N$-almost every $\omega = (\omega^1,\ldots,\omega^N) \in \Omega^N$

$$\begin{aligned}
\frac{d\mathbb{P}^N}{d\mathbb{Q}_\eta^N}(\omega) &= \sum_{i=1}^N \int_0^T (b[\omega_t^i, m(\omega_t)] - b[\omega_t^i,\eta_t])^t\ \ dB_t^\eta(\omega_i) \\
&\qquad -\frac{1}{2}\sum_{i=1}^N \int_0^T \left\|b[\omega_t^i, m(\omega_t)] - b[\omega_t^i,\eta_t]\right\|^2\ ds \\
&= N[S_N(\omega,\eta) + C_1(\pi_N(\omega),\eta)]
\end{aligned}$$

with for any $\omega \in \Omega^N$ and $\mu,\eta \in M$

$$\begin{aligned}
S_N(\omega,\eta) &= \int_0^T \int m(\omega_t)(du) m(\omega_t)(dv)\ (b(u,v) - b[u,\eta_t])\ dB_t^\eta(u) \\
C_1(\mu,\eta) &= -\frac{1}{2}\int_0^T \int \mu_t(du)\ \|b[u,\mu_t] - b[u,\eta_t]\|^2\ ds
\end{aligned}$$

Here, the superscript t denotes transpose. Arguing as in the pure jump situation we define a family of probability measures $\mathbb{P}_{\alpha,\eta}^N$, with $\alpha \in \mathbb{R}$, by setting

$$\frac{d\mathbb{P}_{\alpha,\eta}^N}{d\mathbb{Q}_\eta^N}(\omega) = \exp N[\alpha S_N(\omega,\eta) + C_\alpha(\pi_N(\omega),\eta)] \qquad \text{with} \qquad C_\alpha = \alpha^2\ C_1$$

Note that under $\mathbb{P}_{\alpha,\eta}^N$ the N-IPS model $(\xi_t)_{t\in I}$ is the N-IPS model associated to the time inhomogeneous generators $L_{t,\mu}^{(\alpha)}$ defined for any $t \in I$ and $\varphi \in D(L) = \mathcal{C}_b^2(\mathbb{R}^d)$ by

$$L_{t,\mu}^{(\alpha)}(\varphi)(x) = L_{\eta_t}(\varphi)(x) + \alpha \sum_{i=1}^d (b_i[x,\mu] - b_i[x,\eta_t])$$

For any pair $(\mu,\eta) \in \mathcal{P}(E)^2$ we have that

$$\int \mu(du)\ \|b[u,\mu] - b[u,\eta])\|^2 \ = \ \sum_{i=1}^d \int \mu(du)\ [\mu(b_i(u,.)) - \eta(b_i(u,.))]^2 \tag{13}$$

Noting that

$$[\mu(b_i(u,.)) - \eta(b_i(u,.))]^2 = \mu(b_i(u,.))^2 + \eta(b_i(u,.))^2 - 2\mu(b_i(u,.))\eta(b_i(u,.))$$

and

$$\begin{aligned}\mu(b_i(u,.))\eta(b_i(u,.)) &= \int(\mu\otimes\eta)(dv,dv')b_i(u,v)b_i(u,v') \\ &= (\mu\otimes\eta)(b_i(u,.)\otimes b_i(u,.))\end{aligned}$$

we can rewrite (13) as

$$\int \mu(du)\ \|b[u,\mu] - b[u,\eta]\|^2$$

$$= \sum_{i=1}^{d} \int \mu(du)[(\mu\otimes\mu) + (\eta\otimes\eta) - 2(\mu\otimes\eta)](b_i(u,.)\otimes b_i(u,.))$$

$$= \mu\otimes[(\mu\otimes\mu) + (\eta\otimes\eta) - 2(\mu\otimes\eta)](B)$$

with $B \in \mathcal{C}_b(E^3)$ given by

$$B(u,v,v') = \sum_{i=1}^{d} b_i(u,v)\ b_i(u,v')$$

This implies that for any pair $(\mu,\eta) \in M^2$

$$\begin{aligned}C_\alpha(\mu,\eta) &= -\frac{\alpha^2}{2}\int_0^T \mu_t\otimes[(\mu_t\otimes\mu_t) + (\eta_t\otimes\eta_t) - 2(\mu_t\otimes\eta_t)](B)ds \\ &= -\frac{\alpha^2}{2}\ \mu\otimes[(\mu\otimes\mu) + (\eta\otimes\eta) - 2(\mu\otimes\eta)](\mathcal{B})\end{aligned}$$

with $\mathcal{B} \in \mathcal{C}_b(\Omega^3)$ given by

$$\mathcal{B}(\omega,\omega',\omega'') = \int_0^T\ B(\omega_t,\omega'_t,\omega''_t)\ dt$$

Arguing as in jump case the end of the proof is clear.

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