Exact and Efficient Simulation of Correlated Defaults

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Corporate defaults cluster

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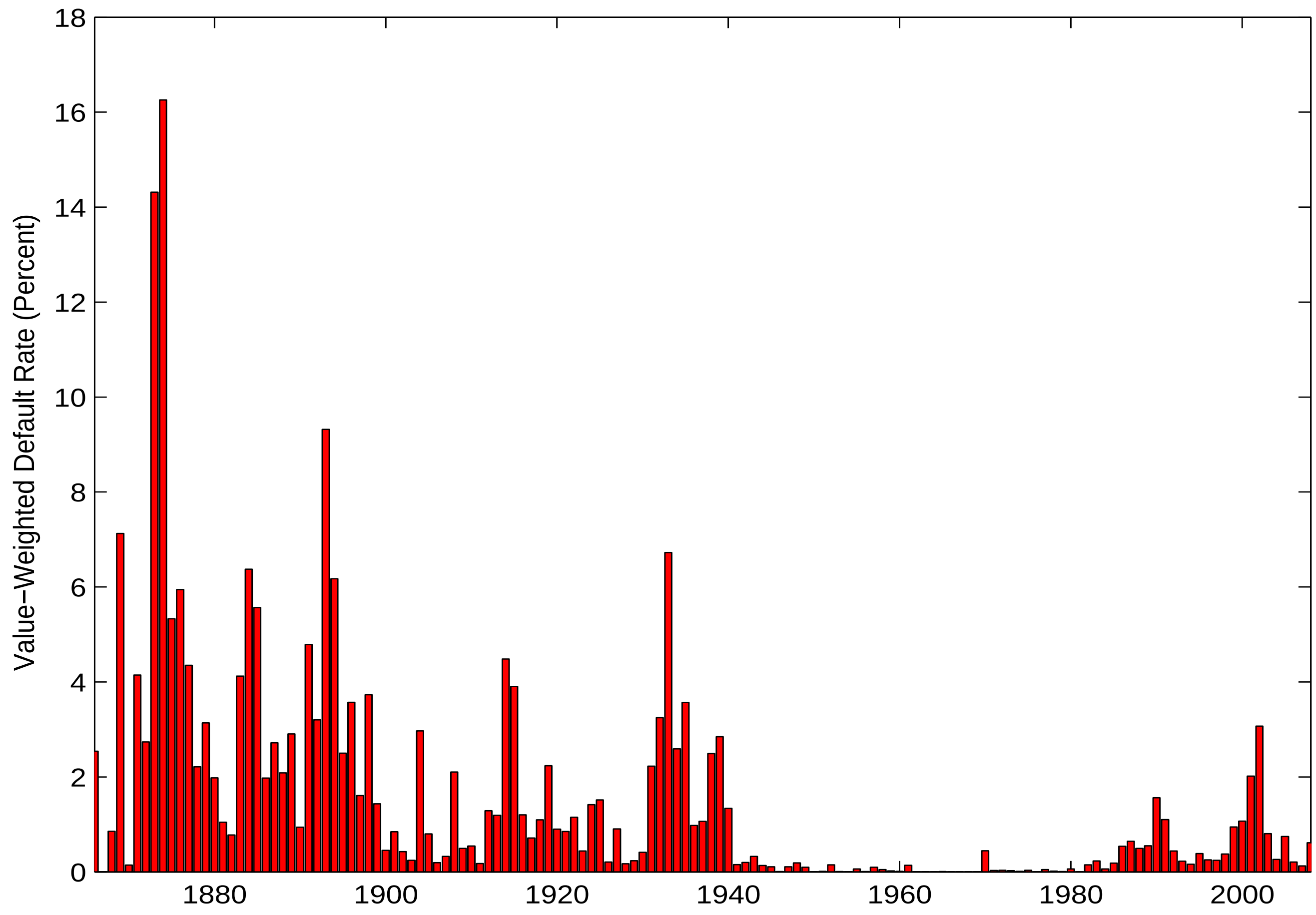


Correlated default risk

Important applications

- Risk management of credit portfolios
  - Prediction of correlated defaults and losses
  - Portfolio risk measures: VaR etc.
- Optimization of credit portfolios
- Risk analysis, valuation, and hedging of portfolio credit derivatives
  - Collateralized debt obligations (CDOs)

Default timing

- Consider a portfolio of n defaultable assets
  - Default stopping times τ^i relative to $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{F}$
  - Default indicators $N_t^i = I(\tau^i \leq t)$
  - Vector of default indicators $N = (N^1, \ldots, N^n)$
- The **portfolio default process** $1_n \cdot N$ counts defaults
  - At the center of many applications

Bottom-up model of default timing

- Name i defaults at intensity λ^i
  - A martingale is given by $N^i - \int_0^{\cdot}(1 - N_s^i)\lambda_s^i ds$
  - λ^i represents the conditional default rate: for small $\Delta > 0$

$$\lambda_t^i \Delta \approx \mathbb{P}(i \text{ defaults during } (t, t + \Delta] \,|\, \mathcal{F}_t)$$

- The vector process $\lambda = (\lambda^1, \ldots, \lambda^n)$ is the modeling primitive
  - Component processes are correlated: diffusion, common or correlated or feedback jumps
  - Large literature

Model computation

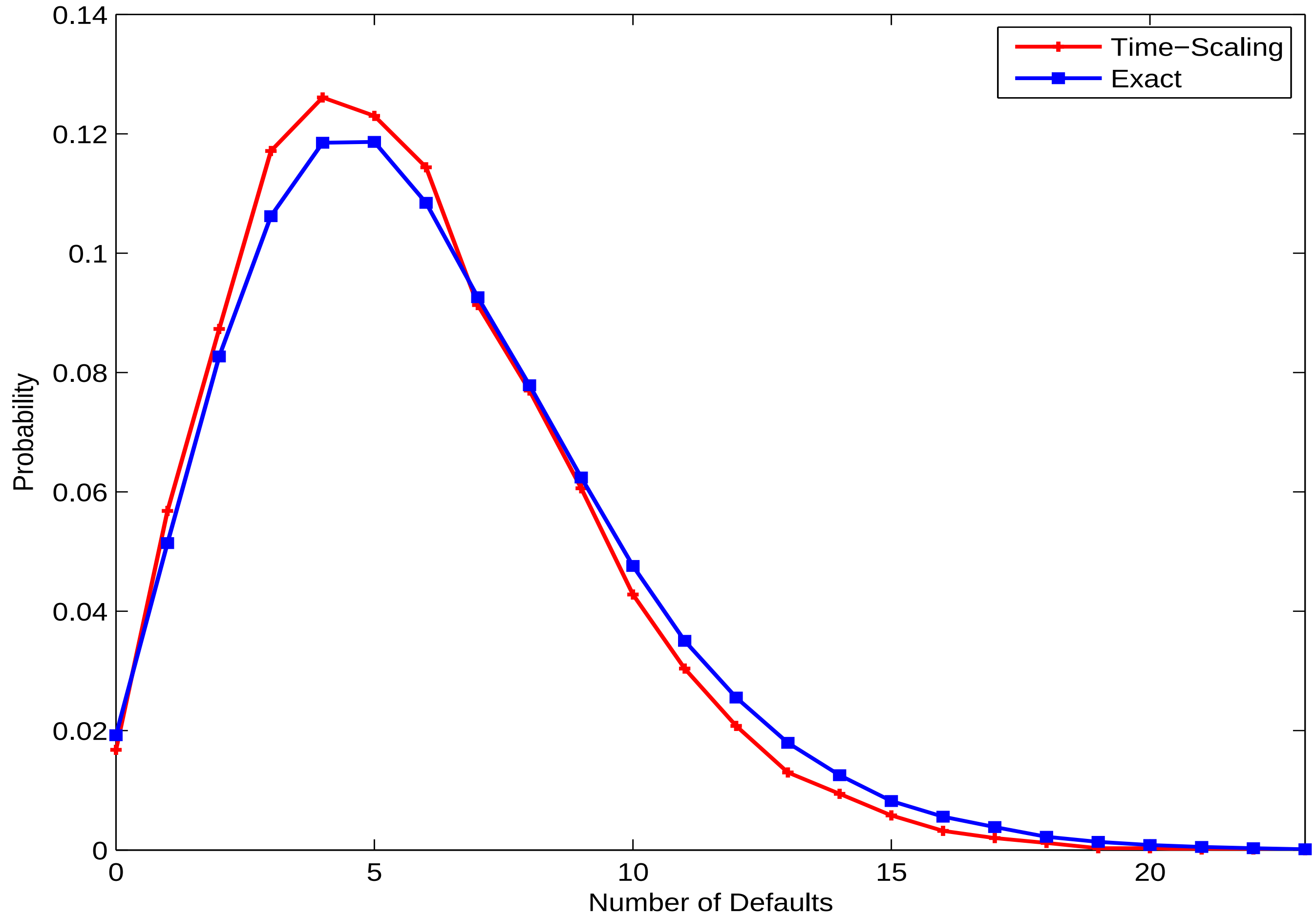
- We require $\mathbb{E}(f(N_T))$ for $T > 0$ and real-valued f on $\{0,1\}^n$
  - $\mathbb{P}(1_n \cdot N_T = k)$
  - $\mathbb{P}(\tau^i > t)$ for constituents i
- Semi-analytical transform techniques
  - Limited to (one-) factor doubly-stochastic intensity models
- Monte Carlo simulation
  - Larger class of intensity models
  - Treatment of more complex instruments such as cash CDOs

Simulation by time-scaling

- Widely used
  - τ^i has the same distribution as $\inf\{t : \int_0^t \lambda_s^i ds = \mathsf{Exp}(1)\}$
  - In practice: approximate λ^i on discrete-time grid, integrate, and record the hitting time of the integrated process
- Potential problems
  - Discretization may introduce **bias**
    * Magnitude?
    * Computational effort
    * Allocation of resources
  - Can be computationally burdensome (often $n \geq 100$)

Time-scaling vs. exact methods

Distribution of $1_{100} \cdot N_2$



Exact and efficient simulation

- Our approach has two parts
  1. Construct a time-inhomogeneous, continuous-time Markov chain $M \in \{0,1\}^n$ with the property that $M_t = N_t$ in law
  2. Estimate $\mathbb{E}(f(N_T)) = \mathbb{E}(f(M_T))$ by simulating M
     - **Exact**: avoids intensity discretization
     - **Efficient**: adaptive variance reduction scheme
- Powerful simulation engine applicable to many intensity models in the literature

Multivariate Markovian projection

Proposition

- Let M be a Markov chain that takes values in $\{0,1\}^n$, starts at 0_n, has no joint transitions in any of its components and whose ith component has transition rate $h^i(\cdot, M)$ where

$$h^i(t,B) = \mathbb{E}(\lambda_t^i I(\tau^i > t) \,|\, N_t = B), \quad B \in \{0,1\}^n$$

  Then $M_t = N_t$ in distribution:

$$\mathbb{P}(M_t = B) = \mathbb{P}(N_t = B), \quad B \in \{0,1\}^n$$

- Related univariate results in Brémaud (1980), Arnsdorf & Halperin (2007), Cont & Minca (2008), Lopatin & Misirpashaev (2007)

Markov counting process

- M is a Markov point process in its own filtration $\mathbb{G}$
- The Markov counting process $1_n \cdot M$ has $\mathbb{G}$-intensity

$$1_n \cdot h(t, M_t) = \sum_{k=0}^{n-1} H(t,k) I(T_k \leq t < T_{k+1})$$

where $h = (h^1, \ldots, h^n)$, and (T_k) is the strictly increasing sequence of event times of $1_n \cdot M$, and

$$H(t,k) = 1_n \cdot h(t, M_{T_k}), \quad t \geq T_k$$

- Compare: original portfolio default process $1_n \cdot N$ has $\mathbb{F}$-intensity

$$\sum_{i=1}^{n} \lambda_t^i I(\tau^i > t)$$

Markov counting process

- The $\mathbb{G}$ inter-arrival intensities $H(t,k)$ of the Markov counting process $1_n \cdot M$ are deterministic
- Exact simulation of arrival times of $1_n \cdot M$
  - Time-scaling method based on $H(t,k)$
  - Equivalently, inverse method based on

$$\mathbb{P}(T_{k+1} - T_k > s \,|\, \mathcal{G}_{T_k}) = \exp\Big(\int_{T_k}^{T_k+s} H(t,k)dt \Big)$$

  - **Sequential acceptance/rejection** based on $H(t,k)$
- Exact simulation of the component $I_k \in \{1, 2, \ldots, n\}$ of M in which the kth transition took place:

$$\mathbb{P}(I_k = i \,|\, \mathcal{G}_{T_k-}) = \frac{h^i(T_k, M_{T_{k-1}})}{H(T_k, k-1)}$$

Variance reduction

- Interested in $\mathbb{P}(1_n \cdot N_T = k)$ for large k
  - Need to force mimicking chain M into rare-event regime
- **Selection/mutation scheme**
  - Evolve R copies (V_p^r) of M over grid $p = 0, 1, \ldots, m$ under $\mathbb{P}$
  - At each p, select R particles by sampling with replacement

$$\mathbb{P}(\text{particle } r \text{ selected}) = \frac{1}{R\eta_p} \exp\left[\delta 1_n \cdot (V_p^r - V_{p-1}^r)\right]$$

where $\eta_p = \frac{1}{R}\sum_{r=1}^R \exp[\delta 1_n \cdot (V_p^r - V_{p-1}^r)]$ and $\delta > 0$

  - Final estimator of $\mathbb{P}(1_n \cdot N_T = k)$ corrects for selections

$$\frac{\eta_0 \cdots \eta_{m-1}}{R} \sum_{r=1}^R I(1_n \cdot V_m^r = k) \exp\left(-\delta 1_n \cdot V_m^r\right)$$

Selection/mutation scheme

- The selection mechanism **adaptively** forces the mimicking Markov chain M into the rare-event regime
  - Del Moral & Garnier (2005, AAP)
  - Carmona & Crepey (2009, IJTAF)
  - Carmona, Fouque & Vestal (2009, FS)
  - Twisting of Feynman-Kac path measures
  - Well-suited to deal with different model specifications
- Mutations are generated under the reference measure $\mathbb{P}$ via the exact A/R scheme
- Estimators are **unbiased**
- Choice of R, m and δ

Calculating the projection

- Need $h^i(t,B) = \mathbb{E}(\lambda_t^i I(\tau^i > t) \,|\, N_t = B)$ for given $(\lambda^1, \ldots, \lambda^n)$
- We show how to calculate $h^i(t,B)$ for a range of
  - Multi-factor doubly-stochastic models $\lambda_t^i = X_t^i + \alpha^i \cdot Y_t$
  - Multi-factor frailty models $\lambda_t^i = X_t^i + \mathbb{E}(\alpha^i \cdot Y_t \,|\, \mathcal{F}_t)$
  - Self-exciting models $\lambda_t^i = X_t^i + c^i(t, N_t)$

  in terms of the transform

$$\phi(t,u,z,Z) = \mathbb{E}\left[\exp\left(-u\int_0^t Z_s ds - zZ_t\right)\right], \quad Z \in \{X^i, Y\}$$

- This extends the reach of our exact method to most models in the literature, and beyond

Numerical results

Self-exciting intensity model for $n = 100$

- Suppose the intensities $\lambda^i = X^i + \sum_{j \neq i}^n \beta^{ij} N^j$
  - Extends Jarrow & Yu (2001), Kusuoka (1999), Yu (2007)
  - Feedback specification can be varied
  - Analytical solutions not known
- Suppose the idiosyncratic factor follows the Feller diffusion

$$dX_t^i = \kappa_i(\theta_i - X_t^i)dt + \sigma_i \sqrt{X_t^i} dW_t^i$$

  where $(W^1, \ldots, W^n)$ is a standard Brownian motion

- Parameters selected randomly (relatively high credit quality)

Numerical results

Projection for self-exciting intensity model

- The projected intensity is given by

$$\begin{aligned} h^i(t,B) &= \mathbb{E}(\lambda_t^i I(\tau^i > t) \,|\, N_t = B) \\ &= (1 - B^i)\Big\{ -\frac{\partial_z \phi(t,1,z,X^i)|_{z=0}}{\phi(t,1,0,X^i)} + \sum_{j\neq i}^{n} \beta^{ij} B^j \Big\} \end{aligned}$$

- The transform $\phi(t,u,z,X^i)$ is in closed form, and so is $h^i(t,B)$
  - Can add compound Poisson jumps without reducing tractability
  - General affine jump diffusion dynamics

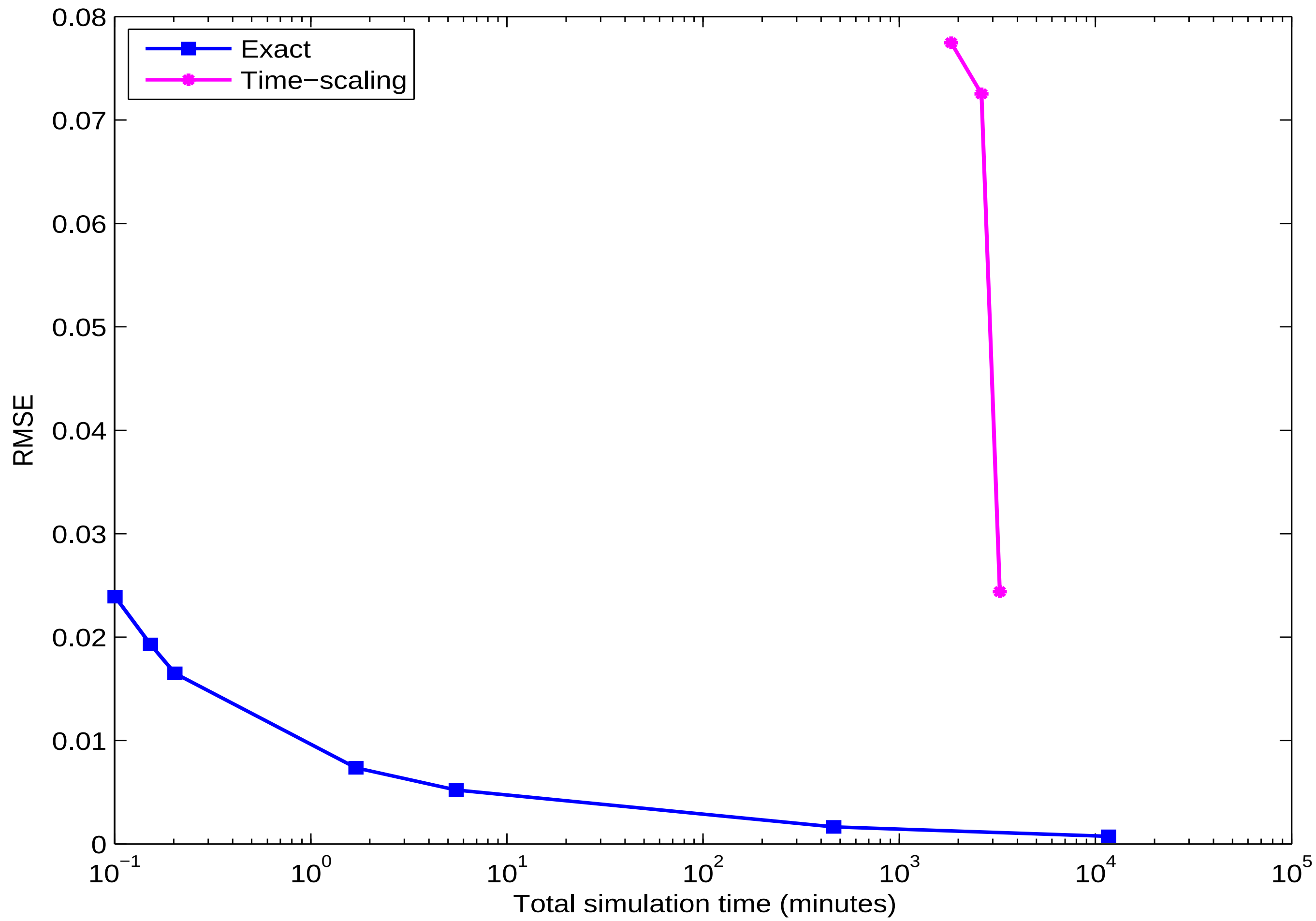
Numerical results

Simulation results for $\mathbb{E}((C_1 - 3)^+)$ where $C_1 = N_1 \cdot 1_n$

Method	Trials	Steps	Bias	SE	RMSE	Time
Exact	5,000	N/A	0	0.0239	0.0239	0.10 min
	7,500	N/A	0	0.0193	0.0193	0.15
	10,000	N/A	0	0.0165	0.0165	0.20
	50,000	N/A	0	0.0073	0.0073	1.69
	100,000	N/A	0	0.0052	0.0052	5.51
	1,000,000	N/A	0	0.0016	0.0016	463.78
Time	5,000	71	0.0735	0.0246	0.0775	1842.15
Scaling	7,500	87	0.0697	0.0199	0.0725	2628.33
	10,000	100	0.0174	0.0171	0.0244	3255.12

Numerical results

Convergence of RMS errors



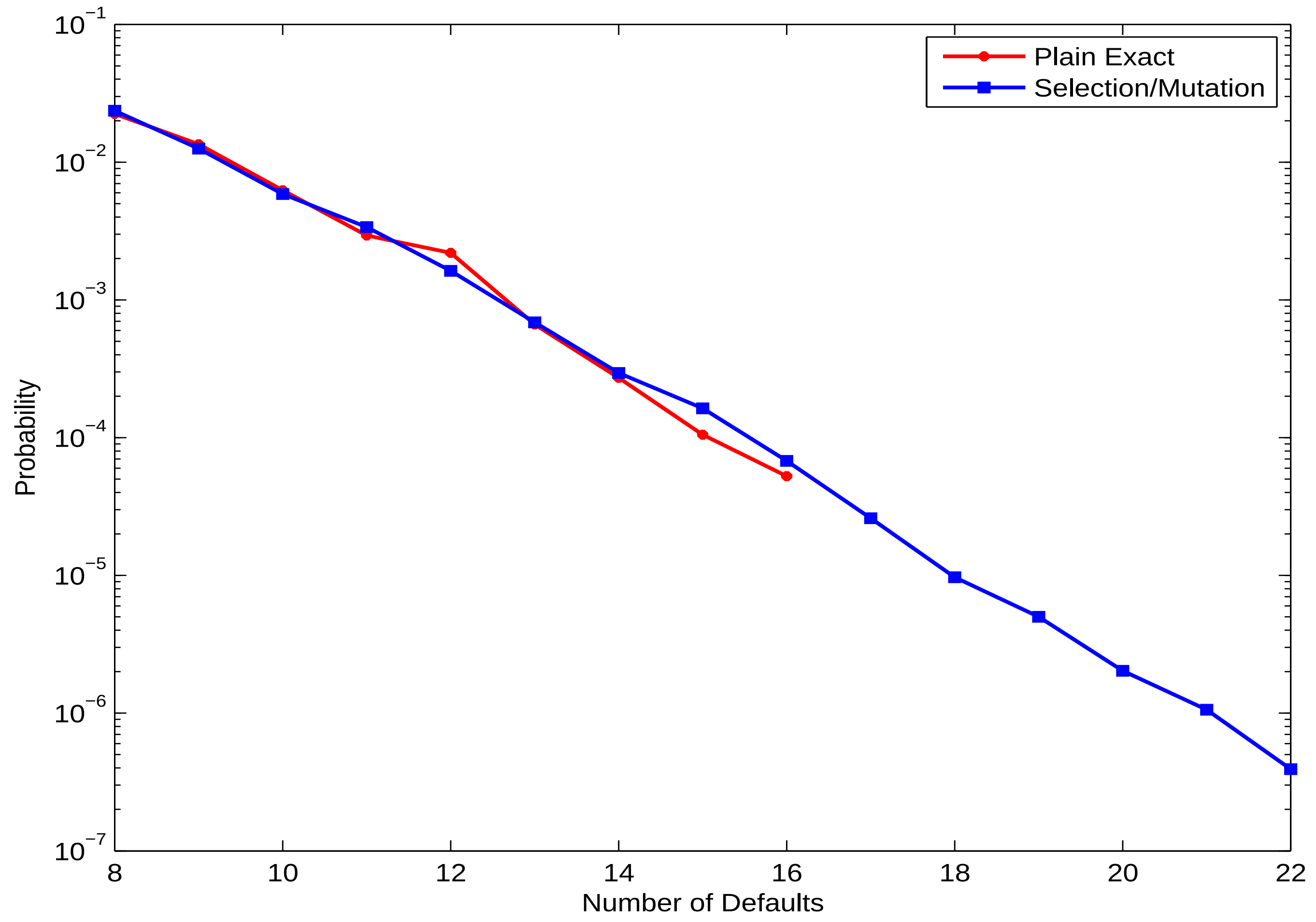
Numerical results

Variance reduction for $\mathbb{P}(C_1 = k)$, $R = 10,000$ particles, $m = 4$

	Selection/Mutation			**Plain Exact**		
k	δ	Particles	$P(C_1 = k)$	Trials	$P(C_1 = k)$	VarRatio
12	0.8	10,000	0.00162340	17,742	0.00220	10.14
13	0.85	10,000	0.00068818	18,065	0.00066	12.33
14	0.85	10,000	0.00029433	18,387	0.00027	63.88
15	1.05	10,000	0.00016310	19,032	0.00011	121.80
16	1.05	10,000	0.00006790	19,032	0.00005	236.92
17	1.15	10,000	0.00002597	19,355	0	
18	1.15	10,000	0.00000970	19,355	0	
19	1.15	10,000	0.00000500	19,355	0	
20	1.15	10,000	0.00000203	19,355	0	
21	1.15	10,000	0.00000106	19,355	0	
22	1.3	10,000	0.00000039	19,677	0	

Numerical results

Probabilities $\mathbb{P}(C_1 = k)$, $R = 10,000$ particles, $m = 4$ selections



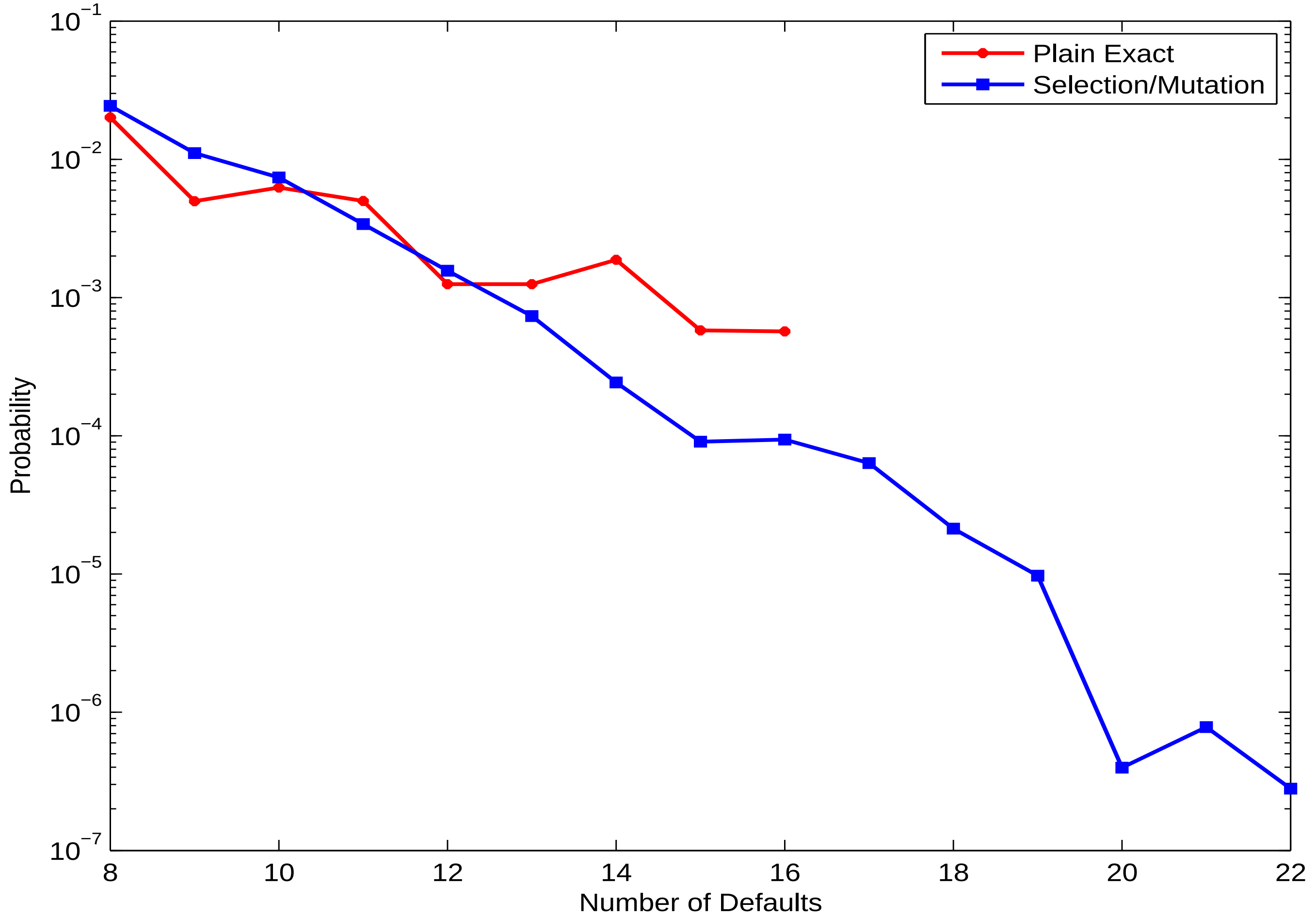
Numerical results

Variance reduction for $\mathbb{P}(C_1 = k)$, $R = 1,000$ particles, $m = 4$

	Selection/Mutation			**Plain Exact**		
k	δ	Particles	$P(C_1 = k)$	Trials	$P(C_1 = k)$	VarRatio
12	0.8	1,000	0.00156206	1,600	0.00125	14.16
13	0.85	1,000	0.00073476	1,600	0.00125	32.62
14	0.85	1,000	0.00024347	1,600	0.00188	565.20
15	1.05	1,000	0.00009063	1,726	0.00058	1562.75
16	1.05	1,000	0.00009381	1,759	0.00057	2951.99
17	1.15	1,000	0.00006339	1,790	0	
18	1.15	1,000	0.00002132	1,823	0	
19	1.15	1,000	0.00000972	1,887	0	
20	1.15	1,000	0.00000040	1,887	0	
21	1.15	1,000	0.00000078	1,918	0	
22	1.3	1,000	0.00000028	1,983	0	

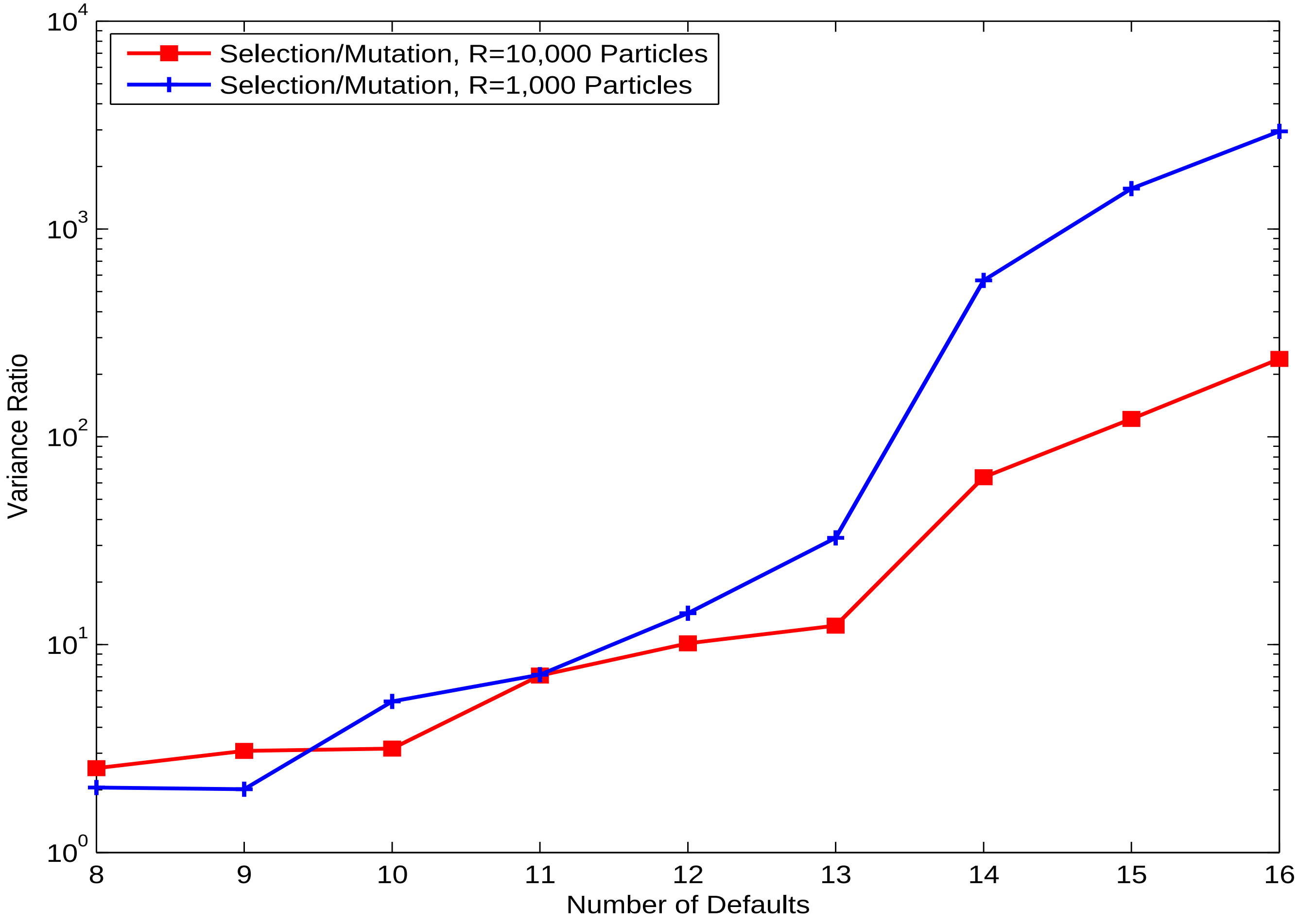
Numerical results

Probabilities $\mathbb{P}(C_1 = k)$, $R = 1,000$ particles, $m = 4$ selections



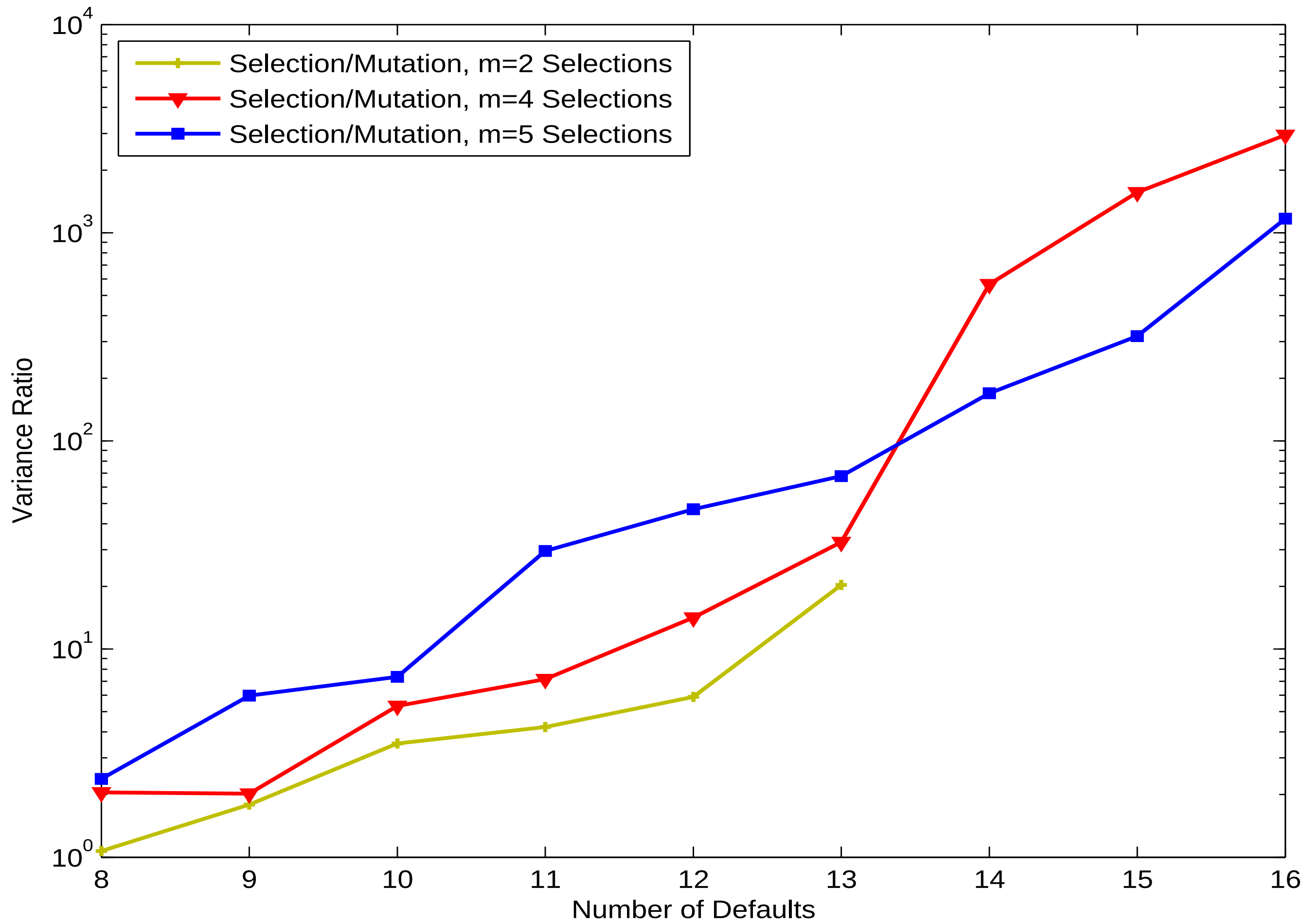
Numerical results

Variance ratios for $\mathbb{P}(C_1 = k)$, varying R, $m = 4$ selections



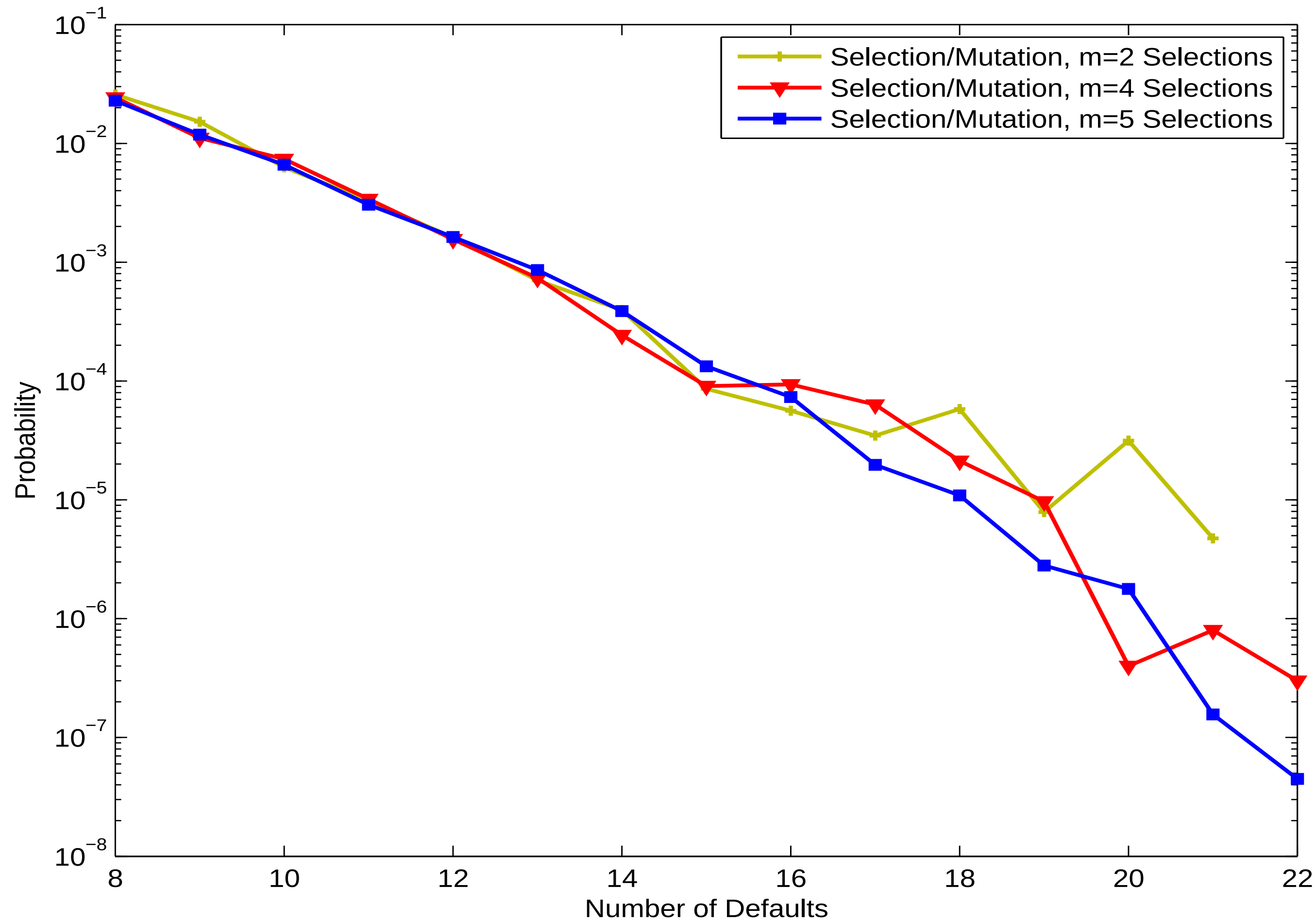
Numerical results

Variance ratios for $\mathbb{P}(C_1 = k)$, $R = 1,000$ particles, varying m



Numerical results

Probabilities $\mathbb{P}(C_1 = k)$, $R = 1,000$ particles, varying m



Conclusions

- Exact and efficient simulation engine for portfolio credit risk
  - Based on multivariate Markovian projection
  - Variance reduction via selection/mutation scheme
- Broadly applicable
  - Multi-factor doubly-stochastic models
  - Multi-factor frailty models
  - Self-exciting models
- Full portfolio and single-name functionality

Conclusions

- Our results address a gap in the literature on intensity-based models of portfolio credit risk
  - Bassamboo & Jain (2006, WSC)
- Our results complement the simulation methods developed for copula-based models of portfolio credit risk
  - Bassamboo, Juneja & Zeevi (2008, OR)
  - Chen & Glasserman (2008, OR)
  - Glasserman & Li (2005, MS)
- Our results are relevant in several other application areas, including reliability, insurance, queuing

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