Simulation Methods for Nonlinear and Non-Gaussian Models in Finance

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Financial variables, such as asset returns in international stock and bond markets or interest rates in the liquidity market, often exhibit a heterogeneous time evolution, with a unconditional density characterised by heavy tails, skewness, multimodality and time changing volatility. Through an empirical study, all these features appear clearly in some financial indexes sampled with monthly frequency and become more evident when data are collected with a higher frequency (i.e. weekly, daily or intra-day frequencies). Gaussian distribution and linear dynamic assumptions reveal unsatisfactory in many financial applications like asset pricing, risk measurement and management. Nonlinear and non-Gaussian models have been introduced in finance in order to come to more attractive results.

Many stochastic models are now available as alternatives to the linear and Gaussian ones. But all of them are generally difficult to handle and represent challenging problems in applied mathematics. Some recent works (see for example Doucet, Freitas and Gordon [7], Robert and Casella [9] and Del Moral [6]) highlight the ability of the Monte Carlo simulation methods in solving optimisation and integration problems, which arise in treating complex probabilistic models and suggest moreover a Bayesian approach to optimal decision and inference making. Within the simulation based inference framework the Bayesian approach has been widely applied in many recent studies, due to the natural way the Monte Carlo approximation can enter into the inference procedure. The Bayesian framework accounts for prior information about the parameters and allows to treat complex models, such as mixtures of distributions, stochastic volatility and stochastic trend models. For an introduction to the basic and more advanced simulation methods we refer the interested reader to Robert and Casella [9], Doucet, Freitas and Gordon [7] and Liu [8]. In the following we evidence the relation between some particular financial models and some general class of problems which can be solved by means of the Monte Carlo simulation methods.

Consider a financial asset portfolio problem with given constraints on the lowest return allowed

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r^{low} and risk level α, to fail the minimum target (shortfall probability). The portfolio model becomes more realistic if we remove the usual Gaussian distribution assumption for the asset return r. Following the empirical evidence we assume n Student-t distributions, with different degrees of freedom for each asset class (see Billio, Casarin and Toniolo [2]). The resulting problem is a stochastic optimisation problem

$$\max_{x \in \mathcal{X}} \mathbb{E}_{\mathbb{P}} \{f(x,r)\}, \quad \text{with } \mathcal{X} = \left\{ x \in \mathbb{R}^n_+ \,|\, \mathbb{P}\left(\sum_{i=1}^{n} x_i r_i < r^{low} \right) \leq \alpha, \sum_{i=1}^{n} x_i = 1 \right\}. \tag{1}$$

where x is the vector of portfolio weights and $\mathbb{P}$ a probability measure on $\mathbb{R}^n$. A Monte Carlo simulation approach can be used to approximate and solve the optimisation problem and also to evaluate the portfolio risk-level in presence of parameter estimation errors or misspecified tails behaviour.

A second application of the simulation methods relates the inference procedure on parametric models for financial asset returns. In many studies the hypothesis of α-stable distributed asset returns have been successfully tested. It is possible to generalize this model by assuming mixture of α-stable distributions. The α-stable mixture is able to capture not only skewness and excess of kurtosis, but also the multimodality in the probability distribution of many financial variables. The mixture model is difficult to estimate and a simulation based approach is needed. In Casarin [3] a Bayesian inference model is developed following a data augmentation principle. The estimation method is based on the use of Monte Carlo Markov Chain (MCMC), such as Gibbs sampling algorithm, to solve the integration problem

$$\hat{\theta}_B \triangleq \mathbb{E}_{p(\theta|Y)}(\theta) = \int_{\Theta} \theta\, p(\theta|Y) d\theta \approx \frac{1}{N} \sum_{i=1}^{N} \theta_i, \quad \text{with}\, \theta_i \sim p(\theta_i|Y) \tag{2}$$

where $\hat{\theta}_B$ is the Bayes estimator of the mixture parameter vector, θ, under the quadratic loss assumption and conditionally on the observation vector Y. The validity of the inference tool has been verified on synthetic data and an application to financial indexes has been provided.

The estimation of latent factor dynamic models, such as stochastic volatility models, represents another important area, where the Bayesian simulation based approach can be successfully applied. In latent variable models, hidden variable filtering and parameter estimation may be rather difficult to carry out. Once the latent variable model has been redefined in terms of a Bayesian dynamic model, the related inference problems of parameters estimation and hidden states filtering can be solved within a Bayesian simulation based approach. See for example Casarin [3] for a comparison between different simulation techniques for volatility filtering and parameter estimation. Within the simulation based filtering techniques, sequential Monte Carlo (i.e. *particle filters*) recently reveals one of the most general and powerful filtering method. It allows to perform on-line prediction and filtering of the unobservable states, x_t, and estimation of the parameter, θ, for a general probabilistic dynamic model. Assume we are interested in filtering the hidden variables x_t, as a new observation y_t becomes available. By means of a weighted sample (particle set), $\{x_{t-1}^i, w_{t-1}^i\}_{i=1}^N$, drawn at time $t-1$ from the posterior density $p(x_{t-1}|y_{1:t-1}, \theta)$, it is possible to approximate the filtering density

$$p(x_t|y_{1:t}, \theta) = \int_{\mathcal{X}} p(x_t|x_{t-1}, y_{1:t-1}, \theta)p(x_{t-1}|y_{1:t-1}, \theta)dx_{t-1} \approx \sum_{i=1}^{N} w_{t-1}^i p(x_t|x_{t-1}^i, y_{1:t-1}, \theta) \quad (3)$$

Applications of sequential Monte Carlo methods for Bayesian inference on Markov switching stochastic volatility and on business cycle models are provided in Casarin [5] and Billio, Casarin and Sartore [1]. These works are in a promising wider research area, which relates the applications of more advanced and efficient Monte Carlo methods in economics and financial modelling.

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