Mean field simulation of (quasi-)invariant measures and related topics

P. Del Moral

INRIA Bordeaux - Sud Ouest

Genetic Models and Quasi-stationarity, CIRM Luminy, March 2013

Some hyper-references

- Branching and interacting particle systems. **(with L. Miclo) Sém. Proba. de Strasbourg (2000).**
- A Moran particle system approximation of Feynman-Kac formulae. **(with L. Miclo) SPA (2000).**
- On the stability of interacting processes **(with A. Guionnet) IHP (2001).**
- On the Stability of Feynman-Kac sg. **(with L. Miclo) Annales de la Fac. Sci. Toulouse (2002)**
- Particle Lyapunov exponents connected to Schrödinger op. **(with L. Miclo) ESAIM PS (2003).**
- Particle Motions in Absorbing Medium with Hard and Soft Obst. **(with A. Doucet) SAA (2004).**

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- Feynman-Kac formulae, Genealogical & Interacting Particle Systems, **Springer (2004).**
- On the concentration of interacting processes. **FTML (with P. Hu & L. Wu) (2012)**
- Mean field simulation for Monte Carlo integration. **Chapman & Hall CRC Press (2013)**

Introduction

Absorption models

Extended path integration models

Feynman-Kac models

Stochastic analysis

How & Why it works

Continuous time models

Introduction
Some basic notation
Boltzmann-Gibbs transformation
Nonlinear transport models

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Basic notation

- **Lebesgue integral** Measures μ, functions f on E

$$\mu(f) = \int \mu(dx)\ f(x)$$

- **Integral operators** $Q(x_1, dx_2)$, $x_1 \in E_1 \rightsquigarrow x_2 \in E_2$

$$\begin{aligned} Q(f)(x_1) &= \int Q(x_1, dx_2) f(x_2) \\ [\mu Q](dx_2) &= \int \mu(dx_1) Q(x_1, dx_2) \qquad (\Longrightarrow [\mu Q](f) = \mu[Q(f)]\ ) \end{aligned}$$

- **Composition**

$$(Q_1 Q_2)(x_1, dx_3) = \int\ Q_1(x_1, dx_2)\ Q_2(x_2, dx_3)$$

- **Semigroups**

$$Q_{p,n} = Q_{p+1} Q_{p+1} \ldots Q_n$$

Boltzmann-Gibbs transformation

Boltzmann-Gibbs transformation :

- G positive and bounded potential function on E
- μ positive bounded measure on E

$$\Psi_G \ : \ \mu(dx) \ \mapsto \ \Psi_G(\mu)(dx) = \frac{1}{\mu(G)} \ G(x) \ \mu(dx)$$

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Important observation : $\exists$ a (nonlinear) Markov transport eq.

$$\Psi_G(\mu) = \mu S_\mu \quad \left(\Leftrightarrow \int \mu(dx) \; S_\mu(x, dy) = \Psi_G(\mu)(dy) \right)$$

for some (non unique) collection Markov transition S_μ from E into itself.

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Later $\mu \simeq \mu^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{X^i} \rightsquigarrow$ Mean field selection transition

$$X^i \rightsquigarrow \widehat{X}^i \;\; \text{with law} \;\; S_{\mu^N}(X^i, dx)$$

Example 1 : $\forall \epsilon$ s.t. $\epsilon G \leq 1$

$$S_\mu(x, dy) = \epsilon G(x)\ \delta_x(dy) + (1 - \epsilon G(x))\ \Psi_G(\mu)(dy)$$

Some choices :

$$\begin{aligned} \epsilon^{-1} &= \mu - \text{ess-sup}\, G \quad \epsilon^{-1} = \|G\| \\ \epsilon &= 0, \quad \text{or} \quad \epsilon = 1 \quad \text{when} \quad G \leq 1 \end{aligned}$$

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Example 2 : $\forall a \geq 0$ s.t. $G > a$

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Example 3 : $\forall G$

$$S_\mu(x, dy) = \alpha(x)\ \delta_x(dy) + (1 - \alpha(x))\ \Psi_{(G-G(x))_+}(\mu)(dy)$$

with the acceptance rate

$$\alpha(x) = \mu(G \wedge G(x))/\mu(G)$$

Introduction

Absorption models
- Hard obstacles
- Soft obstacles
- A brief review on genetic type models
- MCMC absorption models

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Absorption models

Example 1 : Markov chain models $\mathbf{X_n} \in \mathbf{E_n}$ restricted to subsets A_n

$$\mathbf{X} = (X_0, \ldots, X_n) \in \mathbf{A} = (A_0 \times \ldots \times A_n)$$

$$\Downarrow$$

Non absorption conditional distributions

$$\mathrm{Law}\,(\mathbf{X} \mid \mathbf{X} \in \mathbf{A}) = \mathrm{Law}((X_0, \ldots, X_n) \mid X_p \in A_p, \quad p < n) = \mathbb{Q}_n$$

and

$$\mathrm{Proba}(X_p \in A_p, \quad p < n) = \mathcal{Z}_n$$

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$$\mathbb{P}_n = \mathrm{Law}(X_0, \ldots, X_n) \quad \text{and} \quad G_p = 1_{A_p}, \quad p < n$$

Particle absorption models

N-Particle system $(\xi_n^i)_{1\leq i\leq N}$ with selection-mutation transitions

- N iid $(\xi_0^i)_{1\leq i\leq N}$ copies of X_0, and set $P_0^N = \frac{1}{N}\sum_{1\leq i\leq N} 1_{A_0}(\xi_0^i)$.

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Example 2 : Absorbed Markov chain with rate $(1 - G_n)$ **on** E_n

$$X_n^c \in E_n^c \xrightarrow{\text{absorption} \sim (1-G_n)} \widehat{X}_n^c \xrightarrow{\text{exploration} \sim M_n} X_{n+1}^c$$

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n-th time marginals:

$$\eta_n(f) := \gamma_n(f)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f) = \mathbb{E}\left(f(X_n) \prod_{0 \leq p < n} G_p(X_p) \right)$$

$$\gamma_n(1) = \mathbb{E}\left(\prod_{0 \leq p < n} G_p(X_p) \right) = \mathbb{P}\left(T^{absorption} \geq n \right)$$

and

$$\eta_n(f) = \mathbb{E}\left(f(X_n^c) \mid T^{absorption} \geq n \right)$$

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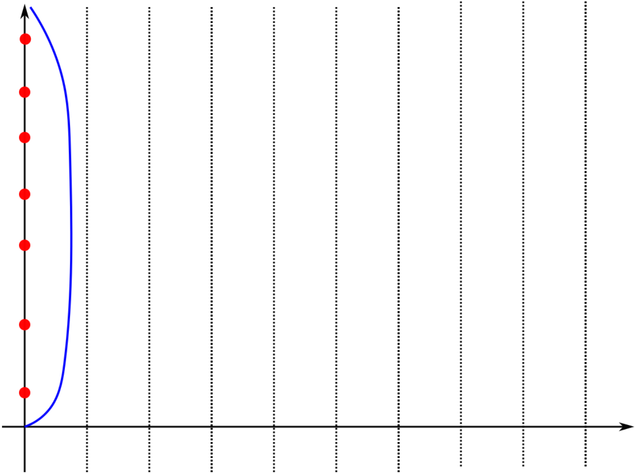

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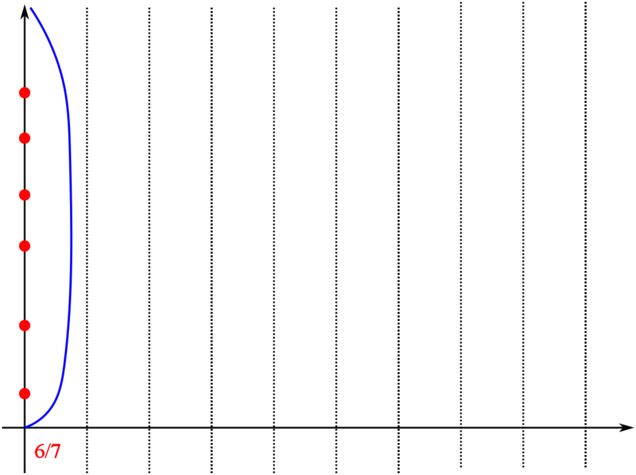

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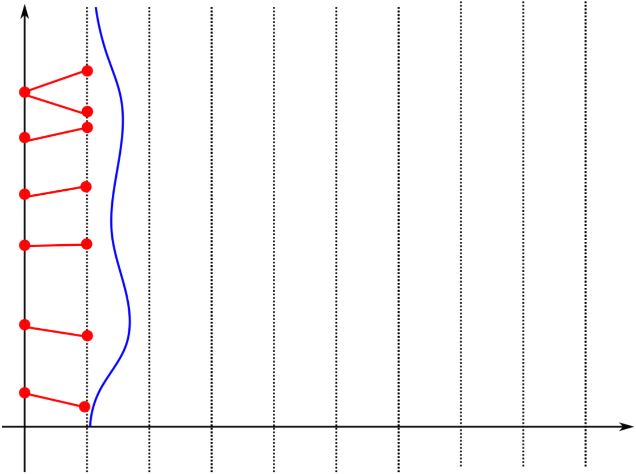

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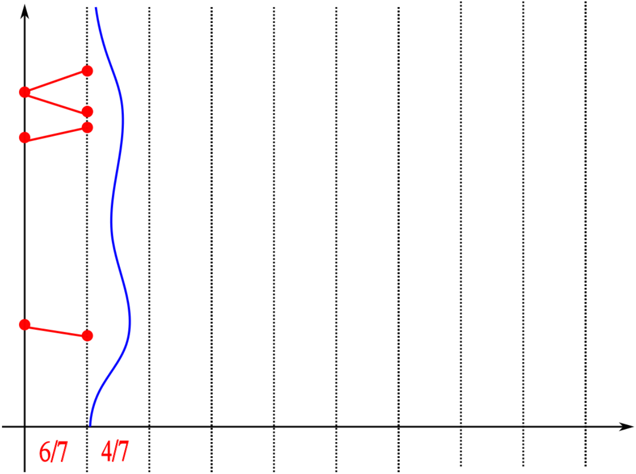

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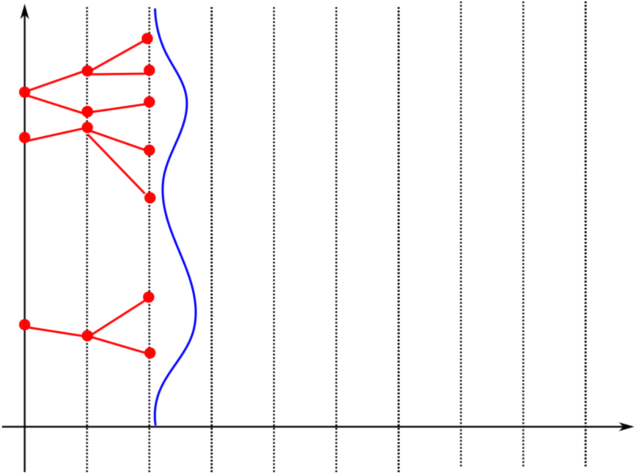

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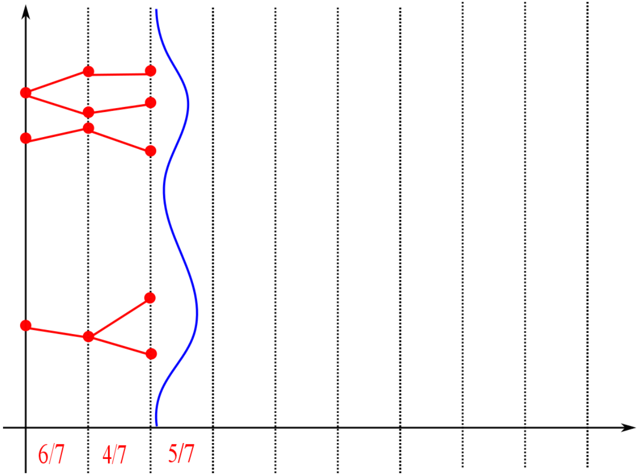

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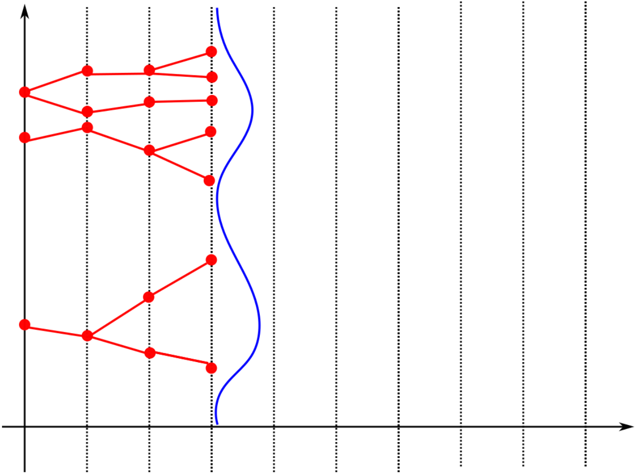

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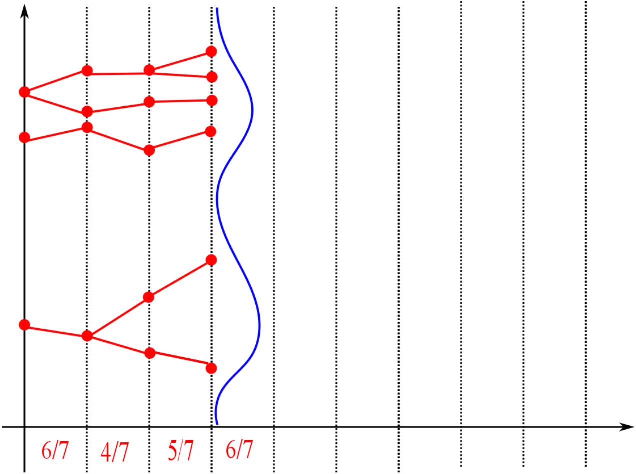

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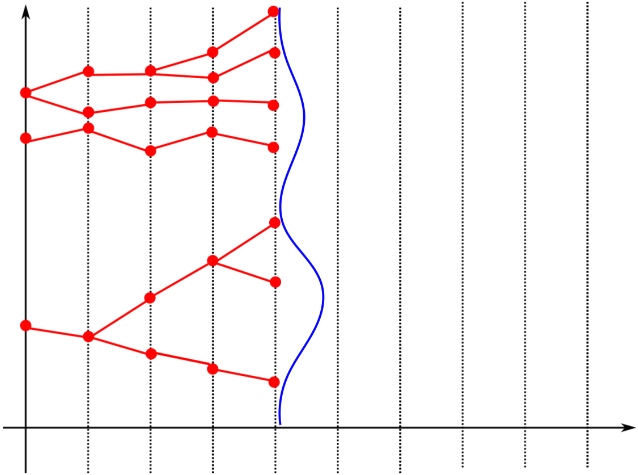

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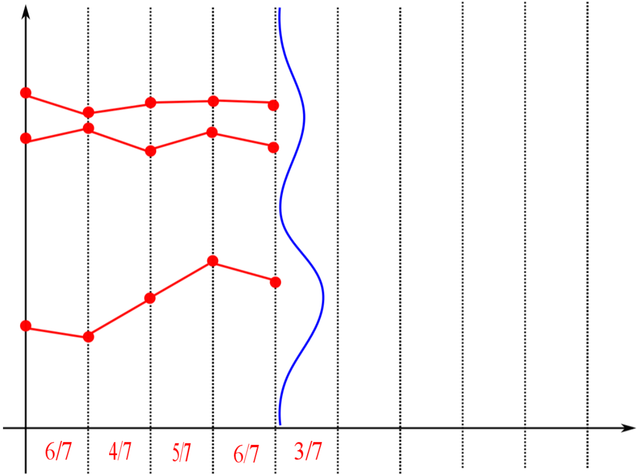

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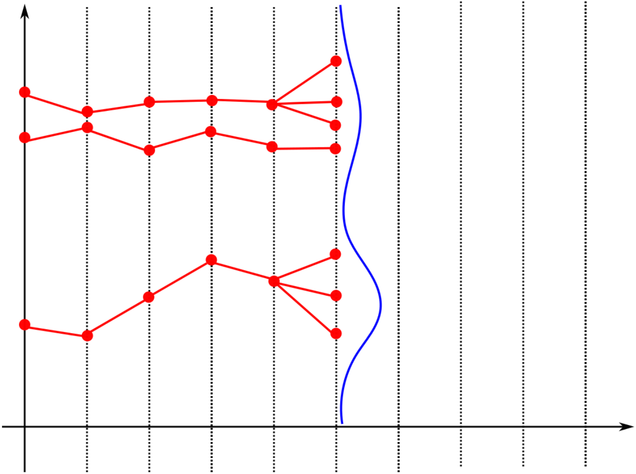

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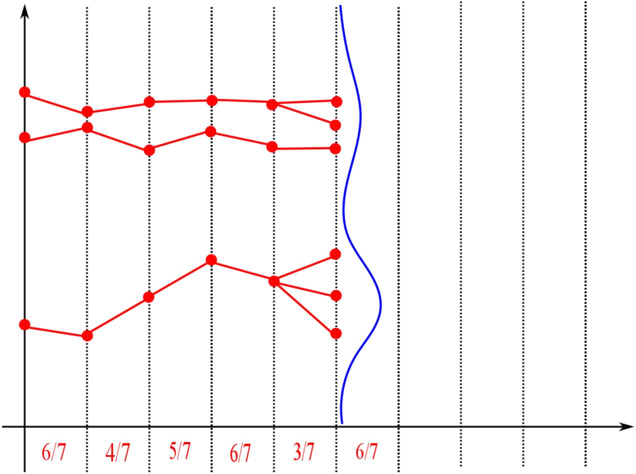

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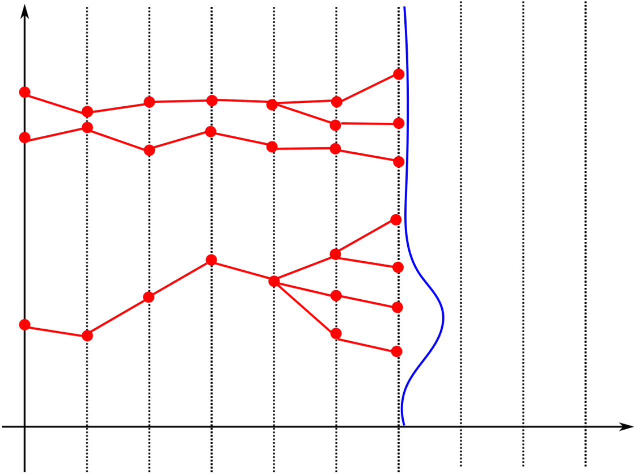

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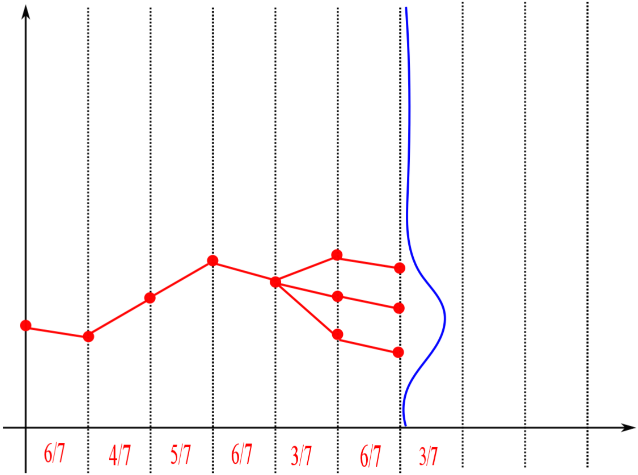

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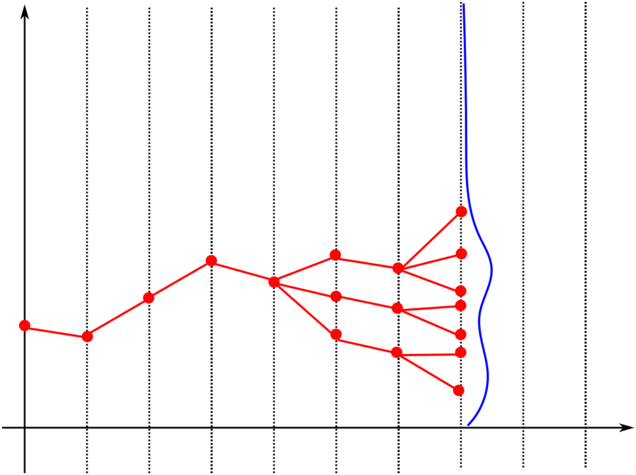

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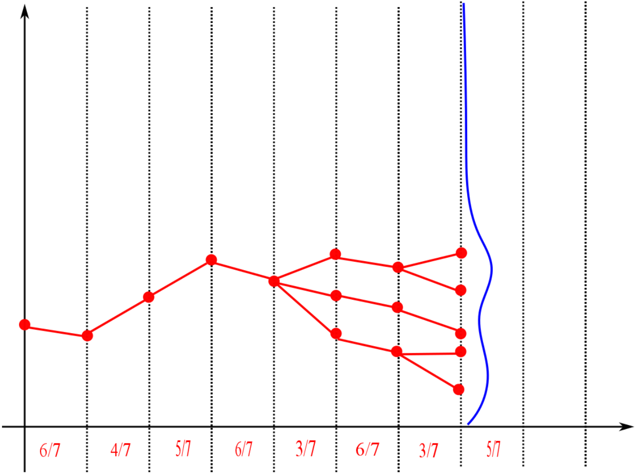

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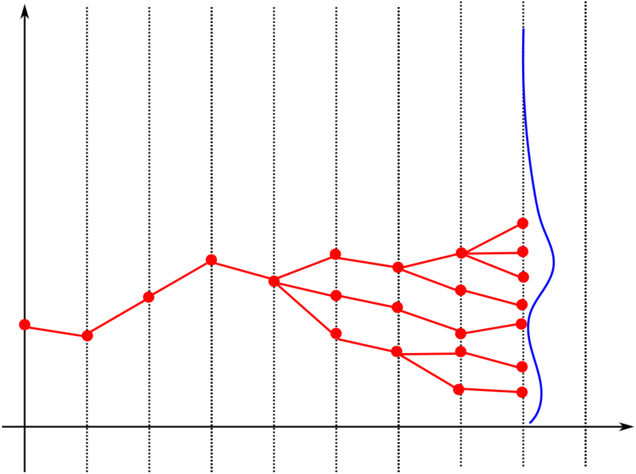

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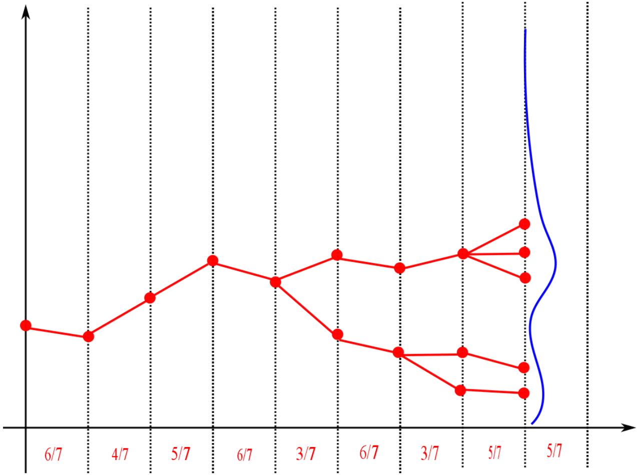

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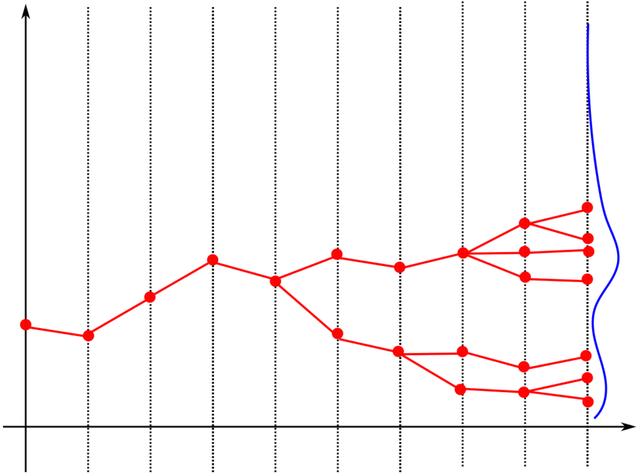

Continuous time models X'_t, $t \in \mathbb{R}_+$

- $X_n = X'_{[t_n, t_{n+1}]}$ and $G_n(X_n) = \exp\left\{\int_{t_n}^{t_{n+1}} V_t(X'_t)dt\right\}$

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- Euler/Milstein/... discrete time approximations

Equivalent heuristic like particle algorithms $\in [1950 - 1996]$

Sequential Monte Carlo	Sampling	Resampling
Particle Filters	Prediction	Updating
Genetic Algorithms	Mutation	Selection
Evolutionary Population	Exploration	Branching-selection
Diffusion Monte Carlo	Free evolutions	Absorption
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Many other lively buzzwords :
bootstrapping, spawning, cloning, pruning, replenish, splitting, enrichment, go with the winner, look-ahead, weighted dynamics, ...

Remarks

- **Geo. accept. rates** $e^{-V\ \Delta t}$ $\rightsquigarrow$ Continuous time interact. jumps

$$\frac{d}{dt}\eta_t(f) = \eta_t(L_{t,\eta_t}(f)) = \eta_t(L_t^X) - [\eta_t(fV) - \eta_t(f)\eta_t(V)]$$

with $L_{t,\eta_t} = L_t^X + \widehat{L}_{t,\eta_t}$ the jump generator:

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- **Fleming-Viot and Dawson-Watanable :**
  - Different scaling : higher jump rate $N \rightsquigarrow N^2$

    Ex.: finite state space

    genetic selection $\longrightarrow$ diffusions (at the level of the proportions)
  - Neutral and/or symmetric adaptation $V(x,y) = V(y,x)$

Some open questions

Finite population model:

- Invariant measure, limiting occupation measures.
- Long time behavior : relaxation times, spectral analysis, ...
- k-Times de common ancestors, population size at each level.
- Occupation measures of the complete ancestral tree.
- Effects of multiple energy well envrionement.

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Some answers $[1996 - \ldots]$**:**

- Occupation meas. complete ancestral tree $\rightarrow_{N\uparrow\infty}$ McKean process.
- Occupation meas. genealogical tree $\rightarrow_{N\uparrow\infty}$ Feynman-Kac model
- Long time behavior (under mixing and regularity conditions) $\Rightarrow \lim_{t\rightarrow\infty} \lim_{N\rightarrow\infty} = \lim_{N\rightarrow\infty} \lim_{t\rightarrow\infty}$
- Propagations of chaos expansions, CLT, LDP, $\mathbb{L}_p$-estimates, Empirical processes, Moderate deviations, stability and contraction inequalities, exponential concentration analysis

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n-th time marginals:

$$\eta_n(f) = \mathbb{E}\left(f(X_n^c) \mid T^{absorption} \geq n\right) \quad \text{and} \quad \gamma_n(1) = \mathbb{P}\left(T^{absorption} \geq n\right)$$

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An MCMC absorption model

Target measures:

$$\eta_n(dx) := \frac{1}{\mathcal{Z}_n} \left\{ \prod_{p=0}^{n} h_p(x) \right\} \lambda(dx) \quad \text{with} \quad 0 \leq h_p \leq 1$$

A couple of examples:

- $h_n = 1_{A_n}$ with $A_n \downarrow \Rightarrow d\eta_n = \frac{1}{\lambda(A_n)} \; 1_{A_n} \; d\lambda$
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Absorption model $\rightsquigarrow$ exact sampling & Mean field simulation:

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$$\Downarrow$$

$$\begin{aligned}\eta_n &= \mathrm{Law}(X_n^c \mid T^{absorption} \geq n) \quad \text{and} \quad \mathcal{Z}_n = \mathbb{P}\left(T^{absorption} \geq n\right) \\ &= \mathrm{Law}(\text{MCMC at time n} \mid h\text{-rejection time} \geq n)\end{aligned}$$

Absorption models : A couple of bad tempting ideas

1. **Acceptance-Rejection simulation :** X_n^i iid copies of X_n^c

$$\mathcal{Z}_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} 1_{T^i \geq n} \quad \simeq_{N \uparrow \infty} \quad \mathcal{Z}_n$$

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$\rightsquigarrow$ **Exact sampling but with extremely poor estimates:**

$$N\ \mathrm{Var}\left(P_n^N / P_n\right) = (1 - P_n)\ P_n^{-1} \quad (\text{for Mean field IPS} \quad \leq c \ \times n)$$

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2. **Weighted models** $\stackrel{G_n = 1_{A_n}}{\Leftrightarrow}$ **Acceptance-Rejection simulation :**

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Introduction

Absorption models

Extended path integration models
- Branching processes
- Non commutative models
- Lyapunov weighted dynamics model
- Path integration and sensitivity measures
- Interacting Island models

Feynman-Kac models

Stochastic analysis

How & Why it works

Continuous time models

Branching processes when $G_n \geq 1$

$$\mathcal{X}_n = \sum_{1 \leq i \leq N_n} \delta_{X_n^i} \xrightarrow[\mathbb{E}(g_n(x)) = G_n(x)]{branching w.r.t.} \widehat{\mathcal{X}}_n = \sum_{1 \leq i \leq \widehat{N}_n} \delta_{\widehat{X}_{n+1}^i} \xrightarrow[M_{n+1}]{exploration} \mathcal{X}_{n+1}$$

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Path space first moments given by the Feynman-Kac measures

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with

$$\mathbb{P}_n = \text{Law}(X_0, \ldots, X_n) \quad \text{and} \quad \mathcal{Z}_n = \mathbb{E}\left(N_n\right)$$

Some questions

n-th time marginals:

$$\eta_n(f) = \mathbb{E}\left(\mathcal{X}_n(f)\right) / \mathbb{E}\left(\mathcal{X}_n(1)\right) \quad \text{and} \quad \gamma_n(1) = \mathbb{E}\left(\mathcal{X}_n(1)\right)$$

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Adding mass (notation : $Q_{n+1}(f) = G_n \; M_{n+1}(f)$)

First moment evolution equation

$$
\begin{aligned}
\mathbb{E}\left(\mathcal{X}_{n+1}(f)\right) &= \gamma_{n+1}(f) = \gamma_n(G_n M_{n+1}(f)) + \mu_{\mathbf{n}}(\mathbf{f}) \quad \textbf{with} \quad \mu_{\mathbf{n}} \textbf{ positive} \\
\eta_n(f) &:= \gamma_n(f)/\gamma_n(1) \quad \textbf{Normalized measures}
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$$\Downarrow$$

Three typical scenarios when

$$(G_n, M_n, Q_n, \mu_n, \gamma_0) = (G, M, Q, \mu, \mu) \quad \text{and} \quad g_- := \inf G \leq \sup G := g_+$$

Adding mass (notation : $Q_{n+1}(f) = G_n\ M_{n+1}(f)$)

First moment evolution equation

$$\begin{aligned}\mathbb{E}\left(\mathcal{X}_{n+1}(f)\right) &= \gamma_{n+1}(f) = \gamma_n(G_n M_{n+1}(f)) + \mu_n(\mathbf{f}) \quad \textbf{with} \quad \mu_n \textbf{ positive} \\ \eta_n(f) &:= \gamma_n(f)/\gamma_n(1) \quad \textbf{Normalized measures}\end{aligned}$$

$$\Downarrow$$

Three typical scenarios when

$$(G_n, M_n, Q_n, \mu_n, \gamma_0) = (G, M, Q, \mu, \mu) \quad \text{and} \quad g_- := \inf G \leq \sup G := g_+$$

1. $G = 1$ & $\eta_\infty := \eta_\infty M$ (independent of μ)

$$\gamma_n(1) = \gamma_0(1) + \mu(1)\ n \quad \text{and} \quad \|\eta_n - \eta_\infty\|_{\mathrm{tv}} = O\left(1/n\right)$$

2. $g_+ < 1$ & $\eta_\infty := \gamma_\infty / \gamma_\infty(1)$ with γ_∞ given by

$$\gamma_\infty := \sum_{n \geq 0} \mu Q^n \Longleftrightarrow \text{Poisson equation} \quad \gamma_\infty (Id - Q) = \mu$$

and

$$|\gamma_n(f) - \gamma_\infty(f)| \vee |\eta_n(f) - \eta_\infty(f)| \leq c \; g_+^n \; \|f\|$$

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Continuous time models $G = e^{-V\Delta t}$ **&** $M = Id + L \; \Delta t$

$$\gamma_t = \int_0^t \; \mathbb{E}_\mu \left(f(X_s) \exp\left(- \int_0^s \; V(X_r) dr \right) \right) \; ds$$

$t \longrightarrow \infty \rightsquigarrow$ Poisson equation $\gamma_\infty L^V = \mu$ with $L^V = L + V$

3. $g_- > 1$ & $\eta_\infty(f) := \eta_\infty Q(f)/\eta_\infty Q(1)$ (independent of μ)

$\eta_\infty =$ Fixed point of FK-sg$\supset$[quasi-inv. meas., ground states, etc.]

$$\lim_{n\to\infty} \frac{1}{n} \log \gamma_n(1) = \log \eta_\infty(G) \quad \text{and} \quad \|\eta_n - \eta_\infty\|_{\rm tv} \leq c\ e^{-\lambda n}$$

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Hyper-references (including Mean field simulation schemes) :

- Particle approximations of a class of branching distribution flows arising in multi-target tracking (with Caron, Doucet, Pace) SIAM (2011).
- Mean field simulation for Monte Carlo integration. Chapman & Hall CRC Press (2013)

Non commutative models

- $G_n(x_n) \in \mathbb{R}^{\mathbf{d\times d}}$ s.t. $\forall u \in \mathbb{S}^{d-1} := \{\|u\| = 1\}$ we have $\|G_n(x)\ u\| > 0$
- $f_n(x_0, \ldots, x_n) \in \mathbb{R}^d$ and $\prod_{0\leq p\leq n} A_p = A_0 A_1 \ldots A_n$

$$\Gamma_n(f_n).u_0 \quad := \quad \mathbb{E}\left(f_n(X_0, \ldots, X_n) \prod_{0\leq p<n} G_p(X_p)\ .u_0\right)$$

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$$\begin{aligned}
\Gamma_n(f_n).u_0 \quad &:= \quad \mathbb{E}\left(f_n(X_0, \ldots, X_n) \prod_{0\leq p<n} G_p(X_p)\ .u_0 \right) \\
&= \quad \mathbb{E}\left(\mathbf{f_n(X_0, \ldots, X_n)} \prod_{\mathbf{0\leq p<n}} \mathbf{G_p(X_p)} \right)
\end{aligned}$$

with

$$\mathbf{X_n} = (X_n, U_n) \in \left(E_n \times \mathbb{S}^{d-1}\right) \quad \text{and} \quad \mathbf{G_n(X_n)} = \|G_n(X_n).U_n\|$$

and the walk on the sphere model

$$U_{n+1} = \frac{G_n(X_n).U_n}{\|G_n(X_n).U_n\|}$$

$\bigtriangledown$ of $P_n(\varphi)(x) = \mathbb{E}_x(\varphi(Y_n))$ with $Y_{n+1} = F_n(Y_n, W_n)$

First variational equation

$$\begin{aligned}
\mathrm{Jac}\,(Y_{n+1}) &= G_n(Y_n, W_n)\ \mathrm{Jac}\,(Y_n) \quad && \text{with} \quad G_n^{i,j} = \partial_{x^j} F_n^i \\
&= \prod_{0 \le p \le n} G_p(X_p) && \text{with} \quad X_n = (Y_n, W_n)
\end{aligned}$$

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$$\Rightarrow \bigtriangledown P_n(\varphi)(x) \cdot u_0 \quad := \quad \mathbb{E}_x\left(\underbrace{f_n(X_n)}_{=\bigtriangledown(\varphi)(Y_n)} \prod_{0 \le p < n} G_p(X_p)\ \cdot u_0 \right)$$

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$$
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&= \quad \mathbb{E}\left(\underbrace{\mathbf{f_n(X_n)}}_{=\bigtriangledown(\varphi)(Y_n)\,.\,U_n} \times \prod_{\mathbf{0\leq p<n}} \underbrace{\mathbf{G_p(X_p)}}_{=\|G_p(X_p)\,.\,U_p\|} \right)
\end{aligned}
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$$\begin{aligned} \Rightarrow \nabla P_n(\varphi)(x)\,.\,u_0 &:= \mathbb{E}_x\left(\underbrace{f_n(X_n)}_{=\nabla(\varphi)(Y_n)} \prod_{0\leq p<n} G_p(X_p)\ .\,u_0\right) \\ &= \mathbb{E}\left(\underbrace{\mathbf{f_n(X_n)}}_{=\nabla(\varphi)(Y_n)\,.\,U_n} \times \prod_{\mathbf{0\leq p<n}} \underbrace{\mathbf{G_p(X_p)}}_{=\|G_p(X_p)\,.\,U_p\|}\right) \end{aligned}$$

$\Leftrightarrow$ **FK model w.r.t.** Y_n **weighted with the directional Lyap. exp.**

$$\prod_{\mathbf{0\leq p\leq n}} \mathbf{G_p(X_p)} = \|\mathrm{Jac}\,(Y_n)\ .\,u_0\| = \prod_{0\leq p\leq n} \frac{\|\mathrm{Jac}\,(Y_p)\ .\,u_0\|}{\|\mathrm{Jac}\,(Y_{p-1})\,.\,u_0\|}$$

Related Feynman-Kac model

$$\mathbf{X_n} = (X_n, X_{n+1}) \quad \text{and} \quad \mathbf{G_n}(\mathbf{X_n}) = \|\mathrm{Jac}\,(X_{n+1})\|^{\alpha} / \|\mathrm{Jac}\,(X_n)\|^{\alpha}$$

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$$\Downarrow$$

Feynman-Kac model = The Lyapunov weighted dynamics model

$$\mathbf{dQ_n} = \frac{1}{\mathcal{Z}_\mathbf{n}} \; \|\mathrm{Jac}\,(X_n)\|^{\alpha} \; \mathbf{dP_n}$$

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- $\alpha > 0 \Leftrightarrow$ $\mathbf{dQ_n}$ favors high Lyapunov trajectories

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Hyper-references :

- T Laffargue, K.D. Nguyen Thu Lam, J. Kurchan, J. Tailleur LDP of Lyapunov exp. (2013)
- S. Tanese Nicola, J. Kurchan. Metastable states, transitions, basins and borders at finite temperature J. Stat. Phys. (2004).
- J. Tailleur, S. Tanese Nicola, J. Kurchan. Kramers equations an supersymmetry J. Stat. Phys. (2006).
- J. Tailleur, J. Kurchan. Probing rare physical trajectories with Lyapunov weighted dynamics, Nature Physics (2007)
- $\subset$ Genealogical particle analysis of aare events (joint work with J. Garnier)AAP (2005).

Sensitivity measures

hypothesis : $\theta \in \mathbb{R}^d \mapsto G_{\theta,n-1}(x)M_{\theta,n}(x, dy) = H_{\theta,n}(x, y)\ \lambda_n(dy)$

$$\Gamma_{\theta,n}(\mathbf{f_n}) \quad = \quad \mathbb{E}\left(\mathbf{f_n}(X_0^{(\theta,c)}, \ldots, X_n^{(\theta,c)}) \mid T^{(\theta,\mathrm{absorption})} \geq n\right)$$

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and the *additive functional*

$$\mathbb{L}_{\theta,n}(x_0,\ldots,x_n) := \sum_{p=1}^{n} \log\left(H_{\theta,p}(x_{p-1},x_p)\right)$$

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$$\Downarrow$$

Derivation = Integration of additive functionals

$$\begin{aligned}\nabla \Gamma_{\theta,n}(\mathbf{f_n}) &= \Gamma_{\theta,n}(\mathbf{f_n} \nabla \mathbb{L}_{\theta,n}) \\ \nabla^2 \Gamma_{\theta,n}(\mathbf{f_n}) &= \Gamma_{\theta,n}\left[\mathbf{f_n}(\nabla \mathbb{L}_{\theta,n})' \nabla \mathbb{L}_{\theta,n} + \mathbf{f_n} \nabla^2 \mathbb{L}_{\theta,n}\right], \ \ldots\end{aligned}$$

Some examples

Potential perturbation:

$$\log G_n = [V_n + \theta V'_n]$$

$$\Downarrow$$

$$\frac{\partial}{\partial \theta} \sum_{1 \leq p \leq n} \log \left(H_{\theta,p}(x_{p-1}, x_p) \right) = - \sum_{0 \leq p < n} V'_p(x_p)$$

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$$\Downarrow$$

Derivative of the non absorption probability:

$$\frac{1}{n}\frac{\partial}{\partial\theta}\log\Gamma_{\theta,n}(1) = \frac{1}{n}\frac{\partial}{\partial\theta}\log\mathbb{P}\left(\ T^{(\theta,\text{absorption})} \geq n\right) = -\mathbb{Q}_{\theta,n}\left(f_n\right)$$

with the normalized additive functional

$$f_n(x_0,\ldots,x_n) = \frac{1}{n}\sum_{0\leq p<n} V'_p(x_p)$$

Some examples

Diffusion perturbation:

$$X_n^{(\theta)} - X_{n-1}^{(\theta)}$$

$$= b\left(X_{n-1}^{(\theta)}\right)\ \Delta + \left[\sigma\left(X_{n-1}^{(\theta)}\right) + \theta\ \sigma'\left(X_{n-1}^{(\theta)}\right)\right]\ \left(W_{t_n} - W_{t_{n-1}}\right) \in \mathbb{R}$$

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$$\Downarrow$$

$$\frac{\partial}{\partial\theta}\sum_{p=1}^{n} \log\left(H_{\theta,p}(x_{p-1}, x_p)\right)$$

$$= \sum_{p=1}^{n} \frac{\sigma'(x_{p-1})}{\sigma(x_{p-1}) + \theta\sigma'(x_{p-1})}\ \left[\left(\frac{(x_p - x_{p-1}) - b(x_{p-1})\Delta}{(\sigma(x_{p-1}) + \theta\sigma'(x_{p-1}))\sqrt{\Delta}}\right)^2 - 1\right]$$

Some examples

Drift perturbation:

$$X_n^{(\theta)} - X_{n-1}^{(\theta)}$$

$$= \left[b\left(X_{n-1}^{(\theta)}\right) + \theta b'\left(X_{n-1}^{(\theta)}\right)\right] \Delta + \sigma\left(X_{n-1}^{(\theta)}\right) \left(W_{t_n} - W_{t_{n-1}}\right) \in \mathbb{R}$$

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$$\frac{\partial}{\partial\theta} \sum_{p=1}^{n} \log\left(H_{\theta,p}(x_{p-1}, x_p)\right)$$

$$= \sum_{p=1}^{n} \left[(x_p - x_{p-1}) - [b(x_{p-1}) + \theta b'(x_{p-1})]\,\Delta\right] \times b'(x_{p-1})/\sigma^2(x_{p-1}) \ .$$

FK (absorption) model : $\theta \mapsto (M_{\theta,n}, G_{\theta,n})$ and $\Theta \sim \nu(d\theta)$

$$\mathbb{Q}_{\theta,n} = \text{Law}\left((X_0^c, \ldots, X_n^c) \mid T^{absorption} \geq n,\ \Theta = \theta\right) \rightsquigarrow n\text{-th marginal } \eta_{\theta,n}$$

$$\Downarrow$$

Multiplicative formula

$$\mathcal{Z}_{\theta,n} = \mathbb{P}\left(T^{absorption} \geq n,\ \Theta = \theta\right) = \prod_{0 \leq p < n} \underbrace{\eta_{\theta,p}(G_{\theta,p})}_{=h_p(\theta)}$$

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$$\Downarrow$$

Conditional distribution of the environment w.r.t. non absorption:

$$\mathbb{P}\left(\Theta \in d\theta \mid T^{absorption} \geq n\right) = \frac{1}{\mathcal{Z}_n} \left[\prod_{0 \leq p < n} h_p(\theta)\right] \times \nu(d\theta)$$

when h_n are known :
$\rightsquigarrow$ *use the MCMC absorption model $\oplus$ Mean field particle approximation*

Interacting Island models

$$\xi_{\theta,n} = \text{particle Feynman-Kac model} \sim (M_{\theta,n}, G_{\theta,n}) \quad \text{and} \quad \Theta \sim \nu(d\theta)$$

$$\left.\begin{array}{rcl} x & = & \left(\theta, (\xi_{\theta,n})_{n\in[0,T]}\right) \\ h_n(x) & = & \eta^N_{\theta,n}(G_{\theta,n}) \end{array}\right\} \rightarrow \mu_n(dx) = \frac{1}{\mathcal{Z}_n} \left\{ \prod_{0\leq p<n} h_p(x) \right\} \lambda(dx)$$

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By the unbiased property

$$\mu_n \circ \Theta^{-1} = \frac{1}{\mathcal{Z}_n} \left\{\prod_{0\leq p<n} \eta_{\theta,n}(G_{\theta,n})\right\} \nu(d\theta)$$

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By the unbiased property

$$\mu_n \circ \Theta^{-1} = \frac{1}{\mathcal{Z}_n} \left\{ \prod_{0\leq p<n} \eta_{\theta,n}(G_{\theta,n}) \right\} \nu(d\theta)$$

- MCMC shaking moves in (parameter-island)-spaces

$$\mathbb{P}\left(X_n \in dx \mid X_{n-1}\right) = M_n(X_{n-1}, dx) \quad \text{s.t.} \quad \mu_n M_n = \mu_n$$

- Updating w.r.t. the average fitness of the islands $\eta^N_{\theta,n}(G_{\theta,n})$

Introduction

Absorption models

Extended path integration models

Feynman-Kac models
- Nonlinear evolution equations
- Historical processes
- Mean field particle models
- Some particle estimates

Stochastic analysis

How & Why it works

Continuous time models

FK model: $\forall G_n \geq 0$ **and** $M_n =$ **Markov transition of** X_n

$$d\mathbb{Q}_n := \frac{1}{\mathcal{Z}_n} \left\{ \prod_{0 \leq p < n} G_p(X_p) \right\} d\mathbb{P}_n$$

η_n=n-**marginal measures of** $\mathbb{Q}_n$ **and the unnormalized** $\gamma_n = \mathcal{Z}_n \times \eta_n$

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Key formula $\rightsquigarrow \quad \mathcal{Z}_n = \prod_{0 \leq p < n} \eta_p(G_p)$

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$$\eta_{n+1} = \Psi_{G_n}(\eta_n) M_{n+1} = \eta_n K_{n+1,\eta_n} \quad \text{with} \quad K_{n+1,\eta_n} = S_{n,\eta_n} M_{n+1}$$

Two more key observations

Time marginal

$$\eta_n(f) \propto \mathbb{E}\left(f(X_n) \prod_{0 \leq p < n} G_p(X_p) \right)$$

Two more key observations

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Example

$$G_n = 1_{A_n} = \text{Hard} \rightsquigarrow \text{Soft obstacles} = \widehat{G}_{n-1}(x) = \mathbb{P}(X_n \in A_n \mid X_{n-1} = x)$$

Mean field and Interacting particle models

- **Nonlinear McKean Markov models** $\eta_{n+1} = \eta_n K_{n+1,\eta_n}$

$$\mathbb{P}\left(\overline{X}_{n+1} \in dx \mid \overline{X}_n\right) = K_{n+1,\eta_n}\left(\overline{X}_n, dx\right) \quad \text{with} \quad \eta_n = \text{Law}\left(\overline{X}_n\right)$$

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Mean field particle model when $K_{n+1,\eta} = S_{n,\eta} M_{n+1}$

Mean field simulation:

$$K_{n+1,\eta_n^N} = \underbrace{S_{n,\eta_n^N}}_{selection} \quad \underbrace{M_{n+1}}_{mutation} \Leftrightarrow \textbf{Genetic type interacting particle system}$$

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Genealogical tree occupation measures

$$\eta_n^N = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \longrightarrow_{N \uparrow \infty} \mathbb{Q}_n$$

Some particle estimates

- Individuals ξ_n^i "almost" iid with law $\eta_n \simeq_{N\uparrow\infty} \eta_n^N = \frac{1}{N}\sum_{1\leq i\leq N} \delta_{\xi_n^i}$

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$$\gamma_n = \mathcal{Z}_n \times \eta_n \simeq_{N\uparrow\infty} \gamma_n^N = \mathcal{Z}_n^N \times \eta_n^N \quad (\text{Unbiased})$$

Important observation

Exponential rate of the normalizing constants

$$\frac{1}{n}\log \mathcal{Z}_n = \frac{1}{n}\sum_{0\le p<n}\log \eta_p(G_p) \simeq \frac{1}{n}\log \mathcal{Z}_n^N = \frac{1}{n}\sum_{0\le p<n}\log \eta_p^N(G_p)$$

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Time homogeneous models $(G_n, M_n) = (G, M)$**:**

Link to the long time behavior of η_n and/or η_n^N

$$\frac{1}{n}\log \mathcal{Z}_n \to_{n\uparrow\infty} \log \eta_\infty(G) \simeq_{N\uparrow\infty} \log \eta_\infty^N(G) \leftarrow_{n\uparrow\infty} \frac{1}{n}\log \mathcal{Z}_n^N$$

Introduction

Absorption models

Extended path integration models

Feynman-Kac models

Stochastic analysis
- Stability and contraction properties
- Uniform concentration inequalities
- Coalescent tree based expansions
- Ground states and h-processes
- Som derivation properties
- Backward particle models

How & Why it works

Continuous time models

Long time behavior of the FK-sg $\Phi_{p,n}(\eta_p) = \eta_n$

Theorem:

- M_n-mixing conditions and G_n unif. lower-upper bounded
- or $\widehat{M}_n$-mixing conditions and $\widehat{G}_n$ unif. lower-upper bounded

$$\widehat{G}_n(x)\ \widehat{M}_{n+1}(x,dy) = M_{n+1}(G_{n+1})(x) \times \frac{M_{n+1}(x,dy)G_{n+1}(y)}{M_{n+1}(G_{n+1})(x)}$$

$$\Downarrow$$

$$\exists (a,b) \in \mathbb{R}_+ \quad \sup_{\mu_1,\mu_2} \|\Phi_{p,n}(\mu_1) - \Phi_{p,n}(\mu_2)\|_{\mathrm{tv}} \leq a\ e^{-b\ (n-p)}$$

$$\beta(P_{p,n}) := \sup_{\mu_1,\mu_2} \|\Phi_{p,n}(\mu_1) - \Phi_{p,n}(\mu_2)\|_{\mathrm{tv}}$$

with the Dobrushin ergodic coefficient of the Markov transition

$$P_{p,n}(x, dy) = \mathbb{P}_{p,x}\left(X_n^c \in dy\right)$$

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$$P_{p,n}(x,dy) = \mathbb{P}_{p,x}\left(X_n^c \in dy\right) = \left(R_{p+1}^{(n)} R_{p+2}^{(n)} \ldots R_n^{(n)}\right)(x,dy)$$

and the non absorption transitions (for absorption type models)

$$R_{p+1}^{(n)}(x,dy) = \mathbb{P}_{p,x}\left(X_{p+1}^c \in dy \mid X_p^c = x,\ T \geq n\right)$$

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Example :

$$\begin{aligned}
\epsilon\ \nu(dy) \leq M(x,dy) \leq \epsilon^{-1}\nu(dy) \quad &\Rightarrow \quad R_{p+1}^{(n)}(x,dy) \geq \epsilon^2\ \nu_{p,n}(dy) \\
&\Rightarrow \quad \beta\left(R_{p+1}^{(n)}\right) \leq (1-\epsilon^2) \\
&\Rightarrow \quad \beta(P_{p,n}) \leq (1-\epsilon^2)^{(n-p)}
\end{aligned}$$

$$\beta(P_{p,n}) := \sup_{\mu_1,\mu_2} \|\Phi_{p,n}(\mu_1) - \Phi_{p,n}(\mu_2)\|_{\rm tv}$$

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\end{aligned}$$

$\rightsquigarrow$ **nice extensions/characterizations of exponential stability by N. Champagnat & D. Villemonais**

Time homogeneous models $\Phi_{p,n} = \Phi^{(n-p)}$

Corollary:

$$\exists! \; \eta_\infty = \Phi(\eta_\infty) \quad \text{and} \quad \left\| \Phi^{(n)}(\mu_1) - \Phi^{(n)}(\eta_\infty) \right\|_{\mathrm{tv}} \leq a \; e^{-b \; n}$$

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$$\eta_{n+1} = \Phi(\eta_n) = \text{Law}\left(X^c_{n+1} \mid T^{absorption} > n \right) \longrightarrow_{n \uparrow \infty} \eta_\infty = \Phi(\eta_\infty)$$

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and

$$\frac{1}{n} \log \mathbb{P}\left(T^{absorption} > n\right) = \log \eta_\infty(G) + \mathrm{O}\left(\frac{1}{n}\right)$$

Some hyper-references [$\supset$ Continuous time models; ex.: non degenerate diffusion $\subset$ compact]

- `Branching and interacting particle systems.` (with L. Miclo) Sém. Proba. de Strasbourg (2000).
- `On the stability of interacting processes` (with A. Guionnet) IHP (2001).
- `On the Stability of Feynman-Kac sg.` (with L. Miclo) Annales de la Fac. Sci. Toulouse (2002)
- `Particle Lyapunov exponents connected to Schrödinger op.` (with L. Miclo) ESAIM PS (2003).
- `Particle Motions in Absorbing Medium with Hard and Soft Obst.` (with A. Doucet) SAA (2004).
- `Feynman-Kac formulae, Genealogical & Interacting Particle Systems`, Springer (2004).

Some consequences

Cts $(c, c_1, c_2) \sim$ (bias,variance,a,b), $\|f_n\| \leq 1$, $\forall\ (x \geq 0, n \geq 0, N \geq 1)$.

- The probability of the next events is greater than $1 - e^{-x}$

$$\left[\eta_n^N - \eta_n\right](f) \quad \leq \quad \frac{c_1}{N}\ \left(1 + x + \sqrt{x}\right) + \frac{c_2}{\sqrt{N}}\ \sqrt{x}$$

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- For the genealogical tree

$$\left[\eta_n^N - \mathbb{Q}_n\right](f) \leq c_1\ \frac{n+1}{N}\ \left(1 + x + \sqrt{x}\right) + c_2\ \sqrt{\frac{(n+1)}{N}}\ \sqrt{x}$$

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$$\left[\eta_n^N - \eta_\infty\right](f) \quad \leq \quad a\ e^{-bn} + \frac{c_1}{N}\ \left(1 + x + \sqrt{x}\right) + \frac{c_2}{\sqrt{N}}\ \sqrt{x}$$

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$$|\frac{1}{n} \log \mathcal{Z}_n^N - \log \eta_\infty(G)| \quad \leq \quad \frac{c}{n} + \frac{c_1}{N}\ \left(1 + x + \sqrt{x}\right) + \frac{c_2}{\sqrt{N}}\ \sqrt{x}$$

Coalescent tree based expansions

Weak propagation of chaos Taylor's type expansions

$$\begin{aligned}\mathbb{P}^{(N,q)} &= \text{Law of the first } q \leq N \text{ ancestral lines} \\ &= \mathbb{Q}^{\otimes q} + \sum_{1\leq l\leq m} \frac{1}{N^l}\, d_l\mathbb{P}_n^{(q)} + \mathrm{O}\left(\frac{1}{N^{m+1}}\right)\end{aligned}$$

with signed measures $d_l\mathbb{P}_n^{(q)}$ expressed in terms of **l-coalescent trees.**

Coalescent tree based expansions

Weak propagation of chaos Taylor's type expansions

$$\begin{aligned}\mathbb{P}^{(N,q)} &= \text{Law of the first } q \leq N \text{ ancestral lines} \\ &= \mathbb{Q}^{\otimes q} + \sum_{1\leq l\leq m} \frac{1}{N^l}\, d_l\mathbb{P}_n^{(q)} + \mathrm{O}\left(\frac{1}{N^{m+1}}\right)\end{aligned}$$

with signed measures $d_l\mathbb{P}_n^{(q)}$ expressed in terms of **l-coalescent trees.**

$$\Downarrow$$

Romberg-Richardson interpolation: For any order $l \geq 1$

$$\sum_{1\leq m\leq l} \frac{(-1)^{l-m}}{m!}\,\frac{m^l}{(l-m)!}\,\mathbb{P}^{(mN,q)} = \mathbb{Q}^{\otimes q} + \mathrm{O}\left(\frac{1}{N^l}\right)$$

Some hyper-references

- Coalescent tree based functional representations for some Feynman-Kac particle models (with F. Patras, S. Rubenthaler) AAP (2009)
- U-statistics for interacting particle systems (with F. Patras, S. Rubenthaler) JTP (2011).

Time homogeneous models $(G_n, M_n) = (G, M)$

$$Q(x, dy) = G(y)\ M(x, dy) \quad \text{with} \quad G(x) \leq 1 \ \ (\rightsquigarrow \text{Sub-Markov})$$

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- Reversibility condition : $\mu(dx)M(x,dy) = \mu(dy)M(y,dx)$

$$\frac{1}{n} \log \mathbb{P}\left(T^{absorption} \geq n\right) = \frac{1}{n} \sum_{0 \leq p < n} \log \eta_p(G) \simeq \log \lambda = \log \eta_\infty(G)$$

with $\lambda =$ top eigenvalue of

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- $Q(h) = \lambda h \rightsquigarrow$ Doob h-process X^h

$$M^h(x, dy) = \frac{1}{\lambda} h^{-1}(x) Q(x, dy) h(y) = \frac{Q(x, dy)h(y)}{Q(h)(x)} = \frac{M(x, dy)h(y)}{M(h)(x)}$$

Homogeneous models $(G_n, M_n) = (G, M)$

$$\mathbb{Q}_n(d(x_0, \ldots, x_n)) \propto \mathbb{P}((X_0^h, \ldots, X_n^h) \in d(x_0, \ldots, x_n))\ h^{-1}(x_n)$$

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- *Invariant measure $\mu_h = \mu_h M^h$ & normalized additive functionals*

$$\overline{F}_n(x_0, \ldots, x_n) = \frac{1}{n+1} \sum_{0 \le p \le n} f(x_p) \Longrightarrow \mathbb{Q}_n(\overline{F}_n) \simeq_n \mu_h(f)$$

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- If $G = G^\theta$ depends on some $\theta \in \mathbb{R} \rightsquigarrow f := \frac{\partial}{\partial\theta} \log G^\theta$

$$\underbrace{\frac{\partial}{\partial\theta}\log\lambda^\theta}_{\text{derivation}} \simeq_n \frac{1}{n+1}\frac{\partial}{\partial\theta}\log\mathcal{Z}_{n+1}^\theta = \underbrace{\mathbb{Q}_n(\overline{F}_n)}_{\text{path-integration}}$$

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NB : Similar expression when M^θ *depends on some* $\theta \in \mathbb{R}$.

The last key

- Backward Markov models

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) \propto \eta_0(dx_0)Q_1(x_0,dx_1)\ldots Q_n(x_{n-1},dx_n)$$

with

$$\begin{aligned} Q_n(x_{n-1},dx_n) &:= G_{n-1}(x_{n-1})M_n(x_{n-1},dx_n) \\ &\overset{hyp}{=} H_n(x_{n-1},x_n)\;\nu_n(dx_n) \end{aligned}$$

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&\Rightarrow \eta_{n+1}(dx) = \frac{1}{\eta_n(G_n)}\ \eta_n\left(H_{n+1}(\,.\,,x)\right)\ \nu_{n+1}(dx)
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If we set

$$\mathbb{M}_{n+1,\eta_n}(x_{n+1},dx_n) = \frac{\eta_n(dx_n)\ H_{n+1}(x_n,x_{n+1})}{\eta_n\left(H_{n+1}(\,.\,,x_{n+1})\right)}$$

then we find the backward equation

$$\eta_{n+1}(dx_{n+1})\ \mathbb{M}_{n+1,\eta_n}(x_{n+1},dx_n) = \frac{1}{\eta_n(G_n)}\ \eta_n(dx_n)\ Q_{n+1}(x_n,dx_{n+1})$$

The last key (continued)

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) \propto \eta_0(dx_0)Q_1(x_0,dx_1)\ldots Q_n(x_{n-1},dx_n)$$

$\oplus$

$$\eta_{n+1}(dx_{n+1})\ \mathbb{M}_{n+1,\eta_n}(x_{n+1},dx_n) \propto \eta_n(dx_n)\ Q_{n+1}(x_n,dx_{n+1})$$

The last key (continued)

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$$\Downarrow$$

Backward Markov chain model :

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) = \eta_n(dx_n)\ \mathbb{M}_{n,\eta_{n-1}}(x_n,dx_{n-1})\ldots\mathbb{M}_{1,\eta_0}(x_1,dx_0)$$

with the dual/backward Markov transitions

$$\mathbb{M}_{n+1,\eta_n}(x_{n+1},dx_n) \propto \eta_n(dx_n)\ H_{n+1}(x_n,x_{n+1})$$

How to use the full ancestral tree model ?

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) = \eta_n(dx_n) \quad \underbrace{\mathbb{M}_{n,\eta_{n-1}}(x_n, dx_{n-1})}_{\propto\ \eta_{n-1}(dx_{n-1})\ H_n(x_{n-1},x_n)} \quad \ldots \mathbb{M}_{1,\eta_0}(x_1, dx_0)$$

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$$\Downarrow$$

Particle approximation

$$\mathbb{Q}_n^N(d(x_0,\ldots,x_n)) = \eta_n^N(dx_n)\ \mathbb{M}_{n,\eta_{n-1}^N}(x_n, dx_{n-1}) \ldots \mathbb{M}_{1,\eta_0^N}(x_1, dx_0)$$

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Ex.: Additive functionals $\quad \mathbf{f_n}(x_0,\ldots,x_n) = \frac{1}{n+1}\sum_{0\leq p\leq n} f_p(x_p)$

$$\underbrace{\mathbb{Q}_n^N(\mathbf{f_n})}_{\textbf{path-integration}} := \frac{1}{n+1}\sum_{0\leq p\leq n}\eta_n^N\underbrace{\mathbb{M}_{n,\eta_{n-1}^N}\ldots\mathbb{M}_{p+1,\eta_p^N}(f_p)}_{\textbf{recursive matrix operations}}$$

Backward particle models

Cts (c_1, c_2) related to (bias,variance,a,b) $\mathbf{f_n}$ normalized additive functional with $\|f_p\| \leq 1$, $\forall \; (x \geq 0, n \geq 0, N \geq 1)$

$$\Downarrow$$

The probability of the event

$$\left[\mathbb{Q}_n^N - \mathbb{Q}_n\right](\overline{\mathbf{f}}_n) \leq c_1 \; \frac{1}{N} \; (1 + (x + \sqrt{x})) + c_2 \; \sqrt{\frac{x}{N(n+1)}}$$

is greater than $1 - e^{-x}$.

Introduction

Absorption models

Extended path integration models

Feynman-Kac models

Stochastic analysis

How & Why it works
- A local fluctuation theorem
- Second order decompositions
- Uniform concentration w.r.t. time
- Particle free energy expansions

Continuous time models

How & Why it works (general mean field models)

- (Computer Sci.) Stochastic adaptive grid approximation.

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$\Longrightarrow$ Uniform control w.r.t. the time horizon

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$\rightsquigarrow$ **New concentration inequalities for (general) interacting processes**

A stochastic perturbation analysis

Key telescoping decomposition

$$\eta_n^N - \eta_n = \sum_{p=0}^{n} \left[\Phi_{p,n}(\eta_p^N) - \Phi_{p,n}\left(\Phi_p(\eta_{p-1}^N)\right)\right]$$

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⊕ First order expansion

$$\sqrt{N} \left[\Phi_{p,n}(\eta_p^N) - \Phi_{p,n}\left(\Phi_p(\eta_{p-1}^N)\right)\right]$$

$$= \sqrt{N} \left[\Phi_{p,n}\left(\Phi_p(\eta_{p-1}^N) + \frac{1}{\sqrt{N}} \; V_p^N\right) - \Phi_{p,n}\left(\Phi_p(\eta_{p-1}^N)\right)\right]$$

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$$\simeq V_p^N D_{p,n} + \frac{1}{\sqrt{N}}\ R_{p,n}^N$$

with $\underbrace{\text{a predictable } D_{p,n} - \text{first order operator}}_{\text{fluctuation term}} \oplus \underbrace{\text{2nd-order measure } R_{p,n}^N}_{\text{bias-term}}$

Uniform concentration w.r.t. time

Stochastic perturbation model

$$W_n^{\eta,N} := \sqrt{N}\left[\eta_n^N - \eta_n\right] = \sum_{0\leq p\leq n} V_p^N D_{p,n} + \frac{1}{\sqrt{N}}\, R_n^N$$

Under some mixing condition on the limiting FK semigroups $\Phi_{p,n}$

$$\operatorname{osc}\left(D_{p,n}(f)\right) \leq Cte\ e^{-(n-p)\alpha}$$

and

$$\mathbb{E}\left(|R_n^N(f)|^m\right) \leq Cte\ 2^{-m}(2m)!/m!$$

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$$\Downarrow$$

Uniform concentration estimates w.r.t. the time parameter

Particle free energy expansions

Multiplicative formulae

$$\mathcal{Z}_n^N = \prod_{0 \leq p < n} \eta_p^N(G_p) = \gamma_n^N(1) \longrightarrow_{N \uparrow \infty} \gamma_n(1) = \prod_{0 \leq p < n} \eta_p(G_p)$$

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Taylor first order expansion

$$\forall x, y > 0 \qquad \log y - \log x = \int_0^1 \frac{(y-x)}{x+t(y-x)}\ dt$$

$$\Downarrow$$

$$\log\left(\gamma_n^N(1)/\gamma_n(1)\right)$$

$$= \sum_{0\leq p<n}\left(\log \eta_p^N(G_p) - \log \eta_p(G_p)\right)$$

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Particle free energy expansions

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$$= \tfrac{1}{\sqrt{N}} \sum_{0\leq p<n} \int_0^1 \frac{W_p^{\eta,N}(G_p)}{\eta_p(G_p) + \tfrac{t}{\sqrt{N}}\; W_p^{\eta,N}(G_p)}\; dt$$

$\rightsquigarrow$ first order expansion [exercice]

Introduction

Absorption models

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How & Why it works

Continuous time models

- Discrete time formulations
- Fully continuous time models
- Some examples of McKean models

⊃ Continuous time models

- ⊃ **Continuous time models with $X_n := X'_{[t_n, t_{n+1}[}$**

$$G_n(X_n) = \exp \int_{t_n}^{t_{n+1}} V_s(X'_s) ds$$

or

$$G_n(X_n) = \exp \int_{t_n}^{t_{n+1}} \left[V^1_s(X'_s) dW_s + V^2_s(X'_s) ds \right]$$

- ⊃ **Euler style approximations**

Fully continuous time Feynman-Kac models

$$d\mathbb{Q}_t := \frac{1}{\mathcal{Z}_t} \; \exp\left\{\int_0^t V_s(X_s)ds\right\} d\mathbb{P}_t \quad \text{with} \quad \mathbb{P}_n = \text{Law}(X_{[0,t]})$$

and

$$\mathcal{Z}_t = \mathbb{E}\left(\exp\int_0^t V_s(X_s)ds\right)$$

η_t=t-marginal measures of $\mathbb{Q}_t$ and the unnormalized $\gamma_t = \mathcal{Z}_t \times \eta_t$

$$\Downarrow$$

Key formula: $\displaystyle \frac{1}{t}\log\mathcal{Z}_t = \frac{1}{t}\int_0^t \eta_s(V_s)ds$

$$\Downarrow \;\; (L_t = \text{generator of } X_t)$$

$$\begin{aligned}
\frac{d}{dt}\gamma_t(f) &= \gamma_t(L_t^V(f)) \quad \text{with} \quad L_t^V = L_t + V_t \\
\frac{d}{dt}\eta_t(f) &= \eta_t(L_{t,\eta_t}(f)) \quad \text{with} \quad L_{t,\eta_t} = L_t + V_t - \eta_t(V_t)
\end{aligned}$$

Example : $V_t = -U \leq 0$ and $L_t = L$

Absorption model $E^c = E \cup \{c\}$:

$$L^V(f)(x) = L(f)(x) + \underbrace{U(x)}_{\text{absorption rate}} \int (f(y) - f(x))\ \delta_c(dy)$$

Interacting jump generator

$$L_{\eta_t}(f)(x) = \underbrace{L(f)(x)}_{\text{free exploration}} + \underbrace{U(x)}_{\text{acceptance/jump rate}} \int (f(y) - f(x)) \underbrace{\eta_t(dy)}_{\text{interacting jump law}}$$

$$\Downarrow$$

Particle model when $\eta_t \simeq \eta_t^N = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_t^i}$

Survival-acceptance rates $\oplus$ Interacting-recycling jumps

Other examples (non uniqueness of McKean models)

$V_t = U > 0$ and $L_t = L$ Interacting jump generator

$$L_{\eta_t}(f)(x) = \underbrace{L(f)(x)}_{\text{free exploration}} + \underbrace{\eta_t(U)}_{\text{acceptance/jump rate}} \int (f(y) - f(x)) \underbrace{\Psi_{U_t}(\eta_t)(dy)}_{\text{interacting jump law}}$$

$\forall V_t$ and $L_t = L$ Interacting jump generator

$$L_{\eta_t}(f)(x) = \underbrace{L(f)(x)}_{\text{free exploration}} + \underbrace{\eta_t((V - V(x))_+)}_{\text{acceptance/jump rate}} \int (f(y) - f(x)) \underbrace{\Psi_{(V-V(x))_+}(\eta_t)(dy)}_{\text{interacting jump law}}$$

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In all cases, in practice we work with the discrete time models

- **Geometric clocks (discrete time) $\rightsquigarrow$ Poisson interacting jump rates (continuous time)**

  Feynman-Kac particle integration with geometric interacting jumps (with P. Jacob, A. Lee, L. Murray, G.W. Peters). ArXiv preprint arXiv:1211.7191 (2012).