A backward particle interpretation of Feynman-Kac path measures with applications

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Some hyper-refs

- Feynman-Kac formulae, Genealogical & Interacting Particle Systems with appl., Springer (2004)
- Sequential Monte Carlo Samplers JRSS B. (2006). (joint work with A. Doucet & A. Jasra)
- A Backward Particle Interpretation of Feynman-Kac Formulae M2AN (2010). (joint work with A. Doucet & S.S. Singh)
- On the concentration of interacting processes. Foundations & Trends in Machine Learning [170p.] (2012). (joint work with Peng Hu & Liming Wu) [+ Refs]
- More references on the website http://www.math.u-bordeaux1.fr/∼delmoral/index.html [+ Links]

Basic notation

$\mathcal{P}(E)$ probability meas., $\mathcal{B}(E)$ bounded functions on E.

- $(\mu, f) \in \mathcal{P}(E) \times \mathcal{B}(E) \quad \longrightarrow \quad \mu(f) = \int \mu(dx)\ f(x)$
- $Q(x_1, dx_2)$ **integral operators** $x_1 \in E_1 \rightsquigarrow x_2 \in E_2$

$$\begin{aligned} Q(f)(x_1) &= \int Q(x_1, dx_2) f(x_2) \\ [\mu Q](dx_2) &= \int \mu(dx_1) Q(x_1, dx_2) \qquad (\Longrightarrow [\mu Q](f) = \mu[Q(f)]\ ) \end{aligned}$$

- **Boltzmann-Gibbs transformation**

  [Positive and bounded potential function G]

$$\mu(dx) \quad \mapsto \quad \Psi_G(\mu)(dx) = \frac{1}{\mu(G)}\ G(x)\ \mu(dx)$$

Feynman-Kac measures

- Markov chain X_n on E_n, with transitions M_n :

$$\mathbb{P}_n\left(d(x_0,\ldots,x_n)\right) = \eta_0(dx_0)M_1(x_0,dx_1)\ldots M_n(x_{n-1},dx_n)$$

- Potential functions $G_n(x_n) \in [0,1]$

$$d\mathbb{Q}_n := \frac{1}{\mathcal{Z}_n}\left\{\prod_{0\leq p<n} G_p(X_p)\right\} d\mathbb{P}_n$$

Flow of *n*-marginals

$$\eta_n(f) = \gamma_n(f)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f) := \mathbb{E}\left(f(X_n)\prod_{0\leq p<n} G_p(X_p)\right)$$

Nonlinear equation:

$$\eta_{n+1} = \Psi_{G_n}(\eta_n)M_{n+1} := \Phi_{n+1}(\eta_n)$$

$\supset$ *Continuous time models*

$$X_n := X'_{[t_n, t_{n+1}[} \quad \& \quad G_n(X_n) = \exp \int_{t_n}^{t_{n+1}} V_s(X'_s) ds$$

$$\Downarrow$$

$$\prod_{0 \leq p < n} G_p(X_p) = \exp \left\{ \int_{t_0}^{t_n} V_s(X'_s) ds \right\} d\mathbb{P}'_{t_n}$$

$$\Downarrow$$

$$d\mathbb{Q}_n = \frac{1}{\mathcal{Z}_n} \exp \left\{ \int_{t_0}^{t_n} V_s(X'_s) ds \right\}$$

$\supset$ *Continuous time models*

$$X_n := X'_{[t_n, t_{n+1}[} \quad \& \quad G_n(X_n) = \exp \int_{t_n}^{t_{n+1}} V_s(X'_s) ds$$

$$\Downarrow$$

$$\prod_{0 \le p < n} G_p(X_p) = \exp \left\{ \int_{t_0}^{t_n} V_s(X'_s) ds \right\} \, d\mathbb{P}'_{t_n}$$

$$\Downarrow$$

$$d\mathbb{Q}_n = \frac{1}{\mathcal{Z}_n} \, \exp \left\{ \int_{t_0}^{t_n} V_s(X'_s) ds \right\}$$

or using a simple "Euler's scheme" $X'_{t_p} = X_p$

$$e^{\int_{t_0}^{t_n} \left[V_s(X'_s) ds + W_s(X'_s) dB_s \right]} \simeq \prod_{0 \le p < n} e^{V_{t_p}(X_p) \, \Delta t + W_{t_p}(X_p) \sqrt{\Delta t} \, N_p(0,1)}$$

Self avoiding walks

$$\mathbf{X_n} = (X_0, \ldots, X_n) \quad \& \quad \mathbf{G_n}(\mathbf{X_n}) = 1_{X_n \notin \{X_0, \ldots, X_{n-1}\}}$$

$$\Downarrow$$

$$\begin{aligned} \mathbb{Q}_n \quad &:= \quad \frac{1}{\mathcal{Z}_n} \left\{ \prod_{0 \leq p < n} G_p(X_p) \right\} d\mathbb{P}_n \\ &= \quad \mathrm{Law}\left((\mathbf{X_0}, \ldots, \mathbf{X_n}) \mid X_p \neq X_q, \ \forall 0 \leq p < q < n \right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{Z}_n \quad &= \quad \gamma_n(1) \\ &= \quad \mathrm{Proba}\left(X_p \neq X_q, \ \forall 0 \leq p < q < n \right) \end{aligned}$$

Absorption models

- Sub-Markov semigroups

$$Q_n(x,dy) = G_{n-1}(x)\ M_n(x,dy) \rightsquigarrow E_n^c = E_n \cup \{c\}$$

- Absorbed Markov chain

$$X_n^c \in E_n^c \xrightarrow{\text{absorption } \sim (1-G_n)} \widehat{X}_n^c \xrightarrow{\text{exploration } \sim M_n} X_{n+1}^c$$

$$\Downarrow$$

$$\mathbb{Q}_n = \mathrm{Law}((X_0^c, \ldots, X_n^c) \mid T^{absorption} \geq n)$$

and

$$\mathcal{Z}_n = \mathrm{Proba}\left(T^{absorption} \geq n\right)$$

Homogeneous models $(G_n, M_n) = (G, M)$

- Reversibility condition : $\mu(dx)M(x,dy) = \mu(dy)M(y,dx)$

$$\mathrm{Proba}\left(T^{absorption} \geq n\right) \simeq \lambda^n$$

with $\lambda =$ top eigenvalue of

$$Q(x,dy) = G(x)\ M(x,dy)$$

- $Q(h) = \lambda h \rightsquigarrow$ Doob h-process X^h

$$M^h(x,dy) = \frac{1}{\lambda} h^{-1}(x) Q(x,dy) h(y) = \frac{Q(x,dy)h(y)}{Q(h)(x)} = \frac{M(x,dy)h(y)}{M(h)(x)}$$

Homogeneous models $(G_n, M_n) = (G, M)$

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) \propto \mathbb{P}((X_0^h,\ldots,X_n^h) \in d(x_0,\ldots,x_n))\ h^{-1}(x_n)$$

- *Invariant measure $\mu_h = \mu_h M^h$ & normalized additive functionals*

$$\overline{F}_n(x_0,\ldots,x_n) = \frac{1}{n+1}\sum_{0\leq p\leq n} f(x_p) \Longrightarrow \mathbb{Q}_n(\overline{F}_n) \simeq_n \mu_h(f)$$

- If $G = G^\theta$ depends on some $\theta \in \mathbb{R} \rightsquigarrow f := \frac{\partial}{\partial\theta}\log G^\theta$

$$\frac{\partial}{\partial\theta}\log\lambda^\theta \simeq_n \frac{1}{n+1}\frac{\partial}{\partial\theta}\log\mathcal{Z}_{n+1}^\theta = \mathbb{Q}_n(\overline{F}_n)$$

NB : Similar expression when M^θ depends on some $\theta \in \mathbb{R}$.

Filtering and Hidden Markov Chains

- Signal-Observation $(X, Y) =$ Markov chain

  $$\mathbb{P}\left((X_n, Y_n) \in d(x,y) | (X_{n-1}, Y_{n-1})\right) := M_n(X_{n-1}, dx)\, g_n(x,y)\, \nu_n(dy)$$

- Conditional distributions : $Y = y$ fixed, $G_n(x_n) \propto g_n(x_n, y_n)$

  $$\begin{array}{rcl} \mathbb{Q}_n & = & \mathrm{Law}((X_0, \ldots, X_n) \mid \forall 0 \leq p < n \quad Y_p = y_p) \\ \mathcal{Z}_{n+1} & = & p_n(y_0, \ldots, y_n) \end{array}$$

- Hidden Markov chains $\theta \mapsto (X^\theta, Y^\theta)$

  Pb : $\mathrm{Arg\text{-}max}_\theta p_n(y_0, \ldots, y_n \mid \theta)$ $\rightsquigarrow$ Algo. EM $\oplus$ Gradient

  $$\theta_n = \theta_{n-1} + \tau_n \, \nabla \log \mathcal{Z}_n^{\theta_{n-1}}$$

  Posterior models : $\mathbb{P}(\Theta \in d\theta \mid Y_p, \; p < n) \propto \mathcal{Z}_n^\theta \times \mathbb{P}(\Theta \in d\theta)$

Rare event simulation (1)

Ruin and default probabilities = Level crossing probabilities

$$\mathbb{P}\left(V_n(X_n) \geq a\right) \quad \text{or} \quad \mathbb{P}\left(X \text{ hits } A_n \text{ before } B\right)$$

- Level crossing at a fixed given time

$$\begin{aligned} \mathbb{P}\left(V_n(X_n) \geq a\right) &= \mathbb{E}\left(f_n(X_n)\; e^{V_n(X_n)}\right) \\ &= \mathbb{E}\left(\mathbf{f_n}(\mathbf{X}_n) \prod_{0\leq p<n} G_p(\mathbf{X}_p)\right) \end{aligned}$$

with

- The Markov chain on transition space

$$\mathbf{X}_n = (X_n, X_{n+1}) \quad \text{and} \quad G_n(\mathbf{X}_n) = \exp\left[V_{n+1}(X_{n+1}) - V_n(X_n)\right]$$

- The test functions

$$f_n(\mathbf{X}_n) = 1_{V_n(X_n)\geq a}\; e^{-V_n(X_n)}$$

Rare event simulation (2)

- Excursion level crossing $A_n \downarrow$, with B non critical recurrent subset.

$$\mathbb{P}(X \text{ hits } A_n \text{ before } B) \quad = \quad \mathbb{E}\left(\prod_{0\leq p\leq n} 1_{A_p}(X_{T_p})\right)$$

$$T_n := \inf\{p \geq T_{n-1} \ : \ X_p \in (A_n \cup B)\}$$

Feynman-Kac model

$$\mathbb{E}\left(\prod_{0\leq p\leq n} 1_{A_p}(X_{T_p})\right) = \mathbb{E}\left(\prod_{0\leq p<n} G_p(\mathbf{X}_p)\right)$$

with

$$\mathbf{X}_n = (X_p)_{p\in[T_n,T_{n+1}]} \quad \& \quad G_n(\mathbf{X}_n) = 1_{A_{n+1}}(X_{T_{n+1}})$$

Path measures and Historical processes

- Time marginal measures = Path space measures:

$$\gamma_n(f_n) = \mathbb{E}\left(f_n(\mathbf{X_n}) \prod_{0 \leq p < n} \mathbf{G_p}(\mathbf{X_p}) \right)$$

with the historical process

$$[\ \mathbf{X_n} := (X_0, \ldots, X_n) \ \ \& \ \ \mathbf{G_n}(\mathbf{X_n}) = G_n(X_n)]$$

$$\Downarrow$$

$$\eta_n = \mathbb{Q}_n$$

Free energy models

- Normalizing constants (= Free energy models):

$$\mathcal{Z}_n = \gamma_n(1) = \mathbb{E}\left(\prod_{0\leq p<n} G_p(X_p)\right) = \prod_{0\leq p<n} \eta_p(G_p)$$

and

$$\gamma_n(f) = \eta_n(f) \times \gamma_n(1) = \eta_n(f) \times \prod_{0\leq p<n} \eta_p(G_p)$$

$\triangledown$ *Product formula*

$$\gamma_n(1) = \gamma_{n-1}(G_{n-1}) = \eta_{n-1}(G_{n-1})\ \gamma_{n-1}(1)$$

$\triangle$

A backward Markov chain model

- Notice that

$$\mathbb{Q}_n(d(x_0, \ldots, x_n)) \propto \eta_0(dx_0) Q_1(x_0, dx_1) \ldots Q_n(x_{n-1}, dx_n)$$

with

$$\begin{aligned} Q_n(x_{n-1}, dx_n) &:= G_{n-1}(x_{n-1}) M_n(x_{n-1}, dx_n) \\ &\overset{hyp}{=} H_n(x_{n-1}, x_n)\ \nu_n(dx_n) \\ &\Rightarrow \eta_{n+1}(dx) = \frac{1}{\eta_n(G_n)}\ \eta_n\left(H_{n+1}(\,.\,, x)\right)\ \nu_{n+1}(dx) \end{aligned}$$

If we set

$$\mathbb{M}_{n+1,\eta_n}(x_{n+1}, dx_n) = \frac{\eta_n(dx_n)\ H_{n+1}(x_n, x_{n+1})}{\eta_n\left(H_{n+1}(\,.\,, x_{n+1})\right)}$$

then we find the backward equation

$$\eta_{n+1}(dx_{n+1})\ \mathbb{M}_{n+1,\eta_n}(x_{n+1}, dx_n) = \frac{1}{\eta_n(G_n)}\ \eta_n(dx_n)\ Q_{n+1}(x_n, dx_{n+1})$$

A backward Markov chain model (continued)

In summary

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) \propto \eta_0(dx_0)Q_1(x_0,dx_1)\ldots Q_n(x_{n-1},dx_n)$$

$\bigoplus$

$$\eta_{n+1}(dx_{n+1})\ \mathbb{M}_{n+1,\eta_n}(x_{n+1},dx_n) \propto \eta_n(dx_n)\ Q_{n+1}(x_n,dx_{n+1})$$

$$\Downarrow$$

Backward Markov chain model :

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) = \eta_n(dx_n)\ \mathbb{M}_{n,\eta_{n-1}}(x_n,dx_{n-1})\ldots\mathbb{M}_{1,\eta_0}(x_1,dx_0)$$

with the dual/backward Markov transitions

$$\mathbb{M}_{n+1,\eta_n}(x_{n+1},dx_n) \propto \eta_n(dx_n)\ H_{n+1}(x_n,x_{n+1})$$

Feynman-Kac evolution semigroups

Evolution equations

$$\eta_n \xrightarrow{\text{Correction-Selection}} \widehat{\eta}_n = \Psi_{G_n}(\eta_n) \xrightarrow{\text{Prediction-Mutation}} \eta_{n+1} = \widehat{\eta}_n M_{n+1}$$

with the nonlinear mass transport equation

$$\Psi_{G_n}(\eta_n)(dy) := \frac{1}{\mu(G_n)}\; G_n(y)\; \eta_n(dy) = \int \eta_n(dx) S_{n,\eta_n}(x, dy)$$

where

$$S_{n,\eta_n}(x, .) := G_n(x)\; \delta_x + (1 - G_n(x))\;\; \Psi_{G_n}(\eta_n)$$

$$\Downarrow$$

$$\eta_{n+1} = \eta_n\left(S_{n,\eta_n} M_{n+1}\right) := \eta_n K_{n+1,\eta_n} = \mathrm{Law}(\overline{X}_n)$$

McKean Markov chain model $\overline{X}_n$ = Perfect sampler

$$\mathbb{P}(\overline{X}_{n+1} \in dx \mid \overline{X}_n) = K_{n+1,\eta_n}(\overline{X}_n, dx)$$

$\rightsquigarrow$ **McKean measure**

$$\overline{\mathbb{P}}_n(d(x_0,\ldots,x_n)) = \eta_0(dx_0)\; K_{1,\eta_0}(x_0,dx_1)\ldots K_{n,\eta_{n-1}}(x_n,dx)$$

Mean field particle model = Stochastic population dynamic model

$\rightsquigarrow$ $\xi_n = (\xi_n^1,\ldots,\xi_n^N) \in E_n^N$ **such that**

$$\eta_n^N := \frac{1}{N}\sum_{1\leq i\leq N} \delta_{\xi_n^i} \;\simeq_{N\uparrow\infty}\; \eta_n$$

$\rightsquigarrow$ Natural approximated transitions

$$\xi_n^i \rightsquigarrow \xi_{n+1}^i \sim K_{n+1,\eta_n^N}(\xi_n^i, dx)$$

$\Leftrightarrow$ Stochastic population dynamics

$$
\left.\begin{matrix} \xi_n^1 \\ \vdots \\ \xi_n^i \\ \vdots \\ \xi_n^N \end{matrix}\right] \xrightarrow{S_{n,\eta_n^N}} \left[\begin{matrix} \widehat{\xi}_n^1 & \xrightarrow{M_{n+1}} & \xi_{n+1}^1 \\ \vdots & & \vdots \\ \widehat{\xi}_n^i & \longrightarrow & \xi_{n+1}^i \\ \vdots & & \vdots \\ \widehat{\xi}_n^N & \longrightarrow & \xi_{n+1}^N \end{matrix}\right]
$$

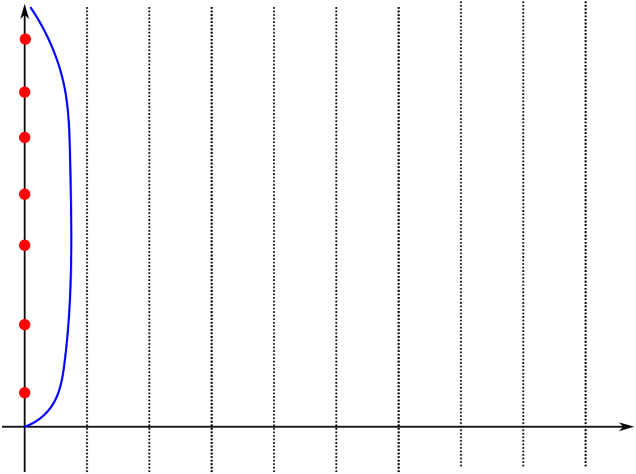
Accept/Reject-Recycle=Selection =Geometric recycling interacting jumps

$$
S_{n,\eta_n^N}(\xi_n^i, dx)
$$

$$
:= G_n(\xi_n^i)\ \delta_{\xi_n^i}(dx) + \left(1 - G_n(\xi_n^i)\right)\ \sum_{j=1}^N \frac{G_n(\xi_n^j)}{\sum_{k=1}^N G_n(\xi_n^k)} \delta_{\xi_n^j}(dx)
$$

Ex. : $G_n = 1_A \rightsquigarrow G_n(\xi_n^i) = 1_A(\xi_n^i)$

Illustration : $\eta_n \simeq \eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i}$



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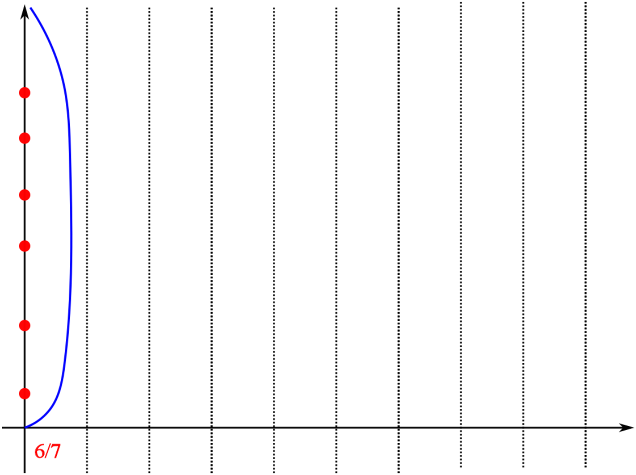

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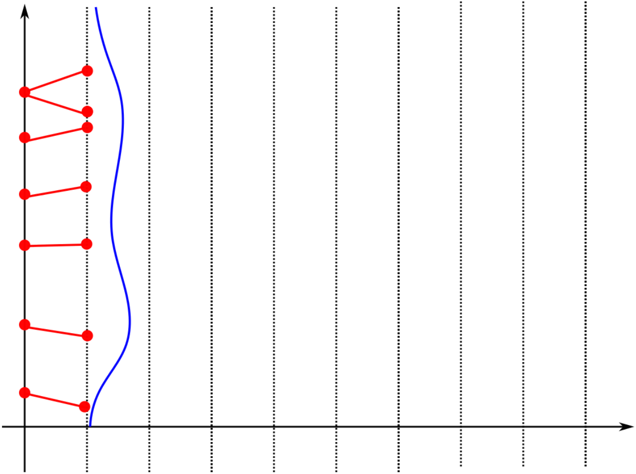

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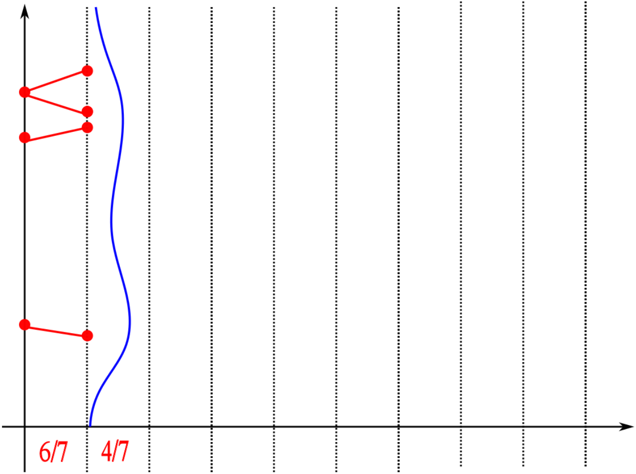

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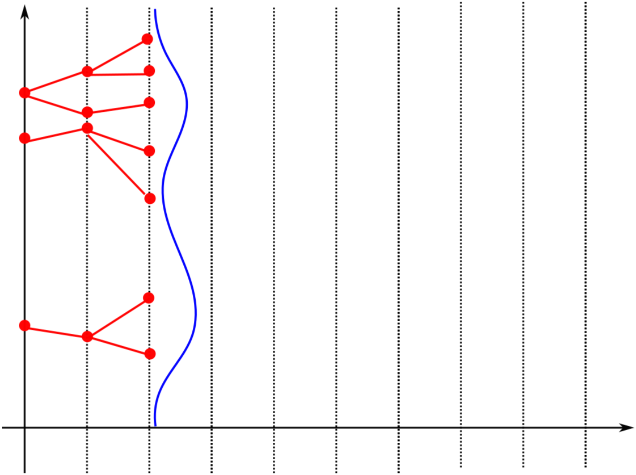

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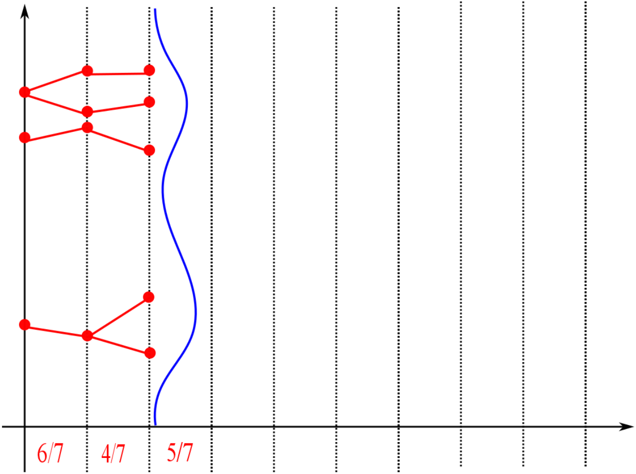

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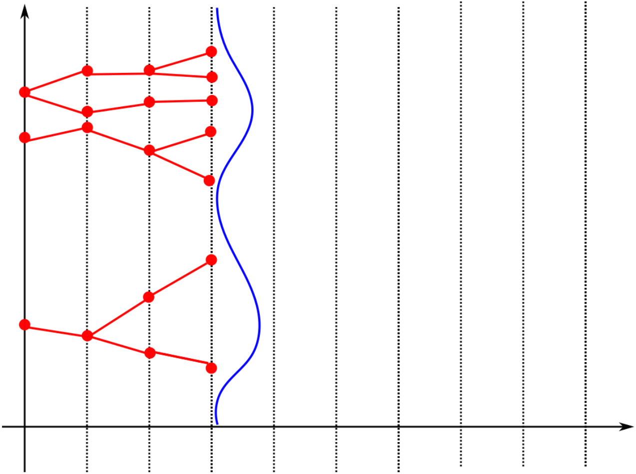

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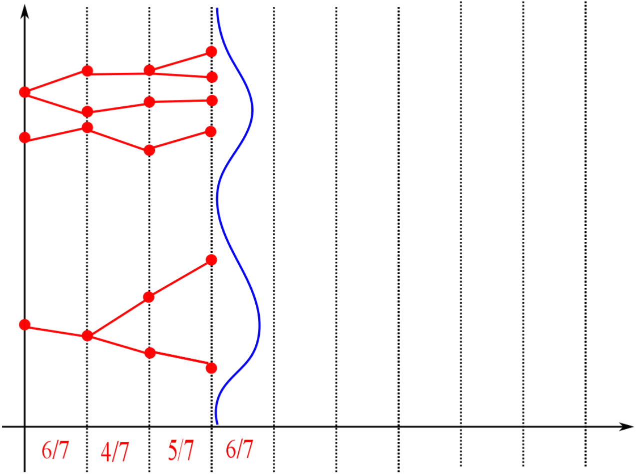

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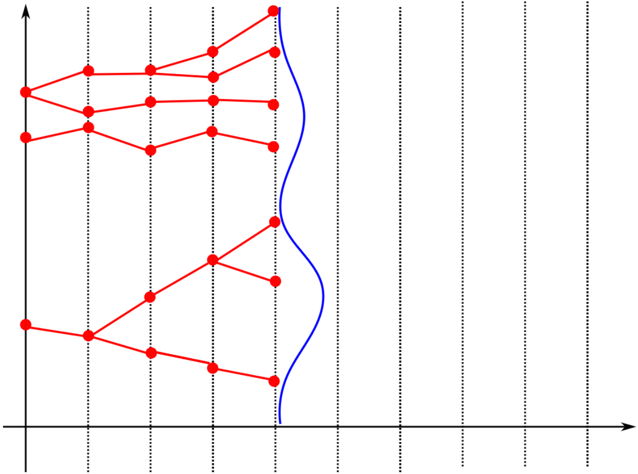

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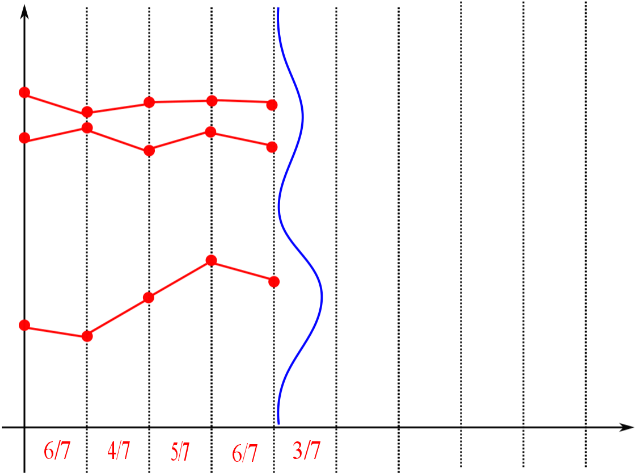

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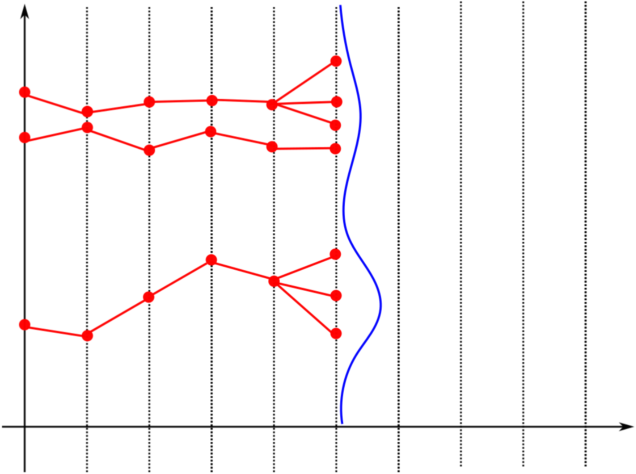

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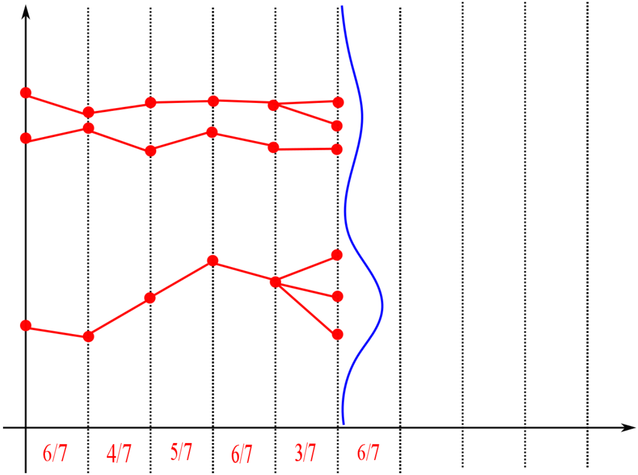

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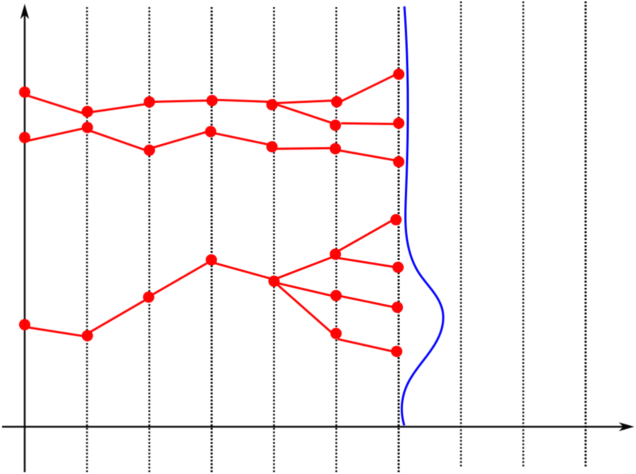

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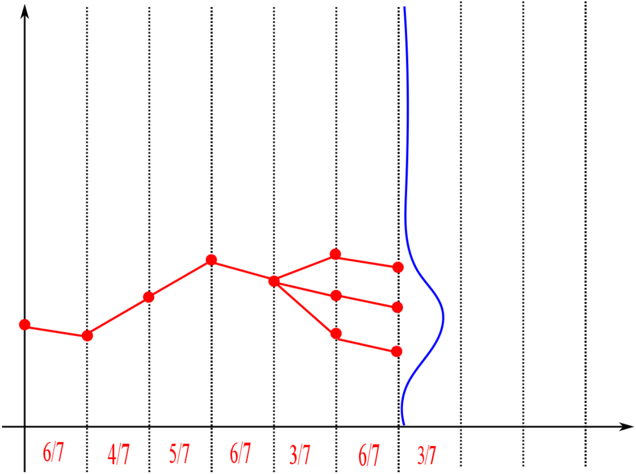

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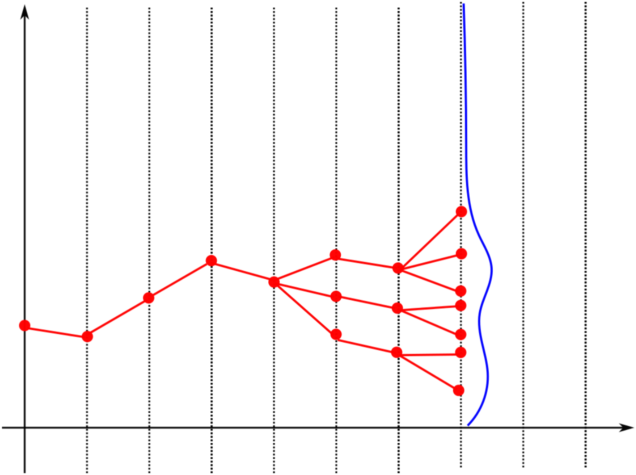

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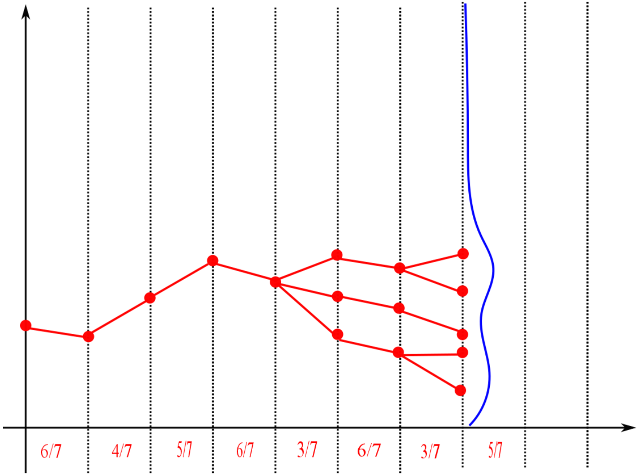

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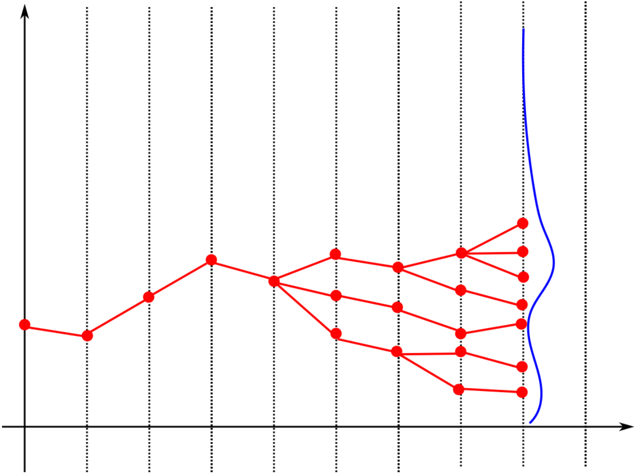

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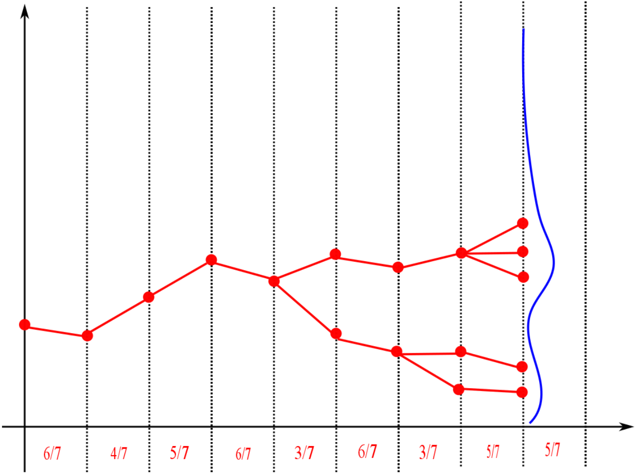

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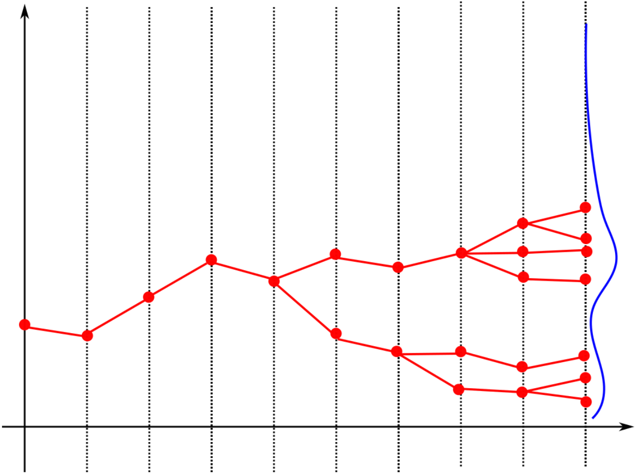


Some key advantages

- Mean field models=Stochastic linearization/perturbation technique

$$\eta_n^N = \eta_{n-1}^N K_{n,\eta_{n-1}^N} + \frac{1}{\sqrt{N}} \; W_n^N$$

  with (**Theorem**) $W_n^N \simeq W_n$ independent centered Gaussian fields .

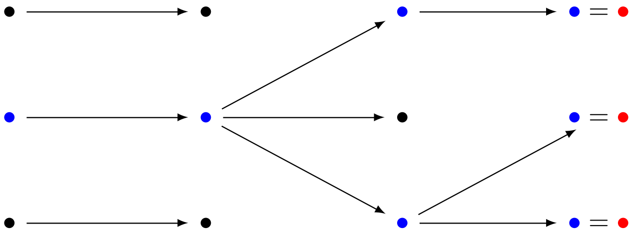
- $\eta_n = \eta_{n-1} K_{n,\eta_{n-1}}$ stable $\Rightarrow$ Non propagation of local sampling errors

  $\Longrightarrow$ Uniform control w.r.t. the time horizon

- "No burning, no need to study the stability of MCMC models".
- Stochastic adaptive grid approximation
- Nonlinear system $\rightsquigarrow$ positive - beneficial interactions.
- Simple and natural sampling algorithm.
- Local conditional iid samples $\oplus$ Stability of nonlinear sg
  $\rightsquigarrow$ New concentration and empirical process theory

The 5 particle estimates

Genealogical tree evolution $(N, n) = (3, 3)$

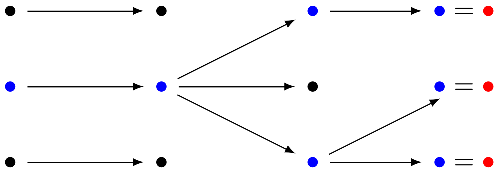


- Individuals in the current population ($\simeq$ i.i.d. η_n)

$$\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \longrightarrow_{N \to \infty} \eta_n = \textbf{solution of a nonlinear m.v.p.}$$

$$\overline{\mathbb{P}}_n^N = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{(\xi_0^i, \xi_1^i, \ldots, \xi_n^i)} \longrightarrow_{N \to \infty} \overline{\mathbb{P}}_n = \textbf{McKean measure.}$$

Two more particle estimates



- Ancestral lines = *Almost* i.i.d. sampled paths w.r.t. $\mathbb{Q}_n$.

$$\left(\xi^i_{0,n}, \xi^i_{1,n}, \ldots, \xi^i_{n,n}\right) := i\text{-th ancetral line of the } i\text{-th current individual} = \xi^i_n$$

$$\frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\left(\xi^i_{0,n}, \xi^i_{1,n}, \ldots, \xi^i_{n,n}\right)} \longrightarrow_{N \to \infty} \mathbb{Q}_n$$

- Unbiased particle free energy functions

$$\mathcal{Z}^N_n = \prod_{0 \leq p < n} \eta^N_p(G_p) \longrightarrow_{N \to \infty} \mathcal{Z}_n = \prod_{0 \leq p < n} \eta_p(G_p)$$

... and the last particle measure

$$\mathbb{Q}_n^N(d(x_0,\ldots,x_n)) := \eta_n^N(dx_n)\ \mathbb{M}_{n,\eta_{n-1}^N}(x_n,dx_{n-1})\ldots\mathbb{M}_{1,\eta_0^N}(x_1,dx_0)$$

with the random particle matrices:

$$\mathbb{M}_{n+1,\eta_n^N}(x_{n+1},dx_n) \propto \eta_n^N(dx_n)\ H_{n+1}(x_n,x_{n+1})$$

Example: Normalized additive functionals

$$\mathbf{f_n}(x_0,\ldots,x_n) = \frac{1}{n+1}\sum_{0\leq p\leq n} f_p(x_p)$$

$$\Downarrow$$

$$\mathbb{Q}_n^N(\mathbf{f_n}) := \frac{1}{n+1}\sum_{0\leq p\leq n}\eta_n^N\underbrace{\mathbb{M}_{n,\eta_{n-1}^N}\ldots\mathbb{M}_{p+1,\eta_p^N}(f_p)}_{\text{matrix operations}}$$

Current population models

Constants (c_1, c_2) related to (bias,variance), c universal constant
Test funct. $\|f_n\| \leq 1$, $\forall\ (x \geq 0, n \geq 0, N \geq 1)$.

- The probability of the event

$$\left[\eta_n^N - \eta_n\right](f) \leq \frac{c_1}{N}\ \left(1 + x + \sqrt{x}\right) + \frac{c_2}{\sqrt{N}}\ \sqrt{x}$$

  is greater than $1 - e^{-x}$.

- $x = (x_i)_{1 \leq i \leq d} \rightsquigarrow (-\infty, x] = \prod_{i=1}^{d}(-\infty, x_i]$ cells in $E_n = \mathbb{R}^d$.

$$F_n(x) = \eta_n\left(1_{(-\infty,x]}\right) \quad \text{and} \quad F_n^N(x) = \eta_n^N\left(1_{(-\infty,x]}\right)$$

  The probability of the following event

$$\sqrt{N}\ \left\|F_n^N - F_n\right\| \leq c\ \sqrt{d\ (x+1)}$$

  is greater than $1 - e^{-x}$.

Particle free energy models

Constants (c_1, c_2) related to (bias,variance), c universal constant
$\forall\ (x \geq 0, n \geq 0, N \geq 1)$

- Unbiased property

$$\mathbb{E}\left(\eta_n^N(f_n) \prod_{0\leq p<n} \eta_p^N(G_p)\right) = \mathbb{E}\left(f_n(X_n) \prod_{0\leq p<n} G_p(X_p)\right)$$

- For any $\epsilon \in \{+1, -1\}$, the probability of the event

$$\frac{\epsilon}{n} \log \frac{\mathcal{Z}_n^N}{\mathcal{Z}_n} \leq \frac{c_1}{N}\ \left(1 + x + \sqrt{x}\right) + \frac{c_2}{\sqrt{N}}\ \sqrt{x}$$

is greater than $1 - e^{-x}$.

note $(0 \leq \epsilon \leq 1 \Rightarrow (1 - e^{-\epsilon}) \vee (e^{\epsilon} - 1) \leq 2\epsilon)$

$$e^{-\epsilon} \leq \frac{z^N}{z} \leq e^{\epsilon} \Rightarrow \left|\frac{z^N}{z} - 1\right| \leq 2\epsilon$$

Genealogical tree models $:= \eta_n^N$ (in path space)

Constants (c_1, c_2) related to (bias,variance), c universal constant
$\mathbf{f_n}$ test function $\|\mathbf{f_n}\| \leq 1$, $\forall\ (x \geq 0, n \geq 0, N \geq 1)$.

- The probability of the event

$$\left[\eta_n^N - \mathbb{Q}_n\right](f) \leq c_1\ \frac{n+1}{N}\ \left(1 + x + \sqrt{x}\right) + c_2\ \sqrt{\frac{(n+1)}{N}}\ \sqrt{x}$$

  is greater than $1 - e^{-x}$.

- $\mathcal{F}_n =$ indicator fct. $\mathbf{f_n}$ of cells in $\mathbf{E_n} = \left(\mathbb{R}^{d_0} \times \ldots, \times \mathbb{R}^{d_n}\right)$
  The probability of the following event

$$\sup_{\mathbf{f_n} \in \mathcal{F}_n} \left|\eta_n^N(\mathbf{f_n}) - \mathbb{Q}_n(\mathbf{f_n})\right| \leq c\ (n+1)\ \sqrt{\frac{\sum_{0 \leq p \leq n} d_p}{N}\ (x+1)}$$

  is greater than $1 - e^{-x}$.

Backward particle models

Constants (c_1, c_2) related to (bias,variance), c universal constant.
$\mathbf{f_n}$ normalized additive functional with $\|f_p\| \leq 1$, $\forall\ (x \geq 0, n \geq 0, N \geq 1)$

- The probability of the event

$$\left[\mathbb{Q}_n^N - \mathbb{Q}_n\right](\bar{\mathbf{f}}_n) \leq c_1\ \frac{1}{N}\ (1 + (x + \sqrt{x})) + c_2\ \sqrt{\frac{x}{N(n+1)}}$$

- $\mathbf{f}_{a,n}$ normalized additive functional w.r.t. $f_p = 1_{(-\infty,a]}$, $a \in \mathbb{R}^d = E_n$

  .

  The probability of the following event

$$\sup_{a \in \mathbb{R}^d} \left|\mathbb{Q}_n^N(\mathbf{f_{a,n}}) - \mathbb{Q}_n(\mathbf{f_{a,n}})\right| \leq c\ \sqrt{\frac{d}{N}(x+1)}$$

  is greater than $1 - e^{-x}$.