Some theoretical aspects of Ensemble Kalman Filters (and Particle Filters)

P. Del Moral

Imperial College, 25th March 2024.

Continuous time **Linear+Gaussian filtering problem**

$$\begin{cases} dX_t &= & A\, X_t\, dt \;+\; R^{1/2}\, dW_t \;\in \mathbb{R}^r \\ dY_t &= & C\, X_t\, dt \;+\; \Sigma^{1/2}\, dV_t \end{cases} \quad \rightsquigarrow \mathcal{Y}_t := \sigma(Y_s,\ s \leqslant t).$$

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Optimal $\mathbb{L}_2$-estimate = Kalman-Bucy filter

$$\widehat{X}_t := \mathbb{E}(X_t \mid \mathcal{Y}_t) \quad \text{and} \quad P_t := \mathbb{E}\left(\left(X_t - \widehat{X}_t\right)\left(X_t - \widehat{X}_t\right)'\right)$$

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State estimate

$$d\widehat{X}_t = A\, \widehat{X}_t\, dt + P_t\, C'\Sigma^{-1}\, \left(dY_t - C\widehat{X}_t\, dt\right)$$

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$$d\widehat{X}_t = A\,\widehat{X}_t\, dt + P_t\, C'\Sigma^{-1}\left(dY_t - C\widehat{X}_t\, dt\right)$$

with the gain given by the matrix Riccati equation

$$\partial_t P_t = \mathrm{Ricc}(P_t) := AP_t + P_tA' - P_t\mathbf{S}P_t + R \quad \text{with} \quad \mathbf{S} := \mathbf{C}'\mathbf{\Sigma}^{-1}\mathbf{C}$$

Reformulation $\rightsquigarrow$ Nonlinear Kalman-Bucy diffusion

Stoch. version of KB = McKean-Vlasov-type diffusions $\overline{X}_t$:

$$\eta_t := \mathrm{Law}(\overline{X}_t \mid \mathcal{Y}_t) \ = \ \mathcal{N}\left[\widehat{X}_t, P_t\right]$$

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$\rightsquigarrow$ **Interacting with their conditional mean and covariance matrices**

$$\mathcal{P}_{\eta_t} = \eta_t\left[(e-\eta_t(e))(e-\eta_t(e))'\right] \quad \text{with} \quad e(x) := x.$$

and their mean

$$\mathbf{m_t} := \eta_t(e)$$

2 classes of McKean-Vlasov type diffusions

1) "Vanilla EnKF" ($\rightsquigarrow$ (corrected) discrete time - Evensen 94)

$$d\overline{X}_t = A\,\overline{X}_t dt + R^{1/2}\, d\overline{W}_t + \mathcal{P}_{\eta_t} C'\Sigma^{-1}\left[dY_t - \left(C\,\overline{X}_t dt + \Sigma^{1/2}\, d\overline{V}_t\right)\right]$$

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Caution:

Euler-discrete versions not consistent with discrete-time Kalman

More classes of McKean-Vlasov type diffusions

3) Pure transport equation ($\rightsquigarrow$ **discrete time** - **Reich-Cotter 13**)

$$d\overline{X}_t = A\,\overline{X}_t dt$$

$$+\ \frac{1}{2}\ (R - \mathcal{P}_{\eta_t} S \mathcal{P}_{\eta_t})\mathcal{P}_{\eta_t}^{-1}(\overline{X}_t - \eta_t(e))\ dt + \mathcal{P}_{\eta_t} C' \Sigma^{-1}\ [dY_t - C\eta_t(e)dt]$$

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$\oplus$ **Many others, adding** $\mathcal{Q}_{\eta_t}\mathcal{P}_{\eta_t}^{-1}(\overline{X}_t - \eta_t(e))\ dt$ **for any** $\mathcal{Q}'_{\eta_t} = -\mathcal{Q}_{\eta_t}$.

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$\rightsquigarrow$ **Nonlinear models:**

Tempting to replace "A x" and "C x" by A(x), C(x) (often done)

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Tempting to replace "A x" and "C x" by $\mathbf{A}(\mathbf{x}), \mathbf{C}(\mathbf{x})$ (often done)

BUT <u>NOT</u> CONSISTENT WITH THE OPTIMAL FILTER

The Ensemble Kalman-Bucy filter

(Case 1) Mean field interpretation $\rightsquigarrow$ $N+1$ interacting diffusions

$$d\xi_t^i = A\,\xi_t^i dt + R^{1/2} d\overline{W}_t^i + p_t C' \Sigma^{-1} \left[dY_t - \left(C\xi_t^i dt + \Sigma^{1/2}\, d\overline{V}_t^i \right) \right]$$

with the rescaled particle covariance matrices

$$p_t := \frac{1}{N} \sum_{1 \leqslant i \leqslant N+1} \left(\xi_t^i - m_t\right)\left(\xi_t^i - m_t\right)'$$

and the sample mean

$$m_t := \frac{1}{N+1} \sum_{1 \leqslant i \leqslant N+1} \xi_t^i.$$

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Where are the Kalman-Bucy filter and the Riccati equations ?

Th1: The EnKF equations

$$dm_t = A\ m_t dt + p_t\ C'\Sigma^{-1}\ (dY_t - Cm_t\ dt) + \frac{1}{\sqrt{N+1}}\ d\overline{M}_t$$

$\rightsquigarrow$ **r-dim. martingale** $\overline{M}_t = \left(\overline{M}_t(k)\right)_{1\leqslant k\leqslant r}$ **with angle-brackets**

$$\partial_t\langle \overline{M} \mid \otimes \mid \overline{M}\rangle_t = U + p_t V p_t.$$

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With

1) $(U,V) = (R,S)$ 2) $(U,V) = (R,0)$ 3) $(U,V) = (0,0)$

Th2: The particle/ensemble Riccati equation

$$dp_t = \mathrm{Ricc}(p_t)\ dt + \frac{1}{\sqrt{N}}\ dM_t$$

$\rightsquigarrow$ **Symmetric matrix-valued martingale** $M_t = (M_t(k,l))_{1 \leqslant k,l \leqslant r}$

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Angle brackets $=$ the Wick-type formula $((\centerdot \otimes \ \centerdot)^{\sharp}$:= entrywise)

$$\partial_t \langle M \mid \otimes \mid M \rangle_t^{\sharp} = p_t \otimes_{sym} (U + p_t V p_t)$$

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Orthogonality property

$$\forall 1 \leqslant k, l, l' \leqslant r \qquad \left\langle M(k,l), \overline{M}(l') \right\rangle_t = 0.$$

In terms of random matrices with $\epsilon := \frac{2}{\sqrt{N}}$

$$\mathcal{W}_t = (\mathcal{W}_t(i,j))_{1\leqslant i,j\leqslant r} \quad \text{independent Brownian motions}$$

$$\downarrow$$

$$dp_t = \left[Ap_t + p_t A' + R - p_t S p_t\right] \, dt + \epsilon \, \left(p_t^{1/2} d\mathcal{W}_t \left(U + p_t V p_t\right)^{1/2}\right)_{sym}$$

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$$p_t \rightsquigarrow \textit{non colliding eigenvalues} \quad \lambda_r(t) < \ldots < \lambda_2(t) < \lambda_1(t)$$

satisfying the Dyson equation

$$d\lambda_i(t) =$$

$$\left[2\alpha\lambda_i(t) + \beta - \lambda_i(t)^2\gamma + \frac{\epsilon^2}{4} \sum_{j\neq i} \frac{\lambda_i(t) + \lambda_j(t)}{\lambda_i(t) - \lambda_j(t)}\right] dt + \epsilon \sqrt{\lambda_i(t)} \; dW_t^i$$

Analysis/Performance/Convergence/... Crude Monte Carlo

$$\text{Sample mean} \quad m_t := \frac{1}{N} \sum_{1 \leqslant i \leqslant N} Z_t^i \quad \text{with } \textbf{iid} \text{ copies } Z_t^i \text{ of some } Z_t$$

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$\rightsquigarrow$ **stochastic perturbation formulation**

$$\mathbf{m_t} := \mathbb{E}(\mathbf{Z_t}) + \frac{1}{\sqrt{N}} \, M_t^N$$

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$$\mathbb{E}(M_t^N) = 0 \quad \& \quad \mathbb{E}((M_t^N)^2) = \mathbb{E}((Z_t - \mathbb{E}(Z_t))^2)$$

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Key Observation:

$$Z_t \text{ stable} \implies \sup_{t \geqslant 0} \mathbb{E}((m_t - \mathbb{E}(Z_t))^2) \leqslant c/N$$

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$$\lim_{t \to \infty} \mathbb{E}((Z_t - \mathbb{E}(Z_t))^2) = \infty \implies \sup_{t \geqslant 0} \mathbb{E}((m_t - \mathbb{E}(Z_t))^2) = \infty$$

$\rightsquigarrow$ **Need to study the stability of Kalman-Bucy filters**

(conditional means $\oplus$ Riccati matrix eq.)

1d $\rightsquigarrow$ Closed form Riccati semigroups/tangent proc.

Deterministic Riccati flow $\phi_t(Q)$ on $\mathbb{R}_+$: $\mathrm{Ricc}(\varpi_\pm) = 0$ for

$$S\varpi_- := A - \lambda/2 < 0 < S\varpi_+ := A + \lambda/2$$

with

$$\lambda = 2\sqrt{A^2 + RS}$$

$$\Downarrow$$

$\forall t \geqslant \upsilon > 0$

$$\sup_{Q \geqslant 0} \left[|\phi_t(Q) - \varpi_+| \vee exp\left(2\int_0^t [A - \phi_s(Q)S]ds\right)\right] \leqslant c_\upsilon \ \exp\left(-\lambda t\right)$$

Stochastic Riccati flow

$$dp_t = \mathrm{Ricc}(p_t)dt + \frac{2}{\sqrt{N}} \sqrt{p_t(U + p_t V p_t)}\, dW_t$$

$$\text{with} \quad U < RN/2 \quad \Longrightarrow \quad \text{origin repellent}$$

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Reversible measures $\pi_{\mathbf{N}}(dz)$ on $\mathbb{R}_+$:

- $U \wedge V > 0 \rightsquigarrow$ Heavy tails

$$\propto \frac{z^{\frac{\mathbf{N}}{2}\frac{R}{U}-1}}{[U + Vz^2]^{1+\frac{\mathbf{N}}{4}\left(\frac{R}{U}+\frac{S}{V}\right)}} \exp\left[\mathbf{N}\, \frac{A}{\sqrt{UV}}\, \tan^{-1}\left(z\sqrt{\frac{V}{U}}\right)\right] dx.$$

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- $U > V = 0 \rightsquigarrow$ Gaussian-type tails

$$\propto z^{\frac{\mathbf{N}}{2}\frac{R}{U}-1} \exp\left[-\frac{\mathbf{N}S}{4U}\left(z - 2\frac{A}{S}\right)^2\right] dx.$$

$\rightsquigarrow$ **Stability/Time-uniform estimates**/...+Bishop, Kamatani, Rémillard Arxiv17/AAP19

Multivariate KB : Observability + Controllability

$$d(\widehat{X}_t - X_t)$$

$$= (\mathbf{A} - \mathbf{P_t S})\ (\widehat{X}_t - X_t)dt - \boldsymbol{R}^{1/2}\ \boldsymbol{d W_t} + \boldsymbol{P_t C'}\boldsymbol{\Sigma}^{-1/2}\ \boldsymbol{d V_t}$$

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Steady state: $\exists! P_\infty > 0$ **s.t.** $\mathrm{Ricc}(P_\infty) = 0$ **and spectral abscissa**

$$\varsigma(\mathbf{A} - \mathbf{P_\infty S}) := \max\left\{\mathrm{Re}(\lambda)\ :\ \lambda \in \mathrm{Spec}(A - P_\infty S)\right\} < 0$$

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$$\Downarrow$$

STABLE EVEN WHEN A is unstable.

$\rightsquigarrow$ `SIAM Control & Opt.-17` $\oplus$ `Arxiv-18` (+ Bishop)
Review on the stability of Kalman-Bucy filters and Riccati matrix semigroups $\oplus$ *Floquet representation of exponential semigroups*

Floquet representations

$$P_t = \phi_t(P_0) \quad \text{flow of the Riccati equation} \quad \partial_t P_t = \mathrm{Ricc}(P_t)$$

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in the sense that (with $\mathcal{E}_{t,t}(P) = Id$)

$$\partial_t \mathcal{E}_{s,t}(P) = (A-\phi_t(P)S)E_{s,t}(P) \quad \text{and} \quad \partial_s \mathcal{E}_{s,t}(P) = -E_{s,t}(P)(A-\phi_s(P)S)$$

Floquet representations

$$P_t = \phi_t(P_0) \quad \text{flow of the Riccati equation} \quad \partial_t P_t = \mathrm{Ricc}(P_t)$$

$$\downarrow$$

Exponential semigroups/Fundamental matrices:

$$\mathcal{E}_{s,t}(P) = \exp \oint_s^t (A - \phi_u(P)S)\ du$$

in the sense that (with $\mathcal{E}_{t,t}(P) = Id$)

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Nb.:

$$P = P_\infty \Longrightarrow \mathcal{E}_{s,t}(P_\infty) = e^{(t-s)(A - P_\infty S)} \quad \text{with} \quad A - P_\infty S \quad \text{stable}$$

Floquet representations 2/2

Theo.:(+ Bishop - Arxiv18/IJC19)

$$\mathcal{E}_t(P) = e^{t(A-P_\infty S)}\ \mathbb{C}_t(P)^{-1}$$

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Cor. $(t \geqslant \delta > 0)$:

$$\|\phi_t(P_1)-\phi_t(P_2)\| \leqslant c_\delta\ \|e^{t(A-P_\infty S)}\|\ \|P_1-P_2\|$$

$\oplus$ **same type of estimates for the time varying linear process**

$$d(\widehat{X}_t - X_t)$$

$$= (\mathbf{A}-\mathbf{P_t S})\ (\widehat{X}_t - X_t)\ dt - \boldsymbol{R}^{1/2}\ \boldsymbol{dW_t} + \boldsymbol{P_t C'\Sigma^{-1/2}\ dV_t}$$

Multivariate : EnKF

$$(m_t, X_t, \mathbf{p_t}) = (\text{sample mean}, \text{true signal}, \textbf{sample covariance})$$

$$\Downarrow$$

$$d(m_t - X_t) = (A - \mathbf{p_t} S)\,(m_t - X_t)\,dt - \boldsymbol{R}^{1/2} \boldsymbol{dW_t} + \boldsymbol{p_t}\ \boldsymbol{C'\Sigma}^{-1/2}\ \boldsymbol{dV_t} + \frac{\mathbf{d\overline{M}_t}}{\sqrt{\mathbf{N}}}$$

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- **Time varying $\oplus$ stochastic type linear diffusion**

DRIVEN BY A STOCH. MATRIX-RICCATI DIFFUSION $\mathbf{p_t}$

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- **Always under-biased**

  $$\forall t > 0 \qquad 0 < p_t \quad \text{but} \quad 0 < \mathbb{E}(p_t) < P_t$$

When A stable and $S > 0$ (i.e. full observability)

Theo `[+ Tugaut (Arxiv16/AAP18)]` $\forall n \geqslant 1\ \exists N_n \geqslant 1 : \forall N \geqslant N_n$

$$\sup_{t \geqslant 0} \left[\mathbb{E}\left(\|p_t - P_t\|^n\right)^{1/n} \vee \mathbb{E}(\|m_t - \widehat{X}_t\|^n)^{1/n} \right] < c_n/\sqrt{N}$$

Under "only" : Full obs. $(S > 0)$ + Controllability

Time-uniform Riccati estimates + stability/invariant meas. Riccati diffusions + non asymptotic CLT rates+ Bias-Taylor type expansions + Perturbations analysis (inflation, masking, shrinkage, projections),...

$\subset$ **Some refs:**

– `Review article + Bishop Arxiv20`/`MCSS23`

– `log-likelihood/normalizing cts+Crisan-Jasra-Ruzayqat AdAP22`

- `Arxiv17/SPA18 (+ Bishop, Pathiraja):`
  Time-uniform robustness properties : inflation and localisation techniques.
- `Arxiv18/EJP19 (+ Bishop):`
  Stability of **matrix Riccati diffusions (= EnKF covariances).**
- `Arxiv17/IHP20 (+ Bishop, Niclas):`
  Prop. chaos expansions **Riccati diffusions,** non asymp. bias + CLT.
- `Arxiv18/SIAM19 (+ Bishop):`
  Stability of **stochastic**+time-varying linear diffusions.

Extended Kalman-Bucy Ensemble filter

$$\begin{cases} d\widehat{X}_t &= A(\widehat{X}_t)\; dt + P_t C' \Sigma^{-1} \; \left(dY_t - C\widehat{X}_t \; dt \right) \\ \partial_t P_t &= \partial A(\widehat{X}_t)\; P_t + P_t \; \partial A(\widehat{X}_t)' - P_t S P_t + R \end{cases}$$

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Nonlinear/McKean-Vlasov-type diffusion

$$\begin{aligned} d\overline{X}_t &= \mathcal{A}\left(\overline{X}_t, \eta_t(e)\right)\ dt + R^{1/2}\ d\overline{W}_t \\ &\qquad + \mathcal{P}_{\eta_t} C' \Sigma^{-1}\ \left(dY_t - \left(C\overline{X}_t\ dt + \Sigma^{1/2}\ d\overline{V}_t\right)\right) \end{aligned}$$

$$\mathcal{A}(x, m) := A(m) + \partial A(m)\ (x - m)$$

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Refs:

- SIAM17/Arxiv16(Extended EnKBF)+ Kurtzmann, Tugaut.
- EJP18/Arxiv16(Stab. Extended KBF)+ Kurtzmann, Tugaut.

Extended-EnKF = Mean field particle model

$$
\begin{aligned}
d\xi_t^i &= \mathcal{A}(\xi_t^i, m_t)\ dt + R^{1/2} d\overline{W}_t^i \\
&\qquad + p_t C' \Sigma^{-1} \left[dY_t - \left(C\xi_t^i\ dt + \Sigma^{1/2}\ d\overline{V}_t^i \right) \right]
\end{aligned}
$$

the drift

$$
\mathcal{A}(\xi_t^i, m_t) := A\left[m_t\right] + \underbrace{\partial A\left[m_t\right]\ \left(\xi_t^i - m_t\right)}_{\text{Repulsion/Attraction w.r.t. } m_t}
$$

and the rescaled particle covariance matrices

$$
p_t := \frac{1}{N} \sum_{1 \leqslant i \leqslant N+1} \left(\xi_t^i - m_t\right) \left(\xi_t^i - m_t\right)'
$$

and the sample mean

$$
m_t := \frac{1}{N+1} \sum_{1 \leqslant i \leqslant N+1} \xi_t^i.
$$

Some illustrations

Langevin type signal processes

$$R = \sigma^2 \; Id \quad \text{and} \quad (A, \partial A) = (-\partial \mathcal{V}, -\partial^2 \mathcal{V})$$

Non quadratic potential $(q \in \mathbb{R}^r, \mathcal{Q}_1, \mathcal{Q}_2 \geqslant 0)$

$$\mathcal{V}(x) = \frac{1}{2} \langle \mathcal{Q}_1 x, x \rangle + \langle q, x \rangle + \frac{1}{3} \langle \mathcal{Q}_2 x, x \rangle^{3/2}$$

Interacting diffusion gradient flows

$$\mathcal{V}(x) = \sum_{1 \leqslant i \leqslant r} \mathcal{U}_1(x_i) + \sum_{1 \leqslant i \neq j \leqslant r} \mathcal{U}_2(x_i, x_j)$$

for some convex confining potential $\mathcal{U}_i : \mathbb{R}^i \mapsto [0, \infty[$

Regularity conditions

Full observation $S = s\ Id$ and

$$-\lambda_{\partial A} \quad := \quad \sup_{x\in\mathbb{R}^r} \lambda_{max}(\partial A(x) + \partial A(x)') < 0$$

$$\|\partial A(x) - \partial A(y)\| \quad \leqslant \quad \kappa_{\partial A}\ \|x-y\|$$

Examples: Langevin signal-diffusion

$$(\lambda_{\partial A}, \kappa_{\partial A}) = \beta\ \left(2^{-1}\lambda_{min}(\mathcal{Q}_1),\ 2\lambda_{max}^{3/2}(\mathcal{Q}_2)\right).$$

more generally $\partial^2\mathcal{V} \geqslant v\ Id \oplus$ *Lipschitz condition*

Stability theorem

$$(\overline{X}_t, \overline{X}'_t) := \text{McKean-Vlasov starting at} \quad (\overline{X}_0, \overline{X}'_0)$$

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$$\Downarrow$$

Theo [+Kurtzmann-Tugaut Arxiv16/EJP19]

When $\lambda_{\partial A}$ is sufficiently large we have

$$\mathbb{W}_2(\mathrm{Law}(\overline{X}_t), \mathrm{Law}(\overline{X}'_t)) \leqslant c \;\; \exp\left[-\lambda t\right] \quad \text{for some} \quad \lambda > 0.$$

$\exists$ *more explicit description in terms of* $(R, S, \kappa_{\partial A})$.

Some estimates

$$\mathbb{P}_t^N := \mathrm{Law}(m_t, p_t) \qquad \mathbb{P}_t := \mathrm{Law}(\widehat{X}_t, P_t)$$

and

$$\mathbb{Q}_t^N := \mathrm{Law}(\xi_t^1) \qquad \mathbb{Q}_t := \mathrm{Law}(\overline{X}_t)$$

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$$\Downarrow$$

Theo [+Kurtzmann-Tugaut Arxiv16/SIAM17]

When $\lambda_{\partial A}$ is sufficiently large, $\exists \beta \in]0, 1/2]$ s.t.

$$\sup_{t \geqslant 0} \mathbb{W}_2\left(\mathbb{P}_t^N, \mathbb{P}_t\right) \vee \sup_{t \geqslant 0} \mathbb{W}_2\left(\mathbb{Q}_t^N, \mathbb{Q}_t\right) \leqslant c\ N^{-\beta}$$

as soon as $\mathrm{tr}(P_0)$ is not too large and N large enough...

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2nd part $\rightsquigarrow$ "some" answers these 2 ? for 1D linear/Gaussian

- `1d-discrete time/EnKF (+ Horton, Arxiv21/AAP23)`
- $\rightsquigarrow$ `EnKF Review article (+ Bishop, Arxiv20/MCSS23)`

Continuous time EnKF

Performance analysis

Nonlinear filtering

Discrete time EnKF/Particle filters

Linear+Gaussian+discrete 1d-filtering problem

$$\begin{cases} X_{n+1} = A\,X_n + B\,W_{n+1} & X_0 \sim \mathcal{N}(\widehat{X}_0^-, P_0) \\ Y_n = C\,X_n + D\,V_n & n \in \mathbb{N} := \{0, 1, 2 \ldots\} \end{cases}$$

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$$\Downarrow \quad \mathcal{Y}_n := (Y_0, \ldots, Y_n)$$

One-step predictor & Optimal filter = Gaussian

$$\mathrm{Law}(X_n \mid \mathcal{Y}_{n-1}) = \mathcal{N}(\widehat{X}_n^-, P_n) \quad \& \quad \mathrm{Law}(X_n \mid \mathcal{Y}_n) = \mathcal{N}(\widehat{X}_n, \widehat{P}_n)$$

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$\rightsquigarrow$ **Kalman filter (1960s') = Gauss-Legendre regression (1800s')**

$$(\widehat{X}_n^-, P_n) \xrightarrow{\text{updating}} (\widehat{X}_n, \widehat{P}_n) \xrightarrow{\text{prediction}} (\widehat{X}_{n+1}^-, P_{n+1})$$

$\rightsquigarrow$ P_n and $(\widehat{X}_n - X_n)$ are stable for any A (Kalman/Bucy-Stab Theory) !

Particle filters = GA = SMC = DMC = ...

$$\left(\xi_n^{i-}\right)_{1\leqslant i\leqslant N} \in \mathbb{R}^N \xrightarrow{\textbf{Selection}} \left(\xi_n^{\mathbf{j}}\right)_{\mathbf{1\leqslant j\leqslant N}} \xrightarrow{\textbf{Mutation}} \left(\xi_{\mathbf{n+1}}^{\mathbf{i-}}\right)_{\mathbf{1\leqslant i\leqslant N}}$$

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Selection/ **Mutation**:

$$\xi_n^j \sim \sum_{1\leqslant i\leqslant N} \frac{e^{-(Y_n - C\xi_n^{i-})^2/(2D^2)}}{\sum_{1\leqslant j\leqslant N} e^{-(Y_n - C\xi_n^{j-})^2/(2D^2)}} \, \delta_{\xi_n^{i-}} \quad \text{and set} \quad \xi_{n+1}^{j-} := A\,\xi_n^j + B\,W_{n+1}^j$$

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$$\Downarrow$$

Sample means $\simeq$ Conditional expectations:

$$\forall n \in \mathbb{N} \qquad \widehat{X}_n^{\mathrm{PF}} := \frac{1}{N} \sum_{1\leqslant i\leqslant N} \xi_n^i \simeq_{N\to\infty} \widehat{X}_n$$

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BUT for any $A > 1$

$$\xi_0^{i-} = x_0^i > \frac{B}{A-1}\sqrt{2\log N} \Longrightarrow \lim_{n\to\infty} \mathbb{E}\left[\left|\widehat{X}_n^{\mathrm{PF}} - \widehat{X}_n\right|\right] = +\infty$$

Kalman filter

$$\begin{cases} \widehat{X}_n &= \widehat{X}_n^- + Gain_n \left(Y_n - C\widehat{X}_n^- \right) \quad \text{with} \quad Gain_n := CP_n/(C^2P_n + D^2) \\ \widehat{X}_{n+1}^- &= A\,\widehat{X}_n \end{cases}$$

Offline Riccati equations

$$\begin{cases} \widehat{P}_n &= (1 - G_nC)P_n = P_n/(1 + SP_n) \quad \text{with} \quad S := (C/D)^2 \\ P_{n+1} &= A^2\widehat{P}_n + R \quad \text{with} \quad R = B^2 \end{cases}$$

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$$\rightsquigarrow P_{n+1} = \phi(P_n) := \frac{aP_n + b}{cP_n + d} \quad \text{with} \quad (a, b, c, d) := \left(A^2 + RS, R, S, 1 \right)$$

Conditional-Nonlinear Markov chain ($\mathfrak{Perfect\ Sampler}$)

$$\begin{cases} \widehat{\mathfrak{X}}_n &= \mathfrak{X}_n + \mathfrak{gain}_n \left(Y_n - (C\,\mathfrak{X}_n + D\,\mathfrak{V}_n)\right) \quad \text{with} \quad \mathfrak{gain}_n := C\mathfrak{P}_n/(C^2\mathfrak{P}_n + D^2) \\ \mathfrak{X}_{n+1} &= A\,\widehat{\mathfrak{X}}_n + B\,\mathfrak{W}_{n+1}. \end{cases}$$

$(\mathfrak{V}_n, \mathfrak{W}_n)$ copies of (V_n, W_n) and $\mathfrak{P}_n$ variance of the state $\mathfrak{X}_n$.

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Consistency property (given obs.):

$$\mathfrak{X}_n \sim \mathcal{N}\left(\mathfrak{m}_n = \widehat{X}_n^-, \mathfrak{P}_n = P_n\right) \quad \text{and} \quad \widehat{\mathfrak{X}}_n \sim \mathcal{N}\left(\widehat{X}_n, \widehat{P}_n\right).$$

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Clearly simpler with "deterministic" updating:

$\rightsquigarrow$ **square root/adjustments/transforms,...**

$$\widehat{\mathfrak{X}}_n = \mathfrak{m}_n + \mathfrak{gain}_n \left(Y_n - C\,\mathfrak{m}_n\right) + \left(1 - C\mathfrak{gain}_n\right)^{1/2} \left(\mathfrak{X}_n - \mathfrak{m}_n\right)$$

($\rightsquigarrow$ cf. some nonlinear drifts in Lange-Stannat AIMS/Arxiv21)

EnKF = Mean field interacting particle sampler

$$\left\{\begin{array}{rcl}\widehat{\xi}^i_n &=& \xi^i_n + g_n\left(Y_n - (C\xi^i_n + D\mathcal{V}^i_n)\right) \quad \text{with} \quad g_n := C p_n/(C^2 p_n + D^2) \\ \xi^i_{n+1} &=& A\,\widehat{\xi}^i_n + B\,\mathcal{W}^i_{n+1} \qquad i \in \{1, \ldots, N+1\}\end{array}\right.$$

EnKF = Mean field interacting particle sampler

$$\left\{ \begin{array}{rcl} \widehat{\xi}^i_n & = & \xi^i_n + g_n \left(Y_n - (C\xi^i_n + D\mathcal{V}^i_n)\right) \quad \text{with} \quad g_n := C p_n/(C^2 p_n + D^2) \\ \xi^i_{n+1} & = & A\,\widehat{\xi}^i_n + B\,\mathcal{W}^i_{n+1} \qquad i \in \{1, \ldots, N+1\} \end{array} \right.$$

$(\mathcal{V}^i_n, \mathcal{W}^i_n)$ *copies of* (V_n, W_n) *and re-scaled sample variance*

$$p_n := \frac{1}{N} \sum_{1 \leqslant i \leqslant N+1} (\xi^i_n - m_n)^2$$

with the sample mean

$$m_n := \frac{1}{N+1} \sum_{1 \leqslant i \leqslant N+1} \xi^i_n$$

Perturbation theo.

$$\begin{cases} \widehat{m}_n = m_n + g_n \left(Y_n - C m_n\right) + \dfrac{1}{\sqrt{N+1}}\, \widehat{\upsilon}_n \\ \widehat{p}_n = \left(1 - g_n C\right)\, p_n + \dfrac{1}{\sqrt{N}}\, \widehat{\nu}_n \end{cases} \quad \begin{cases} m_{n+1} = A\ \widehat{m}_n + \dfrac{1}{\sqrt{N+1}}\, \upsilon_{n+1} \\ p_{n+1} = A^2\ \widehat{p}_n + R + \dfrac{1}{\sqrt{N}}\, \nu_{n+1}. \end{cases}$$

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local perturbations υ_n, ν_n and $\widehat{\nu}_n, \widehat{\upsilon}_n$ in terms of non central χ^2

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local perturbations υ_n, ν_n and $\widehat{\nu}_n, \widehat{\upsilon}_n$ in terms of non central χ^2

Corollary: p_n is a Markov chain $\rightsquigarrow$ Stochastic Riccati equation

$$p_{n+1} = \phi(p_n) + \frac{1}{\sqrt{N}}\,\delta_{n+1} \quad \text{with} \quad \delta_{n+1} := A^2\,\widehat{\nu}_n + \nu_{n+1}.$$

Time uniform estimates **for any** A

Theo 1 [(Under) Bias]: $\forall k \geqslant 1 \; \exists \iota_k < \infty$ s.t. $\forall N \geqslant 1 \; \forall n \geqslant 0$

$$0 \leqslant P_n - \mathbb{E}(p_n) \leqslant \iota_1 / N$$

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& Time-uniform control of the bias

$$\sup_{n\geqslant 0} \mathbb{E}\left(|\mathbb{E}(\widehat{m}_n \mid \mathcal{Y}_n) - \widehat{X}_n|^k\right)^{1/k} \leqslant \iota_k/N$$

Time uniform estimates **for any A**

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$$\sup_{n\geqslant 0}\left(\mathbb{E}\left(\left|\widehat{m}_n - \widehat{X}_n\right|^k\right)^{1/k} \vee \mathbb{E}\left(|p_n - P_n|^k\right)^{1/k}\right) \leqslant \iota_k/\sqrt{N}.$$

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Many other results/fairly complete analysis: multivariate central limit theorems, exponential decays random products,...

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Multivariate case $\rightsquigarrow$ *working paper in preparation*