

- Simplex algorithm -

- Exercise 1 - Batteries. A battery factory wants to add two new products to its catalog: the Everlast III and the Xeros dry-cell. The Everlast III contains 2g of Cadmium and 4g of Nickel, whereas the Xeros needs 3g of Nickel and 4g of powder Zinc. The total quantity of metal available on the market is 1 ton of Cadmium, and 3 tons for Nickel. Zinc is unlimited and its sputtering is a mere formality. The production of 1000 Everlast III needs 2 hours on a Glunt II press, and the production of 1000 Xeros dry-cells needs 3 hours. The press is available 2400 hours per year. The company expects a profit of 1000 euros per thousand for the Everlast, and 1200 euros per thousand for the Xeros.

- a. Translate this into a canonical linear program.
- b. Solve the problem using a graphical method.
- c. Maximize the gain over a year with the simplex method. Iterate for all possible entering variable for the first pivot.
- d. Draw on the graph the evolution of the decision variables for each iteration of the simplex.
- e. A study shows that the high noxiousness of the Xeros push the company to increase the advertisement credits for this product. The profit for the Xeros is then impacted and reduced to 750 € per thousand Xeros. Recalculate the optimum.

- Exercise 2 - Nutritionnist. A nutritionnist is in charge of working out a diet from the following ingredients: Eggs, Milk, Cheese and Bread. The compositions (in mg) of those products in Cadmium, Nickel and Zinc are respectively of: Eggs: 6,2,1. Milk: 8,1,3. Cheese: 5,1,1. Bread: 9,3,2. A recent study showed the high noxiousness of Nickel and Zinc, and it is assessed that the daily consumption should not exceed 15mg for the Nickel and 10mg for the Zinc. The study shows however that Cadmium is a trace element that is highly beneficial.

- a. Use the simplex method to compute a diet that contains as much Cadmium as possible.
- b. Show the unicity of the solution.
- c. An error made its way into the report and swapped the roles of the Zinc and the Cadmium (Cadmium being highly toxic). It is also assessed that a diet has to contain at least one unit of bread and at most three of eggs. Recompute an optimal solution.

- Exercise 3 - Thief. A thief having a backpack with a capacity of 60 liters is confronted to the problem of choosing objects to steal among seven available. Respective volumes (in liters) and sell prices are given in the following table:

	object 1	object 2	object 3	object 4	object 5	object 6	object 7
volume	20	16	7	10	42	4	12
price	25	18	10	12	50	5	14

- a. Solve the problem by hand. Try to show the optimality of your solution.
- b. Model your problem with an integer linear program.
- c. Solve the linear relaxation of this problem using a greedy algorithm.
- d. Solve the linear relaxation using the simplex algorithm.

- **Exercise 4 - Morra Game.** This game opposes two players A and B . Each turn, every player hide one or two coins, and then tries to guess aloud the number of coins hidden by the other player. If at the end of the turn, only one player correctly guessed, this player receives from the other player as many coins as the total number of hidden coins. In all other cases, the game is null. For example:

- If A hides 1 and guess 2 and if B hides 2 and guess 1, the game is null.
- If A hides 1 and guess 2 and if B hides 2 and guess 2, then B gives 3 coins to A .

The goal of this exercise is to find a mixed strategy for player A , *i.e.* a probability distribution on the four possible moves that ensure player A a non negative expected gain against any strategy of B .

- a. Model the problem with a linear program.
- b. Solve it.

- **Exercise 5 - Desks.** A school director wants to change all desks of all classrooms. He managed to obtains tablets with an interesting price and has now to buy metal rods to build the legs. The company offers him cheap rods of 2.10 meters.

Depending on the age of the kids (and especially on their height!), the height of the desks varies. The director wants to build 60 small desks (height: 50cm), 80 medium desks (height: 80cm) and 65 high desks (height: 1.10m). Hence, to build a small desk he has to cut 4 rods of 50cm.

The director wonders how to cut the rods of 2.10m to obtain the required number of legs minimizing the scrap metal. Translate this problem into a linear problem.

- **Exercise 6 - Power plants.** Consider three power plants of respectively 700, 400 and 500 megawatt. Those power plants power two cities, both needing 800 megawatt. Each power plant can provide all or a percentage of its production to each of the city.

The costs (per megawatt) to route the power in the electric network are given in the following table:

	City 1	City 2
Power plant 1	20	25
Power plant 2	15	10
Power plant 3	10	15

The problem is to power each city minimizing the cost. Model this with a linear program. (We do not ask to solve the problem.)

- **Exercise 7 - Paths.** Let $D = (V, A)$ be a digraph and two vertices s and t of D .

- a. Write the fractional relaxation of the shortest path from s to t . (*Show that a shortest path is an admissible solution. Furthermore, show that an integer solution to your relaxation is a shortest path.*)
- b. Write the fractional relaxation of the maximum number of arc-disjoint paths from s to t .

- **Exercise 8 - Separation.** Let X and Y be two sets of points of the plane.

- a. Write a linear program deciding if there exists a separating line between points of X and points of Y .
- b. Generalize for higher dimensions, for example to separate two tetrahedra in \mathbb{R}^4 .

- **Exercise 9 - Matching.** Let $G = (V, E)$ be a graph.

- a. Write the fractional relaxation of the problem of maximum matching in G .
- b. Is the solution of your program equal to the size of the maximum matching? *Harder: what about when there are no odd cycles?*

- **Exercise 10 - 3-SAT.** Write the fractional relaxation of the 3-SAT problem. Deduce from it that computing the optimal solution of a linear program with integers (*i.e.* where variables are chosen among integers) is NP-hard.

- **Exercise 11 - Fraction.** We want to maximize the following fraction:

$$\frac{3 + 2x_1 + 3x_2 + x_3}{x_1 + 3x_1 + x_2 + 4x_3}$$

subject to the constraints $5x_1 + x_2 + 6x_3 \leq 10$ and $x_1 + 2x_2 + x_3 \leq 2$ and non negative x_i .

Show that this problem can be modeled with the following linear program:

$$\begin{array}{rllllll} \text{Maximize} & 3t & + & 2y_1 & + & 5y_2 & + & y_3 \\ \text{Subject to} & t & + & 3y_1 & + & y_2 & + & y_3 = 1 \\ & -10t & + & 5y_1 & + & y_2 & + & 6y_3 \leq 0 \\ & -2t & + & y_1 & + & 2y_2 & + & y_3 \leq 0 \\ & & & t, y_1, y_2, y_3 & & \geq & & 0 \end{array}$$

- **Exercise 12 - Trucking company.** The daily needs of a trucking company are of 13 truck drivers on Mondays, 18 on Tuesdays, 21 on Wednesdays, 16 on Thursdays, 12 on Fridays, 25 on Saturdays, and 9 on Sundays. What is the smallest number of truck drivers that must be hired to ensure the company needs assuming that a driver works five days straight? Write the linear program.