Interacting particle processes and approximation of Markov processes conditioned to not be killed.

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Abstract

We prove an approximation method for general strong Markov processes conditioned to not be killed. The method is based on a Fleming-Viot type interacting particle system, whose particles evolve as independent copies of the original strong Markov process and jump onto each others instead of being killed. We only assume that the number of jumps of the Fleming-Viot type system doesn't explode in finite time almost surely, and that the survival probability at fixed time of the original process is positive. We also give a speed of convergence for the approximation method.

A criterion for the non-explosion of the number of jumps is then given for general systems of time and environment dependent diffusion particles, which includes the case of the Fleming-Viot type system of the approximation method. The proof of the criterion uses an original non-attainability of (0,0) result for a pair of non-negative semi-martingales with positive jumps.

Key words : diffusion process, interacting particle system, empirical process, quasi-stationary distribution, Yaglom limit.
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1 Introduction

Let F be a Banach space and ∂ be a point which doesn't belong to F. Let P be the semi-group of a strong Markov process $\mathcal{Z}$ which evolves in $F \cup \{\partial\}$ and denote by τ_∂ the hitting time of $\{\partial\}$. We assume that ∂ is a cemetery point for $\mathcal{Z}$, which means that $\mathcal{Z}_t = \partial$ for all $t \geq \tau_\partial$, and we call τ_∂ the *killing time* of $\mathcal{Z}$.

Killed Markov processes are commonly used in a large area of applications in biology, demography, chemistry or finance, where there is two natural ways of *killing* a Markov process, which correspond to different interpretations. The first way is to kill the process when it reaches a given set. For instance, a demographic's model is stopped when the size of the population hits 0, since it corresponds to the extinction of the population. The second way of killing a process is to stop it at an exponential time. For example, a chemical

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particle typically disappears by reacting with another one after an exponential time, whose rate depends on the concentration of reactant in the medium. If the killing time τ_∂ is given by the time at which the process reaches a set, we call it a *hard* killing time. If it is given by an exponential clock, we call it a *smooth* killing time. While the distribution of the process after its killing time is of poor interest, numerous studies concentrate on the behavior of the process conditioned to not be killed (see [5] and references therein). The main motivation of this paper is to provide an approximation method for the distribution of Markov processes evolving in a random/time dependent environment and conditioned to not be killed.

The main tool of the approximation method is given by a Fleming-Viot type interacting particle system introduced by Burdzy, Holyst, Ingermann and March in [3] and [4]: the N particles of the system evolve as independent Brownian motions in an open subset D of $\mathbb{R}^d$, and, when a particle hits the boundary ∂D, it jumps onto the position of an other particle chosen uniformly between the $N-1$ other ones; then the particles evolve as independent particles and so on. When N goes to infinity, the empirical measure of the process converges to the distribution of a standard multi-dimensional Brownian motion conditioned to not be killed at the current time. Such an approximation method has been proved by Grigorescu and Kang in [11] for a standard multi-dimensional Brownian motion, in [22] for Brownian motions with drift and by Del Moral and Miclo for smoothly killed Markov processes (see [6] and references therein). Let us also mention the work of Ferrari and Marič [9], which regards continuous time Markov chains in discrete spaces.

In Section 2, we prove that this method works in a very general setting. Namely, let $(\mathcal{Z}^N)_{N\geq 2}$ be a sequence of strong Markov processes which evolve in $F \cup \{\partial\}$, where ∂ is the cemetery point for each $\mathcal{Z}^N$. We fix $T \geq 0$ and we assume that the sequence $(\mathcal{Z}^N_T)_N$ converges to $\mathcal{Z}_T$ in the sens of Hypothesis 2.1 . For each $N \geq 2$, we build a Fleming-Viot type system of N interacting particles as above: the particles evolve as independent copies of $\mathcal{Z}^N$ until one of them is killed; at this time, the killed particle jumps onto the position of another particle, chosen between the $N-1$ remaining ones. We assume that the number of jumps in the N particles system doesn't explode up to time T, and we prove in Theorem 2.1 that the associated sequence of empirical stationary distributions converges when $N \to \infty$ to the distribution of the process $\mathcal{Z}$ conditioned to not be killed at time T. We also give a speed of convergence for the method, which only depends on the survival probability of the Markov processes $\mathcal{Z}^N$, $N \geq 2$.

This result comes as an important generalization of the previously cited ones. Firstly, we allow both hard and soft killings, which is a natural setting in applications: typically, a species can disappear because of a lack of born of new specimens (which corresponds to a hard killing at 0) or because of a brutal natural catastrophe (which typically happens following an exponential time). Secondly, we implicitly allow time and environment dependency, which is also quite natural in applications, where individual paths are influenced by external stochastic factors (as the weather) whose distribution varies with time (because of the seasons by instance). Finally, we allow the process $\mathcal{Z}^N$ which drives the particles to depend on N, and we only require the non-explosion of the number of jumps of the Fleming-Viot type system build on $\mathcal{Z}^N$. As a consequence, one can apply the approximation method to a process $\mathcal{Z}$, without requiring that the Fleming-Viot process based on $\mathcal{Z}$ is well defined. This is typically the case for degenerate diffusions, or for diffusions with hard killing at the boundary of a non-regular domain, or for Markov processes with smooth killing given by an unbounded rate function. In our case, the three irregularities can be combined, by successive approximations of the coefficients, domain and rate of killing respectively.

Since the method works in a very general setting, it only remains us to prove the non-explosion of the number of jumps. This problem is studied in Section 3. Such non-explosion results have been recently obtained by Löbus in [17] and by Bienek, Burdzy and Finch in [2] for Brownian particles killed at the boundary of a given open set, by Grigorescu and Kang in [13] for time-homogeneous particles driven by a stochastic equation with regular coefficients killed at the boundary of a non-smooth domain (a survey of the previous results is done in the introduction of [13]) and in [22] for Brownian particles with drift. Other models of diffusions with jumps from a boundary have been introduced in [1], with a continuity condition on the jump measure that isn't fulfilled in our case, in [12], where fine properties of a Brownian motion with rebirth have been established, and in [15], [16], where Kolb and Wükber have studied the spectral properties of this model. In Section 3, we state the non-explosion of an interacting particle process, whose construction is a generalization of the previous ones. Indeed we consider particles which evolve as Itô diffusion processes in a random/time dependent environment with both hard and soft killings, with a different space of values for each particle. Moreover, at each killing time, we allow very general jump locations for the killed particle. In particular, this validates the approximation method described above for time/environment dependent diffusions with hard and soft killing.

The proof of the non-explosion is based on an original non-attainability of (0,0) result for semi-martingales, which is stated in the last section of this paper.

2 Approximation of a Markov process conditioned to not be killed

Let F be a polish space and let $\mathcal{Z}$ be a càdlàg strong Markov process which evolves in F until it is killed. When it is killed, it jumps to a cemetery point $\partial \notin F$. The killing time is denoted by $\tau_\partial = \inf\{t \geq 0,\ \mathcal{Z}_t = \partial\}$. In this section, we fix $T \geq 0$ and we prove an approximation method for the distribution of the process $\mathcal{Z}_T$ starting with distribution $\mu_0 \in \mathcal{M}_1(F)$ and conditioned to the event $\{T < \tau_\partial\}$.

The approximation method is based on a sequence of Fleming-Viot type systems $\mathbb{X}^{(N)} = \left(X^{1,(N)},\ldots,X^{2,(N)}\right)$ with values in F^N, $N \geq 2$. A natural choice for the dynamic of $\mathbb{X}^{(N)}$, $N \geq 2$, should be the following: the particles evolve independently as N independent copies of $\mathcal{Z}$ until one of them is killed; at this time, the killed particle jumps from ∂ to the position of one of the $N-1$ remaining particles; then the particles evolve as N independent copies of $\mathcal{Z}$ until one of them is killed and so on. Unfortunately, for a general choice of $\mathcal{Z}$, the number of jumps of the system could explode in finite time, or the N particles could be killed at the same time (see [2, Example 5.3] for an example of explosion in a non-trivial setting). When this happens, the approximation method can no longer operate. In order to overcome this difficulty, we assume that we're given a sequence $\left(\mathcal{Z}^N\right)_{N\geq 2}$ of strong Markov processes which converges to $\mathcal{Z}$ at time T (Hypothesis 2.1 below) and such that, for all $N \geq 2$, the Fleming-Viot system with N particles driven by $\mathcal{Z}^N$ between the killings doesn't explode before time T (Hypothesis 2.2). Theorem 2.1

below states that the empirical measure at time T of the system $\mathbb{X}^{(N)}$ (whose particles are driven by $\mathcal{Z}^N$ between the killings) converges, when N goes to infinity, to the distribution of $\mathcal{Z}$ conditioned to $\{T < \tau_\partial\}$. A rate of convergence of the approximation method is also given, which only depends on the survival probability of $\mathcal{Z}^N$ at time $T \geq 0$.

Let $\left(\mathcal{Z}^N\right)_{N\geq 2}$ be a sequence of càdlàg strong Markov processes which evolve in $F \cup \{\partial\}$, where ∂ is a cemetery point for each $\mathcal{Z}^N$. We denote the killing time of $\mathcal{Z}^N$ by $\tau_\partial^N = \inf\{t \geq 0,\ \mathcal{Z}_t^N = \partial\}$. For each $N \geq 2$, we define the interacting particle system $\mathbb{X}^{(N)} = (X^{1,(N)},...,X^{N,(N)})$ with values in F^N as follows:

- Let $m^{(N)} \in \mathcal{M}_1\left(F^N\right)$ be the initial distribution of the system.
- The N particles evolve as N independent copies of $\mathcal{Z}^N$ until one of them is killed. This killing time is denoted by $\tau_1^{(N)}$.
- At time $\tau_1^{(N)}$, the process is modified:
  - If there exists more than one particle which is killed at time $\tau_1^{(N)}$, we stop the interacting particle system itself and this time is denoted by $\tau_{stop}^{(N)}$ (In fact, we will assume that this kind of event doesn't happen almost surely).
  - Otherwise the unique killed particle jumps instantaneously onto the position of another particle, chosen uniformly between the $N-1$ remaining ones.
- At time $\tau_1^{(N)}$ and after proceeding to the jump, the process lies in F^N. Then the system evolves as N independent copies of $\mathcal{Z}^N$, until the next killing time, denoted by $\tau_2^{(N)}$.
- At this time, the process jumps with the same mechanism as above (and could be stopped at a time denoted by $\tau_{stop}^{(N)}$, as above).
- Then the particles evolve as N independent copies of $\mathcal{Z}^N$, and so on.

We set $\tau_{stop}^{(N)} = +\infty$ if $X^{i,(N)}$ and $X^{j,(N)}$ are never killed at the same time, for all $i \neq j$. On the event $\{\tau_{stop}^{(N)} = +\infty\}$, we denote by $\tau_1^{(N)} < \tau_2^{(N)} < ... < \tau_n^{(N)} < ...$ the sequence of jump times and we set

$$\tau_\infty^{(N)} = \lim_{n\to\infty} \tau_n^{(N)}. \tag{2.1}$$

If $\tau_{stop}^{(N)} < +\infty$, we set $\tau_\infty^{(N)} = +\infty$. The interacting particle system is then well defined for all time $t < \tau_{stop}^{(N)} \wedge \tau_\infty^{(N)}$.

We denote by $A_t^{i,(N)}$ the number of jumps of the i^{th} particle up to time t, $t < \tau_{stop}^{(N)} \wedge \tau_\infty^{(N)}$. We denote the total number of jumps of the system by $A_t^{(N)}$:

$$A_t^{(N)} = \sum_{i=1}^N A_t^{i,(N)},$$

and by $\mu_t^{(N)}$ the empirical distribution of $\mathbb{X}_t^{(N)}$:

$$\mu_t^{(N)} = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,(N)}} \in \mathcal{M}_1(F),$$

where $\mathcal{M}_1(F)$ denotes the space of probability measures on F.

The first assumption concerns the convergence of $\mathcal{Z}^N_T$ starting with initial random distribution $\mu_0^{(N)}$ to $\mathcal{Z}_T$ starting with (possibly random) distribution μ_0.

Hypothesis 2.1. *We assume that, for all bounded and continuous functions $f : F\cup\{\partial\} \mapsto \mathbb{R}_+$ such that $f(\partial) = 0$,*

$$\mu_0^{(N)}\left(P^N_T f\right) \xrightarrow[N\to\infty]{law} \mu_0\left(P_T f\right).$$

where $P^N_{\cdot}$ (respectively $P_{\cdot}$) denotes the semi-group of the process with killing $\mathcal{Z}^N$ (respectively $\mathcal{Z}$).

Remark 2.1. A typical situation where Hypothesis 2.1 is fulfilled is the following: we're given μ_0, $\mathcal{Z}$, and a sequence $\mathcal{Z}^N$ such that, for all $x \in F$ and all continuous and bounded function $f : F \mapsto \mathbb{R}_+$,

$$P^N_T f(x) \xrightarrow[N\to\infty]{} P_T f(x). \tag{2.2}$$

If we assume that $m^{(N)} = \mu_0^{\otimes N}$, then Hypothesis 2.1 is fulfilled. Indeed, we have

$$\mu_0^{(N)}\left(P^N_T f\right) \stackrel{law}{=} \frac{1}{N}\sum_{i=1}^{N}\left[P^N_T f(x_i) - \mu_0\left(P^{(N)}_T f\right)\right] + \mu_0\left(P^{(N)}_T f\right),$$

where $(x_i)_{i\geq 1}$ is an *iid* sequence of random variables with law μ_0. By the law of large numbers, the first right term converges to 0 almost surely. By the convergence assumption (2.2) and by dominated convergence, the second right term converges almost surely to $\mu_0\left(P_T f\right)$, so that Hypothesis 2.1 is fulfilled.

The second assumption concerns the non-explosion of the number of jumps for the system with N particles driven by $\mathcal{Z}^N$ between the killings.

Hypothesis 2.2. *We assume that, for all $N \geq 2$, the process $\mathbb{X}^N$ is well defined up to time T, which means that*

$$P_{m^{(N)}}\left(T < \tau_{stop} \wedge \tau^{(N)}_\infty\right) = 1.$$

Hypothesis 2.2 is clearly fulfilled if $\mathcal{Z}^N$ is only subject to smooth killing events happening with uniformly bounded killing rates (the question has not been answered to in the case of unbounded killing rates). In the case of an Itô'S diffusion driven by time-homogeneous stochastic differential equations and hardly killed when it hits the boundary of an open set, the problem is much harder and has been extensively studied recently (see [22], [13] and references therein for different and quite general criteria of non-explosion). The case of Itô diffusions driven by stochastic differential equations with time/environment dependent coefficients subject to soft and hard killings is treated in Section 3 of this paper.

Theorem 2.1. *We assume that the survival probability of $\mathcal{Z}$ at time T is strictly positive, which means that*

$$\mu_0\left(P_T \mathbf{1}_F\right) > 0, \text{ almost surely.} \tag{2.3}$$

Assume that Hypotheses 2.1 and 2.2 are fulfilled. Then, for any continuous and bounded function $f : F \mapsto \mathbb{R}_+$,

$$\mu_T^{(N)}(f) \xrightarrow[N\to\infty]{law} \frac{\mu_0\left(P_T f\right)}{\mu_0\left(P_T \mathbf{1}_F\right)}.$$

Moreover, for any bounded measurable function $f : F \mapsto \mathbb{R}_+$, we have the inequality

$$E\left(\left|\mu_T^{(N)}(f) - \frac{\mu_0^{(N)}\left(P_T^{(N)} f\right)}{\mu_0^{(N)}\left(P_T^{(N)}\mathbf{1}_F\right)}\right|\right) \leq \frac{4\|f\|_\infty}{\sqrt{N}} \sqrt{E\left(\frac{1}{\left(\mu_0^{(N)}\left(P_T^N \mathbb{1}_F\right)\right)^2}\right)}.$$

Remark 2.2. In Section 3, we give a non-explosion criterion for systems whose particles are driven by diffusions evolving in a random/time dependent environment, killed after exponential times or when they hit the boundary of a given open set. In particular, this criterion requires that the rate of killing is bounded and that the killing boundary and the coefficients of the diffusions are smooth. If $\mathcal{Z}$ is a diffusion in random environment, with unbounded killing rate, irregular coefficients and non-smooth killing boundary, one can define a sequence of strong Markov processes $(\mathcal{Z}^N)_{N\geq 2}$ which approximates $\mathcal{Z}$ and fulfills the criterion of Section 3 for all $N \geq 2$, proceeding by successive approximations of the rate of killing, the killing boundary and the coefficients of the diffusion $\mathcal{Z}$. It yields that Theorem 2.1 gives an approximation method for $\mathcal{Z}$ conditioned to $\{T < \tau_\partial\}$, while $\mathcal{Z}$ is degenerate. This example illustrates that allowing an approximating sequence $\mathcal{Z}^N$ for $\mathcal{Z}$ gives a great generality to the approximation method of Theorem 2.1.

Remark 2.3. In the particular case of a process $\mathcal{Z}$ with a uniformly bounded killing rate and without hard killing, a uniform rate of convergence over all times T can be obtained, using the stability of the underlying Feynman-Kac semi-group (we refer the reader to Rousset's work [20] and references therein).

Proof of Theorem 2.1. The proof consists of three steps. In a first step, we fix $N \geq 2$ and we prove that, for any bounded and measurable function $f : F \cup \{\partial\}$ such that $f(\partial) = 0$, there exists a martingale $M_t^{(N)}$ such that

$$\mu_t^{(N)}\left(P_{T-t}^N f\right) = \mu_0^{(N)}\left(P_T^N f\right) + M_t^{(N)} + \frac{1}{N}\sum_{i=1}^{N}\sum_{n=1}^{A_t^{i,(N)}}\left[\frac{1}{N-1}\sum_{j\neq i} P_{T-\tau_n^{i,(N)}}^N f(X_{\tau_n^{i,(N)}}^{j,(N)})\right] \quad (2.4)$$

where $\tau_n^{i,(N)}$ is the n^{th} killing time of the i^{th} particle. In a second step, we define the measure $\nu_t^{(N)}$ on F by

$$\nu_t^{(N)}(dx) = \left(\frac{N-1}{N}\right)^{A_t^{(N)}} \mu_t^{(N)}(dx),$$

where a loss of mass is introduced at each jump, in order to compensate the last right term in (2.4): we prove that $\nu_T^{(N)}(f) - \mu_0^{(N)}\left(P_T^N f\right)$ is the sum of two martingales. Then we prove that the L^2 norm of each of these martingales is bounded by $\|f\|_\infty/\sqrt{N}$, which yields us to

$$\sqrt{E\left(\left|\nu_T^{(N)}(f) - \mu_0^{(N)}(P_T^N f)\right|^2\right)} \leq \frac{2\|f\|_\infty}{\sqrt{N}}.$$

In the third step of the proof, we remark that $\nu_T^{(N)}$ and $\mu_T^{(N)}$ are proportional measures, which allows us to conclude the proof of Theorem 2.1 by renormalizing $\nu_T^{(N)}$ and $\mu_0^{(N)}\left(P_T^N .\right)$.

Step 1: Fix $N \geq 2$ and let $f : F \cup \{\partial\} \mapsto \mathbb{R}_+$ be a measurable bounded function such that $f(\partial) = 0$. Let us prove (2.4). We define, for all $t \in [0,T]$ and $z \in F \cup \{\partial\}$,

$$\psi_t^N(z) = P_{T-t}^N f(z).$$

The process $\left(\psi_t^N(\mathcal{Z}_t^N)\right)_{t\in[0,T]}$ is a martingale which is equal to 0 at time τ_∂^N almost surely, as soon as $\tau_\partial^N \leq T$. Indeed, for all $s,t \geq 0$ such that $s + t \leq T$, we have by the Markov property and the fact that P^N is a semi-group:

$$E\left(\psi_{t+s}^N(\mathcal{Z}_{t+s}^N) | \left(\mathcal{Z}_u^N\right)_{u\in[0,t]}\right) = P_s^N \psi_{t+s}^N(\mathcal{Z}_t^N) = \psi_t^N(\mathcal{Z}_t^N).$$

Moreover ∂ is an absorbing state and $f(\partial) = 0$, then

$$\psi_{\tau_\partial^N \wedge T}^N(\mathcal{Z}_{\tau_\partial^N \wedge T}^N) = \psi_{\tau_\partial^N}^N(\partial)\mathbb{1}_{\tau_\partial \leq T} + \psi_{\tau_\partial^N}^N(\mathcal{Z}_T^N)\mathbb{1}_{\tau_\partial > T} = \psi_{\tau_\partial^N}^N(\mathcal{Z}_T^N)\mathbb{1}_{\tau_\partial > T}.$$

Fix $i \in \{1,...,N\}$ and denote by $\tau_n^{i,(N)}$ the n^{th} jump time of the particle i. For all $n \geq 0$, we define the process $\left(\mathbb{M}_t^{i,n,(N)}\right)_{t\in[0,T]}$ by

$$\mathbb{M}_t^{i,n,(N)} = \mathbb{1}_{t<\tau_{n+1}^{i,(N)}} \psi_{t\wedge\tau_{n+1}^{i,(N)}}^N(X_{t\wedge\tau_{n+1}^{i,(N)}}^{i,(N)}) - \psi_{t\wedge\tau_n^{i,(N)}}^N(X_{t\wedge\tau_n^{i,(N)}}^{i,(N)}) \text{ (with } \tau_0^{i,(N)} = 0 \text{).}$$

Since $X^{i,(N)}$ evolves as $\mathcal{Z}^N$ in the time interval $[\tau_n^{i,(N)},\tau_{n+1}^{i,(N)}[$, $\mathbb{M}_t^{i,(N),n}$ is a martingale which fulfills almost surely

$$\mathbb{M}_t^{i,n,(N)} = \begin{cases} -\psi_{\tau_n^{i,(N)}}^N(X_{\tau_n^{i,(N)}}^{i,(N)}), \text{ if } n < A_t^{i,(N)}, \\ \psi_t^N(X_t^{i,(N)}) - \psi_{\tau_n^{i,(N)}}^N(X_{\tau_n^{i,(N)}}^{i,(N)}), \text{ if } n = A_t^{i,(N)}, \\ 0, \text{ if } n > A_t^{i,(N)}, \end{cases}$$

since $n < A_t^{i,(N)}$ is equivalent to $\tau_{n+1}^{i,(N)} < t$, while $n > A_t^{i,(N)}$ is equivalent to $\tau_n^{i,(N)} > t$. Summing over all jumps, we get

$$\psi_t^N(X_t^{i,(N)}) = \psi_0(X_0^{i,(N)}) + \sum_{n=0}^{A_t^{i,(N)}} \mathbb{M}_t^{i,n,(N)} + \sum_{n=1}^{A_t^{i,(N)}} \psi_{\tau_n^{i,(N)}}^N(X_{\tau_n^{i,(N)}}^{i,(N)}). \tag{2.5}$$

Defining

$$\mathbb{M}_t^{i,(N)} = \sum_{n=0}^{A_t^{i,(N)}} \mathbb{M}_t^{i,n,(N)} \text{ and } \mathbb{M}_t^{(N)} = \frac{1}{N}\sum_{i=1}^{N} \mathbb{M}_t^{i,(N)}$$

and summing over $i \in \{1,...,N\}$, we get

$$\mu_t^{(N)}(\psi_t^N) = \mu_0^{(N)}(\psi_0^N) + \mathbb{M}_t^{(N)} + \frac{1}{N}\sum_{i=1}^{N}\sum_{n=1}^{A_t^{i,(N)}} \psi_{\tau_n^{i,(N)}}^N(X_{\tau_n^{i,(N)}}^{i,(N)}).$$

At each jump time $\tau_n^{i,(N)}$, the position of the particle $X^{i,(N)}$ after the jump is chosen with respect to the empirical measure of the other particles. The expectation of $\psi_{\tau_n^{i,(N)}}^N(X_{\tau_n^{i,(N)}}^{i,(N)})$

conditionally to the position of the other particles at the jump time is then the average value $\frac{1}{N-1}\sum_{j\neq i}\psi^N_{\tau_n^{i,(N)}\text{-}}(X^{j,(N)}_{\tau_n^{i,(N)}\text{-}})$. We deduce that

$$\mathcal{M}^{(N)}_t = \frac{1}{N}\sum_{i=1}^{N}\sum_{n=1}^{A_t^{i,(N)}}\left(\psi^N_{\tau_n^{i,(N)}}(X^{i,(N)}_{\tau_n^{i,(N)}}) - \frac{1}{N-1}\sum_{j\neq i}\psi^N_{\tau_n^{i,(N)}\text{-}}(X^{j,(N)}_{\tau_n^{i,(N)}\text{-}})\right).$$

is a local martingale. We finally get

$$\mu^{(N)}_t(\psi^N_t) = \mu^{(N)}_0(\psi^N_0) + \mathbb{M}^{(N)}_t + \mathcal{M}^{(N)}_t + \frac{1}{N}\sum_{i=1}^{N}\sum_{n=1}^{A_t^{i,(N)}}\left[\frac{1}{N-1}\sum_{j\neq i}\psi^N_{\tau_n^{i,(N)}\text{-}}(X^{j,(N)}_{\tau_n^{i,(N)}\text{-}})\right], \quad (2.6)$$

which is exactly (2.4).

Step 2: Let us now explain why $\nu^{(N)}_{T\wedge\tau_\alpha^{(N)}}(\psi^N_{T\wedge\tau_\alpha^{(N)}}) - \nu^{(N)}_0(\psi^N_0)$ is the sum of two martingales. Since N is fixed and in order to clarify the calculus, we remove the superscripts N and (N) when there is no risk of confusion. Denoting by $\mathbb{M}^C$ the continuous part of $\mathbb{M} = \mathbb{M}^{(N)}$, we deduce from (2.6) that

$$\nu_T(\psi_T) - \nu_0(\psi_0) = \int_0^T \left(\frac{N-1}{N}\right)^{A_{t\text{-}}} d\mathbb{M}^C_t + \sum_{n=1}^{A_T}\nu_{\tau_n}(\psi_{\tau_n}) - \nu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}}).$$

Let us compute each term in the right side sum. For all $n\geq 1$,

$$\begin{aligned}\nu_{\tau_n}(\psi_{\tau_n}) - \nu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}}) = \left(\frac{N-1}{N}\right)^{A_{\tau_n}}&(\mu_{\tau_n}(\psi_{\tau_n}) - \mu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}}))\\ &+ \mu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}})\left(\left(\frac{N-1}{N}\right)^{A_{\tau_n}} - \left(\frac{N-1}{N}\right)^{A_{\tau_n\text{-}}}\right).\end{aligned}$$

On the one hand, we have

$$\left(\frac{N-1}{N}\right)^{A_{\tau_n}} - \left(\frac{N-1}{N}\right)^{A_{\tau_n\text{-}}} = -\frac{1}{N-1}\left(\frac{N-1}{N}\right)^{A_{\tau_n}}.$$

On the other hand, denoting by i the index of the killed particle at time τ_n, we have

$$\mu_{\tau_n}(\psi_{\tau_n}) - \mu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}}) \;=\; \frac{1}{N(N-1)}\sum_{j\neq i}\psi_{\tau_n^i\text{-}}(X^j_{\tau_n^i\text{-}}) + \mathbb{M}_{\tau_n} - \mathbb{M}_{\tau_n\text{-}} + \mathcal{M}_{\tau_n} - \mathcal{M}_{\tau_n\text{-}},$$

where

$$\frac{1}{N(N-1)}\sum_{j\neq i}\psi_{\tau_n^i\text{-}}(X^j_{\tau_n^i\text{-}}) = \frac{1}{N-1}\mu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}}) - \frac{1}{N(N-1)}\psi_{\tau_n\text{-}}(X^i_{\tau_n\text{-}})$$

and, by the definition of $\mathbb{M} = \mathbb{M}^{(N)}$,

$$-\frac{1}{N(N-1)}\psi_{\tau_n\text{-}}(X^i_{\tau_n\text{-}}) = \frac{1}{N-1}\left(\mathbb{M}_{\tau_n} - \mathbb{M}_{\tau_n\text{-}}\right).$$

We then have

$$\mu_{\tau_n}(\psi_{\tau_n}) - \mu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}}) \;=\; \frac{1}{N-1}\mu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}}) + \frac{N}{N-1}\left(\mathbb{M}_{\tau_n} - \mathbb{M}_{\tau_n\text{-}}\right) + \mathcal{M}_{\tau_n} - \mathcal{M}_{\tau_n\text{-}},$$

Finally, we get

$$\nu_{\tau_n}(\psi_{\tau_n}) - \nu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}}) \;=\; \left(\frac{N-1}{N}\right)^{A_{\tau_n\text{-}}}\left(\mathbb{M}_{\tau_n} - \mathbb{M}_{\tau_n\text{-}}\right) + \left(\frac{N-1}{N}\right)^{A_{\tau_n}}\left(\mathcal{M}_{\tau_n} - \mathcal{M}_{\tau_n\text{-}}\right).$$

The process $\nu_t(\psi_t) - \nu_0(\psi_0)$ is then the sum of two local martingales and we have

$$\nu_T(\psi_T) - \nu_0(\psi_0) = \int_0^T \left(\frac{N-1}{N}\right)^{A_{t\text{-}}} d\mathbb{M}_t + \frac{N-1}{N}\int_0^T \left(\frac{N-1}{N}\right)^{A_{t\text{-}}} d\mathcal{M}_t \tag{2.7}$$

Let us bound both terms on the right-hand side (where N is still fixed). We do not have any control on the moments of the number of jumps, while we would like to deal with real martingales instead of local ones. In order to do this, we fix an integer $\alpha \geq 1$ and we stop the interacting particle system when the number of jumps A_t reaches α, which is equivalent to stop the process at time $\tau_\alpha = \tau_\alpha^{(N)}$. By the optional stopping time theorem, the processes $\mathbb{M}$ and $\mathcal{M}$ stopped at time $\tau_\alpha^{(N)}$ are true martingales, almost surely bounded by $\alpha\|f\|_\infty$.

On the one hand, the martingale jumps $\mathcal{M}_{\tau_n} - \mathcal{M}_{\tau_n\text{-}}$ are bounded by $\|f\|_\infty/N$, while the martingale is constant between the jumps, then

$$\begin{aligned} E\left(\left|\frac{N-1}{N}\int_0^{T\wedge\tau_\alpha}\left(\frac{N-1}{N}\right)^{A_{t\text{-}}} d\mathcal{M}_t\right|^2\right) &= E\left[\sum_{n=1}^{A_T\wedge\alpha}\left(\frac{N-1}{N}\right)^{2A_{\tau_n\text{-}}}\left(\mathcal{M}_{\tau_n} - \mathcal{M}_{\tau_n\text{-}}\right)^2\right] \\ &\leq \frac{\|f\|_\infty^2}{N}. \end{aligned} \tag{2.8}$$

On the other hand, we have

$$\begin{aligned} E\left(\left(\int_0^{T\wedge\tau_\alpha}\left(\frac{N-1}{N}\right)^{A_{t-}} d\mathbb{M}_t\right)^2\right) &\leq E\left((\mathbb{M}_{T\wedge\tau_\alpha})^2\right) \\ &= \frac{1}{N^2}\sum_{i,j=1}^{N} E\left(\mathbb{M}^i_{T\wedge\tau_\alpha}\mathbb{M}^j_{T\wedge\tau_\alpha}\right) \end{aligned}$$

where

$$E\left(\mathbb{M}^i_{T\wedge\tau_\alpha}\mathbb{M}^j_{T\wedge\tau_\alpha}\right) = \sum_{m=0,n=0}^{\alpha} E\left(\mathbb{M}^{i,m}_{T\wedge\tau_\alpha}\mathbb{M}^{j,n}_{T\wedge\tau_\alpha}\right).$$

If $i \neq j$, then the expectation of the product of the martingales $\mathbb{M}^{i,n}$ and $\mathbb{M}^{j,m}$ is 0, since the particles are independent between the jumps and do not jump simultaneously. Assume $i = j$ and fix $m < n$. By definition, we have

$$M^{i,m}_{T\wedge\tau_\alpha} = M^{i,m}_{T\wedge\tau_\alpha\wedge\tau^i_{m+1}},$$

which is measurable with respect to $\mathbb{X}_{T\wedge\tau_\alpha\wedge\tau^i_{m+1}}$, then

$$\begin{aligned}E\left(M^{i,m}_{T\wedge\tau_\alpha}M^{i,n}_{T\wedge\tau_\alpha}|\mathbb{X}_{T\wedge\tau_\alpha\wedge\tau^i_{m+1}}\right) &= M^{i,m}_{T\wedge\tau_\alpha\wedge\tau^i_{m+1}}E\left(M^{i,n}_{T\wedge\tau_\alpha}|\mathbb{X}_{T\wedge\tau_\alpha\wedge\tau^i_{m+1}}\right)\\ &= M^{i,m}_{T\wedge\tau_\alpha\wedge\tau^i_{m+1}}M^{i,n}_{T\wedge\tau_\alpha\wedge\tau^i_{m+1}} = 0,\end{aligned}$$

using the optional sampling theorem with the martingale $M^{i,n}_{T\wedge\tau_\alpha}$ and the uniformly bounded stopping time $T\wedge\tau_\alpha\wedge\tau^i_n$. We deduce that

$$\begin{aligned}E\left(\left(M^i_{T\wedge\tau_\alpha}\right)^2\right) &= E\left(\sum_{n=0}^{\alpha}\left(M^{i,n}_{T\wedge\tau_\alpha}\right)^2\right)\\ &\leq E\left(\sum_{n=0}^{\alpha}\psi_{T\wedge\tau^i_n}(X^i_{T\wedge\tau^i_n})^2\right)\\ &\leq \|f\|_\infty E\left(\sum_{n=0}^{\alpha}\psi^N_{T\wedge\tau^i_n}(X^i_{T\wedge\tau^i_n})\right).\end{aligned}$$

By (2.5), we have

$$E\left(\sum_{n=0}^{\alpha}\psi_{T\wedge\tau^i_n}(X^i_{T\wedge\tau^i_n})\right)\leq\|f\|_\infty,$$

and we deduce that

$$E\left(\left(M^i_{T\wedge\tau_\alpha}\right)^2\right)\leq\|f\|^2_\infty.$$

Finally, we have

$$E\left(\left(\int_0^{T\wedge\tau_\alpha}\left(\frac{N-1}{N}\right)^{A_{t-}}d\mathbb{M}_t\right)^2\right)\leq\frac{\|f\|^2_\infty}{N}. \tag{2.9}$$

The formula (2.7) and inequalities (2.8) and (2.9) lead us to

$$\sqrt{E\left(\left|\nu^{(N)}_{T\wedge\tau^{(N)}_\alpha}(P^N_{T-T\wedge\tau^{(N)}_\alpha}f)-\mu^{(N)}_0(P^N_{T\wedge\tau^{(N)}_\alpha}f)\right|^2\right)}\leq\frac{2\|f\|_\infty}{\sqrt{N}}.$$

The number of jumps of the interacting particle system remains bounded up to time T by Hypothesis 2.2, so that $T\wedge\tau^{(N)}_\alpha$ is equal to T for α big enough almost surely. As a consequence, making α go to infinity in the inequality above, we get by the dominated convergence theorem

$$\sqrt{E\left(\left|\nu^{(N)}_T(f)-\mu^{(N)}_0(P^N_Tf)\right|^2\right)}\leq\frac{\sqrt{2}\|f\|_\infty}{\sqrt{N}}. \tag{2.10}$$

Step 3: Let us now conclude the proof of Theorem 2.1. By Hypothesis 2.1, $\mu^{(N)}_0(P^N_T.)$ converges in distribution to $\mu_0(P^N_T.)$. It yields that, for each continuous and bounded function $f:F\to\mathbb{R}_+$, the sequence of random variables $\left(\mu^{(N)}_0(P^N_T\mathbf{1}_F),\mu^{(N)}_0(P^N_Tf)\right)$ converges in distribution to the random variable $(\mu_0(P_T\mathbf{1}_F),\mu_0(P_Tf))$. By (2.10), we deduce that the sequence of random variables $\left(\nu_T^{(N)}(\mathbf{1}_F),\nu_T^{(N)}(f)\right)$ converges in distribution to the random variable $(\mu_0(P_T\mathbf{1}_F),\mu_0(P_Tf))$. Finally, using that $\mu_0(P_T\mathbf{1}_F)$ never vanishes almost surely, we get

$$\mu_T^{(N)}(f)=\frac{\nu_T^{(N)}(f)}{\nu_T^{(N)}(\mathbf{1}_F)}\xrightarrow[N\to\infty]{law}\frac{\mu_0(P_Tf)}{\mu_0(P_T\mathbf{1}_F)},$$

for any continuous and bounded function $f:F\to\mathbb{R}_+$, which implies the first part of Theorem 2.1.

We can also deduce from (2.10) that

$$\sqrt{E\left(\left|\left(\frac{N-1}{N}\right)^{A_T^{(N)}}-\mu_0^{(N)}\left(P_T^N\mathbb{1}_F\right)\right|^2\right)}\leq\frac{2}{\sqrt{N}},$$

then

$$\sqrt{E\left(\left|\mu_0^{(N)}(P_T^N\mathbb{1}_F)\mu_T^{(N)}(f)-\mu_0^{(N)}(P_T^Nf)\right|^2\right)}\leq\frac{4\|f\|_\infty}{\sqrt{N}}.$$

Using the Cauchy Schwartz inequality, we deduce that

$$E\left(\left|\mu_T^{(N)}(f)-\frac{\mu_0^{(N)}(P_T^Nf)}{\mu_0^{(N)}P_T^N\mathbb{1}_F}\right|\right)\leq\sqrt{E\left(\frac{1}{\left(\mu_0^{(N)}\left(P_T^N\mathbb{1}_F\right)\right)^2}\right)}\frac{4\|f\|_\infty}{\sqrt{N}},$$

which concludes the proof of Theorem 2.1 . □

3 Criterion for the non-explosion of the number of jumps

Fix $N\geq 2$. The aim of this section is to give a criterion for the non-explosion assumption of Hypothesis 2.2 (Section 2) when the process $\mathcal{Z}^N$ is driven by a stochastic differential equation in a random time/dependent environment, with a uniformly bounded smooth killing rate and a hard killing set given by the boundary of an open set. While this problem is the main motivation for proving our non-explosion result, Theorem 3.1 below is stated in a far more general setting. Firstly, we do not require that the particles follow the same dynamic between the killings: the i^{th} particle will be driven by the dynamic of a strong Markov process $\mathcal{Z}^{i,N}$, *a priori* different for each $i\in\{1,...,N\}$. Secondly, the jump position of the killed particle is chosen with respect to a general jump measure, not necessarily supported by the positions of the $N-1$ remaining particles.

For all $i\in\{1,...,N\}$, we assume that the process $\mathcal{Z}^{i,N}$ is a strong Markov process equal to a 3-tuple $(t,e_t^i,Z_t^i)_{t\in[0,\tau_\partial[}$ up to its killing time, where t is the time, e_t^i is the environment and Z_t^i is the actual position of the diffusion. The environment e_t^i evolves in an open set $E_i\subset\mathbb{R}^{d_i}$ ($d_i\geq 1$), the position Z_t^i evolves in an open set $D_i\subset\mathbb{R}^{d_i'}$ ($d_i'\geq 1$), and we assume that there exist four measurable functions

$$\begin{aligned}
s_i&:[0,T]\times E_i\times D_i\mapsto\mathbb{R}^{d_i}\times\mathbb{R}^{d_i}\\
m_i&:[0,T]\times E_i\times D_i\mapsto\mathbb{R}^{d_i}\\
\sigma_i&:[0,T]\times E_i\times D_i\mapsto\mathbb{R}^{d_i'}\times\mathbb{R}^{d_i'}\\
\mu_i&:[0,T]\times E_i\times D_i\mapsto\mathbb{R}^{d_i'},
\end{aligned}$$

such that $\mathcal{Z}^{i,N} = (.,e^i,Z^i)$ fulfills the stochastic differential equation

$$de_t^i = s_i(t,e_t^i,Z_t^i)d\beta_t^i + m_i(t,e_t^i,Z_t^i)dt,\ e_0^i \in E_i,$$
$$dZ_t^i = \sigma^i(t,e_t^i,Z_t^i)dB_t^i + \mu^i(t,e_t^i,Z_t^i)dt,\ Z_0^i \in D_i,$$

where (β^i,B^i) is a standard d_i+d_i' Brownian motion. We also assume that the process $\mathcal{Z}^{i,N}$ is hardly killed when Z_t^i hits ∂D_i and smoothly killed with a rate of killing $\kappa_i(t,e_t^i,Z_t^i) \geq 0$, where

$$\kappa_i : [0,+\infty[\times E_i \times D_i \mapsto \mathbb{R}_+$$

is a measurable function. We recall that the distribution of the smooth killing time produced by the rate of killing κ_i is given by

$$P\left(\tau_\partial^{smooth} > t\right) = E\left(e^{-\int_0^t \kappa_i(\mathcal{Z}_s^{i,N})ds}\right).$$

Each particle in the system is a 3-tuple $(t,o_t^i,X_t^i) \in [0,+\infty[\times E_i \times D_i$ and we denote the whole system by $(t,\mathbb{O}_t,\mathbb{X}_t)$, where

$$\mathbb{O}_t = (o_t^1,...,o_t^N) \in E \stackrel{def}{=} E_1 \times ... \times E_N \text{ and}$$
$$\mathbb{X}_t = (X_t^1,...,X_t^N) \in D \stackrel{def}{=} D_1 \times ... \times D_N,$$

denote respectively the vector of environments and the vector of positions. Let $S : [0,+\infty[\times E^N \times D^N \to \mathcal{M}_1(E^N \times D^N)$ and $\mathcal{H} : [0,+\infty[\times E^N \times \partial(D^N) \to \mathcal{M}_1(E^N \times D^N)$ be two given measurable jump measures, which will be used to choose the jump location after the smooth killing and hard killing respectively. We define the dynamics of the system $(t,\mathbb{O}_t,\mathbb{X}_t)$ starting from $(0,\mathbb{O}_0,\mathbb{X}_0)$ as follows:

- For all $i \in \{1,...,N\}$, the 3-tuple (t,o_t^i,X_t^i) starts from $(0,o_0^i,X_0^i)$ and evolves as $\mathcal{Z}^{i,N} = (.,e^i,Z^i)$ independently of the rest of the system until one of the particles is killed. This first killing time is denoted by τ_1.
- At time τ_1, the process jumps to a new position, whose choice depends on the kind of killing (the time component isn't changed):
  - if it is a smooth killing event, then the process jumps to a position chosen with respect to the jump measure $\mathcal{S}(\tau,\mathbb{O}_{\tau\text{-}},\mathbb{X}_{\tau\text{-}})$,
  - if it is a hard killing event and there exists one and only one element $i_1 \in \{1,...,N\}$ such that $X_{\tau_1\text{-}}^{i_1}$ belongs to ∂D_{i_1}, then the position of $(\mathbb{O},\mathbb{X})$ at time τ_1 is chosen with respect to the probability measure $\mathcal{H}(\tau,\mathbb{O}_{\tau\text{-}},\mathbb{X}_{\tau\text{-}})$.
  - if it is a hard killing event and there exist more than one element which hits its corresponding boundary ∂D_i, we stop the process and this time is denoted by τ_{stop} (in fact, we will prove that this kind of event doesn't happen almost surely under our hypotheses).
- At time τ_1 and after proceeding to the jump, the process lies in $\{\tau_1\} \times E \times D$. Then each 3-tuple (t,o^i,X^i) evolves as $(.,e^i,Z^i)$ starting from $(\tau_1,o_{\tau_1}^i,X_{\tau_1}^i)$, independently of the rest of the system and until one of them is killed. This second killing time is denoted by τ_2.

- At this time, the process jumps with the same mechanism as above (and could be stopped at a time denoted by τ_{stop}, as above).
- Then each 3-tuple (t,o^i,X^i) evolves as $(.,e^i,Z^i)$ starting from $(\tau_2,o^i_{\tau_2},X^i_{\tau_2})$, independently of the rest of the system, and so on.

We set $\tau_{stop} = +\infty$ if (X^i,X^j) never reaches $\partial D_i \times \partial D_j$, for all $i \neq j$. On the event $\{\tau_{stop} = +\infty\}$, we denote by $\tau_1 < \tau_2 < ... < \tau_n < ...$ the sequence of jump times and we set

$$\tau_\infty = \lim_{n\to\infty} \tau_n. \tag{3.1}$$

The number of jumps of the system explodes in finite time if and only if $\tau_\infty < +\infty$. We prove in Theorem 3.1 below, that this doesn't happen almost surely under the two following conditions: Hypothesis 3 and Hypothesis 4.

In the following hypothesis, the function ϕ_i is the Euclidean distance from the boundary ∂D_i, which means that

$$\phi_i(x) = \min_{z\in\partial D_i} \|x - z\|_2, \ \forall x \in D_i,$$

where $\|.\|$ denotes the Euclidean distance. For all $a > 0$, D_i^a will denote the boundary's neighborhood

$$D_i^a = \{x \in D_i, \ \phi_i(x) < a\}.$$

Hypothesis 3.1. *We assume that, for all $i \in \{1,...,N\}$ and all $T \geq 0$, there exists $a > 0$ such that*

1. *ϕ_i is of class C_b^2 on D_i^a,*
2. *the smooth killing rate κ_i is uniformly bounded on $[0,T] \times E_i \times D_i$*
3. *s_i,σ_i,m_i and μ_i are uniformly bounded on $[0,T] \times E_i \times D_i^a$,*
4. *there exist two measurable functions $f_i : [0,T] \times E_i \times D_i^a \mapsto \mathbb{R}_+$ and $g_i : [0,T] \times E_i \times D_i^a \mapsto \mathbb{R}$ such that $\forall (t,\epsilon,z) \in [0,T] \times E \times D_i^a$,*

$$\sum_{k,l} \frac{\partial \phi_i}{\partial x_k}(z) \frac{\partial \phi_i}{\partial x_l}(z) [\sigma_i \sigma_i^*]_{kl}(t,\epsilon,z) = f_i(t,\epsilon,z) + g_i(t,\epsilon,z), \tag{3.2}$$

and such that

(a) *f_i is of class C^1 in time and of class C^2 in environment/space, and the derivatives of f_i are uniformly bounded,*

(b) *there exists a positive constant $k_g > 0$ such that, for all $(t,\epsilon,z) \in [0,T] \times E_i \times D_i^a$,*

$$|g_i(t,\epsilon,z)| \leq k_g \phi_i(z),$$

(c) *there exists two positive constants $0 < c_\pi < C_\pi$ such that, for all $(t,\epsilon,z) \in [0,T] \times E_i \times D_i^a$,*

$$c_\pi < f_i(t,\epsilon,z) + g_i(t,\epsilon,z) < C_\pi.$$

The last point of Hypothesis 3.1 says that the term (3.2), which naturally appears in the quadratic variation of $\phi_i(Z_t^i)$, is well approximated by a smooth positive function f_i near the boundary ∂D_i. However, we do not require any strict regularity assumption on σ_i, since g_i is only required to be measurable.

Remark 3.1. 1. We recall that the C^k regularity of ϕ_i near the boundary is equivalent to the C^k regularity of the boundary ∂D_i itself, for all $k \geq 2$ (see [8, Chapter 5, Section 4]).

2. In particular, if each D_i is bounded and has a boundary of class C^3, and if σ_i is of class C^2, then the first point and the last point of Hypothesis 3.1 are fulfilled. Indeed, the regularity of D_i implies that ϕ_i is of class C^3 on a neighborhood of ∂D_i, and the regularity of σ_i implies that (3.2) happens, with $g_i = 0$.

We introduce now a condition on the jump measure $\mathcal{H}$, which will ensure that $\tau_\infty < +\infty$ implies that at least two particles converge to the boundary when the time goes to τ_∞. we denote by $\mathcal{D}_i$ the set

$$\mathcal{D}_i = D_1 \times ... \times D_{i-1} \times \partial D_i \times D_{i+1} \times ... \times D_N.$$

Since we decide to stop the process when more than two particles hit simultaneously their corresponding boundaries, it is sufficient to define the jump measure $\mathcal{H}$ on $\cup_{i=1}^N \mathcal{D}_i$.

Hypothesis 3.2. *1. There exists a non-decreasing continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$ vanishing only at 0 such that, $\forall i \in \{1,...,N\}$,*

$$\inf_{(t,e,(x_1,...,x_N))\in[0,+\infty[\times E\times\mathcal{D}_i} \mathcal{H}(t,e,x_1,...,x_N)(E \times A_i) \geq p_0,$$

where $p_0 > 0$ is a positive constant and $A_i \subset D$ is the set defined by

$$A_i = \{(y_1,...,y_N) \in D \,|\, \exists j \neq i \text{ such that } \phi_i(y_i) \geq h(\phi_j(y_j))\}.$$

2. We have

$$\inf_{(t,e,(x_1,...,x_N))\in[0,+\infty[\times E\times\mathcal{D}_i} \mathcal{H}(t,e,x_1,...,x_N)(E \times B_{x_1,...,x_n}) = 1,$$

where

$$B_{x_1,...,x_n} = \{(y_1,...,y_N) \in D \,|\, \forall i,\ \phi_i(y_i) \geq \phi_i(x_i)\}$$

Informally, $h(\phi_j)$ is a kind of distance from the boundary and we assume in the first point that, if all the not-killed particles are far from their respective boundaries at time τ_n, then the jump position $X^i_{\tau_n}$ is chosen far from ∂D_i with probability $p_0 > 0$. The second point ensures that each particle lies farther from its boundary after than before a hard killing jump.

Remark 3.2. 1. The model of interacting particles system introduced above is very general, even if e^i is required to be continuous up to the killing time. Indeed, it also includes the case of a diffusion evolving in an environment given by a continuous time Markov Chain. By instance, if one set s_i and m_i equal to 0, κ_i equal to 1 and $\mathcal{S} = \frac{1}{2}\left(\delta_{(t,\epsilon_i+1,z_i)} + \delta_{(t,\epsilon_i-1,z_i)}\right)$, then the particle X^i will evolve as a diffusion with an environment o^i defined as a simple continuous time random walk.

2. Hypothesis 3.2 is very general and allows a lot of choices for $\mathcal{H}$. For instance:

   (a) For all $\mu \in \mathcal{M}_1(E \times D)$, one can find a compact set $K \subset E \times D$ such that $\mu(K) > 0$. Then $\mathcal{H} = \mu$ fulfills the assumption with $p_0 = \mu(K)$ and $h(\phi_j) = \phi_j \wedge d(K, E \times \partial D)$.

   (b) Hypothesis 3.2 also includes the case studied by Grigorescu and Kang in [13], where
$$\mathcal{H} = \sum_{j \neq i} p_{ij}(x_i)\delta_{x_j},\ \forall (x_1,\dots,x_N) \in \mathcal{D}_i.$$
   with $\sum_{j\neq i} p_{ij}(x_i) = 1$ and $\inf_{i\in\{1,\dots,N\}, j\neq i, x_i \in \partial D} p_{ij}(x_i) > 0$. In that case, the particle on the boundary jumps to the position of another one, with positive weights. It yields that Hypothesis 3.2 is fulfilled with $p_0 = 1$ and $h(\phi_j) = \phi_j$. This is also the case for the Fleming-Viot type system used in the approximation method proved in Section 2.

We're now able to state the main result of this section:

Theorem 3.1. *Assume that Hypotheses 3.1 and 3.2 are satisfied. Then $\tau_\infty = +\infty$ almost surely.*

Remark 3.3. Another model of diffusions killed at the boundary of an open set can be defined as follows: the particle is reflected on the boundary until its local time on this boundary reaches an independent exponentially distributed random variable, then it is killed. We emphasize that the statement of Theorem 3.1 is still valid if the particles are driven by such diffusions with reflecting/killing boundaries. Indeed, the only difference with our proof is that the reflection on the boundary makes appear an additional increasing local time in the decomposition of the semi-martingale Y^i (see (3.4) in the proof).

The long-time behavior of diffusions with reflecting/killing boundaries conditioned to not be killed has been studied in [14] by Kolb and Steinsaltz and in [21] by Evans and Steinsaltz. The approximation method proved in Section 2 can be used to compute the distribution of diffusions with reflecting/killing boundaries conditioned to not be killed.

Proof of Theorem 3.1. Since κ_i is uniformly bounded for all $i \in \{1,\dots,N\}$ in finite time almost surely, there is no accumulation of soft killing events almost surely. As a consequence, we only have to prove the non-accumulation of hard killing events and we assume until the end of the proof that $\kappa_i = 0$ for all $i \in \{1,\dots,N\}$.

The proof is organised as follows. For each particle X^i, we compute the Itô's decomposition of the semi-martingale $\phi_i(X^i)$ when X^i is in D_i^a. Then we prove that $\tau_{stop} \wedge \tau_\infty < +\infty$ implies that at least two particles X_t^i and X_t^j converge to their respective boundaries when $t \to \tau_{stop} \wedge \tau_\infty$. Denoting by
$$T_0^{ij} = \inf\{t \geq 0,\ \phi_i(X_{t-}^i) = \phi_j(X_{t-}^j) = 0\},$$
we deduce that
$$P\left(\tau_{stop} \wedge \tau_\infty < +\infty\right) \leq \sum_{1\leq i<j\leq N} P\left(T_0^{ij} < +\infty\right)$$
This allows us to reduce the problem of non-explosion of the number of jumps to a problem of non-attainability of (0,0) for a pair of semi-martingales.$(\phi_i(X^i),\phi_j(X^j))$ fulfills a criterion which implies its non-attainability of (0,0) in finite time almost surely, concluding

the proof of Theorem 3.1. The above-mentioned criterion of non-attainability is proved in the last section of the present paper (Proposition 4.1).

By definition, if $\tau_{stop} < +\infty$, then at least two particles X^{i_0} and X^{j_0} hit their respective boundaries at time τ_{stop}. It yields that $\phi_i(X^i_{\tau_{stop}\text{-}}) = \phi_j(X^j_{\tau_{stop}\text{-}}) = 0$. Now, we define the event

$$\mathcal{E} = \{\tau_\infty < T \text{ and } \tau_{stop} = +\infty\}.$$

Conditionally to $\mathcal{E}$, the total number of jumps of the system goes to $+\infty$ up to time τ_∞. Since there is only a finite number of particles, at least one of them, say i_0, jumps infinitely many times up to time τ_∞. For each jumping time τ_n, we denote by $\sigma_n^{i_0}$ the next jump time of i_0, with $\tau_n < \sigma_n^{i_0} < \tau_\infty$. Conditionally to the event $\mathcal{E}$, we get $\sigma_n^{i_0} - \tau_n \to 0$ when $n \to \infty$. Let $\gamma :]0,a[\mapsto \mathbb{R}_+$ be a C^2 function with compact support in $]0,a[$. The Itô's formula applied to the semi-martingale $\gamma(\phi_i(X^{i_0}))$ and Hypothesis 3.1 immediately imply that $\gamma(\phi_i(X^{i_0}))$ is a continuous diffusion process with bounded coefficients between τ_n and $\sigma_n^{i_0}$-. Moreover, $\phi_i(X_t^{i_0})$ goes to 0 when t goes to $\sigma_n^{i_0}$, then $\gamma(\phi_i(X^{i_0}_{\sigma_n^{i_0}\text{-}})) = 0$. We deduce that

$$\sup_{t\in[\tau_n,\sigma_n^{i_0}[} \gamma(\phi_i(X_t^{i_0})) = \sup_{t\in[\tau_n,\sigma_n^{i_0}[} \gamma(\phi_i(X_t^{i_0})) - \gamma(\phi_i(X^{i_0}_{\sigma_n^{i_0}\text{-}})) \xrightarrow[n\to\infty]{} 0, \ a.s.$$

Since the process $\phi_{i_0}(X^{i_0})$ is continuous between τ_n and $\sigma_n^{i_0}-$, we conclude that $\phi_{i_0}(X^{i_0}_{\tau_n})$ doesn't lie above the support of γ, for n big enough, almost surely. But the support of γ can be chosen arbitrarily close to 0, it yields that $\phi_{i_0}(X^{i_0}_{\tau_n})$ goes to 0 almost surely conditionally to $\mathcal{E}$. Let us denote by $(\tau_n^{i_0})_n$ the sequence of jumping times of the particle i_0. We denote by $\mathcal{A}_n$ the event

$$\mathcal{A}_n = \left\{\exists j \neq i_0 \,|\, \phi_{i_0}(X^{i_0}_{\tau_n^{i_0}}) \geq h(\phi_j(X^j_{\tau_n^{i_0}}))\right\},$$

where h is the function of Hypothesis 3.2. We have, for all $1 \leq k \leq l$,

$$\begin{aligned} P\left(\bigcap_{n=k}^{l+1} \mathcal{A}_n^c\right) &= E\left(E\left(\prod_{n=k}^{l+1} \mathbb{1}_{\mathcal{A}_n^c} \,|\, (X_t^1,...X_t^N)_{0\leq t<\tau_{l+1}^{i_0}}\right)\right) \\ &= E\left(\prod_{n=k}^{l} \mathbb{1}_{\mathcal{A}_n^c} E\left(\mathbb{1}_{\mathcal{A}_{l+1}^c} \,|\, (X_t^1,...X_t^N)_{0\leq t<\tau_{l+1}^{i_0}}\right)\right). \end{aligned}$$

By definition of the jump mechanism of the interacting particle system and by the first point of Hypothesis 3.2,

$$\begin{aligned} E\left(\mathbb{1}_{\mathcal{A}_{l+1}^c} \,|\, (X_t^1,...X_t^N)_{0\leq t<\tau_{l+1}^{i_0}}\right) &= \mathcal{H}(t,\mathbb{O}_{\tau_{l+1}^{i_0}},\mathbb{X}_{\tau_{l+1}^{i_0}})\left(A_{i_0}^c\right) \\ &\leq 1 - p_0, \end{aligned}$$

where A_{i_0} and p_0 are defined in Hypothesis 3.2. By induction on l, we get

$$P\left(\bigcap_{n=k}^{l} \mathcal{A}_n^c\right) \leq (1-p_0)^{l-k}, \ \forall 1 \leq k \leq l.$$

Since $p_0 > 0$, it yields that

$$P\left(\bigcup_{k\geq 1}\bigcap_{n=k}^{\infty}\mathcal{A}_n^c\right) = 0.$$

It means that, for infinitely many jumps τ_n almost surely, one can find a particle j such that $\phi_{i_0}(X^{i_0}_{\tau_n}) \geq h(\phi_j(X^j_{\tau_n}))$. Because there is only a finite number of other particles, one can find a particle, say j_0 (which is a random variable), such that

$$\phi_{i_0}(X^{i_0}_{\tau_n}) \geq h(\phi_{j_0}(X^{j_0}_{\tau_n})), \text{ for infinitely many } n \geq 1.$$

In particular, $\lim_{n\to\infty}\phi_{j_0}(X^{j_0}_{\tau_n}) = 0$ almost surely. We deduce that

$$\lim_{n\to\infty}(\phi_{i_0}(X^{i_0}_{\tau_n}),\phi_{j_0}(X^{j_0}_{\tau_n})) = (0,0).$$

This immediately imply that

$$\left(\phi_{i_0}(X^{i_0}_{\tau_\infty\text{-}}),\phi_{j_0}(X^{j_0}_{\tau_\infty\text{-}})\right) = (0,0).$$

We finally conclude that

$$P\left(\tau_{stop}\wedge\tau_\infty < +\infty\right) \leq \sum_{1\leq i<j\leq N} P\left(T_0^{ij} < +\infty\right). \tag{3.3}$$

Fix $i \neq j \in \{1,...,N\}$ and let us prove that $P\left(T_0^{ij} < +\infty\right) = 0$. We begin to divide the time into a sequence of intervals $[t_n,t_{n+1}[$ such that, for each interval, or the pair $(\phi_i(X^i),\phi_j(X^j))$ is far from $(0,0)$, or the distance functions ϕ_i and ϕ_j are of class C^2 (which will allow us to use the Itô's formula). Let $(t_n)_{n\geq 0}$ be the sequence of stopping times defined by

$$t_0 = \inf\{t \in [0,\tau_{stop}\wedge\tau_\infty[,\ \phi^i(X^i_t) + \phi^j(X^j_t) \leq a/2\}$$

and, for all $n \geq 0$,

$$\begin{aligned} t_{2n+1} &= \inf\{t \in [t_{2n},\tau_{stop}\wedge\tau_\infty[,\ \phi_i(X^i_t) + \phi_j(X^j_t) \geq a\} \\ t_{2n+2} &= \inf\{t \in [t_{2n+1},\tau_{stop}\wedge\tau_\infty],\ \phi_i(X^i_t) + \phi_j(X^j_t) \leq a/2\}. \end{aligned}$$

By construction, we have for all $n \geq 0$,

$$\begin{cases} \phi_i(X^i_t) < a \text{ and } \phi_j(X^j_t) < a,\ \forall t \in [t_{2n},t_{2n+1}[, \\ \phi_i(X^i_t) \geq a/2 \text{ or } \phi_j(X^j_t) \geq a/2 \text{ otherwise.} \end{cases}$$

We emphasize that $T_0^{ij} \notin [t_{2n+1},t_{2n+2}[$ almost surely, while, for all $t \in [t_{2n},t_{2n+1}[$, ϕ_i and ϕ_j are of class C^2 at X^i_t and X^j_t, which will allow us to use the Itô's formula during these intervals of time. In particular, Hypothesis 3.1 and the Itô's formula immediately implies that $\phi_i(X^i) + \phi_j(X^j)$ is an Itô diffusion process with bounded coefficients between times t_{2n} and t_{2n+1} for all $n \geq 0$. Since $\phi_i(X^i) + \phi_j(X^j)$ goes from $a/2$ to a between times t_{2n} and t_{2n+1}, we deduce that $(t_n)_{n\geq 0}$ converges to $+\infty$ almost surely. We deduce that

$$P\left(T_0^{ij} < +\infty\right) \leq \sum_{n=0}^{+\infty} P\left(T_0^{ij} \in [t_{2n},t_{2n+1}[\right).$$

It remains us to prove that $P\left(T_0^{ij} \in [t_{2n},t_{2n+1}[\right) = 0$ for all $n \geq 0$.

Fix $n \geq 0$. We define the positive semi-martingale Y^i by

$$Y_t^i = \begin{cases} \phi_i(X^i_{t_{2n}+t}) \text{ if } t < t_{2n+1} - t_{2n}, \\ a/2 + |W_t^i| \text{ if } t \geq t_{2n+1} - t_{2n}, \end{cases} \tag{3.4}$$

where W^i is a standard one dimensional Brownian motion, which allows us to define Y_t^i for all time $t \in [0, +\infty[$. We define similarly the semi-martingale Y^j. It is clear that

$$P(T_0^{ij} \in [t_{2n},t_{2n+1}[) \leq P(\exists t \geq 0,\, (Y^i_{t\text{-}},Y^j_{t\text{-}}) = (0,0)).$$

The problem of non-explosion of our interacting process is then reduced to the problem of the attainability of (0,0) by a given semi-martingale. In order to prove the non-attainability of (0,0) by (Y^i, Y^j), we need to compute the Itô's decomposition of Y^i and Y^j.

Let us set

$$\pi_t^i = \begin{cases} f_i(t,o_t^i,X_t^i), & \text{if } t < t_{2n+1} - t_{2n}, \\ 1, & \text{if } t \geq t_{2n+1} - t_{2n} \end{cases} \text{ and } \rho_t^i = \begin{cases} g(t,o_t^i,X_t^i), & \text{if } t < t_{2n+1} - t_{2n}, \\ 0, & \text{if } t \geq t_{2n+1} - t_{2n}, \end{cases}$$

where f_i and g_i are given by Hypothesis 3.1. By the Itô's formulas applied to Y^i, we have

$$dY_t^i = dM_t^i + b_t^i dt + dK_t^i + Y_t^i - Y^i_{t\text{-}},$$

where M^i is a local martingale such that

$$d\langle M^i \rangle_t = (\pi_t^i + \rho_t^i)dt,$$

b^i is the adapted process given by

$$b_t^i = \begin{cases} \sum_{k=1}^{d'i} \frac{\partial \phi_i}{\partial x_k}(X_t^i)[\mu_i]_k(t,o_t^i,X_t^i) + \frac{1}{2}\sum_{k,l=1}^{d_i} \frac{\partial^2 \phi}{\partial x_k \partial x_l}(X_t^i)[\sigma_i\sigma_i^*]_{kl}(t,o_t^i,X_t^i), & \text{if } t < t_{2n+1} - t_{2n}, \\ 0 & \text{if } t \geq t_{2n+1} - t_{2n}, \end{cases}$$

and K^i is a non-decreasing process given by the local time of $|W_t|$ at 0 after time $t_{2n+1}-t_{2n}$. By the 4^{th} point of Hypothesis 3.1, we have, for all $t \geq 0$,

$$c_\pi \wedge 1 \leq \pi_t^i + \rho_t^i \leq C_\pi \vee 1, \text{ and } |\rho_t^i| \leq k_0 Y_t^i \tag{3.5}$$

The regularity of ϕ_i in D_a (1^{st} point of Hypothesis 3.1) and the boundedness of μ_i,σ_i (3^{rd} point of Hypothesis 3.1), implies that there exists $b_\infty > 0$ such that, for all $t \geq 0$,

$$b_t^i \geq -b_\infty. \tag{3.6}$$

Similarly, we get the decomposition of Y^j, with π^j, ρ^j and b^j fulfilling inequalities (3.5) and (3.6) (without loss of generality, we keep the same constants c_π, C_π, k_0 and b_∞).

The previous decomposition isn't *a priori* sufficient to prove the non-attainability of (0,0) by (Y^i,Y^j): we also need to compute the decomposition of π^i and π^j. We deduce from the Itô's formula that there exists a local martingale N^i and a finite variational process L^i such that, for all $t \geq 0$,

$$d\pi_t^i = dN_t^i + dL_t^i + \pi_t^i - \pi^i_{t\text{-}}.$$

We emphasize that we do not need the explicit computation of L^i. Let us set, for all $t < t_{2n+1} - t_{2n}$,

$$\xi^i_t = \sum_{k=1,l}^{d_i} \frac{\partial f_i}{\partial e_k}(t,o^i_t,X^i_t)\frac{\partial f_i}{\partial e_l}(t,o^i_t,X^i_t)[s_i s_i^*]_{kl}(t,o^i_t,X^i_t)$$
$$+ \sum_{k=1,l}^{d'_i} \frac{\partial f_i}{\partial x_k}(t,o^i_t,X^i_t)\frac{\partial f_i}{\partial x_l}(t,o^i_t,X^i_t)[\sigma_i \sigma_i^*]_{kl}(t,o^i_t,X^i_t)$$

and, for all $t \geq t_{2n+1} - t_{2n}$, $\xi^i_t = 0$. Then we have

$$\langle N^i \rangle_t = \xi^i_t dt.$$

Thanks to the regularity assumptions on f_i and the boundedness of s_i,σ_i, there exists $C_\xi > 0$ such that

$$\xi^i_t \leq C_\xi. \tag{3.7}$$

Of course, the same holds for π^j.

Since the particles are independent between the jumps, we have for all $i \neq j$,

$$\langle M^i, M^j \rangle = 0 \text{ and } \langle N^i, N^j \rangle = 0 \text{ a.s.} \tag{3.8}$$

We claim that the decompositions of Y^i, Y^j, π^i, π^j, together with the inequalities (3.5), (3.6), (3.7) and equation (3.8), imply that (Y^1,Y^2) never converges to $(0,0)$ almost surely. This is proved in the next section, where a criterion for non-attainability of $(0,0)$ for semi-martingales is given (Hypothesis 4.1 and Proposition 4.1 of Section 4). In particular, we deduce that $T^{ij}_0 \notin [t_{2n},t_{2n+1}[$ almost surely, for all $n \geq 0$.

We then have $T^{ij}_0 = +\infty$ almost surely, for all $i \neq j \in \{1,...,N\}$, which imply, by (3.3), that $\tau_{stop} \wedge \tau_{infty} = +\infty$. This concludes the proof of Theorem 3.1.

□

4 Non-attainability of $(0,0)$ for semi-martingales

Fix $T > 0$ and let $(Y^i_t)_{t\in[0,T]}$, $i = 1,2$, be two non-negative one-dimensional semi-martingales such that,

$$dY^i_t = dM^i_t + b^i_t dt + dK^i_t + I^i_t - I^i_{t\text{-}},\ Y^i_0 > 0,$$

where $(M^i_t)_{t\in[0,T]}$ is a continuous local martingale , $(b^i_t)_{t\in[0,T]}$ is an adapted process, $(K^i_t)_{t\in[0,T]}$ is a continuous and non-decreasing adapted process, and I^i_t is a pure-jump càdlàg process. The aim of this section is to give some conditions, which ensure that (Y^1,Y^2) doesn't hit $(0,0)$ up to time T. The problem has been solved for time homogeneous stochastic differential equations by Friedman [10], Ramasubramanian [18] and the proof of Proposition 4.1 below is inspired by the recent work of Delarue [7], which obtains lower and higher bound for the hitting time of a corner for a diffusion driven by a time homogeneous SDE reflected in the square. In our case, time-dependency is allowed and we don't require any Markovian property. This generalization finds an important application in the previous section, where the non-explosion of a very general interacting particle system with jumps from a boundary is proved.

Hypothesis 4.1. *For each $i = 1{,}2$, there exists a non-negative local semi-martingale π^i such that*

$$d\pi_t^i = dN_t^i + dL_t^i + J_t^i - J_{t\text{-}}^i,$$

where N^i is a continuous local martingale and L^i is a continuous finite variational adapted process and J_t^i is a pure-jump càdlàg process. Moreover, there exist two adapted processes ρ_t^i and ξ_t^i, and some positive constants $b_\infty, k_0, c_\pi, C_\pi, C_\xi$ such that, almost surely,

1. $d\langle M^i\rangle_t = (\pi_{t\text{-}}^i + \rho_{t\text{-}}^i)dt$ *and* $d\langle N^i\rangle_t = \xi_{t\text{-}}^i dt$,
2. $c_\pi \le \pi_t^i + \rho_t^i \le C_\pi$, $|\rho_t^i| \le k_0 Y_t^i$, $\xi_t \le C_\xi$ *and* $b_t^i \ge -b_\infty$ *for all* $t \in [0,T]$
3. $\langle M^1, M^2\rangle$ *and* $\langle N^1, N^2\rangle$ *are non-increasing processes.*
4. I^i *and* J^i *are such that, for all jump time t of the processes I and J,*

$$\frac{Y_t^i}{\sqrt{\pi_t^i}} - \frac{Y_{t\text{-}}^i}{\sqrt{\pi_{t\text{-}}^i}} \ge 0.$$

The third point of Hypothesis 4.1 has the following geometrical interpretation: when an increment of M^1 is non-positive (that is when M^1 goes closer to 0), the increment of M^2 is non-negative (so that M^2 goes farther from 0), as a consequence (M^1, M^2) remains away from 0. A nice graphic representation of this phenomenon is given by Delarue's [7, Figure 1].

Remark 4.1. An example of a pair of semi-martingales which fulfills Hypothesis 4.1 is given in the proof of Theorem 3.1 in Section 3, where (Y^1, Y^2) is given by a smooth function of a pair of diffusion processes. In this typical case, checking the validity of our assumption is a simple application of the Itô's formula.

The process

$$\Phi_t \overset{def}{=} -\frac{1}{2}\log\left(\frac{(Y_t^1)^2}{\pi_t^1} + \frac{(Y_t^2)^2}{\pi_t^2}\right) \tag{4.1}$$

goes to infinity when (Y_t^1, Y_t^2) goes to $(0{,}0)$, since π_t^i is uniformly bounded below by c_π. For all $\epsilon > 0$, we define the stopping time $T_\epsilon = \inf\{t \in [0,T],\ \Phi_t \ge \epsilon^{-1}\}$. We denote the hitting time of $(0{,}0)$ by $T_0 = \inf\{t \in [0,T],\ (Y_{t\text{-}}^1, Y_{t\text{-}}^2) = (0{,}0) \text{ or } (Y_t^1, Y_t^2) = (0{,}0)\}$. In particular, we have

$$T_0 = \lim_{\epsilon \to 0} T_\epsilon, \text{ almost surely.}$$

We are now able to state our non-attainability result.

Proposition 4.1. *Assume that Hypothesis 4.1 is fulfilled. Then (Y^1, Y^2) doesn't go to $(0{,}0)$ in $[0,T]$ almost surely, which means that T_0 is equal to $+\infty$ almost surely.*

Moreover, there exists a positive constant C which only depends on $b_\infty, k_0, c_\pi, C_\pi, C_\xi$ such that, for all $\epsilon^{-1} > \Phi_0$,

$$P(T_\epsilon \le T) \le \frac{1}{\epsilon^{-1} - \Phi_0} C\left(E(|L^1|_T + |L^2|_T) + T\right),$$

where $|L^i|_T$ is the total variation of L^i at time T and Φ_0 is defined in (4.1).

Proof of Proposition 4.1: Let $(\theta'_n)_{n\in\mathbb{N}}$ and $(\theta''_n)_{n\in\mathbb{N}}$ be two increasing sequences of stopping times which converge to T such that $(M^i_t)_{t\in[0,\theta'_n]}$ and $(N^i_t)_{t\in[0,\theta'_n]}$ are true martingales and such that $\theta''_n = \inf\{t \in [0,T], \int_0^{\theta''_n} d|L^i|_t \geq n\} \wedge T$. The whole proof is based on an application of the Itô's formula to the semi-martingale

$$\left(\int_0^{\Phi_t} \exp(e^{C_F} e^{-u})du\right)_{t\in[0,T_\epsilon\wedge\theta'_{n'}\wedge\theta''_{n''}]}, \ n',n'' \in \mathbb{N},$$

where $C_F > 0$ is a constant which only depends on the parameters $b_\infty, k_0, c_\pi, C_\pi, C_\xi$. We prove that, for a good choice of C_F, there exists a constant C which doesn't depend on ϵ, n' and n'' such that

$$E\left(\int_{\Phi_0}^{\Phi_{T_\epsilon\wedge\theta'_{n'}\wedge\theta''_{n''}}} \exp(e^{C_F} e^{-u})du\right) \leq C(E(|L^1|_{\theta''_{n''}} + |L^2|_{\theta''_{n''}}) + T). \tag{4.2}$$

Assume that this inequality has been proved. We notice that $\Phi_{t\wedge T_\epsilon\wedge\theta'_{n'}\wedge\theta''_{n''}}$ reaches ϵ^{-1} if and only if $T_\epsilon \leq \theta'_{n'} \wedge \theta''_{n''}$, then, by the right continuity of Y^1, Y^2, π^1 and π^2,

$$\begin{aligned} P\left(T_\epsilon \leq \theta'_{n'} \wedge \theta''_{n''}\right) &= P\left(\Phi_{T_\epsilon\wedge\theta'_{n'}\wedge\theta''_{n''}} - \Phi_0 \geq \epsilon^{-1} - \Phi_0\right) \\ &\leq P\left(\int_{\Phi_0}^{\Phi_{T_\epsilon\wedge\theta'_{n'}\wedge\theta''_{n''}}} \exp(e^{C_F} e^{-u})du \geq \epsilon^{-1} - \Phi_0\right), \end{aligned}$$

since $r - q \leq \int_q^r \exp(e^{C_F} e^{-u})du$ for all $0 \leq q \leq r$. Finally, using the Markov inequality and (4.2), we get, for all $\epsilon^{-1} > \Phi_0$,

$$P\left(T_\epsilon \leq \theta'_{n'} \wedge \theta''_{n''}\right) \leq \frac{1}{\epsilon^{-1} - \Phi_0} C\left(E(|L^1|_{\theta''_{n''}} + |L^2|_{\theta''_{n''}}) + T\right).$$

Letting n' go to ∞, then ϵ go to 0 and finally n'' go to ∞, we deduce that $P\left(T_0 \leq T\right) = 0$, which is the first point of Proposition 4.1. Since $\theta'_{n'}$ and $\theta''_{n''}$ converge to T almost surely, letting n' and n'' go to ∞ implies the second part of Proposition 4.1, which concludes the proof.

It remains us to prove inequality (4.2). We assume in a first time that $\langle M^1, M^2\rangle = \langle N^1, N^2\rangle = 0$. We define the function

$$\begin{aligned} \Phi : \ \mathbb{R}^*_+ \times \mathbb{R}^*_+ \times \mathbb{R}_+ \times \mathbb{R}^*_+ &\to \mathbb{R} \\ (\alpha_1, \alpha_2, x_1, x_2) &\mapsto -\log\left(\frac{x_1^2}{\alpha_1} + \frac{x_2^2}{\alpha_2}\right). \end{aligned}$$

We have $\Phi_t = Phi(\pi^1_t, \pi^2_t, Y^1_t, Y^2_t)$. We will apply the Itô's formula to the semi-martingale $(\Phi_t)_{t\in[0,T_\epsilon\wedge\theta'_{n'}\wedge\theta''_{n''}[}$. The successive derivatives of the function Φ are

$$\begin{aligned} \frac{\partial\Phi}{\partial x_i} &= -\alpha_i^{-1} x_i e^{2\Phi}, \ \frac{\partial^2\Phi}{\partial x_i^2} = -\alpha_i^{-1} e^{2\Phi} + 2\alpha_i^{-2} x_i^2 e^{4\Phi}, \\ \frac{\partial\Phi}{\partial\alpha_i} &= \frac{1}{2}\alpha_i^{-2} x_i^2 e^{2\Phi}, \ \frac{\partial^2\Phi}{\partial\alpha_i^2} = -\alpha_i^{-3} x_i^2 e^{2\Phi} + \alpha_i^{-4} x_i^4 e^{4\Phi}, \\ \frac{\partial^2\Phi}{\partial x_i \alpha_i} &= \alpha_i^{-2} x_i e^{2\Phi} - \alpha_i^{-3} x_i^3 e^{4\Phi}, \ \frac{\partial^2\Phi}{\partial x_i \alpha_j} = -\alpha_i^{-1}\alpha_j^{-2} x_i x_j^2 e^{4\Phi} \text{ with } i \neq j. \end{aligned}$$

In particular, one can check that

$$\sum_{i=1,2} \frac{\partial^2 \Phi}{\partial x_i^2}(\pi^1_{t\text{-}},\pi^2_{t\text{-}},Y^1_{t\text{-}},Y^2_{t\text{-}})\pi^i_{t\text{-}} = 0, \text{ almost surely.}$$

Using the previous equalities and the Itô's formula, we get

$$\begin{aligned}
d\Phi_t = &-\sum_{i=1,2} \frac{Y^i_t}{\pi^i_t} e^{2\Phi_t} dM^i_t + \sum_{i=1,2} \frac{(Y^i_t)^2}{2(\pi^i_t)^2} e^{2\Phi_t} dN^i_t - \sum_{i=1,2} \frac{Y^i_t}{\pi^i_t} e^{2\Phi_t} dK^i_t \\
&- \sum_{i=1,2} \frac{Y^i_t}{\pi^i_t} e^{2\Phi_t} b^i_t dt + \sum_{i=1,2} \frac{(Y^i_t)^2}{2(\pi^i_t)^2} e^{2\Phi_t} dL^i_t \\
&+ \frac{1}{2} \sum_{i=1,2} \left(-\frac{1}{\pi^i_t} e^{2\Phi_t} + 2\frac{(Y^i_t)^2}{(\pi^i_t)^2} e^{4\Phi_t} \right) \rho^i_t dt \\
&+ \frac{1}{2} \sum_{i=1,2} \left(-\frac{(Y^i_t)^2}{(\pi^i_t)^3} e^{2\Phi_t} + \frac{(Y^i_t)^4}{(\pi^i_t)^4} e^{4\Phi_t} \right) d\left\langle N^i \right\rangle_t \\
&+ \frac{1}{2} \sum_{i=1,2} \left(\frac{Y^i_t}{(\pi^i_t)^2} e^{2\Phi_t} - \frac{(Y^i_t)^3}{(\pi^i_t)^3} e^{4\Phi_t} \right) d\left\langle M^i,N^i \right\rangle_t \\
&- \frac{1}{2} \sum_{i\neq j\in\{1,2\}} \frac{Y^i_t (Y^j_t)^2}{\pi^i_t (\pi^j_t)^2} e^{4\Phi_t} d\left\langle M^i,N^j \right\rangle_t + \Phi_t - \Phi_{t\text{-}}
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
d\left\langle \Phi \right\rangle_t = &\sum_{i=1,2} \frac{(Y^i_t)^2}{(\pi^i_t)^2} e^{4\Phi_t} (\rho^i_t + \pi^i_t) dt + \sum_{i=1,2} \frac{(Y^i_t)^4}{4(\pi^i_t)^4} e^{4\Phi_t} d\left\langle N^i \right\rangle_t \\
&- \sum_{i=1,2} \frac{(Y^i_t)^3}{2(\pi^i_t)^3} e^{4\Phi_t} d\left\langle M^i,N^i \right\rangle_t - \sum_{i\neq j\in\{1,2\}} \frac{Y^i_t (Y^j_t)^2}{2\pi^i_t (\pi^j_t)^2} e^{4\Phi_t} d\left\langle M^i,N^j \right\rangle_t .
\end{aligned}$$

Let $C_F > 0$ be a positive constant that will be fixed later in the proof and define the function $F : \mathbb{R} \mapsto \mathbb{R}$ by

$$F(r) = \int_0^r \exp\left(C_F e^{-s}\right) ds.$$

We check that

$$r \leq F(r),\ 1 \leq F'(r) \leq e^{C_F} \text{ and } F''(r) = -C_F e^{-r} F'(r),\ \forall r \in \mathbb{R}_+.$$

We deduce from Itô's formula that

$$F(\Phi_t) - F(\Phi_0) = \int_0^t F'(\Phi_s) d\Phi^c_s - \frac{C_F}{2} \int_0^t e^{-\Phi_s} F'(\Phi_s) d\left\langle \Phi \right\rangle_s + \sum_{0\leq s\leq t} F(\Phi_s) - F(\Phi_{s\text{-}}), \tag{4.4}$$

where $d\Phi^c_s$ is the continuous part of $d\Phi_s$.

Using equation (4.3), we begin to prove a higher bound for $\int_0^t F'(\Phi_s)d\Phi^c_s$. We define the local martingale

$$M_t = -\sum_{i=1,2} \int_0^t \frac{Y^i_s}{\pi^i_s} e^{2\Phi_s} F'(\Phi_s) dM^i_s + \sum_{i=1,2} \int_0^t \frac{(Y^i_s)^2}{2(\pi^i_s)^2} e^{2\Phi_s} F'(\Phi_s) dN^i_s.$$

Since K^i is non-decreasing, we have

$$-\sum_{i=1,2}\int_0^t \frac{Y_s^i}{\pi_s^i}e^{2\Phi_s}F'(\Phi_s)dK_s^i \le 0.$$

One can easily check that, for all $t \in [0,T_0[$, $Y_t^i e^{\Phi_t} \le \sqrt{\pi_t^i}$, then

$$\frac{Y_t^i}{\pi_t^i}e^{\Phi_t} \le \frac{1}{\sqrt{c_\pi}}.$$

Since $b_t^i \ge -b_\infty$ for all $t \in [0,T_0[$, we have

$$-\sum_{i=1,2}\int_0^t \frac{Y_s^i}{\pi_s^i}F'(\Phi_s)e^{2\Phi_s}b_s^i ds \le \frac{2b_\infty}{\sqrt{c_\pi}}\int_0^t e^{\Phi_s}F'(\Phi_s)ds.$$

The inequality $F'(\Phi_s) \le e^{C_F}$ yields to

$$\sum_{i=1,2}\int_0^t \frac{(Y_s^i)^2}{2(\pi_s^i)^2}e^{2\Phi_s}F'(\Phi_s)dL_s^i \le \frac{e^{C_F}}{2c_\pi}\left(|L^1|_t + |L^2|_t\right).$$

We deduce from

$$|\rho_t^i|e^{\Phi_t} \le k_0 Y_t^i e^{\Phi_t} \le k_0\sqrt{\pi_t^i} \le k_0\sqrt{C_\pi}$$

that

$$\frac{1}{2}\sum_{i=1,2}\int_0^t \left(-\frac{1}{\pi_s^i}e^{2\Phi_s} + 2\frac{(Y_s^i)^2}{(\pi_s^i)^2}e^{4\Phi_s}\right)\rho_s^i F'(\Phi_s)ds \le \frac{3e^{C_F}k_0\sqrt{C_\pi}}{c_\pi}\int_0^t e^{\Phi_s}F'(\Phi_s)ds.$$

Since $d\langle N^i\rangle_t = \xi_t^i dt$, with $0 \le \xi_t^i \le C_\xi$, we have

$$\frac{1}{2}\sum_{i=1,2}\int_0^t \left(-\frac{(Y_s^i)^2}{(\pi_s^i)^3}e^{2\Phi_s} + \frac{(Y_s^i)^4}{(\pi_s^i)^4}e^{4\Phi_s}\right)F'(\Phi_s)d\left\langle N^i\right\rangle_s \le \frac{e^{C_F}C_\xi t}{c_\pi^2}.$$

By the Kunita-Watanabe inequality (see [19, Corollary 1.16 of Chapter IV]), we get, for all predictable process h_s,

$$\left|\int_0^t h_s\left\langle M^i,N^j\right\rangle_s\right| \le \sqrt{\int_0^t h_s\left\langle M^i\right\rangle_s}\sqrt{\int_0^t h_s\left\langle N^j\right\rangle_s} \le \sqrt{C_\pi C_\xi}\int_0^t h_s ds,$$

so that

$$\frac{1}{2}\sum_{i=1,2}\int_0^t \left(\frac{Y_s^i}{(\pi_s^i)^2}e^{2\Phi_s} - \frac{(Y_s^i)^3}{(\pi_s^i)^3}e^{4\Phi_s}\right)F'(\Phi_s)d\left\langle M^i,N^i\right\rangle_s \le \frac{2\sqrt{C_\pi C_\xi}}{c_\pi^{3/2}}\int_0^t e^{\Phi_s}F'(\Phi_s)ds$$

and

$$-\frac{1}{2}\sum_{i\neq j\in\{1,2\}}\int_0^t \frac{Y_s^i(Y_s^j)^2}{\pi_s^i(\pi_s^j)^2}e^{4\Phi_s}F'(\Phi_s)d\left\langle M^i,N^j\right\rangle_s \le \frac{\sqrt{C_\pi C_\xi}}{c_\pi^{3/2}}\int_0^t e^{\Phi_s}F'(\Phi_s)ds.$$

We finally get

$$\int_0^t F'(\Phi_s)d\Phi_s^c \le M_t + C'\int_0^t e^{\Phi_s}F'(\Phi_s)ds + \frac{e^{C_F}}{2c_\pi}\left(|L^1|_t+|L^2|_t\right) + \frac{e^{C_F}C_\xi t}{c_\pi^2}. \tag{4.5}$$

where

$$C' = \frac{2b_\infty}{\sqrt{c_\pi}} + \frac{3k_0\sqrt{C_\pi}}{c_\pi} + \frac{3\sqrt{C_\pi C_\xi}}{c_\pi^{3/2}} > 0.$$

We prove now a lower bound for $\int_0^t e^{-\Phi_s}F'(\Phi_s)d\left\langle\Phi\right\rangle_s$. We have

$$\frac{e^{2\Phi_s}}{C_\pi} \le \sum_{i=1,2}\frac{(Y_s^i)^2}{(\pi_s^i)^2}e^{4\Phi_s} \le \frac{2e^{2\Phi_s}}{c_\pi},\ \pi_s^i \ge c_\pi \text{ and } \rho_s^i \ge -k_0\sqrt{C_\pi}e^{-\Phi_s}$$

then

$$\int_0^t \sum_{i=1,2}\frac{(Y_s^i)^2}{(\pi_s^i)^2}e^{4\Phi_s}(\pi_s^i+\rho_s^i)e^{-\Phi_s}F'(\Phi_s)ds \ge \frac{c_\pi}{C_\pi}\int_0^t e^{\Phi_s}F'(\Phi_s)ds - \frac{2k_0\sqrt{C_\pi}}{c_\pi}e^{C_F}t.$$

The process $\langle N^i\rangle$ being non-decreasing, we have

$$\sum_{i=1,2}\int_0^t \frac{(Y_s^i)^4}{4(\pi_s^i)^4}e^{4\Phi_s}F'(\Phi_s)e^{-\Phi_s}d\left\langle N^i\right\rangle_s \ge 0.$$

The same argument as above leads us to

$$-\sum_{i=1,2}\int_0^t \frac{(Y_s^i)^3}{2(\pi_s^i)^3}e^{4\Phi_s}F'(\Phi_s)e^{-\Phi_s}d\left\langle M^i,N^i\right\rangle_s \ge -\frac{\sqrt{C_\pi C_\xi}}{c_\pi^{3/2}}e^{C_F}t$$

and

$$-\sum_{i\neq j\in\{1,2\}}\int_0^t \frac{Y_s^i(Y_s^j)^2}{2\pi_s^i(\pi_s^j)^2}e^{4\Phi_s}F'(\Phi_s)e^{-\Phi_s}d\left\langle M^i,N^j\right\rangle_s \ge -\frac{\sqrt{C_\pi C_\xi}}{c_\pi^{3/2}}e^{C_F}t.$$

We finally deduce that

$$\int_0^t e^{-\Phi_s}F'(\Phi_s)d\left\langle\Phi\right\rangle_s \ge \frac{c_\pi}{C_\pi}\int_0^t e^{\Phi_s}F'(\Phi_s)ds - \left(\frac{2k_0\sqrt{C_\pi}}{c_\pi} + \frac{2\sqrt{C_\pi C_\xi}}{c_\pi^{3/2}}\right)e^{C_F}t \tag{4.6}$$

Since the jumps of Φ_t are negative and F is non-decreasing, we get

$$\sum_{0\le s\le t} F(\Phi_s) - F(\Phi_{s\text{-}}) \le 0. \tag{4.7}$$

By (4.5), (4.6) and (4.7), we deduce from (4.4) that

$$F(\Phi_t) - F(\Phi_0) \le M_t + \left(C' - \frac{C_F c_\pi}{2C_\pi}\right)\int_0^t e^{\Phi_s}F'(\Phi_s)ds + \frac{e^{C_F}}{2c_\pi}\left(|L^1|_t+|L^2|_t\right)$$
$$+\left(\frac{C_\xi}{c_\pi^2} - \frac{2k_0C_F\sqrt{C_\pi}}{2c_\pi} - \frac{C_F\sqrt{C_\pi C_\xi}}{c_\pi^{3/2}}\right)e^{C_F}t.$$

Choosing $C_F = 2C_\pi C'/c_\pi$, we've proved that there exists $C > 0$ such that

$$F(\Phi_t) - F(\Phi_0) \leq M_t + C\left(|L^1|_t + |L^2|_t + t\right).$$

This yields to (4.2), since the process M_t stopped at $T_\epsilon \wedge \theta'_n \wedge \theta''_n$ is a true martingale. The proposition is then proved when $\langle M^1,M^2\rangle = \langle N^1,N^2\rangle = 0$.

Assume now that $\langle M^1,M^2\rangle$ and $\langle N^1,N^2\rangle$ are non-increasing. We define Φ'_t as the process starting from Φ_0 and whose increments are defined by the right term of (4.3). On the one hand, the same calculation as above leads to

$$F(\Phi'_t) \leq M_t + \frac{e^{C_F}}{c_\pi}\left(|L^1|_{\theta''_{n''}} + |L^2|_{\theta''_{n''}}\right) + \left(e^{C_F}C' + \frac{C_F}{2}C'''\right)t. \tag{4.8}$$

On the other hand,

$$d\Phi_t = d\Phi'_t + \frac{\partial^2\Phi}{\partial x_1 \partial x_2}(\pi^1_t,\pi^2_t,Y^1_t,Y^2_t)d\left\langle M^1,M^2\right\rangle_t + \frac{\partial^2\Phi}{\partial \alpha_1 \partial \alpha_2}(\pi^1_t,\pi^2_t,Y^1_t,Y^2_t)d\left\langle N^1,N^2\right\rangle_t,$$

and we can check that $\frac{\partial^2\Phi}{\partial x_1\partial x_2}$ and $\frac{\partial^2\Phi}{\partial \alpha_1\partial \alpha_2}$ are non-negative functions. We deduce from the third point of Hypothesis 4.1 that $\Phi_t \leq \Phi'_t$. But F is increasing, so that (4.8) leads us to (4.2) in the general case. □

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