Interacting Particle Filtering With Discrete Observations

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1 Introduction

We consider a pair of processes (X, Y), where X represents the "state" of a system (or "signal") and Y represents the observation: X may take its values in an arbitrary measurable space $(E, \mathcal{E})$, but it is important for what follows that Y takes its values in $\mathbb{R}^q$ for some $q \geq 1$.

A distinctive feature of our study is that although the pair (X, Y) might be indexed by $\mathbb{R}_+$, the actual observations take place at **discrete times** only: this is not for mathematical convenience, but because discrete time observations arise in a natural way as soon as real data are concerned. We will also assume that the pair (X, Y) is Markov (see below). So, up to a change of time, it is no restriction to suppose that the observation times are the nonnegative integers, and the basic assumption will then be that the process $(X_n, Y_n)_{n \in \mathbb{N}}$ is a — possibly non-homogeneous — Markov chain.

In this paper we are interested in the nonlinear filtering problem (in short **NLF**). That is, we want to find the one step predictor conditional probability given for each $n \in \mathbb{N}$ and each measurable function f on E such that $f(X_n)$ is integrable by

$$\eta_{n,Y} f \;=\; \mathbb{E}(f(X_n) | Y_0, \ldots, Y_{n-1})$$

(with the convention that $\eta_{0,Y}$ is the law of X_0), and the so-called filter conditional distribution given by

$$\widehat{\eta}_{n,Y} f \;=\; \mathbb{E}(f(X_n) | Y_0, \ldots, Y_n).$$

Throughout all the paper we fix the observations $Y_n = y_n,\ n \in I\!N$, and instead of $\eta_{n,Y}$ and $\widehat{\eta}_{n,Y}$ we simply write η_n and $\widehat{\eta}_n$.

Our aim is the design of an interacting particle system approach (**IPS** in short) for numerical computation of the distributions η_n and $\widehat{\eta}_n$.

Let us now single out the two types of NLF problems covered by our work.

- **Case A**:
  The state signal $(X_n)_{n\in I\!N}$ is an E-valued non-homogeneous Markov chain with 1-step transition probabilities $(Q_n)_{n\geq 1}$ (i.e. Q_n is the transition from time $n-1$ to time n) and initial law η_0. The observation sequence $(Y_n)_{n\in I\!N}$ takes the form

  $$Y_n \ = \ H_n(X_n, V_n) \tag{1.1}$$

  for some measurable functions H_n from $E\times F$ (with $(F,\mathcal{F})$ an auxiliary measurable space) into $I\!R^q$, and where the sequence (V_n) is i.i.d. with values in $(F,\mathcal{F})$. The basic assumption concerning H and the law of the V_n's is that for any $x\in E$ and any n the variable $H_n(x,V_n)$ admits a *strictly positive* and *bounded* density $y\mapsto \bar{g}_n(x,y)$ w.r.t. Lebesgue measure on $I\!R^q$.

- **Case B**:
  The pair signal/observation process (X,Y) is an $E\times I\!R^q$-valued non-homogeneous Markov chain with 1-step transition probabilities $(P_n)_{n\geq 1}$ and initial law μ_0 having the form (dy denotes Lebesgue measure):

  $$\left.\begin{aligned} \mu_0(dx,dy) \ &= \ \eta_0(dx)\bar{G}_0(x,y)dy \\ P_n(x,y;dx',dy') \ &= \ Q_n(x,dx')\bar{G}_n(x,y;x',y')dy' \end{aligned}\right\} \tag{1.2}$$

  where each Q_n is a transition probability and η_0 is a probability and each function $\bar{G}_n$ is nonnegative measurable and *bounded.* For homogeneity of notation we can consider $\bar{G}_0$ to be a function on $E\times I\!R^q\times E\times I\!R^q$, with the convention $\bar{G}_0(x,y;x',y') = \bar{G}_0(x',y')$. Note that necessarily $\int \bar{G}_n(x,y;x',y')dy' = 1$.

Clearly Case A is a particular case of Case B, with

$$\bar{G}_n(x,y;x',y') \ = \ \bar{g}_n(x',y'). \tag{1.3}$$

However, we single out Case A: it is easier to handle, and in fact the filtering problem in Case B will be reduced to an associated Case A problem. Observe also that in Case B the process X itself is Markov.

A crucial practical advantage of the first category of NLF problems is that it leads to a natural IPS-approximating scheme which is easily implemented. The first idea is to consider the equations that sequentially update the flow of distributions $\{\eta_n \; ; \; n \geq 0\}$, which are of the form

$$\eta_n = \Phi_n\left(\eta_{n-1}\right), \qquad n \geq 1, \tag{1.4}$$

with continuous mappings Φ_n (to be defined later) on the set $\mathbf{M}_1(E)$ of all probability measures on E. In this formulation the NLF problem is now reduced to the problem of solving a dynamical system taking values into the infinite dimensional state space $\mathbf{M}_1(E)$. To overcome this problem a natural idea is to approximate η_n for $n \geq 1$ by a sequence of empirical measures

$$\eta_n^N = \frac{1}{N} \sum_{i=1}^N \delta_{\xi_n^i}, \tag{1.5}$$

(δ_a stands for the Dirac measure at $a \in E$) associated with a system of N interacting particles $\xi_n = (\xi_n^1, \ldots, \xi_n^N)$ moving in the set E. It is indeed natural, in view of (1.4) above, to construct the sequence (ξ_n) as a Markov chain taking values in E^N, starting with the initial distribution $\widehat{\eta}_0$ and with 1-step transition probabilities $\widetilde{Q}_n$ given by

$$\widehat{\eta}_0(dx) = \prod_{p=1}^N \eta_0(dx^p), \qquad \widetilde{Q}_n(z, dx) = \prod_{p=1}^N \Phi_n(m(z))(dx^p), \tag{1.6}$$

where $dx \stackrel{\text{def}}{=} dx^1 \times \cdots \times dx^N$ is an infinitesimal neighborhood of the point $x = (x^1, \ldots, x^N) \in E^N$, $z = (z^1, \ldots, z^N) \in E^N$ and where $m(z)$ is the empirical distribution associated with z, i.e. $m(z) = \frac{1}{N}\sum_{i=1}^N \delta_{z^i}$.

The rationale behind this is that η_n^N is the empirical measure associated with N independent variables with common law $\Phi_n(\eta_{n-1}^N)$, so as soon as η_{n-1}^N is a good approximation of η_{n-1} then, in view of (1.4), η_n^N should be a good approximation of η_n.

The paper is organized as follows: Section 2 is devoted to recalling some known facts about NLF problems. In Section 3 we present an IPS to solve Case A. Since such an IPS is based upon Monte-Carlo simulations, we first study in details (in Subsection 3.1) the situation where all laws are explicitly known and where the simulation of random variables according to the laws $Q_n(x, .)$ is possible; then in Subsection 3.2 we consider the case where the laws are "approximately" known only and the simulations can be performed according to laws which are close to the $Q_n(x, .)$'s. In Section 4 we study Case B. Finally we give some applications to the situation where the state process X is the solution of a stochastic differential equation, in Section 5.

Most of the results presented here are taken from (Del Moral, Jacod and Protter 1998), (Del Moral and Jacod 1999), (Del Moral and Miclo 2000) and (Del Moral 1998). However the presentation here is quite different, and some new results have been added, mainly in the way Subcases A2 and B2 are treated, and Case B is more general here than in the above mentioned papers. The ideas of the proofs are provided, as well as full proofs for the new results.

Note that here we restrict our attention to bounded test functions. The analysis for unbounded test functions is more involved: see (Del Moral and Jacod 1999) and (Del Moral and Guionnet 1999).

2 Non linear filtering: general facts

1) In this section we remind the reader of some elementary facts about non-linear filtering. In the sequel, we use the traditional notation for transition kernels: if P and Q are two transition kernels and μ is a measure and f is a measurable functions (all on $(E,\mathcal{E})$), then we have another transition kernel PQ (the usual product, or composition), and a function Pf, and a measure μP, and a number μPf. We also denote by $\mathbf{M}_1(E)$ the set of all probability measures on $(E,\mathcal{E})$.

First we give some facts which holds under Case B, and thus *a fortiori* under Case A. Recall that the observations $y_0, y_1, \cdots$ are given and fixed, and introduce the following functions:

$$G_n(x,x') \;=\; \bar{G}_n(x,y_{n-1};x',y_n) \tag{2.1}$$

(recall that $\bar{G}_0(x,y;x',y')$ does not actually depend on y, so the above is well defined even for $n=0$).

Below, $I\!\!E$ denotes the expectation on the original probability space, on which the chain (X,Y) is Markov with the prescribed initial condition. Then Bayes' formula gives an "explicit" expression for η_n and $\hat{\eta}_n$, namely

$$\eta_n(f) = \frac{\gamma_n(f)}{\gamma_n(1)}, \qquad \widehat{\eta}_n(f) = \frac{\widehat{\gamma}_n(f)}{\widehat{\gamma}_n(1)} \tag{2.2}$$

for any measurable function f for which the following expressions make sense:

$$\left.\begin{array}{l} \gamma_n(f) = I\!\!E\left(f(X_n)\ \prod_{k=0}^{n-1} G_k(X_{k-1},X_k)\right), \\ \widehat{\gamma}_n(f) = I\!\!E\left(f(X_n)\ \prod_{k=0}^{n} G_k(X_{k-1},X_k)\right), \end{array}\right\} \tag{2.3}$$

with the convention $\prod_\emptyset = 1$; since $G_0(x,x')$ does not depend on x, the value X_{-1} does not actually occur in the formula (2.3). Indeed, (2.2) comes from the following fact: observing for example that $\widehat{\gamma}_n(f)$ is in fact a function of the observed values $y_0,\ldots,y_n$, we see that

$$(y_0,\ldots,y_n) \mapsto \widehat{\gamma}_n(1)(y_0,\ldots,y_n)$$

is the density of the variable $(Y_0,\dots,Y_n)$ w.r.t. Lebesgue measure on $(\mathbb{R}^q)^{n+1}$.

Observe that γ_n and $\widehat{\gamma}_n$ can be considered as finite positive measures (since each function G_n is bounded by hypothesis). These formulae are often referred to as *Feynman-Kac formulae*. Note that $\gamma_0 = \eta_0$, so (2.2) gives that the prediction measure at time 0 is indeed the initial measure η_0, as it should be since at this stage we have no observation at all.

2) Now we derive some (well known) formulae which allow to compute the filter and the one-step prediction measure in a recursive way. First, the Markov property in (2.9) gives for $n \geq 1$:

$$\widehat{\gamma}_n(f) = \widehat{\gamma}_{n-1}(\widehat{L}_n f), \tag{2.4}$$

where

$$\widehat{L}_n f(x) = \int Q_n(x,dx')G_n(x,x')f(x').$$

Therefore if we set for $0 \leq p \leq n$:

$$\widehat{L}_{p,n} = \widehat{L}_{p+1}\widehat{L}_{p+2}\dots\widehat{L}_n, \tag{2.5}$$

with the convention $\widehat{L}_{n,n} = Id$, we readily get

$$\widehat{\gamma}_n = \widehat{\gamma}_p\widehat{L}_{p,n}, \qquad \widehat{\gamma}_0(f) = \eta_0(fG_0).$$

In other words, we have

$$\widehat{\gamma}_n(f) = \eta_0(G_0\widehat{L}_{0,n}f), \tag{2.6}$$

and using (2.2) and (2.4), we get

$$\widehat{\eta}_n = \widehat{\Psi}_n(\widehat{\eta}_{n-1}), \qquad \text{where} \quad \widehat{\Psi}_n(\eta)(f) = \frac{\eta\widehat{L}_n f}{\eta\widehat{L}_n 1}.$$

Next, use again the Markov property to obtain for $n \geq 1$:

$$\gamma_n(f) = \widehat{\gamma}_{n-1}(Q_n f), \tag{2.7}$$

and thus by (2.6),

$$\gamma_n(f) = \begin{cases} \eta_0(f) & \text{if } n = 0 \\ \eta_0(G_0\widehat{L}_{0,n-1}Q_n f) & \text{if } n \geq 1. \end{cases} \tag{2.8}$$

It also follows from (2.2) and (2.7) that

$$\eta_n(f) = \widehat{\eta}_{n-1}(Q_n f) \qquad \text{if } n \geq 1.$$

3) Let us now complement the previous facts in the situation of Case A. First, all what precedes remains true in Case A with $\bar{G}_n$ given by (1.3). So we set

$$g_n(x) \;=\; \bar{g}_n(x, y_n),$$

and (2.3) writes as

$$\gamma_n(f) = I\!\!E\left(f(X_n) \prod_{k=0}^{n-1} g_k(X_k) \right) \tag{2.9}$$

and

$$\widehat{\gamma}_n(f) = I\!\!E\left(f(X_n) \prod_{k=0}^{n} g_k(X_k) \right).$$

Next, in (2.4), $\widehat{L}_n$ becomes $\widehat{L}_n(f) = Q_n(g_n f)$. In the present situation we also have a handy recursive formula for γ_n. Namely, if we set

$$L_{p,n} = L_{p+1} L_{p+2} \ldots L_n, \qquad \text{with} \quad L_n f(x) = g_{n-1}(x)\, Q_n f(x) \tag{2.10}$$

for $0 \leq p \leq n$ and with the convention $L_{n,n} = Id$, we deduce from (2.8) that

$$\gamma_0 = \eta_0, \qquad \gamma_n = \gamma_p\, L_{p,n}, \qquad \widehat{\gamma}_n(f) = \gamma_n(f g_n). \tag{2.11}$$

Then if we introduce the following mappings $(\Psi_n)_{n\geq 0}$ and $(\Phi_n)_{n\geq 1}$ from $\mathbf{M}_1(E)$ into itself by

$$\begin{aligned} \Psi_n(\eta)(f) &= \frac{\eta(\, f\, g_n\,)}{\eta(g_n)}, \\ \Phi_n(\eta)(f) &= \Psi_{n-1}(\eta) Q_n f = \frac{\eta L_n f}{\eta L_n 1}, \end{aligned} \tag{2.12}$$

we have

$$\widehat{\eta}_n = \Psi_n\left(\eta_n\right), \qquad \eta_{n+1} = \Phi_{n+1}(\eta_n). \tag{2.13}$$

Finally, in view of (2.7) and (2.11) and (2.2) we get

$$\gamma_{n+1}(1) = \widehat{\gamma}_n(1) = \gamma_n(g_n) = \eta_n(g_n)\gamma_n(1),$$

hence we have for $n \geq 0$:

$$\gamma_n(1) = \prod_{p=0}^{n-1} \eta_p(g_p), \qquad \gamma_n(f) = \eta_n(f) \prod_{p=0}^{n-1} \eta_p(g_p), \tag{2.14}$$

with the usual convention $\prod_{\emptyset} = 1$.

3 An interacting particle system in Case A

3.1 Subcase A1

We label *Subcase A1* the case A, when one knows explicitely the densities $\bar{g}_n$ (hence all functions g_n), as well as η_0 and Q_n, and one also knows how to simulate random variables exactly according to the laws η_0 and $Q_n(x,.)$ for all $x \in E$ and $n \geq 1$.

1) Let us now define our IPS for approximating the flows η_n and $\widehat{\eta}_n$. This is done according to (1.5) and (1.6), but we will be more precise in our description below.

In fact we have two nested particle systems, each with size N: first N particles $\xi_n = (\xi_n^i)_{1\leq i\leq N}$ at time n which are used for approximating η_n via the empirical measure η_n^N given by (1.5); next N particles $\widehat{\xi}_n = (\widehat{\xi}_n^i)_{1\leq i\leq N}$ at time n which are used as an intermediate step. The resulting motion of the particles is decomposed into two separate mechanisms $\xi_n \longrightarrow \widehat{\xi}_n \longrightarrow \xi_{n+1}$, which can be modelled as follows.

The motion of these particles is defined on an auxiliary probability space, with respect to which the measure and expectation are denoted by $\widetilde{I\!P}$ and $\widetilde{I\!E}$, to differentiate them from the measure and expectation defined on the original space on which the basic process (X,Y) lives. We also denote by $\mathcal{G}_n$ the σ-field generated by the variables ξ_p for $p \leq n$ and $\widehat{\xi}_p$ for $p < n$, while $\widehat{\mathcal{G}}_n$ is the σ-field generated by the variables ξ_p and $\widehat{\xi}_p$ for $p \leq n$.

First, the variables ξ_0^i are drawn independently according to the initial law η_0. Next, the mechanism goes, by induction on n, according to the following two steps Markov rule:

Selection/Updating:

$$\widetilde{I\!P}(\widehat{\xi}_n \in (dx_1,\ldots,dx_N)|\mathcal{G}_n) = \prod_{p=1}^N \sum_{i=1}^N \frac{g_n(\xi_n^i)}{\sum_{j=1}^N g_n(\xi_n^j)}\, \delta_{\xi_n^i}(dx_p). \tag{3.1}$$

Mutation/Prediction:

$$\widetilde{I\!P}(\xi_{n+1} \in (dz_1,\ldots,dz_N)\,|\widehat{\mathcal{G}}_n) = \prod_{p=1}^N Q_{n+1}(\widehat{\xi}_n^p, dz_p). \tag{3.2}$$

As said before, the approximated measure η_n^N is then defined by (1.5), and by analogy with Equation (2.13) we approximate $\widehat{\eta}_n$ by

$$\widehat{\eta}_n^N = \Psi_n(\eta_n^N).$$

So in the selection transition, one updates the positions in accordance with the fitness functions $\{g_n;\, n \geq 1\}$ and the current configuration. More precisely, at each time $n \geq 0$, each particle examines the system of particles $\xi_n = \left(\xi_n^1, \ldots, \xi_n^N\right)$ and chooses randomly a site ξ_n^i, $1 \leq i \leq N$, with a probability which depends on the entire configuration ξ_n and given by

$$\frac{g_n\left(\xi_n^i\right)}{\sum_{j=1}^N g_n\left(\xi_n^j\right)}\,.$$

This mechanism is called the Selection/Updating transition as the particles are selected for reproduction, the most fit individuals being more likely to be selected. In other words, this transition allows heavy particles to give birth to some particles at the expense of light particles which die. The second mechanism is called Mutation/Prediction since at this step each particle evolves randomly according to a given transition probability kernel.

The preceding scheme is clearly a system of interacting particles undergoing adaptation in a time non-homogeneous environment represented by the fitness functions $\{g_n;\, n \in \mathbb{N}\}$.

In other words, we approximate the two step transitions of the system

$$\eta_n \xrightarrow{\text{Updating}} \widehat{\eta}_n \xrightarrow{\text{Prediction}} \eta_{n+1}$$

by a two step Markov chain taking values in the set of finitely discrete probability measures with atoms of size some integer multiple of $1/N$, namely,

$$\eta_n^N \stackrel{\text{def.}}{=} \frac{1}{N}\sum_{i=1}^N \delta_{\xi_n^i} \xrightarrow{\text{Selection}} \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{\xi}_n^i} \xrightarrow{\text{Mutation}} \eta_{n+1}^N = \frac{1}{N}\sum_{i=1}^N \delta_{\xi_{n+1}^i}\,.$$

Remark: The above gives us “good” approximations of η_n and $\widehat{\eta}_n$ for all $n \geq 0$. But of course, for η_n^N, this really makes sense only when $n \geq 1$, since we are supposed to know exactly η_0. ∎

In view of (2.12) and (3.3), we have

$$\widehat{\eta}_n^N = \Psi_n(\eta_n^N) = \Psi_n\Big(\frac{1}{N}\sum_{i=1}^N \delta_{\xi_n^i}\Big) = \sum_{i=1}^N \frac{g_n(\xi_n^i)}{\sum_{j=1}^N g_n(\xi_n^j)}\,\delta_{\xi_n^i}. \tag{3.3}$$

Hence another way of stating the selection/updating step consists in saying that, conditional on $\mathcal{G}_n$, the variables $\widehat{\xi}_n^i$ for $i = 1, \ldots, N$ are i.i.d. with law $\Psi_n(\eta_n^N)$. Taking also into account the mutation/prediction step, and again conditional on $\mathcal{G}_n$, the variables ξ_{n+1}^i are also i.i.d., with law $\Phi_{n+1}(\eta_n^N)$. Hence if

$$\delta_{n+1}^N f = \eta_{n+1}^N f - \Phi_{n+1}(\eta_n^N) f$$

we have for $n \geq 0$:

$$\begin{aligned} \widetilde{I\!\!E}(\delta_{n+1}^N f|\mathcal{G}_n) &= 0, \\ \widetilde{I\!\!E}((\delta_{n+1}^N f)^2|\mathcal{G}_n) &= \frac{1}{N}\,\Phi_{n+1}(\eta_n^N)\left((f - \Phi_{n+1}(\eta_n^N)(f))^2\right). \end{aligned} \tag{3.4}$$

Similarly we also have if $\delta_0^N = \eta_0^N f - \eta_0 f$:

$$\widetilde{I\!\!E}(\delta_0^N f) = 0, \qquad \widetilde{I\!\!E}((\delta_0^N f)^2) = \frac{1}{N}\eta_0\left((f - \eta_0(f))^2\right). \tag{3.5}$$

2) For studying the behavior of the IPS as $N \to \infty$, it is simpler to begin by studying the approximations of the measures γ_n, because of the “linear” structure of these measures. Starting with η_n^N above, we can introduce a natural approximation γ_n^N by mimicking the relations (2.14), which gives

$$\gamma_n^N(1) = \prod_{p=0}^{n-1} \eta_p^N(g_p), \qquad \gamma_n^N(f) = \eta_n^N(f)\,\gamma_n^N(1). \tag{3.6}$$

Then, recalling (2.10), observe that for $q \geq 0$ and any bounded measurable φ, and with the conventions that $\gamma_{-1}^N = \gamma_0 = \eta_0$ and $L_0 = Id$ and $\Phi_0(\eta_{-1}^N) = \eta_0$, we get from (3.6) that

$$\begin{aligned} \gamma_q^N(\varphi) - \gamma_q(\varphi) &= \sum_{p=0}^q \left(\gamma_p^N(L_{p,q}\varphi) - \gamma_{p-1}^N(L_p L_{p,q}\varphi)\right) \\ &= \sum_{p=0}^q \gamma_p^N(1)\,\left(\eta_p^N(L_{p,q}\varphi) - \Phi_p\left(\eta_{p-1}^N\right)(L_{p,q}\varphi)\right). \end{aligned} \tag{3.7}$$

By choosing $\varphi = L_{q,n} f$ with $q \leq n$ and by (3.4) and (3.5), one sees that the process

$$M_q^N(f) \stackrel{\text{def.}}{=} \gamma_q^N(L_{q,n}f) - \gamma_q(L_{q,n}f), \qquad 0 \leq q \leq n \tag{3.8}$$

is a $(\mathcal{G}_q)_{0\leq q\leq n}$-martingale with the angle bracket given by

$$\langle M^N(f)\rangle_q = \frac{1}{N}\sum_{p=0}^{q} \left(\gamma_p^N(1)\right)^2 \Phi_p(\eta_{p-1}^N)\left(\left(L_{p,n}f - \Phi_p(\eta_{p-1}^N)L_{p,n}f\right)^2\right) \quad (3.9)$$

We now derive some easy consequences of the martingale properties described above. Below, f is an arbitrary bounded measurable function on $(E,\mathcal{E})$, with sup-norm $\|f\|$. We also denote by α_n the sup-norm of the function g_n.

The first consequence follows from taking expectations in (3.8) and (3.9), with $p=n$ (hence $M_n^N(f)=\gamma_n^N(f)-\gamma_n(f)$), and it gives

$$\widetilde{I\!\!E}\left(\gamma_n^N(f)\right)=\gamma_n(f),$$

$$\begin{aligned}\widetilde{I\!\!E}&\left(\left(\gamma_n^N(f)-\gamma_n(f)\right)^2\right)\\ &=\frac{1}{N}\sum_{p=0}^{n} I\!\!E\left(\left(\gamma_p^N(1)\right)^2 \Phi_p(\eta_{p-1}^N)\left(\left(L_{p,n}f - \Phi_p(\eta_{p-1}^N)L_{p,n}f\right)^2\right)\right).\end{aligned} \quad (3.10)$$

Now, (2.10) yields

$$||L_n f|| \leq \alpha_{n-1}||f||,$$

while (2.14) yields

$$\gamma_n^N(1)\leq\prod_{p=1}^{n}\alpha_p.$$

Henceforth, (3.10) gives

$$\widetilde{I\!\!E}\left(\left(\gamma_n^N(f)-\gamma_n(f)\right)^2\right)^{1/2}\leq C_{n,2}\frac{\|f\|}{\sqrt{N}}, \qquad \text{where} \quad C_{n,2}=\sqrt{n+1}\left(\prod_{k=1}^{n}\alpha_k\right). \quad (3.11)$$

Other classical estimates on martingales give an analogue to (3.11) for the p^{th} moment, with a constant $C_{n,p}$.

Exponential estimates can also be deduced from (3.9) using Bernstein-type martingales inequalities or (Del Moral 1998) for an alternative proof based on (3.7) and Hoeffding's inequality. More precisely there exist some finite constants $C_1(n)$ and $C_2(n)$ such that for any $\epsilon>0$,

$$I\!\!P\left(\sup_{0\leq p\leq n}\left|\gamma_p^N(L_{p,n}f)-\gamma_p(L_{p,n}f)\right|>\epsilon\right)\leq C_1(n)\ \exp-\frac{N\epsilon^2}{C_2(n)\,\|f\|^2} \quad (3.12)$$

3) In fact the measures γ_n^N are just tools for studying η_n^N and $\widehat{\eta}_n^N$. But in view of (2.14), and with a bit of work (see e.g. (Del Moral et al. 1998)) one can deduce from (3.11) and (3.12) similar results for the difference $\eta_n^N f - \eta_n f$. Some more work gives in turn the result for $\widehat{\eta}_n^N f - \widehat{\eta}_n f$. At the end, one can prove the following result:

Proposition 3.1 *There exist constants $C(n)$, $C_1(n)$, $C_2(n)$, which depend on n* and on the observed values $y_0, \ldots, y_n$, *such that*

$$\widetilde{\mathbb{E}}\left(\left|\eta_n^N(f) - \eta_n(f)\right|\right) \leq C(n)\frac{\|f\|}{\sqrt{N}}, \quad \widetilde{\mathbb{E}}\left(\left|\widehat{\eta}_n^N(f) - \widehat{\eta}_n(f)\right|\right) \leq C(n)\frac{\|f\|}{\sqrt{N}}, \tag{3.13}$$

$$\widetilde{\mathbb{P}}\left(\left|\eta_n^N(f) - \eta_n(f)\right| > \epsilon\right) \leq C_1(n)\ \exp - \frac{N\epsilon^2}{C_2(n)\,\|f\|^2},$$

and

$$\widetilde{\mathbb{P}}\left(\left|\widehat{\eta}_n^N(f) - \widehat{\eta}_n(f)\right| > \epsilon\right) \leq C_1(n)\ \exp - \frac{N\epsilon^2}{C_2(n)\,\|f\|^2}.$$

More precise estimates of the exponential rates above are given in (Del Moral and Guionnet 1998).

The conclusion of all these estimates is that the rate of convergence of η_n^N and $\widehat{\eta}_n^N$ to η_n and $\widehat{\eta}_n$ as $N \to \infty$ is $1/\sqrt{N}$. The same holds for the convergence of γ_n^N and $\widehat{\gamma}_n^N$ to γ_n and $\widehat{\gamma}_n$. Further exponential bounds for Zolotarev's semi-norms can be found in (Del Moral and Ledoux 2000).

It is also useful to have estimate for the total variation distance between η_n and the laws of the ξ_n^i's. Observing that

$$\eta_n(f) = \frac{\gamma_n(f)}{\gamma_n(1)}, \quad \eta_n^N(f) = \frac{\gamma_n^N(f)}{\gamma_n^N(1)} \quad \text{and} \quad \widetilde{\mathbb{E}}(\gamma_n^N(f)) = \gamma_n(f),$$

we get

$$\widetilde{\mathbb{E}}(\eta_n^N(f)) - \eta_n(f) = \widetilde{\mathbb{E}}\left(\left(\eta_n^N(f) - \eta_n(f)\right)\left(1 - \frac{\gamma_n^N(1)}{\gamma_n(1)}\right)\right)$$

Using Cauchy-Schwarz's inequality this clearly implies

$$\left|\widetilde{\mathbb{E}}(\eta_n^N(f)) - \eta_n(f)\right| \leq\ C(n)\ \frac{\|f\|}{N} \tag{3.14}$$

for some constant $C(n)$ which depends on the time parameter n and on the observed values $y_0, \ldots, y_n$. Since the variables ξ_n^i for $i = 1, \ldots, N$ have the same law, the expectation of a bounded function f against this law is

$$\widetilde{\mathbb{E}}(f(\xi_n^i)) = \widetilde{\mathbb{E}}(\eta_n^N(f))$$

So a consequence of (3.14) is as follows, where $\|\cdot\|_{\mathrm{tv}}$ denotes the total variation distance:

$$\|\mathcal{L}\mathrm{aw}(\xi_n^i) - \eta_n\|_{\mathrm{tv}} \leq \frac{C(n)}{N} \tag{3.15}$$

In much the same way, for any $i = 1, \ldots, N$ one can check that

$$\widetilde{I\!\!E}(f(\widehat{\xi}_n^i)) = \widetilde{I\!\!E}\left(\widehat{\eta}_n^N(f)\right)$$

and

$$\widetilde{I\!\!E}(\widehat{\eta}_n^N(f)) - \widehat{\eta}_n(f)) = \widetilde{I\!\!E}\left(\left(\widehat{\eta}_n^N(f) - \widehat{\eta}_n(f)\right)\left(1 - \frac{\gamma_n^N(g_n)}{\gamma_n(g_n)}\right)\right),$$

from which one concludes that, for a constant $C(n)$ as before,

$$\|\mathcal{L}\mathrm{aw}(\widehat{\xi}_n^i) - \widehat{\eta}_n\|_{\mathrm{tv}} \leq \frac{C(n)}{N}. \tag{3.16}$$

4) Now, if rates of convergence are important, the constants above like $C(n)$, etc..., are also crucial. In (3.11) we have a concrete estimate for $C_{n,2}$, which is already quite bad, but things are even worse for the constants in Proposition 3.1: one can see (in (Del Moral et al. 1998) for example) that standard estimates give constants $C(n)$ and $C_i(n)$ that grow typically as e^{Cn^2} for some $C > 0$ when $n \to \infty$.

But these estimates are obtained via very coarse majorations, as in (3.11) above. The precise magnitude of the error is better given by central limit theorems. A full discussion of these fluctuations would take us too far, but the form of the angle bracket (3.9) suggests that one can prove that the sequence of random variables

$$U_n^N(f) = \sqrt{N}\ \left(\gamma_n^N(f) - \gamma_n(f)\right)$$

converges in law, as $N \to \infty$, to a centered Gaussian variable $U_n(f)$ with variance

$$\widetilde{I\!\!E}\left(U_n(f)^2\right) = \sum_{p=0}^{n} (\gamma_p(1))^2\ \eta_p\left((L_{p,n}f - \eta_p L_{p,n}f)^2\right) \tag{3.17}$$

for any $f \in \mathcal{B}_{\mathrm{b}}(E)$ (the set of all bounded measurable functions on $(E, \mathcal{E})$). One can even prove that the processes $(U_n^N(f) : n \geq 0, f \in \mathcal{B}_{\mathrm{b}}(E))$ converge finite-dimensionally in law to $(U_n(f) : n \geq 0, f \in \mathcal{B}_{\mathrm{b}}(E))$, where the later is a centered Gaussian field whose covariance may be explicitly computed. Note that in (3.17) we have $\eta_p L_{p,n} f = \eta_n f$: use (2.2) and (2.11).

However, we are more interested in the errors $\eta_n^N(f) - \eta_n(f)$ and $\widehat{\eta}_n^N(f) - \widehat{\eta}_n(f)$. To this effect, we observe that by (2.2) and (2.14) on the one hand and by (2.12), (2.13) and (3.3) on the other hand, we get the decompositions

$$W_n^N(f) \stackrel{\text{def.}}{=} \sqrt{N}\ \left(\eta_n^N(f) - \eta_n(f)\right) = \frac{1}{\gamma_n^N(1)} U_n^N\left(f - \eta_n(f)\right),$$

$$\widehat{W}_n^N(f) \stackrel{\text{def.}}{=} \sqrt{N}\ \left(\widehat{\eta}_n^N(f) - \widehat{\eta}_n(f)\right) = \frac{1}{\eta_n^N(1)} W_n^N(g_n(f - \widehat{\eta}_n(f)).$$

Since $\gamma_n^N(1) \to \gamma_n(1)$ and $\eta_n^N(g_n) \to \eta_n(g_n)$ in probability, one easily deduces from the above convergence of the $U_n^N(f)$'s the following central limit theorem:

Theorem 3.2 *For any bounded measurable function f, the sequence of variables $W_n^N(f)$ converges in law to a centered Gaussian variable $W_n(f)$ whose variance is given by*

$$\widetilde{I\!E}\left(W_n(f)^2\right) = \sum_{p=0}^{n} \left(\frac{\gamma_p(1)}{\gamma_n(1)}\right)^2 \eta_p\left(\ (L_{p,n}(f - \eta_n(f)))^2\right). \tag{3.18}$$

Furthermore, the sequence of variables $\widehat{W}_n^N(f)$ converges in law to the variable

$$\widehat{W}_n(f) = \frac{1}{\eta_n(g_n)} W_n(g_n(f - \widehat{\eta}_n f)). \tag{3.19}$$

One can even show that the random fields $W_n^N(f)$ and $\widehat{W}_n^N(f)$, indexed by n and $f \in \mathcal{B}_{\text{b}}(E)$, converge finite-dimensionally in law to a centered Gaussian field. Furthermore, this convergence also holds, under suitable hypotheses, for measurable functions f which are not necessarily bounded: see (Del Moral and Jacod 1999) for details. Glivenko-Cantelli and Donsker theorems can also be found in (Del Moral and Ledoux 2000).

5) Let us present an alternative but somewhat more tractable description of the variances of $W_n(f)$ and $\widehat{W}_n(f)$ in Theorem 3.2. For this, we recall (2.10). First, by (2.2) and (2.11) we have for $0 \le p \le n$:

$$\frac{\gamma_n(1)}{\gamma_p(1)} = \frac{\gamma_p L_{p,n}(1)}{\gamma_p(1)} = \eta_p L_{p,n} 1.$$

Thus we deduce from (3.18) that

$$\widetilde{I\!E}\left(W_n(f)^2\right) = \sum_{p=0}^{n} \frac{\eta_p\left((L_{p,n}(f - \eta_n f))^2\right)}{\left(\eta_p L_{p,n} 1\right)^2}. \tag{3.20}$$

Second, (2.12) and (2.13) yield $\eta_n\left(g_n(f - \widehat{\eta}_n f)\right) = 0$. We also have $L_{p,n}(g_n f) = g_p \widehat{L}_{p,n}(f)$ by (2.4) and (2.10), whence

$$\widehat{\eta}_p \widehat{L}_{p,n} 1 = \frac{\eta_p L_{p,n}(g_n)}{\eta_p(g_p)} = \frac{\gamma_n(g_n)}{\gamma_p(g_p)} = \frac{\eta_n(g_n)\gamma_n(1)}{\eta_p(g_p)\gamma_p(1)} = \frac{\eta_n(g_n)}{\eta_p(g_p)}\ \eta_p L_{p,n} 1.$$

Therefore we deduce from (3.19) and (3.20) that

$$\widetilde{I\!E}\left(\widehat{W}_n(f)^2\right) = \sum_{p=0}^{n} \frac{\eta_p\left(\left(g_p \widehat{L}_{p,n}(f - \widehat{\eta}_n f)\right)^2\right)}{\left(\eta_p(g_p)\right)^2 \left(\widehat{\eta}_p \widehat{L}_{p,n} 1\right)^2}. \tag{3.21}$$

3.2 Subcase A2

1) We label *Subcase A2* the case A, when the three main ingredients: the density $\bar{g}_n$, the initial law η_0 and the transitions Q_n are not explicitly known, and/or when one cannot simulate random variables exactly according to the laws η_0 or $Q_n(x,.)$.

One will replace $\bar{g}_n$, η_0 and Q_n by approximate quantities $\bar{g}_n^{(m)}$, $\eta_0^{(m)}$ and $Q_n^{(m)}$, in such a way that the functions $g_n^{(m)}(x) = \bar{g}_n^{(m)}(x, y_n)$ are explicitly known, and that one can simulate random variables exactly according to the laws $\eta_0^{(m)}$ and $Q_n^{(m)}(x,.)$ for all x. The index m (an integer) is a measure of the quality of the approximation, and in the IPS to be defined below, with N particles, we will make m depend on the number N.

We have thus two filtering schemes: one is relative to the original setting $(\bar{g}_n, \eta_0, Q_n)$, and we have the prediction and filtering measures η_n and $\widehat{\eta}_n$ together will all quantities defined in Section 2; the other one is relative to the approximate setting $(\bar{g}_n^{(m)}, \eta_0^{(m)}, Q_n^{(m)})$, and we have the corresponding prediction and filtering measures $\eta_n^{(m)}$ and $\widehat{\eta}_n^{(m)}$ together with all quantities defined also in Section 2, but with a superscript "$^{(m)}$": for example $\Psi_n^{(m)}$, $\Phi_n^{(m)}$, $\gamma_n^{(m)}$, $\widehat{\gamma}_n^{(m)}$, $L_{p,n}^{(m)}$, and so on ...

As for the IPS, it is of course conducted according to the setting

$$(\bar{g}_n^{(m)}, \eta_0^{(m)}, Q_n^{(m)}),$$

which is the only one available for explicit computations. So in (3.1), (3.2) and (3.3) we replace g_n, Q_{n+1} and Ψ_n by $g_n^{(m)}$, $Q_{n+1}^{(m)}$ and $\Psi_n^{(m)}$ respectively. The approximate prediction and filtering measures are still denoted by η_n^N and $\widehat{\eta}_n^N$, and the approximation for γ_n is γ_n^N, as given by (3.6) with $g_p^{(m)}$ instead of g_p. These approximate measures are good (for big N's) for approximating $\eta_n^{(m)}$, $\widehat{\eta}_n^{(m)}$ and $\gamma_n^{(m)}$ respectively, so we have first to evaluate the differences between these and η_n, $\widehat{\eta}_n$ and γ_n in terms of "differences" between $(\bar{g}_n^{(m)}, \eta_0^{(m)}, Q_n^{(m)})$ and $(\bar{g}_n, \eta_0, Q_n)$.

To this effect, we introduce a number of assumptions. Recall that $\|\mu\|_{\mathrm{TV}}$ denotes the total variation of the signed finite measure μ on $(E, \mathcal{E})$.

Assumption HE: There exist a signed finite measure η_0' on $(E, \mathcal{E})$ and a constant C such that

$$\|m(\eta_0^{(m)} - \eta_0) - \eta_0'\|_{\mathrm{TV}} \leq \frac{C}{m}.$$

Assumption HG: For each n there exist a measurable bounded function g'_n on $(E,\mathcal{E})$ and a constant C_n such that

$$|m(g_n^{(m)}(x) - g_n(x)) - g'_n(x)| \leq \frac{C_n}{m}.$$

Assumption HQ: For all n there exist a signed finite transition measure Q'_n from $(E,\mathcal{E})$ into itself and a constant C_n such that

$$\|m(Q_n^{(m)}(x,.) - Q_n(x,.)) - Q'_n(x,.)\|_{\text{tv}} \leq \frac{C_n}{m}.$$

Proposition 3.3 *Assume that HE, HG and HQ are satisfied, and denote below by C_n a constant which may vary from line to line and which depends only on n and on the observed values $y_0,\ldots,y_n$.*

a) If for $0 \leq p \leq n$ we define the signed transition measure $L'_{p,n}$ by

$$L'_{p,n} = \sum_{q=p+1}^{n} L_{p,q-1}\, L'_q\, L_{q,n} \tag{3.22}$$

with

$$L'_q(f)(x) = g_{q-1}(x)Q'_q(f)(x) + g'_{q-1}(x)Q_q(f)(x)$$

(with $\sum_\emptyset = 0$), we have for $f \in \mathcal{B}_{\rm b}(E)$:

$$\left\| m\,(L_{p,n}^{(m)} f - L_{p,n} f) - L'_{p,n} f \right\| \leq \frac{C_n}{m}\, \|f\|. \tag{3.23}$$

b) If further we define the signed finite measures γ'_n, η'_n and $\widehat{\eta}'_n$ by

$$\left.\begin{aligned} \gamma'_n(f) &= \eta_0 L'_{0,n} f + \eta'_0 L_{0,n} f,\\ \eta'_n(f) &= \frac{1}{\gamma_n(1)}\ \gamma'_n(f - \eta_n f),\\ \widehat{\eta}'_n(f) &= \tfrac{1}{\widehat{\gamma}_n(1)}\ (\gamma'_n(g_n(f - \widehat{\eta}_n f) + \gamma_n(g'_n(f - \widehat{\eta}_n f)), \end{aligned}\right\} \tag{3.24}$$

then we have

$$\|m\left(\gamma_n^{(m)} - \gamma_n\right) - \gamma'_n\|_{tv} \leq \tfrac{C_n}{m}, \tag{3.25}$$

$$\left\|m\left(\eta_n^{(m)} - \eta_n\right) - \eta'_n\right\|_{tv} \leq \tfrac{C_n}{m}, \tag{3.26}$$

$$\left\|m\left(\widehat{\eta}_n^{(m)} - \widehat{\eta}_n\right) - \widehat{\eta}'_n\right\|_{tv} \leq \tfrac{C_n}{m}. \tag{3.27}$$

Proof: a) If R_m is a sequence of signed finite transition measures on $(E,\mathcal{E})$ we write $R_m = O(1/m^2)$ as soon as the sequence $\|m^2 R_m(x,.)\|_{\text{tv}}$ is bounded, uniformly in x.

To prove (3.23) we use a downward induction on the parameter p. For $p = n$, $L^{(m)}_{n,n} = L_{n,n} = Id$ and the result is obvious. Suppose that (3.23) holds at rank $(p+1)$, that is

$$L^{(m)}_{p+1,n} f = L_{p+1,n} f + \frac{1}{m}\, L'_{p+1,n} f + O\left(\frac{1}{m^2}\right) \tag{3.28}$$

with $L'_{p+1,n}$ given by (3.22). Now HG and HQ and (2.10) yield

$$L^{(m)}_{p+1} = L_p + (g^{(m)}_p - g_p)\centerdot Q_{p+1} + g_p \centerdot (Q^{(m)}_{p+1} - Q_{p+1}) + (g^{(m)}_p - g_p)\centerdot (Q^{(m)}_{p+1} - Q_{p+1}).$$

Since $L^{(m)}_{p+1,n} = 0(1)$ by (3.28), we get by (2.10) and (3.22):

$$L^{(m)}_{p,n} = L^{(m)}_{p+1}\, L^{(m)}_{p+1,n} = L_{p+1}\, L^{(m)}_{p+1,n} + \frac{1}{m}\, L'_{p+1} L^{(m)}_{p+1,n} + O\left(\frac{1}{m^2}\right).$$

Then a simple calculation shows that (3.23) holds for p, with $L'_{p,n}$ given again by (3.22).

b) Recalling that $\gamma_n = \eta_0 L_{0,n}$ and

$$\gamma^{(m)}_n = \eta^{(m)}_0 L^{(m)}_{0,n},$$

we get

$$\gamma^{(m)}_n = \gamma_n + (\eta^{(m)}_0 - \eta_0) L_{0,n} + \eta_0 (L^{(m)}_{0,n} - L_{0,n}) + (\eta^{(m)}_0 - \eta_0)(L^{(m)}_{0,n} - L_{0,n}),$$

hence (3.25) follows from HE and (3.23). Similarly, we have

$$\widehat{\gamma}^{(m)}_n(f) = \gamma^{(m)}_n(f g^{(m)}_n)$$

(see (2.11)), hence

$$\widehat{\gamma}^{(m)}_n(f) = \widehat{\gamma}_n(f) + (\gamma^{(m)}_n - \gamma_n)(g_n f) + \gamma_n((g^{(m)}_n - g_n) f) + (\gamma^{(m)}_n - \gamma_n)((g^{(m)}_n - g_n) f),$$

and we deduce from (3.25) and HG that

$$\| m \left(\widehat{\gamma}^{(m)}_n - \widehat{\gamma}_n\right) - \widehat{\gamma}'_n \|_{\rm tv} \le \frac{C_n}{m}, \tag{3.29}$$

where

$$\widehat{\gamma}'_n(f) = \gamma'_n(g_n f) + \gamma_n(g'_n f).$$

For (3.26), let us first recall the properties

$$\eta^{(m)}_n(f) = \frac{\gamma^{(m)}_n(f)}{\gamma^{(m)}_n(1)}, \qquad \eta_n(f) = \frac{\gamma_n(f)}{\gamma_n(1)}, \qquad \gamma_n(1) > 0.$$

Next we observe that it is enough to prove the result for m large enough and (3.25) implies $\gamma^{(m)}_n(1) \to \gamma_n(1) > 0$ as $m \to \infty$. In view of what precedes, we can thus assume that $\gamma_n^{(m)}(1) \geq \varepsilon$ for some $\varepsilon_n > 0$ and all m. Next, a simple computation yields

$$m(\eta_n^{(m)} f - \eta_n f) - \eta_n' f = \frac{1}{(\gamma_n 1)^2 \gamma_n^{(m)} 1} \Big((\gamma_n 1)^2 \left(m(\gamma_n^{(m)} f - \gamma_n f) - \gamma_n' f \right)$$

$$- \gamma_n 1 \gamma_n f \left(m(\gamma_n^{(m)} 1 - \gamma_n 1) - \gamma_n' 1 \right) - (\gamma_n' 1 \gamma_n f - \gamma_n 1 \gamma_n' f)(\gamma_n 1 - \gamma_n^{(m)} 1) \Big).$$

At this stage, (3.26) readily follows from (3.25).

Finally we prove (3.27) in exactly the same way, using (3.29) instead of (3.25). ■

2) The Interacting Particle System.

As seen in the previous proposition, the rates of convergence of $\eta_0^{(m)}$, $g_n^{(m)}$ and $Q_n^{(m)}$ to η_0, g_n and Q_n as $m \to \infty$, as stated in Hypotheses HE, HG and HQ, are so adjusted that they give the rate for the convergence of $\eta_n^{(m)}$ and $\widehat{\eta}_n^{(m)}$ to η_n and $\widehat{\eta}_n$.

Now, the IPS is conducted according to the scheme $(\eta_0^{(m)}, \bar{g}_n^{(m)}, Q_n^{(m)})$. Thus if one wants to evaluate, for example, the difference $\eta_n^N f - \eta_n f$ there are two different errors: one is $\eta_n^N f - \eta_n^{(m)} f$, which is taken care of by (3.13) and is of order of magnitude $1/\sqrt{N}$. The other one is $\eta_n^{(m)} f - \eta_n f$, and is of order of magnitude $1/m$ by (3.26). These two errors should be balanced, so we are lead to choose $m = m(N)$ to depend on N in such a way that $m(N)$ and $\sqrt{N}$ are equivalent as $N \to \infty$: we can take for example $m(N)$ to be the integer part $[\sqrt{N}]$ of $\sqrt{N}$.

Of course, in order to have estimates as in Proposition 3.1, we should also prove that the estimates of $\eta_n^N f - \eta_n^{(m)} f$ in this proposition are uniform in m. This can be proved, and we have the following result:

Proposition 3.4 *Assume HE, HG and HG. If we conduct the IPS with the approximate quantities* $(\eta_0^{(m(N))}, \bar{g}_n^{(m(N))}, Q_n^{(m(N))})$ *where* $m(N) = [\sqrt{N}]$, *then there exist constants* $C(n)$, $C_1(n)$, $C_2(n)$, *which depend on* n and on the observed values $y_0, \ldots, y_n$, *such that*

$$\widetilde{\mathbb{E}}\left(\left| \eta_n^N(f) - \eta_n(f) \right| \right) \leq C(n) \frac{\|f\|}{\sqrt{N}}, \qquad \widetilde{\mathbb{E}}\left(\left| \widehat{\eta}_n^N(f) - \widehat{\eta}_n(f) \right| \right) \leq C(n) \frac{\|f\|}{\sqrt{N}},$$

$$\widetilde{\mathbb{P}}\left(\left| \eta_n^N(f) - \eta_n(f) \right| > \epsilon \right) \leq C_1(n) \ \exp - \frac{N\epsilon^2}{C_2(n) \, \|f\|^2},$$

$$\widetilde{\mathbb{P}}\left(\left| \widehat{\eta}_n^N(f) - \widehat{\eta}_n(f) \right| > \epsilon \right) \leq C_1(n) \ \exp - \frac{N\epsilon^2}{C_2(n) \, \|f\|^2}.$$

Arguing as in section 3.1 if the IPS is conducted according to the scheme $(\eta_0^{(m)}, \bar{g}_n^{(m)}, Q_n^{(m)})$ one can check that for any bounded function f such that $\|f\| \leq 1$ we have that

$$\left|\widetilde{E}(\eta_n^N(f)) - \eta_n^{(m)}(f)\right| \leq \frac{C(n)}{N} \quad \text{and} \quad \left|\widetilde{E}(\widehat{\eta}_n^N(f)) - \widehat{\eta}_n^{(m)}(f)\right| \leq \frac{C(n)}{N}$$

for some constant $C(n)$ which only depends on the time parameter n and on the observed values $y_0, \dots, y_n$. Therefore (3.26) and (3.27) imply that

$$\left|\widetilde{E}(\eta_n^N(f)) - \eta_n(f)\right| \leq \frac{C(n)}{\sqrt{N}} \quad \text{and} \quad \left|\widetilde{E}(\widehat{\eta}_n^N(f)) - \widehat{\eta}_n(f)\right| \leq \frac{C(n)}{\sqrt{N}}$$

for any bounded function f such that $\|f\| \leq 1$ and for some other constant $C(n)$. Since the variables ξ_n^i (respectively $\widehat{\xi}_n^i$) for $i = 1, \dots, N$ have the same law one concludes that

$$\|\mathcal{L}\text{aw}(\xi_n^i) - \eta_n\|_{\text{tv}} \leq \frac{C(n)}{\sqrt{N}} \quad \text{and} \quad \|\mathcal{L}\text{aw}(\widehat{\xi}_n^i) - \widehat{\eta}_n\|_{\text{tv}} \leq \frac{C(n)}{\sqrt{N}}$$

(this should be compared with (3.15) and (3.16)).

More interesting is the central limit theorem 3.2. The same line of argument allows to deduce from Theorem 3.2 the following result:

Theorem 3.5 *Assume HE, HG and HG, and suppose that we conduct the IPS with the approximate quantities*

$$(\eta_0^{(m(N))}, \bar{g}_n^{(m(N))}, Q_n^{(m(N))}) \quad \text{where} \quad m(N) = [\sqrt{N}].$$

a) For any bounded measurable function f, the sequence of random variables

$$W_n^N(f) = \sqrt{N}(\eta_n^N f - \eta_n f)$$

converges in law to a Gaussian variable with mean $\eta_n'(f)$ (see (3.24)) and variance given by (3.18) or equivalently (3.20).

b) For any bounded measurable function f, the sequence of random variables

$$\widehat{W}_n^N(f) = \sqrt{N}(\widehat{\eta}_n^N f - \widehat{\eta}_n f)$$

converges in law to a Gaussian variable with mean $\widehat{\eta}_n'(f)$ (see (3.24)) and variance given by (3.21).

4 An interacting particle system in Case B

4.1 Subcase B1

As we said in the introduction the situation becomes much more involved in Case B, since the noise sources may be related to the signal. Here again we begin with Subcase B1, which means that we know the densities $\bar{G}_n$ (hence all functions G_n of (2.1)), as well as the initial conditions μ_0 and η_0 (connected by (1.2)) and the transitions P_n and Q_n (also connected by (1.2)), and further one can simulate random variables exactly according to the laws μ_0 and $P_n(x, y; .)$ for all $x \in E$, $y \in I\!\!R^d$ and $n \geq 1$.

1) We will in fact modify the setting, in a controlled way, so that the new setting fits in Case A. To this effect, our first objective is to enlarge the state space of the signal and to construct an approximating model which has the same Feynman-Kac formulation as the one in (2.3). To describe precisely this approximating model we need to introduce some additional notation.

- We denote by $\mathcal{X}_n = (\widehat{X}_n, \widehat{Y}_n)$, $n \geq 0$ the time inhomogeneous Markov chain with product state space $E \times I\!\!R^q$, with initial law μ_0 and transition probability kernels $\{\mathcal{Q}_n \ ; \ n \geq 1\}$ given by

$$\left.\begin{array}{l} \mu_0(dx, dy) = \eta_0(dx)\bar{G}_0(x, y)dy, \qquad \text{(see (1.2))} \\ \mathcal{Q}_n\left((x, y), d(x', y')\right) = Q_n(x, dx')\bar{G}_n(x, y_{n-1}, x', y')dy'. \end{array}\right\} \tag{4.1}$$

  The probability measure and the expectation corresponding to this Markov chain are denoted by $\widehat{I\!\!P}$ and $\widehat{I\!\!E}$ respectively. Observe that $\mathcal{Q}_n(x, y; .)$ does not depend on y.

- We also choose a Borel bounded function θ from $I\!\!R^q$ into $(0, \infty)$ such that

$$\int \theta(y)dy \ = \ 1, \qquad \int y \ \theta(y)dy \ = \ 0, \qquad \int \ |y|^3\theta(y) \ dy \ < \ \infty. \tag{4.2}$$

  Then we set for any $\alpha \in (0, \infty)$ and $(x, y) \in E \times I\!\!R^q$

$$g_n^{(\alpha)}(x, y) = \alpha^{-q} \, \theta\left((y - y_n)/\alpha\right). \tag{4.3}$$

By using classical approximation arguments and under appropriate regularity conditions on the functions G_n, a natural choice of α-approximating measures for γ_n and $\widehat{\gamma}_n$ in (2.9) (as $\alpha \to 0$) are the marginals $\gamma_n^{(\alpha)}$ and $\widehat{\gamma}_n^{(\alpha)}$ on the first component of the measures $\nu_n^{(\alpha)}$ and $\widehat{\nu}_n^{(\alpha)}$ on $E \times I\!\!R^q$ defined for any $\varphi \in \mathcal{B}_{\mathrm{b}}(E \times I\!\!R^q)$ by the Feynman-Kac type formulae

$$\nu_n^{(\alpha)}(\varphi) = \widehat{I\!\!E}\left(\varphi(\mathcal{X}_n) \prod_{k=0}^{n-1} g_k^{(\alpha)}(\mathcal{X}_k)\right), \quad \widehat{\nu}_n^{(\alpha)}(\varphi) = \widehat{I\!\!E}\left(\varphi(\mathcal{X}_n) \prod_{k=0}^{n} g_k^{(\alpha)}(\mathcal{X}_k)\right), \tag{4.4}$$

with the convention $\prod_\emptyset = 1$. In other words, for any $f \in \mathcal{B}_{\mathrm{b}}(E)$ we have

$$\gamma_n^{(\alpha)}(f) = \nu_n^{(\alpha)}(f \otimes 1), \qquad \widehat{\gamma}_n^{(\alpha)}(f) = \widehat{\nu}_n^{(\alpha)}(f \otimes 1). \tag{4.5}$$

Another way of interpreting (4.4) is as Feynman-Kac formulae (2.3) for a filtering scheme in Case A1, and which as follows:

$$\left.\begin{array}{ll} \text{the state process is} & \mathcal{X}_n = (\widehat{X}_n, \widehat{Y}_n) \\ \text{the observation is} & \mathcal{Y}_n^{(\alpha)} = \widehat{Y}_n + \alpha\, \mathcal{V}_n, \end{array}\right\} \tag{4.6}$$

where the $\mathcal{V}_n$'s are i.i.d. q-dimensional variables, independent of $\mathcal{X}$ and with distribution $\theta(dz)dz$.

2) In view of this interpretation, we introduce the prediction and filtering measures associated with the scheme (4.6), and with respect to the observations $y_0, \ldots, y_n$ for the true model. Namely, for any measurable function φ on $E \times \mathbb{R}^q$ we set

$$\left.\begin{array}{l} \mu_n^{(\alpha)}(\varphi) = \widehat{\mathbb{E}}\left(\varphi\left(\mathcal{X}_n\right) \Big| \mathcal{Y}_0^{(\alpha)} = y_0, \ldots, \mathcal{Y}_{n-1}^{(\alpha)} = y_{n-1}\right) = \frac{\nu_n^{(\alpha)}(\varphi)}{\nu_n^{(\alpha)}(1)} \\ \widehat{\mu}_n^{(\alpha)}(\varphi) = \widehat{\mathbb{E}}\left(\varphi\left(\mathcal{X}_n\right) \Big| \mathcal{Y}_0^{(\alpha)} = y_0, \ldots, \mathcal{Y}_{n-1}^{(\alpha)} = y_{n-1}, \mathcal{Y}_n^{(\alpha)} = y_n\right) = \frac{\widehat{\nu}_n^{(\alpha)}(\varphi)}{\widehat{\nu}_n^{(\alpha)}(1)}. \end{array}\right\} \tag{4.7}$$

The first marginals of these measures, denoted by $\eta_n^{(\alpha)}$ and $\widehat{\eta}_n^{(\alpha)}$, are thus defined for any $f \in \mathcal{B}_{\mathrm{b}}(E)$ by

$$\eta_n^{(\alpha)}(f) = \mu_n^{(\alpha)}(f \otimes 1) = \frac{\gamma_n^{(\alpha)}(f)}{\gamma_n^{(\alpha)}(1)}, \qquad \widehat{\eta}_n^{(\alpha)}(f) = \widehat{\mu}_n^{(\alpha)}(f \otimes 1) = \frac{\widehat{\gamma}_n^{(\alpha)}(f)}{\widehat{\gamma}_n^{(\alpha)}(1)}. \tag{4.8}$$

Observe that $\nu_0^{(\alpha)} = \mu_0^{(\alpha)} = \mu_0$ and $\eta_0^{(\alpha)} = \eta_0$.

At this stage, we define quantities associated with the filtering scheme (4.6) as in Section 2, except that we only consider the marginals of E of the various transition kernels or measures on $(E \times \mathbb{R}^q)$. Remark once again that if φ is a function on $E \times \mathbb{R}^q$, then $\mathcal{Q}_n \varphi$ can be considered as a function on E as well as on $E \times \mathbb{R}^q$.

First, as in (2.4), we define the kernels $\widehat{L}_n^{(\alpha)}$ on $(E, \mathcal{E})$ and their iterates $\widehat{L}_{p,n}^{(\alpha)}$ for $0 \leq p \leq n$ by

$$\widehat{L}_n^{(\alpha)} f = \mathcal{Q}_n(g_n^{(\alpha)}(f \otimes 1)), \qquad \widehat{L}_{p,n}^{(\alpha)} = \widehat{L}_{p+1}^{(\alpha)} \ldots \widehat{L}_n^{(\alpha)} \quad \text{with} \quad \widehat{L}_{n,n}^{(\alpha)} = Id. \tag{4.9}$$

Next, we set

$$\widehat{G}_0^{(\alpha)}(x) = \int \bar{G}_0(x,y) g_0^{(\alpha)}(x,y) dy. \tag{4.10}$$

Observing that the function G_0 of Section 2 is replaced here by the function $g_0^{(\alpha)}$, and taking into account the fact that $\nu_0(g_0^{(\alpha)}(f \otimes 1)) = \eta_0(\widehat{G}_0 f)$, we can thus deduce from (2.6) and (2.8) that

$$\left.\begin{aligned} &\nu_0^{(\alpha)} = \mu_0, && \widehat{\nu}_0^{(\alpha)}(\varphi) = \mu_0(g_0^{(\alpha)}\varphi), \\ &\nu_n^{(\alpha)}(\varphi) = \eta_0(\widehat{G}_0^{(\alpha)}\widehat{L}_{0,n-1}^{(\alpha)}\mathcal{Q}_n\varphi), && \widehat{\nu}_n^{(\alpha)}(\varphi) = \eta_0(\widehat{G}_0^{(\alpha)}\widehat{L}_{0,n}\mathcal{Q}_n(g_n^{(\alpha)}\varphi)), \quad n \geq 1. \end{aligned}\right\} \tag{4.11}$$

This in turn gives the marginals on E:

$$\left.\begin{aligned} &\gamma_0^{(\alpha)} = \eta_0, && \widehat{\gamma}_0^{(\alpha)}(f) = \eta_0(\widehat{G}_0^{(\alpha)}f), \\ &\gamma_n^{(\alpha)}(f) = \eta_0(\widehat{G}_0^{(\alpha)}\widehat{L}_{0,n-1}^{(\alpha)}Q_n f), && \widehat{\gamma}_n^{(\alpha)}(f) = \eta_0(\widehat{G}_0^{(\alpha)}\widehat{L}_{0,n}f), \qquad n \geq 1. \end{aligned}\right\} \tag{4.12}$$

Now, we evaluate the errors involved by replacing the original scheme by the scheme (4.6), in terms of α. For this, we need a regularity assumption on the functions $\bar{G}_n$, which goes as follows:

Hypothesis R:

a) The function $y \mapsto \bar{G}_0(x,y)$ is three times differentiable, with partial derivatives of order 1, 2 and 3 uniformly bounded in (x,y).

b) Setting $\widetilde{L}_n f(x,y) = \int Q_n(x,dx')f(x')\bar{G}_n(x,y_{n-1};x',y)$, for each $n \geq 1$ and each bounded measurable function f on E, $\|f\| \leq 1$, the function

$$y \mapsto \widetilde{L}_n f(x,y)$$

is three times differentiable, with partial derivatives of order 1, 2 and 3 uniformly bounded in (x,y).

Observe that if the functions $y \mapsto \bar{G}_n(x,y_{n-1};x',y)$ are three times differentiable with partial derivatives of order up to 3 bounded uniformly in (x,x',y) for each $n \geq 1$, then R-b is satisfied.

Below, we denote by $\bar{G}''_{0,jk}(x,y)$ and by $\widetilde{L}''_{n,jk}f(x,y)$ the second order partial derivatives of the functions $y \mapsto \bar{G}_0(x,y)$ and $y \mapsto \widetilde{L}_n f(x,y)$ w.r.t. the components y^j and y^k. We can thus define the finite kernels $\widehat{L}_n^*$ and the function G_0^* on E by

$$\widehat{L}_n^* f(x) = \frac{1}{2}\sum_{j,\ell=1}^{q} \widetilde{L}''_{n,j\ell}f(x,y_n)\int \theta(z)z^j z^\ell dz, \tag{4.13}$$

$$G_0^*(x) = \frac{1}{2}\sum_{j,\ell=1}^{q} \bar{G}''_{0,j\ell}(x,y_0)\int \theta(z)z^j z^\ell dz. \tag{4.14}$$

Then, recalling that $\widehat{L}_n$ is defined by (2.4), the key lemma is:

Lemma 4.1 *Under R we have the following estimates, where C_n denotes a constant which depends only on n and on the observations $y_0, \ldots, y_n$, and f is a bounded measurable function on E:*

$$|\widehat{L}_n^{(\alpha)} f(x) - \widehat{L}_n f(x) - \alpha^2 \widehat{L}_n^* f(x)| \leq C_n \alpha^3 \|f\|, \tag{4.15}$$

$$|\widehat{G}_0^{(\alpha)}(x) - G_0(x) - \alpha^2 G_0^*(x)| \leq C_0 \alpha^3. \tag{4.16}$$

$$|\alpha^q \mathcal{Q}_n((g_n^{(\alpha)})^2 f)(x) - u \widehat{L}_n f(x)| \leq C_n \alpha \|f\|, \tag{4.17}$$

where

$$u = \int \theta(y)^2 dy$$

Proof: Since $\int \theta(y)dy = 1$ and $\widehat{L}_n f(x) = \widetilde{L}_n f(x, y_n)$, we have

$$\widehat{L}_n^{(\alpha)} f(x) - \widehat{L}_n f(x) = \int \left(\widetilde{L}_n f(x, y_n + \alpha y) - \widetilde{L}_n f(x, y_n) \right) \theta(y) dy.$$

Then, taking R into account and using a third order Taylor expansion for the function $y \mapsto \widetilde{L}_n f(x, y)$ around y_n, we readily deduce (4.15) from (4.2).

Since $G_0(x) = \bar{G}_0(x, y_0)$, the proof of (4.16) is similar. Finally

$$\alpha^q \mathcal{Q}_n((g_n^{(\alpha)})^2 f)(x) - u \widehat{L}_n f(x) = \int \left(\widetilde{L}_n f(x, y_n + \alpha y) - \widetilde{L}_n f(x, y_n) \right) \theta(y)^2 dy.$$

So (4.17) follows again from R and from a first order Taylor expansion for the function $y \mapsto \widetilde{L}_n f(x, y)$ around y_n. ∎

Now we define the kernels $\widehat{L}_{p,n}^*$ for $0 \leq p \leq n$ and the measures γ_n^*, $\widehat{\gamma}_n^*$, η_n^* and $\widehat{\eta}_n^*$ on $(E, \mathcal{E})$ for $n \geq 0$ by

$$\widehat{L}_{p,n}^* = \sum_{q=p+1}^{n} \widehat{L}_{p,q-1} \widehat{L}_q^* \widehat{L}_{q,n} \qquad \text{hence} \quad \widehat{L}_{n-1,n}^* = \widehat{L}_n^*,$$

$$\gamma_n^*(f) = \eta_0 \left((G_0^* \widehat{L}_{0,n-1} + G_0 \widehat{L}_{0,n-1}^*) Q_n f \right), \quad \eta_n^*(f) = \frac{\gamma_n^*(f)\gamma_n(1) - \gamma_n^*(1)\gamma_n(f)}{(\gamma_n 1)^2}, \tag{4.18}$$

for $n \geq 1$ and $\gamma_0^* = 0$ and $\eta_0^* = 0$, and for $n \geq 0$:

$$\widehat{\gamma}_n^*(f) = \eta_0 \left((G_0^* \widehat{L}_{0,n} + G_0 \widehat{L}_{0,n}^*) f \right), \qquad \widehat{\eta}_n^*(f) = \frac{\widehat{\gamma}_n^*(f)\widehat{\gamma}_n(1) - \widehat{\gamma}_n^*(1)\widehat{\gamma}_n(f)}{(\widehat{\gamma}_n 1)^2}. \tag{4.19}$$

Then we have an analogue to Proposition 3.3:

Proposition 4.2 *Under R we have the following estimates, where C_n denotes a constant which depends only on n and on the observations $y_0, \ldots, y_n$, and f is a bounded measurable function on E:*

$$|\widehat{L}^{(\alpha)}_{p,n} f(x) - \widehat{L}_{p,n} f(x) - \alpha^2 \widehat{L}^*_{p,n} f(x)| \leq C_n \alpha^3 \|f\|, \qquad 0 \leq p \leq n, \tag{4.20}$$

$$\|\gamma^{(\alpha)}_n - \gamma_n - \alpha^2 \gamma^*_n\|_{tv} \leq C_n \alpha^3, \qquad \|\widehat{\gamma}^{(\alpha)}_n - \widehat{\gamma}_n - \alpha^2 \widehat{\gamma}^*_n\|_{tv} \leq C_n \alpha^3, \tag{4.21}$$

$$\|\eta^{(\alpha)}_n - \eta_n - \alpha^2 \eta^*_n\|_{tv} \leq C_n \alpha^3, \qquad \|\widehat{\eta}^{(\alpha)}_n - \widehat{\eta}_n - \alpha^2 \widehat{\eta}^*_n\|_{tv} \leq C_n \alpha^3. \tag{4.22}$$

Proof: In view of (4.15), the proof of (4.20) is exactly the same as the proof of (3.23) in Proposition 3.3, once observed that $\widehat{L}_{p+1,n}$ is a positive bounded kernel for all $p < n$ and

$$\widehat{L}^{(\alpha)}_{p+1} 1 \leq \widehat{L}_{p+1} 1 + C_{p+1} \alpha^2$$

by (4.15).

The first estimate in (4.21) is trivial for $n = 0$. If $n \geq 1$, comparing (2.8) and (4.12), we get

$$\gamma^{(\alpha)}_n(f) - \gamma_n(f) = \eta_0 \left((\widehat{G}^{(\alpha)}_0 - G_0) \widehat{L}^{(\alpha)}_{0,n-1} Q_n f \right) + \eta_0 \left(G_0 (\widehat{L}^{(\alpha)}_{0,n-1} - \widehat{L}_{0,n-1}) Q_n f \right).$$

Then the first estimate in (4.21) follows from (4.16) and (4.20). The second estimate is proved similarly, using (2.6) instead of (2.8), for $n \geq 0$.

Finally, for (4.22) it is enough to prove the result for α small enough. In view of our standing assumption H and of (4.21), we can then assume that $\gamma_n(1)$, $\gamma^{(\alpha)}_n(1)$, $\widehat{\gamma}_n(1)$ and $\widehat{\gamma}^{(\alpha)}_n(1)$ are all bigger than some $\varepsilon_n > 0$. Then in view of (2.2) and (4.8), the two estimates in (4.22) are deduced from (4.21) exactly as (3.26) and (3.27) were deduced from (3.25) and (3.29). ∎

3) The Interacting Particle System.

For any given N and $\alpha > 0$ we can conduct an IPS of size N for the filtering scheme (4.6). We will choose $\alpha = \alpha(N)$ depending on N, in a way prescribed later. Let us briefly describe how it works here.

We have two nested particle systems: first there are N particles $\kappa_n = (\kappa^i_n)_{1 \leq i \leq N}$ at time n which are used for approximating $\mu^{(\alpha)}_n$ via the empirical measure μ^N_n; next we have N particles $\widehat{\kappa}_n = (\widehat{\kappa}^i_n)_{1 \leq i \leq N}$ at time n and which are used as an intermediate step. All κ^i_n and $\widehat{\kappa}^i_n$ take their values in $E \times I\!R^q$, and we single out their "components" ξ^i_n and ζ^i_n for κ^i_n and $\widehat{\xi}^i_n$ and $\widehat{\zeta}^i_n$ for $\widehat{\kappa}^i_n$, taking values respectively in E and in $I\!R^q$.

The motion of these particles is again defined on an auxiliary probability space, on which we denote by $\mathcal{G}_n$ the σ-field generated by the variables κ_p for $p \leq n$ and $\widehat{\kappa}_p$ for $p < n$, and by $\widehat{\mathcal{G}}_n$ the σ-field generated by the variables κ_p and $\widehat{\kappa}_p$ for $p \leq n$.

First, the variables κ_0^i are drawn independently according to the initial law μ_0 of (4.1). Next, the mechanism goes, by induction on n, according to the following two steps Markov rule:

Selection/Updating:

$$\widetilde{I\!\!P}(\widehat{\kappa}_n \in (dz_1, \ldots, dz_N)|\mathcal{G}_n) = \prod_{p=1}^{N} \sum_{i=1}^{N} \frac{g_n^{(\alpha)}(\kappa_n^i)}{\sum_{j=1}^N g_n^{(\alpha)}(\kappa_n^j)} \, \delta_{\kappa_n^i}(dz_p). \tag{4.23}$$

Mutation/Prediction:

$$\widetilde{I\!\!P}(\kappa_{n+1} \in (dz_1, \ldots, dz_N) \,|\widehat{\mathcal{G}}_n) = \prod_{p=1}^{N} \mathcal{Q}_{n+1}(\widehat{\kappa}_n^p, dz_p). \tag{4.24}$$

Then the approximating measure μ_n^N and η_n^N for $\mu_n^{(\alpha)}$ and $\eta_n^{(\alpha)}$ (see (4.8)) are

$$\mu_n^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(\xi_n^i, \zeta_n^i)}, \qquad \eta_n^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\xi_n^i}. \tag{4.25}$$

Next, comparing with (3.3), we set as approximating measures for $\widehat{\mu}_n^{(\alpha)}$ and $\widehat{\eta}_n^{(\alpha)}$:

$$\widehat{\mu}_n^N = \sum_{i=1}^{N} \frac{g_n^{(\alpha)}(\kappa_n^i)}{\sum_{j=1}^N g_n^{(\alpha)}(\kappa_n^j)} \, \delta_{\kappa_n^i}, \qquad \widehat{\eta}_n^N = \sum_{i=1}^{N} \frac{g_n^{(\alpha)}(\kappa_n^i)}{\sum_{j=1}^N g_n^{(\alpha)}(\kappa_n^j)} \, \delta_{\xi_n^i}. \tag{4.26}$$

4) We will assert the quality of our IPS through a central limit theorem only. For each $\alpha > 0$ we have Theorem 3.2, which we recall in our setting for the marginals η_n^N and $\widehat{\eta}_n^N$ only, since these are the quantities of interest. Set

$$W_n^{N,(\alpha)}(f) = \sqrt{N} \, \left(\eta_n^N(f) - \eta_n^{(\alpha)}(f)\right),$$

and

$$\widehat{W}_n^{N,(\alpha)}(f) = \sqrt{N} \, \left(\widehat{\eta}_n^N(f) - \widehat{\eta}_n^{(\alpha)}(f)\right).$$

Then if $\alpha > 0$ is fixed, Theorem 3.2 yields that, as $N \to \infty$, the sequences $(W_n^{N,(\alpha)}(f))$ and $(\widehat{W}_n^{N,(\alpha)}(f))$ converge in law to centered Gaussian variables $W_n^{(\alpha)}(f)$ and $\widehat{W}_n^{(\alpha)}(f)$ with variances given by (recall (3.20) and (3.21)):

$$\widetilde{I\!\!E}\left(W_n^{(\alpha)}(f)^2\right) = \sum_{p=0}^{n} \frac{\mu_p^{(\alpha)}\left(\left(g_p^{(\alpha)} \widehat{L}_{p,n-1}^{(\alpha)} Q_n(f - \eta_n^{(\alpha)} f)\right)^2\right)}{\left(\mu_p^{(\alpha)}(g_p^{(\alpha)} \widehat{L}_{p,n-1}^{(\alpha)} 1)\right)^2} \tag{4.27}$$

and

$$\widetilde{E}\left(\widehat{W}_n^{(\alpha)}(f)^2\right) = \sum_{p=0}^{n} \frac{\mu_p^{(\alpha)}\left(\left(g_p^{(\alpha)}\widehat{L}_{p,n}^{(\alpha)}(f-\widehat{\eta}_n^{(\alpha)}f)\right)^2\right)}{\left(\mu_p^{(\alpha)}(g_p^{(\alpha)})\right)^2\left(\widehat{\eta}_p^{(\alpha)}\widehat{L}_{p,n}^{(\alpha)}1\right)^2}. \tag{4.28}$$

Now we let $\alpha \to 0$: both quantities (4.27) and (4.28) increase, in a way which is controlled by using Lemma 4.1 and Proposition 4.2. Take for example the summand number p in (4.27). For the denominator, we may write by virtue of (4.7), (4.9), (4.11), (4.12) and (4.21):

$$\mu_p^{(\alpha)}(g_p^{(\alpha)}\widehat{L}_{p,n-1}^{(\alpha)}1) = \frac{\nu_p^{(\alpha)}(g_p^{(\alpha)}\widehat{L}_{p,n-1}^{(\alpha)}1)}{\nu_p^{(\alpha)}(1)} = \frac{\gamma_n^{(\alpha)}(1)}{\gamma_p^{(\alpha)}(1)} = \frac{\gamma_n(1)}{\gamma_p(1)} + O(\alpha^2)$$

As for the numerator, first we can write it as

$$\frac{1}{\nu_p^{(\alpha)}(1)}\ \eta_0\left(\widehat{G}_0^{(\alpha)}\widehat{L}_{0,p-1}^{(\alpha)}\mathcal{Q}_p\left(\left(g_p^{(\alpha)}\widehat{L}_{p,n-1}Q_n(f-\eta_n^{(\alpha)}f)\right)^2\right)\right).$$

Next we can replace above $\widehat{G}_0^{(\alpha)}$ and $\widehat{L}_{0,p-1}^{(\alpha)}$ and $\widehat{L}_{p,n-1}^{(\alpha)}$ and $\eta_n^{(\alpha)}$ by G_0 and $\widehat{L}_{0,p-1}$ and $\widehat{L}_{p,n-1}$ and η_n respectively, up to a relative error of size $O(\alpha^2)$, by using (4.15) and (4.16). Then if we use (4.17), we readily obtain that

$$\alpha^q\ \mu_p^{(\alpha)}\left(\left(g_p^{(\alpha)}\widehat{L}_{p,n-1}^{(\alpha)}Q_n(f-\eta_n^{(\alpha)}f)\right)^2\right)$$

converges to

$$\frac{u}{\gamma_p(1)}\ \eta_0\left(G_0\widehat{L}_{0,p}\left((\widehat{L}_{p,n-1}Q_n(f-\eta_n f))^2\right)\right)$$

as $\alpha \to 0$. Then, using (2.2), (2.6) and (2.7), we finally get that

$$\lim_{\alpha\to 0}\alpha^q E(W_n^{N,(\alpha)}(f)^2) = u\sum_{p=0}^{n}\frac{\gamma_p(1)\gamma_{p+1}(1)}{\gamma_n(1)^2}\widehat{\eta}_p\left((\widehat{L}_{p,n-1}Q_n(f-\eta_n f))^2\right). \tag{4.29}$$

In much the same way, we also obtain that

$$\lim_{\alpha\to 0}\alpha^q E(\widehat{W}_n^{N,(\alpha)}(f)^2) = u\sum_{p=0}^{n}\frac{\gamma_p(1)\gamma_{p+1}(1)}{\widehat{\gamma}_n(1)^2}\widehat{\eta}_p\left((\widehat{L}_{p,n}(f-\widehat{\eta}_n f))^2\right) \tag{4.30}$$

(let us recall that $\gamma_{p+1}(1) = \widehat{\gamma}_p(1)$).

In view of these two results, the way $\alpha = \alpha(N)$ should depend on N is clear: indeed, using (4.22), we get

$$\eta_n^N f - \eta_n f = \frac{1}{\sqrt{N}}W_n^{N,(\alpha(N))}(f) + \alpha(N)^2\eta_n^*(f) + O(\alpha(N)^3)$$

and a similar expression for $\widehat{\eta}_n^N f - \widehat{\eta}_n f$. The first term in the right side above is of order $1/\sqrt{N\alpha(N)^q}$ by (4.29). Optimizing the choice of $\alpha(N)$ leads to

$$\alpha(N) = \frac{1}{N^{\frac{1}{4+q}}}. \tag{4.31}$$

Therefore it is not surprising to have the following result, which is proved in (Del Moral and Jacod 1999) for $\widehat{\eta}_n$, and whose proof for η_n is quite similar:

Theorem 4.3 *Assume R, and take $\alpha = \alpha(N)$ as given by (4.31) in the procedure given by (4.23) and (4.24). Let f be any bounded measurable function on E.*

a) *The sequence of variables*

$$W_n^N(f) = N^{\frac{2}{4+q}} \left(\eta_n^N f - \eta_n f \right)$$

converges in law to a Gaussian variable with mean $\eta_n^* f$ given by (4.18) and variance given by the right side of (4.29).

b) *The sequence of variables*

$$\widehat{W}_n^N(f) = N^{\frac{2}{4+q}} \left(\widehat{\eta}_n^N f - \widehat{\eta}_n f \right)$$

converges in law to a Gaussian variable with mean $\widehat{\eta}_n^* f$ given by (4.19) and variance given by the right side of (4.30).

4.2 Subcase B2

1) Exactly as in Subsection 3.2, we label *Subcase B2* the case B when the initial law η_0 and the transitions P_n are not explicitly known, and/or when one cannot simulate random variables exactly according to the laws μ_0 or $P_n(x, y; .)$. So one will replace μ_0 by

$$\mu_0^{(m)}(dx, dz) = \eta_0^{(m)}(dx) \bar{G}_0^{(m)}(x, y) dy$$

(where $\eta_0^{(m)}$ is a probability measure) and P_n by

$$P_n^{(m)}(x, y; dx', dy') = Q_n^{(m)}(x, dx') \bar{G}_n^{(m)}(x, y; x', y') dy'$$

(where $Q_n^{(m)}$ is a transition probability), as in (1.2). These are chosen in such a way that we can simulate variables according to $\mu_0^{(m)}$ and $P_n^{(m)}(x, y; .)$ for all (x, y). As in Subcase A2, the index m (an integer) measures the quality of the approximation, and will eventually depend on N.

For measuring the quality of this approximation, let us introduce some additional assumptions:

Assumption HG$_n$: There is a measurable bounded function $\bar{G}_n^{\#}$ and a constant C_n such that

$$|m(\bar{G}_n^{(m)}(x, y; x', y') - \bar{G}_n(x, y; x', y')) - \bar{G}_n^{\#}(x, y; x', y')| \leq \frac{C_n}{m}.$$

Furthermore the functions $y' \mapsto \bar{G}_n'(x, y; x', y')$ are equi-continuous; we will use the notation $G_n^{\#}(x, x') = \bar{G}_n^{\#}(x, y_{n-1}; x', y_n)$ (if $n = 0$ we obviously have $\bar{G}_0^{\#}(x, y; x', y') = \bar{G}_0^{\#}(x', y')$ and $G_0^{\#}(x) = \bar{G}_0^{\#}(x, y_0)$).

Assumption HP: Setting

$$\widetilde{L}_n^{(m)} f(x,y) = \int Q_n^{(m)}(x,dx')f(x')\bar{G}_n^{(m)}(x,y_{n-1};x',y)$$

(as in Assumption R), we suppose that for each $n \geq 1$ there is a finite kernel $\widehat{L}_n^{\#}$ from $(E \times I\!R^q, \mathcal{E} \otimes \mathcal{R}^q)$ into $(E, \mathcal{E})$ and a constant C_n depending only on n and on y_{n-1}, such that

$$\|m(\widetilde{L}_n^{(m)}(x,y;.) - \widetilde{L}_n(x,y;.)) - \widehat{L}_n^{\#}(x,y;.)\|_{\text{tv}} \leq \frac{C_n}{m}.$$

Furthermore the functions $y \mapsto \widetilde{L}_n^{\#} f(x,y)$ are equi-continuous for any bounded measurable function f.

Lemma 4.4 *The assumptions R, HQ and HG_n for all $n \geq 1$ imply HP with*

$$\begin{aligned}
&\widehat{L}_n^{\#} f(x,y) \\
&= \int Q_n(x,dx')f(x')\bar{G}_n^{\#}(x,y_{n-1};x',y) + \int Q'_n(x,dx')f(x')G_n(x,y_{n-1};x',y).
\end{aligned} \tag{4.32}$$

Proof: Upon observing that, with the notation (4.32),

$$\begin{aligned}
&m(\widetilde{L}_n^{(m)} f(x,y) - \widetilde{L}_n f(x,y)) - \widehat{L}_n^{\#} f(x,y) \\
&= \int \Big(m(Q_n^{(m)}(x,dx') - Q_n(x,dx')) - Q'_n(x,dx')\Big) f(x')\bar{G}_n(x,y_{n-1};x',y) \\
&\quad + \int Q_n^{(m)}(x,dx')f(x') \\
&\qquad \times \Big\{m\left((\bar{G}_n^{(m)}(x,y_{n-1};x',y) - \bar{G}_n^{(m)}(x,y_{n-1};x',y)\right) - \bar{G}_n^{\#}(x,y_{n-1};x',y)\Big\} \\
&\quad + \int (Q_n^{(m)}(x,dx') - Q_n(x,dx'))f(x')\bar{G}_n^{\#}(x,y_{n-1};x',y),
\end{aligned}$$

the first part of HP is immediate (recall that $\bar{G}_n$ is bounded). The second part follows from the second part of HG_n and from R. ∎

In the sequel we will assume HE, HG_0, R, and HP. Using the notation (4.13) and (4.14), we define a finite kernel $\widehat{L}'_n$ and a bounded function G'_0 by

$$\widehat{L}'_n f(x) = \widehat{L}_n^* f(x) + \widehat{L}_n^{\#} f(x,y_n), \qquad G'_0(x) = G_0^*(x) + G_0^{\#}(x).$$

Now, let $\widehat{L}_n^{(\alpha),(m)}$ and $\widehat{L}_{p,n}^{(\alpha),(m)}$ be the analogue of $\widehat{L}_n^{(\alpha)}$ and $\widehat{L}_{p,n}^{(\alpha)}$ with $\mathcal{Q}_n^{(m)}$ instead of $\mathcal{Q}_n$, and similarly $\widehat{G}_0^{(\alpha),(m)}$ is associated with $\bar{G}_0^{(m)}$ by (4.10). Similarly we define $\gamma_n^{(\alpha),(m)}$ and $\widehat{\gamma}_n^{(\alpha),(m)}$ and $\eta_n^{(\alpha),(m)}$ and $\widehat{\eta}_n^{(\alpha),(m)}$.

Then Lemma 4.1 becomes

Lemma 4.5 *Assume R, HG_0 and HP. Let $\alpha \to 0$ and $m \to \infty$ in such a way that*

$$m\alpha^2 = 1 + o(\alpha). \tag{4.33}$$

Then there are constants $\varepsilon(n,\alpha)$ depending only on n and on the observations $y_0, \dots, y_n$ and on α, with $\varepsilon(n,\alpha) \to 0$ as $\alpha \to 0$, such that for any bounded measurable function f on E and any $x \in E$ we have:

$$|\widehat{L}_n^{(\alpha)} f(x) - \widehat{L}_n f(x) - \alpha^2 \widehat{L}'_n f(x)| \le \varepsilon(n,\alpha)\alpha^2 \|f\|, \tag{4.34}$$

$$|\widehat{G}_0^{(\alpha)}(x) - G_0(x) - \alpha^2 G'_0(x)| \le \varepsilon(0,\alpha)\alpha^2. \tag{4.35}$$

$$|\alpha^q \mathcal{Q}_n^{(m)}((g_n^{(\alpha)})^2 f)(x) - u\widehat{L}_n f(x)| \le \varepsilon(n,\alpha)\|f\|, \tag{4.36}$$

where

$$u = \int \theta(y)^2 dy.$$

Proof: We have

$$\widehat{L}_n^{(\alpha),(m)} f(x) - \widehat{L}_n f(x) = \int \Big(\widetilde{L}_n^{(m)} f(x, y_n + \alpha y) - \widetilde{L}_n f(x, y_n + \alpha y)\Big)\theta(y)dy$$
$$+ \int \Big(\widetilde{L}_n f(x, y_n + \alpha y) - \widetilde{L}_n f(x, y_n)\Big)\theta(y)dy.$$

The first term in the right side is

$$\frac{1}{m}\int \widehat{L}_n^{\#} f(x, y_n + \alpha y)\theta(y)dy$$

plus a remainder smaller than $\varepsilon(n,\alpha)\alpha^2\|f\|$ in view of HP and (4.33); then the last part of HP and (4.33) again yield that it is also $\alpha^2 \widehat{L}_n^{\#} f(x, y_n)$ plus a remainder of the same form.

Next, the second term in the right side is $\alpha^2 \widehat{L}_n^{*} f(x)$, plus a remainder smaller than $C_n \alpha^3 \|f\|$, by the proof of Lemma 4.1. Hence (4.34) holds.

The proofs of (4.35) and (4.36) are similar (and in fact simpler). ∎

Then we define the kernels $\widehat{L}'_{p,n}$ for $0 \le p \le n$ and the measures γ'_n, $\widehat{\gamma}'_n$, η'_n and $\widehat{\eta}'_n$ on $(E, \mathcal{E})$ for $n \ge 0$ by

$$\widehat{L}'_{p,n} = \sum_{q=p+1}^{n} \widehat{L}_{p,q-1} \widehat{L}'_q \widehat{L}_{q,n},$$

$$\gamma'_0(f) = \eta'_0(f),$$

$$\gamma'_{n+1}(f)$$
$$= \eta'_{n+1}(G_0 \widehat{L}_{0,n} Q_{n+1} f) + \eta_0\left(G'_0 \widehat{L}_{0,n} Q_{n+1} f + G_0 \widehat{L}'_{0,n} Q_{n+1} f + G_0 \widehat{L}_{0,n} Q'_{n+1} f\right),$$

and

$$\eta'_n(f) = \frac{\gamma'_n(f)\gamma_n(1) - \gamma'_n(1)\gamma_n(f)}{(\gamma_n 1)^2}, \tag{4.37}$$

$$\widehat{\gamma}'_n(f) = \eta'_0(G_0 \widehat{L}_{0,n} f) + \eta_0\left((G'_0 \widehat{L}_{0,n} + G_0 \widehat{L}'_{0,n})f\right),$$

$$\widehat{\eta}'_n(f) = \frac{\widehat{\gamma}'_n(f)\widehat{\gamma}_n(1) - \widehat{\gamma}'_n(1)\widehat{\gamma}_n(f)}{(\widehat{\gamma}_n 1)^2}. \tag{4.38}$$

Then we deduce from Lemma 4.5 the following result, in exactly the same way Proposition 4.2 was deduced from Lemma 4.1:

Proposition 4.6 *Assume R, HE, HG_0 and HP. Let $\alpha \to 0$ and $m \to \infty$ in such a way that (4.33) holds.*
There are constants $\varepsilon(n,\alpha)$ depending only on n and on the observations $y_0, \ldots, y_n$ and on α, with $\varepsilon(n,\alpha) \to 0$ as $\alpha \to 0$, such that for any bounded measurable function f on E we have:

$$|\widehat{L}^{(\alpha),(m)}_{p,n} f(x) - \widehat{L}_{p,n} f(x) - \alpha^2 \widehat{L}'_{p,n} f(x)| \le \varepsilon(n,\alpha)\alpha^2 \|f\|, \quad 0 \le p \le n, \tag{4.39}$$

$$\|\gamma^{(\alpha),(m)}_n - \gamma_n - \alpha^2 \gamma'_n\|_{tv} \le \varepsilon(n,\alpha)\alpha^2, \quad \|\widehat{\gamma}^{(\alpha),(m)}_n - \widehat{\gamma}_n - \alpha^2 \widehat{\gamma}'_n\|_{tv} \le \varepsilon(n,\alpha)\alpha^2, \tag{4.40}$$

$$\|\eta^{(\alpha),(m)}_n - \eta_n - \alpha^2 \eta'_n\|_{tv} \le \varepsilon(n,\alpha)\alpha^2, \quad \|\widehat{\eta}^{(\alpha),(m)}_n - \widehat{\eta}_n - \alpha^2 \widehat{\eta}'_n\|_{tv} \le \varepsilon(n,\alpha)\alpha^2. \tag{4.41}$$

2) The Interacting Particle System.

Our IPS is of course conducted according to the setting $(\bar{G}^{(m)}_n, \eta^{(m)}_0, Q^{(m)}_n)$, which is the only one available for explicit computations, and using the recipe describe in Subsection 4.1.3: that is, we start with the initial law $\mu^{(m)}_0$, and apply (4.23), and also (4.24) with $\mathcal{Q}^{(m)}_n$ instead of $\mathcal{Q}_n$.

Note that $g^{(\alpha)}_n$ is again given by (4.3). Note also that $\alpha = \alpha(N)$ will be given by (4.31), while $m = m(N)$ will be taken to be equivalent to $1/\alpha(N)^2$; for example we can take

$$m(N) = \left[N^{\frac{2}{4+q}}\right]. \tag{4.42}$$

The IPS gives rise to the approximating measures η^N_n and $\widehat{\eta}^N_n$, as given by (4.25) and (4.26).

Note that with our choice of $\alpha(N)$ and $m(N)$ (see (4.31) and (4.42)), the relation (4.33) is satisfied as $N \to \infty$.

Then the same computations as in Subsection 4.1 give the following theorem:

Theorem 4.7 *Assume R, HE, HG_0 and HP, and take $\alpha = \alpha(N)$ and $m = m(N)$ as in (4.31) and (4.42). Let f be any bounded measurable function on E.*

a) *The sequence of variables*

$$W_n^N(f) = N^{\frac{2}{4+q}} \left(\eta_n^N f - \eta_n f\right)$$

converges in law to a Gaussian variable with mean $\eta_n' f$ given by (4.37) and variance given by the right side of (4.29).

b) *The sequence of variables*

$$\widehat{W}_n^N(f) = N^{\frac{2}{4+q}} \left(\widehat{\eta}_n^N f - \widehat{\eta}_n f\right)$$

converges in law to a Gaussian variable with mean $\widehat{\eta}_n' f$ given by (4.38) and variance given by the right side of (4.30).

5 Discretely observed stochastic differential equations

In this Section, we quickly show how the previous results can be put to use when the state process $X = (X_t)_{t \in \mathbb{R}_+}$ is d-dimensional and is the solution of a stochastic differential equation of the form

$$dX_t \;=\; a(X_t)\; dt \;+\; b(X_t)\; dW_t \;, \qquad \mathcal{L}aw(X_0) \;=\; \eta, \tag{5.1}$$

where η is the initial distribution on $\mathbb{R}^d$, and a, b are known functions, and W is a d-dimensional Wiener process. We suppose that a and b are such that there is a unique solution to (5.1) for every initial distribution. The process X is thus Markov, with a transition semi-group $(\bar{Q}_t)_{t \geq 0}$.

5.1 Case A

We suppose here that the observations occur at times $0, 1, \ldots$, according to the scheme (1.1), with the assumptions of Case A about H_n, V_n and $\bar{g}_n$. Then with the notation of this case we have $Q_n = \bar{Q}_1$ for all n.

We are in Subcase A1 when η_0 and $\bar{g}_n$ and $\bar{Q}_1$ are explicitly known, and when one can simulate variables exactly according to the laws η_0 and $\bar{Q}_1(x, .)$ for any x. Then Proposition 3.1 and Theorem 3.2 are valid (see (Del Moral et al. 1998) and (Del Moral and Jacod 1999)).

But of course such a situation is very rare: in practice, one may know a and b, and perhaps η_0 and the functions $\bar{g}_n$, but $\bar{Q}_1$ is usually not known explicitly. So we have to resort on an approximating scheme for the simulation. An appropriate way consists in using an Euler approximation scheme for the SDE (5.1).

In order to keep things simple, assume that we know $\bar{g}_n$ and η_0, and we also know how to simulate a variable according to the law η_0. Then we can apply Theorem 3.5 as soon as HQ holds (since HG and HE are trivially met with $\eta_0' = 0$ and $g_n' = 0$). The approximate transitions $Q_n^{(m)} = Q_1^{(m)}$ will be the transitions at time 1 of the Euler approximating scheme with stepsize $1/m$ for Equation (5.1).

In other words, for any starting point $x \in I\!\!R^d$ we define (and can easily simulate) by induction on i the following variables:

$$\begin{aligned} X(x)_0^{(m)} &= x, \\ X(x)_{i+1}^{(m)} &= X(x)_i^{(m)} + a(X(x)_i^{(m)})\frac{1}{m} + b(X(x)_i^{(m)})\frac{U_{i+1}}{\sqrt{m}}, \end{aligned} \tag{5.2}$$

where the U_i's are i.i.d. centered Gaussian d-dimensional vectors with covariance matrix equal to the identity, and above we use a vector notation. Then $Q_1^{(m)}(x,.)$ is the law of $X(x)_m^{(m)}$.

Let us introduce the assumption:

Assumption A1:

In (5.1) the functions a and b are twice differentiable with bounded partial derivatives of first and second order, and the matrix $(bb^*)(x)$ is uniformly non-degenerate.

Then, according to (Bally and Talay 1996), the transitions $\bar{Q}_1(x,.)$ and $Q_1^{(m)}(x,.)$ have densities $y \mapsto q(x,y)$ and $y \mapsto q^{(m)}(x,y)$ and there are constants C, C' depending only on the coefficients a and b, such that

$$q(x,x') + q^{(m)}(x,x') \ \le \ Ce^{-C'|x-x'|^2}, \tag{5.3}$$

$$\left.\begin{aligned} q^{(m)}(x,x') - q(x,x') &= \tfrac{\nu(x,x')}{m} + \tfrac{R_m(x,x')}{m^2}, \\ \text{with} \quad |\nu(x,x')| + |R_m(x,x')| &\le Ce^{-C'|x-x'|^2}. \end{aligned}\right\}$$

Then HQ holds trivially with

$$Q_n' f(x) = \int \nu(x,x') f(x') dx' \tag{5.4}$$

for all n.

In other words, we have the following result (see (Del Moral and Jacod 1999)):

Theorem 5.1 *Assume (5.1) and A1, and suppose that we conduct the IPS with the approximate transitions $Q_n^{(m(N))}$, where $m(N) = [\sqrt{N}]$. Define η'_n and $\widehat{\eta}'_n$ by (3.24) with $\eta'_0 = 0$ and $g'_n = 0$ and Q'_n given by (5.4). Let f be any bounded measurable function on E.*

a) *The sequence of random variables*

$$W_n^N(f) = \sqrt{N}(\eta_n^N f - \eta_n f)$$

converges in law to a Gaussian variable with mean $\eta'_n(f)$ and variance given by (3.18) or equivalently (3.20).

b) *The sequence of random variables*

$$\widehat{W}_n^N(f) = \sqrt{N}(\widehat{\eta}_n^N f - \widehat{\eta}_n f)$$

converges in law to a Gaussian variable with mean $\widehat{\eta}'_n(f)$ and variance given by (3.21).

5.2 Case B

Here again the process X is the solution of (5.1), while we observe at times $0, 1, \ldots$ the values of the q-dimensional process Y which is the solution of the equation

$$dY_t \;=\; a'(X_t, Y_t)dt + b'(X_t, Y_t)dW'_t, \qquad Y_0 = 0, \tag{5.5}$$

where a' and b' are known functions and W' is a q-dimensional Wiener process, independent of W.

We assume that the initial law of (X, Y) is of the form μ_0, as given in (1.2), and we make the following assumptions on the coefficients of the two equations (5.1) and (5.5):

Assumption C1: In (5.1) and (5.5) the functions a, b, a' and b' are 3 times differentiable with bounded partial derivatives of orders 1, 2 and 3, and the matrices $(bb^*)(x)$ and $(b'b'^*)(x, y)$ are uniformly non-degenerate.

Then the process (X, Y) is homogeneous Markov with transition semi-group $(\bar{P}_t)_{t\geq 0}$, and the transition $\bar{P}_1(x, y; .)$ admits a positive density $(x', y') \mapsto p(x, y; x', y')$. Further $\bar{P}(x, y; dx' \times I\!R^q)$ does not depend on y and equals $\bar{Q}_1(x, .)$ (the transition of X at time 1). So we are exactly in the situation of Case B, with $Q_n = \bar{Q}_1$ and $P_n = \bar{P}_1$ and

$$\bar{G}_n(x, y; x', y') = \frac{p(x, y; x', y')}{q(x, x')}, \qquad \text{where} \quad q(x, x') = \int p(x, y; x', y')dy' \tag{5.6}$$

for all $n \geq 1$ (note that the integral above does not depend on y).

Here again, to keep things simple, we suppose that we know η_0 and $\bar{G}_0$, which we assume satisfy R-a, and that we can simulate exactly random variables with law μ_0. But we do not know explicitly $\bar{P}_1$ and we have to resort on an Euler scheme to approximately simulate a variable whose law is $\bar{P}_1(x, y; .)$.

More precisely, if the stepsize is $1/m$, we use the scheme (5.2) for (5.1), and similarly define the variables

$$\begin{aligned} &Y(x,y)_0^{(m)} = y, \\ &Y(x,y)_{i+1}^{(m)} = Y(x,y)_i^{(m)} \\ &\qquad + a'(X(x)_i^{(m)}, Y(x,y)_i^{(m)}) \, \frac{1}{m} \; + \; b'(X(x)_i^{(m)}, Y(x,y)_i^{(m)}) \, \frac{U'_{i+1}}{\sqrt{m}} \end{aligned}$$

for approximating equation (5.5): here the U'_i's are i.i.d. centered Gaussian vectors with covariance equal to the identity, and independent of the sequence (U_i). Then $P_1^{(m)}(x, y; .)$ denotes the law of the pair $(X(x)_m^{(m)}, Y(x, y)_m^{(m)})$, and it is easy to simulate variables according to this law.

Now, under C1, the law $P_1^{(m)}(x, y; .)$ admits a density

$$(x', y') \mapsto p^{(m)}(x, y; x', y'),$$

such that $\int p^{(m)}(x, y; x', y')dy'$ does not depend on y and equals the function $q^{(m)}(x, x')$ of (5.3). Furthermore, the functions p and $p^{(m)}$ are three times differentiable, and according again to (Bally and Talay 1996) there exist constants C, C' depending only on the coefficients a, a', b and b' and such that

$$|\nabla_{y'}^j p(x, y; x', y')| + |\nabla_{y'}^j p^{(m)}(x, y; x', y')| \;\leq\; C e^{-C'(|x-x'|^2+|y-y'|^2)}, \qquad (5.7)$$

for $j = 0, 1, 2, 3$ (where $\nabla_{y'}^0 p = p$ and $\nabla_{y'}^j p$ denotes the vector of all partial derivatives of p, of order j, w.r.t. the components of y') and

$$\left.\begin{aligned} &p^{(m)}(x,y;x',y') - p(x,y;x',y') \;=\; \tfrac{\widetilde{\nu}(x,y;x',y')}{m} + \tfrac{\widetilde{R}_m(x,y;x',y')}{m^2}, \\ &\text{with} \quad |\widetilde{\nu}(x,y;x',y')| + |\widetilde{R}_m(x,y;x',y')| \;\leq\; C e^{-C'(|x-x'|^2+|y-y'|^2)}, \end{aligned}\right\} \qquad (5.8)$$

and further the function $\widetilde{\nu}$ is uniformly continuous.

Observe also that we have

$$P_1^{(m)}(x, y; dx', dy') = Q_1^{(m)}(x, dx')\bar{G}_n^{(m)}(x, y; x', y')dy'$$

for all $n \geq 1$, where, as in (5.6):

$$\bar{G}_n^{(m)}(x, y; x', y') = \frac{p^{(m)}(x, y; x', y')}{q^{(m)}(x, x')}.$$

Then we are exactly in the situation of Subcase B2: we have HE with $\eta'_0 = 0$, and R-a and HG$_0$ with $\bar{G}_0^{\#} = 0$ by hypothesis. Furthermore,

$$\widetilde{L}_n f(x,y) = \int p(x, y_{n-1}, x', y) f(x') dx',$$

$$\widetilde{L}_n^{(m)} f(x,y) = \int p^{(m)}(x, y_{n-1}, x', y) f(x') dx'.$$

Then R-b follows from (5.7), and HP follows from (5.8), with

$$\widehat{L}_n^{\#} f x, y) = \int \nu(x, y_{n-1}; x', y) f(x') dx'. \tag{5.9}$$

Henceforth we have the following result:

Theorem 5.2 *Assume (5.1) and (5.5) and C1 and R-a, and take $\alpha = \alpha(N)$ and $m = m(N)$ as given by (4.31) and (4.42). Suppose that we conduct the IPS with the approximate transitions $P_n^{m(N)}$ and the level $\alpha(N)$, and define η'_n and $\widehat{\eta}'_n$ by (4.37) and (4.38) with $\eta'_0 = 0$ and $G_0^{\#} = 0$ and $\widehat{L}_n^{\#}$ given by (5.9). Let f be any bounded measurable function on E.*

a) *The sequence of variables*

$$W_n^N(f) = N^{\frac{2}{4+q}} \left(\eta_n^N f - \eta_n f \right)$$

converges in law to a Gaussian variable with mean $\eta'_n f$ and variance given by the right side of (4.29).

b) *The sequence of variables*

$$\widehat{W}_n^N(f) = N^{\frac{2}{4+q}} \left(\widehat{\eta}_n^N f - \widehat{\eta}_n f \right)$$

converges in law to a Gaussian variable with mean $\widehat{\eta}'_n f$ and variance given by the right side of (4.30).

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