Stochastic Processes

From Applications to Theory
Solutions manual

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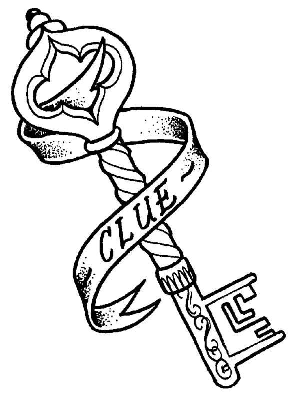
Chapter 1



Solution to exercise 1:
By (1.1), we have

$$\mathbb{E}(X_n) = \mathbb{E}(X_{n-1}) = \ldots = \mathbb{E}(X_0) = 0$$

and

$$\mathrm{Var}(X_n) \quad = \quad \mathrm{Var}\left(\sum_{1\le k\le n} U_k\right) = n\ \mathbb{E}(U_1^2) = n.$$

This ends the proof of the exercise.
■

Solution to exercise 2:
In order to come back to the origin, the walker must take an equal number of positive and negative steps in each direction. Thus, we clearly have $\mathbb{P}(X_{2n+1} = 0) = 0$. In addition, since the variables U_k in (1.1) are independent and identically distributed, each of the $\binom{2n}{n}$ possible paths of the same length $2n$ is equally likely. This implies that

$$\mathbb{P}(X_{2n} = 0) = \binom{2n}{n}\ 2^{-n} \times 2^{-n}.$$

For any $n \geq m$, the walker who starts at the origin and reaches $(2m)$ at time $(2n)$, must have made $(n+m)$ steps upwards and $(n-m)$ steps downwards, so that

$$\mathbb{P}(X_{2n} = 2m) = \binom{2n}{n+m}\ 2^{-(n+m)}\ 2^{-(n-m)} = \binom{2n}{n+m}\ 2^{-2n}.$$

This ends the proof of the exercise. ■

Solution to exercise 3:
If $s < t$ we have

$$N_t = (N_t - N_s) + N_s.$$

By the independence property between $(N_t - N_s) \stackrel{N_0=0}{=}$ and $N_s \stackrel{N_0=0}{=} (N_s - N_0)$, we prove that

$$\mathbb{E}(N_t N_s) = \mathbb{E}\left((N_t - N_s)N_s\right) + \mathbb{E}(N_s^2) = \mathbb{E}\left(N_t - N_s\right)\mathbb{E}\left(N_s\right) + \mathbb{E}(N_s^2).$$

On the other hand $(N_t - N_s) \stackrel{law}{=} N_{t-s}$, so that

$$\mathbb{E}(N_t N_s) = \mathbb{E}\left(N_{t-s}\right)\mathbb{E}\left(N_s\right) + \mathbb{E}(N_s^2).$$

Finally,

$$Cov(N_s, N_t) = \mathbb{E}(N_{t-s})\,\mathbb{E}(N_s) + \mathbb{E}(N_s^2) - \mathbb{E}(N_s)\mathbb{E}(N_t).$$

This ends the proof of the exercise. ■

Solution to exercise 4:

By construction, we have $X_t = (-1)^{N_t} X_0$. This ends the proof of the exercise. ■

Solution to exercise 5:

We clearly have

$$\mathbb{E}(X_t) = \mathbb{E}\left((-1)^{N_t}\right)\ \mathbb{E}(X_0) = 0.$$

In addition, using the fact that

$$N_{t+s} = (N_{t+s} - N_t) + (N_t - N_0)$$

and the independence property between $(N_{t+s} - N_t)$ and $(N_t - N_0)$, we prove that

$$\begin{aligned}\mathbb{E}(X_t X_{t+s}) &= \mathbb{E}\left((-1)^{N_t}(-1)^{(N_{t+s}-N_t)+N_t}\right)\\ &= \mathbb{E}\left((-1)^{(N_{t+s}-N_t)}\right).\end{aligned}$$

Recalling that $(N_{t+s} - N_t)$ has the same law as N_s, we conclude that

$$\begin{aligned}\mathbb{E}(X_t X_{t+s}) &= \mathbb{E}\left((-1)^{N_s}\right)\\ &= e^{-\lambda s}\sum_{n\geq 0}\frac{(\lambda s)^n}{n!}\,(-1)^n = e^{-\lambda s}\sum_{n\geq 0}\frac{(-\lambda s)^n}{n!} = e^{-2\lambda s}.\end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 6:

We let $n_0 < T < n_1$ be the first time a path from P_0 to P_1 hits the time axis. Reflecting the path from P_0 to $(T, 0)$ w.r.t. the time axis we obtain a path from P_0^- to $(T, 0)$, and inversely. This procedure gives a correspondence between the set of paths from P_0 to P_1 that hit the time axis at some time $n_0 < T < n_1$ and the set of all paths from $P_0^- := (n_0, -x_{n_0})$ to P_1 (that necessarily hit the time axis at some time $n_0 < T < n_1$).

This ends the proof of the exercise. ■

Solution to exercise 7:

By construction, the probability density of $Z_{i,n}$ is given by

$$\mathbb{P}(\epsilon_n = 1)\times q(x)\ +\ \mathbb{P}(\epsilon_n = 0)\times\frac{p_i(x) - \rho q(x)}{1-\rho}\ \ =\ \ \rho\, q(x)\ +\ 1-\rho\ \frac{p_i(x) - \rho q(x)}{1-\rho} = p_i(x).$$

This shows that p_i is the probability density of $Z_{i,n}$, for $i = 1, 2$. The independence property of the random variables $(Z_{i,n})_{n\geq 0}$ is immediate, for $i = 1$ or $i = 2$. We let $T = \inf\{n \geq 1\ :\ \epsilon_n = 1\}$ be the first time $\epsilon_n = 1$. We have

$$Z_{1,T} = Z_{2,T} \quad\text{and}\quad \mathbb{P}(T > n) = \mathbb{P}(\epsilon_1 = 0, \ldots, \epsilon_n = 0) = \mathbb{P}(\epsilon_1 = 0)^n = (1-\rho)^n.$$

This shows that T is a geometric random variable and so it is finite.

This ends the proof of the exercise. ∎

Solution to exercise 8:

We let X^i be the Bernoulli random variable taking the value 1 if the i-th guess is correct, and 0 otherwise. Without any information, we have

$$X = \sum_{1\leq i\leq n} X^i \Rightarrow \mathbb{E}(X) = \sum_{1\leq i\leq n} \mathbb{E}(X^i) = \sum_{1\leq i\leq n} \mathbb{P}(X^i = 1) = \sum_{1\leq i\leq n} \frac{1}{n} = 1.$$

If the cards are shown after each guess, the best strategy is to choose one of the cards which has not been shown. Therefore

$$\mathbb{E}(X) = \frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{2} + 1 = \sum_{1\leq i\leq n} \frac{1}{i} \simeq \log n.$$

This ends the proof of the exercise. ∎

Solution to exercise 9:

For $m = 1$ the claim is obvious. We assume that the claim is true at some rank m. At the $(m+1)$-step, the element $a(i)$ (with $1 \leq i \leq m$) is in position $j \leq m$ only if it was there at rank m (with probability $1/m$ under the induction hypothesis), and if it was not swapped with $a(m+1)$ (with probability $m/(m+1)$; the chance to pick $a(i)$ with $1 \leq i \leq m$). Multiplying these two probabilities gives the total probability $1/(m+1)$.

The element at the last position and the location of $a(m+1)$ are obvious.

This ends the inductive proof of the exercise. ∎

Chapter 2

Solution to exercise 11:

- For the Bernoulli distribution $p(x) = p^x(1-p)^{1-x}$, with $p \in [0,1]$ and $x \in \{0,1\}$, we have

$$\varphi(s) \quad = \quad \mathbb{E}(s^X) = p\ s + (1-p)\ s^0 = ps + (1-p).$$

- For the Binomial distribution $p(x) = \binom{n}{x} p^x(1-p)^{n-x}$, with $p \in [0,1]$ and $x \in \{0,\ldots,n\}$ for some $n \in \mathbb{N}$, we have

$$\begin{aligned}\varphi(s) &= \sum_{0\leq x\leq n} \binom{n}{x} s^x\ p^x(1-p)^{n-x} \\ &= \sum_{0\leq x\leq n} \binom{n}{x} (sp)^x(1-p)^{n-x} = ((1-p)+ps)^n\,.\end{aligned}$$

- For the Poisson distribution $p(x) = e^{-\lambda}\lambda^x/x!$, with $\lambda > 0$ and $x \in \mathbb{N}$, we have

$$\begin{aligned}\varphi(s) &= e^{-\lambda} \sum_{x\geq 0} s^x\ \lambda^x/x! \\ &= e^{-\lambda} \sum_{x\geq 0} (s\lambda)^x/x! = e^{-\lambda+s\lambda} = e^{-\lambda(1-s)}.\end{aligned}$$

- For the Geometric distribution $p(x) = (1-p)^{x-1}p$, with $\lambda > 0$ and $x \in \mathbb{N} - \{0\}$, we have

$$\begin{aligned}\varphi(s) &= p\sum_{x\geq 1} s^x\ (1-p)^{x-1} \\ &= ps\sum_{x\geq 1} s^{x-1}\ (1-p)^{x-1} = ps\sum_{x\geq 0}(s(1-p))^x = ps/\left(1-s(1-p)\right).\end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 12:
By construction, we have

$$\begin{aligned}\mathbb{E}(N_{n+1}) &= \mathbb{E}\left(\mathbb{E}\left(\sum_{1\leq i\leq N_n} X_n^i \mid N_n\right)\right) \\ &= \mathbb{E}\left(\sum_{1\leq i\leq N_n} \mathbb{E}\left(X_n^i \mid N_n\right)\right) = \mathbb{E}(N_n)\ m.\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(N_{n+1}) &= \mathbb{E}(N_{n+1}^2) - (\mathbb{E}(N_{n+1}))^2 \\
&= \mathbb{E}\left(\mathbb{E}\left(\left(\sum_{1\leq i\leq N_n} X_n^i\right)^2 \mid N_n\right)\right) - (\mathbb{E}(N_{n+1}))^2 .
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathbb{E}\left(\left(\sum_{1\leq i\leq N_n} X_n^i\right)^2 \mid N_n\right) &= N_n\mathbb{E}(X^2) + N_n(N_n-1)\,(\mathbb{E}(X))^2 \\
&= N_n \ \mathrm{Var}(X) + N_n^2 m^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathrm{Var}(N_{n+1}) &= \mathbb{E}(N_n)\ \mathrm{Var}(X) + \mathbb{E}(N_n^2)m^2 - (\mathbb{E}(N_{n+1}))^2 \\
&= \mathbb{E}(N_n)\ \mathrm{Var}(X) + \mathrm{Var}(N_n)m^2 + \left[(\mathbb{E}(N_n)\ m)^2 - (\mathbb{E}(N_{n+1}))^2\right] \\
&= m^2\ \mathrm{Var}(N_n) + \mathbb{E}(N_n)\ \mathrm{Var}(X).
\end{aligned}$$

We conclude that

$$\begin{aligned}
\mathrm{Var}(N_{n+1}) &= m^2\ \left[m^2\ \mathrm{Var}(N_{n-1}) + \mathbb{E}(N_{n-1})\ \mathrm{Var}(X)\right] + \mathbb{E}(N_n)\ \mathrm{Var}(X) \\
&= m^4\ \mathrm{Var}(N_{n-1}) + \left[m^2\mathbb{E}(N_{n-1}) + \mathbb{E}(N_n)\right]\ \mathrm{Var}(X) \\
&= m^6\ \mathrm{Var}(N_{n-2}) + \left[m^4\mathbb{E}(N_{n-2}) + m^2\mathbb{E}(N_{n-1}) + \mathbb{E}(N_n)\right]\ \mathrm{Var}(X) \\
&= \ldots \\
&= m^{2(n+1)}\ \mathrm{Var}(N_0) + \mathrm{Var}(X) \sum_{0\leq k\leq n} m^{2k}\mathbb{E}(N_{n-k}),
\end{aligned}$$

so that

$$\mathrm{Var}(N_{n+1}) = m^{2(n+1)}\ \mathrm{Var}(N_0) + \mathrm{Var}(X)\ m^n\ (\mathbb{E}(N_0))^n \sum_{0\leq k\leq n} (m/\mathbb{E}(N_0))^k.$$

When $N_0 = 1$ we have $\mathrm{Var}(N_0) = 0$ and $\mathbb{E}(N_0) = 1$. In this case, we have

$$\mathrm{Var}(N_n) = \mathrm{Var}(X)\ m^{n-1} \sum_{0\leq k<n} m^k = \begin{cases} n\ \mathrm{Var}(X) & \text{when} \quad m=1 \\ \mathrm{Var}(X)\ m^{n-1}\ \frac{m^n-1}{m-1} & \text{when} \quad m\neq 1. \end{cases}$$

This ends the proof of the exercise. ■

Solution to exercise 13:
We have

$$\begin{aligned}
\varphi_n(s) &:= \mathbb{E}\left(\mathbb{E}\left(s^{N_n} \mid N_{n-1}\right)\right) = \mathbb{E}\left(\prod_{1\leq i\leq N_{n-1}} \mathbb{E}\left(\ s^{X_n^i} \mid N_{n-1}\right)\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(s^X\right)^{N_{n-1}}\right) = \mathbb{E}\left(\varphi_1(s)^{N_{n-1}}\right) = \varphi_{n-1}\left(\varphi_1(s)\right).
\end{aligned}$$

Recalling that $0^0 = 1$, this implies that

$$\varphi_n(0) = \mathbb{E}(0^{N_n}) = 1 \times \mathbb{P}(N_n = 0) = \mathbb{P}(N_n = 0).$$

For the Bernoulli offspring distribution $p(x) = p^x q^{1-x}$, with $q := (1-p), p \in [0,1]$ and $x \in \{0,1\}$, we have

$$\begin{array}{rcl} \varphi_1(s) & = & q + ps \\ \varphi_2(s) & = & \varphi_1\left(q+ps\right) = q + p(q+ps) = q(1+p) + p^2 s \\ \varphi_3(s) & = & \varphi_2\left(q+ps\right) = q(1+p) + p^2(q+ps) = q(1+p+p^2) + p^3 s \\ \ldots & = & \ldots \\ \varphi_n(s) & = & q(1+p+p^2+\ldots+p^{n-1}) + p^n s = \dfrac{q}{1-p}\,(1-p^n) + p^n s = (1-p^n) + p^n s. \end{array}$$

The last assertion follows from the fact that $\varphi_n(s) = (1-p^n)+p^n s$ is the moment generating function of a Bernoulli random variable N_n with parameter p^n; that is, we have that

$$\mathbb{P}(N_n = 1) = p^n \quad \text{and} \quad \mathbb{P}(N_n = 0) = 1 - p^n.$$

This ends the proof of the exercise. ■

Solution to exercise 14:
We set $g_0^i = \left(g_0^i\,(j)\right)_{j\in S}$. In this notation, we have

$$\mathbb{E}\left(\sum_{1\leq i\leq N_1} f(\widehat{\xi}_0^i) \mid g_0^i,\ \xi_0\right) = \sum_{1\leq i\leq N_0} g_0^i\left(\xi_0^i\right) f(\xi_0^i).$$

This implies that

$$\mathbb{E}\left(\sum_{1\leq i\leq N_1} f(\widehat{\xi}_0^i) \mid \xi_0\right) = \sum_{1\leq i\leq N_0} G\left(\xi_0^i\right)\ f(\xi_0^i)$$

and therefore

$$\mathbb{E}\left(\sum_{1\leq i\leq N_1} f(\widehat{\xi}_0^i)\right) = \sum_{1\leq i\leq N_0} \eta_0(Gf) = N_0\eta_0(Gf). \tag{30.18}$$

In the same vein, we have

$$\mathbb{E}\left(\sum_{1\leq i\leq N_1} f(\xi_1^i) \mid N_1,\ \widehat{\xi}_0\right) = \sum_{1\leq i\leq N_1} \mathbb{E}\left(f(\xi_1^i) \mid \widehat{\xi}_0^i\right) = \sum_{1\leq i\leq N_1} M(f)(\widehat{\xi}_0^i).$$

Using (30.18), we readily deduce that

$$\mathbb{E}\left(\sum_{1\leq i\leq N_1} f(\xi_1^i)\right) = \mathbb{E}\left(\sum_{1\leq i\leq N_1} M(f)(\widehat{\xi}_0^i)\right) = N_0\ \eta_0(GM(f)). \tag{30.19}$$

In much the same way, if we set $g_1^i = \left(g_1^i\,(j)\right)_{j\in S}$ then we have

$$\mathbb{E}\left(\sum_{1\leq i\leq N_2} f(\widehat{\xi}_1^i) \mid g_1^i,\ \xi_1\right) = \sum_{1\leq i\leq N_1} g_1^i\left(\xi_1^i\right) f(\xi_1^i).$$

This implies that

$$\mathbb{E}\left(\sum_{1\leq i\leq N_2} f(\widehat{\xi}_1^i) \mid \xi_1\right) = \sum_{1\leq i\leq N_1} G\left(\xi_1^i\right)\ f(\xi_1^i).$$

Using (30.18), we readily deduce that

$$\mathbb{E}\left(\sum_{1\leq i\leq N_2} f(\widehat{\xi}_1^i)\right) = \mathbb{E}\left(\sum_{1\leq i\leq N_1} G\left(\xi_1^i\right)\ f(\xi_1^i)\right) = N_0\ \eta_0(Q(Gf)). \tag{30.20}$$

Arguing as above we have

$$\mathbb{E}\left(\sum_{1\leq i\leq N_2} f(\xi_2^i) \mid N_2,\ \widehat{\xi}_1\right) = \sum_{1\leq i\leq N_2} \mathbb{E}\left(f(\xi_2^i) \mid \widehat{\xi}_1^i\right) = \sum_{1\leq i\leq N_2} M(f)(\widehat{\xi}_1^i).$$

Using (30.20) we deduce that

$$\mathbb{E}\left(\sum_{1\leq i\leq N_2} f(\xi_2^i)\right) = \mathbb{E}\left(\sum_{1\leq i\leq N_2} M(f)(\widehat{\xi}_1^i)\right) = N_0\ \eta_0(Q(GM(f))) = N_0\ \eta_0(Q^2(f)). \tag{30.21}$$

The last assertion is proved using induction. This ends the proof of the exercise. ■

Solution to exercise 15:
By construction, we have

$$\mathbb{P}\left(X_{n+1} = i \mid X_1, \ldots, X_n\right) = \frac{n}{n+\alpha}\ \frac{1}{n} \sum_{1\leq p\leq n} 1_{X_p}(i) + \frac{\alpha}{n+\alpha}\ \mu(i). \tag{30.22}$$

The number of different tables occupied by the first n customers is defined by

$$T_n := \sum_{1\leq p\leq n} \epsilon_p$$

where ϵ_n stands for a sequence of independent Bernoulli random variables with distribution

$$\mathbb{P}\left(\epsilon_n = 1\right) = 1 - \mathbb{P}\left(\epsilon_n = 0\right) = \frac{\alpha}{\alpha + (n-1)}.$$

This implies that

$$\sum_{1\leq p<n} \int_p^{p+1} \frac{dt}{1+(t/\alpha)} \leq \mathbb{E}\left(T_n\right) = \sum_{0\leq p<n} \frac{\alpha}{\alpha+p} \leq \sum_{1\leq p<n} \int_{p-1}^{p} \frac{dy}{1+(t/\alpha)}.$$

We conclude that

$$\int_1^n \frac{dt}{1+(t/\alpha)} = \alpha\ \log\left(\frac{\alpha+n}{\alpha+1}\right) \leq \mathbb{E}\left(T_n\right) \leq \int_0^{n-1} \frac{dt}{1+(t/\alpha)} = \alpha\ \log\left(1+(n-1)/\alpha\right).$$

The formula

$$\mathbb{P}\left(X_{n+1} = i \mid X_1, \ldots, X_n\right) = \frac{\alpha\mu(i) + V_n(i)}{\alpha+n} \quad \text{with} \quad V_n(i) = \sum_{1\leq p\leq n} 1_{X_p}(i)$$

is a direct consequence of (30.22).

This ends the proof of the exercise. ∎

Solution to exercise 16:

For each $s \in S$ and $x = (x_1, \ldots, x_{n+1})$ we let $t_k(s,x) \in \{1, \ldots, n+1\}$, with $k = 1, \ldots, v_{n+1}(s)$ be the times at which $x_{t_k(s,x)} = s$. In this notation, we have

$$\begin{aligned}\mathbb{P}\left(X_1 = x_1, \ldots, X_{n+1} = x_{n+1}\right) &= \prod_{s \in S} \prod_{0 \le k < v_{n+1}(s)} \frac{\alpha\mu(s) + k}{\alpha + (t_k(s,x) - 1)} \\ &= \left[\prod_{0 \le t \le n} \frac{1}{\alpha + t}\right] \prod_{s \in S} \prod_{0 \le k < v_{n+1}(s)} (\alpha\mu(s) + k).\end{aligned}$$

In the last assertion we have used the fact that $\mathcal{T}(s,x) := \{t_k(s,x)\ ,\ k = 1, \ldots, v_{n+1}(s)\}$, with $s \in S$ is a partition of the set $\{1, \ldots, n+1\}$

$$\cup_{s \in S} \mathcal{T}(s,x) = \{1, \ldots, n+1\}.$$

The formula (2.6) coincides with (4.9) when $a_s = \alpha\mu(s)$. Following the arguments described on page 79, we conclude that $(X_i)_{i \ge 1}$ can be interpreted as a sequence of independent random variables on the set $S := \{1, \ldots, d\}$ with probability distribution given by (2.7). By the law of large numbers, given U check that $\frac{1}{n}\sum_{1 \le p \le n} 1_{X_p}(i)$ converges almost surely to U_i, as $n \uparrow \infty$. In addition, we have

$$\mathbb{E}\left(\frac{1}{n} \sum_{1 \le p \le n} 1_{X_p}(i) \mid U\right) = U_i \quad \text{and} \quad \mathrm{Var}\left(\frac{1}{n} \sum_{1 \le p \le n} 1_{X_p}(i) \mid U\right) = \frac{1}{n}\ (U_i(1 - U_i)).$$

This ends the proof of the exercise. ∎

Solution to exercise 17:

The first assertion is immediate. In addition, we have that

$$\begin{aligned}S_n(f) &= \frac{n}{n+1}\left(\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) + \frac{1}{n}\ f(X_n)\right) \\ &= \frac{n}{n+1}\ S_{n-1}(f) + \frac{1}{n+1}\ f(X_n).\end{aligned}$$

By construction, we have

$$\mathbb{E}(f(X_{n+1}) \mid X_0, \ldots, X_n) = \epsilon\ S_n(f) + (1 - \epsilon)\ \mu(f).$$

In other words, this yields

$$\mathbb{E}([f(X_{n+1}) - \mu(f)] \mid X_0, \ldots, X_n) = \epsilon\ S_n([f - \mu(f)])\ .$$

Thus, for any function f such that $\mu(f) = 0$, we have

$$\mathbb{E}(f(X_{n+1}) \mid X_0, \ldots, X_n) = \epsilon\ S_n(f) \quad \Rightarrow \quad \eta_{n+1}(f) = \epsilon\ \overline{S}_n(f)\ .$$

Recalling that

$$\mathbb{E}(f(X_{n+1})) = \mathbb{E}(\mathbb{E}(f(X_{n+1}) \mid X_0, \ldots, X_n))$$

we prove that

$$\begin{aligned}
\overline{S}_n(f) &= \frac{n}{n+1}\,\overline{S}_{n-1}(f) + \frac{1}{n+1}\,\mathbb{E}(f(X_n)) \\
&= \frac{n+\epsilon}{n+1} \times \overline{S}_{n-1}(f) \\
&= \frac{n+\epsilon}{n+1} \times \frac{(n-1)+\epsilon}{(n-1)+1} \times \overline{S}_{n-2}(f) = \ldots = \left[\prod_{k=1}^{n} \frac{k+\epsilon}{k+1}\right] \times \mathbb{E}(f(X_0)).
\end{aligned}$$

We observe that

$$k \le t \le k+1 \Rightarrow \log\left(1 - \frac{(1-\epsilon)}{k}\right) \le \log\left(1 - \frac{(1-\epsilon)}{t}\right) \le \log\left(1 - \frac{(1-\epsilon)}{k+1}\right).$$

This implies that

$$\sum_{1\le k\le n} \int_k^{k+1} \log\left(1 - \frac{(1-\epsilon)}{t}\right)\,dt \le \log \alpha_\epsilon(n)$$

and

$$\log \alpha_\epsilon(n) \le \sum_{1\le k\le n} \int_{k+1}^{k+2} \log\left(1 - \frac{(1-\epsilon)}{t}\right)\,dt.$$

This ends the proof of (2.8). Using the estimates

$$\forall x \in [0,1[\qquad -\frac{x}{1-x} \le \log(1-x) \le -x$$

we check that

$$\int_2^{n+2} \log\left(1 - \frac{(1-\epsilon)}{t}\right)\,dt \le -(1-\epsilon)\ \log(1+n/2)$$

and

$$\int_1^{n+1} \log\left(1 - \frac{(1-\epsilon)}{t}\right)\,dt \ge -(1-\epsilon)\ \log(1+n/\epsilon).$$

The end of the proof of the exercise is immediate.

■

Solution to exercise 18:
By construction, we have

$$\begin{aligned}
M(f)(i) = \epsilon\ K(f)(i) + (1-\epsilon)\ \nu(f) \quad &\Rightarrow \quad [M(f)(i) - M(f)(j)] = \epsilon\ [K(f)(i) - K(f)(j)] \\
&\Rightarrow \quad \operatorname{osc}(M(f)) \le \epsilon \operatorname{osc}(f).
\end{aligned}$$

Assuming that $\operatorname{osc}(M^n(f)) \le \epsilon^n \operatorname{osc}(f)$ is true at rank n, we have

$$\operatorname{osc}(M^{n+1}(f)) = \operatorname{osc}(M^n(M(f))) \le \epsilon^n \operatorname{osc}(M(f)) \le \epsilon^{n+1} \operatorname{osc}(f).$$

Recall that

$$f = 1_k \Rightarrow M^n(f)(i) = M^n(i,k) = \mathbb{P}(X_n = k | X_0 = i).$$

The end of the proof of the exercise is now clear. ■

Solution to exercise 19: The first assertion is immediate since $\frac{d\overline{W}_t}{dt} = W_t \frac{dW_t}{dt}$. To check the second one, we observe that

$$X_n = a_n X_{n-1} + b_n = \left[\prod_{p=1}^{n} a_p\right] X_0 + \sum_{1\leq p\leq n} \left[\prod_{n\geq q>p} a_q\right] b_p$$

with the sequence of random variables

$$a_n = (1-\epsilon_n) + \epsilon_n \, 4^{-1} = 4^{-\epsilon_n} \quad \text{and} \quad b_n = (1-\epsilon_n)\, h$$

Using the fact that

$$\begin{array}{l} \mathrm{Law}\left((a_1,\ldots,a_{p+1},\ldots,a_n),(b_1,\ldots,b_p,\ldots,b_n)\right) \\ = \\ \mathrm{Law}\left((a_n,\ldots,a_{n-p},\ldots,a_1),(b_n,\ldots,b_{n-p+1},\ldots,b_1)\right). \end{array}$$

we check that

$$\sum_{1\leq p\leq n} \left[\prod_{n\geq q>p} a_q\right] b_p \stackrel{\text{law}}{=} \sum_{1\leq p\leq n} [a_1\ldots a_{n-p}]\, b_{(n-p)+1} = \sum_{0\leq p<n} [a_1\ldots a_p]\, b_{p+1}$$

The end of the proof of the exercise is now clear. ■

Solution to exercise 20:
We have

$$\begin{aligned} &\mathbb{P}(X_T = x_{\max}|X_0 = x) \\ &= \mathbb{E}\left(\mathbb{P}(X_T = x_{\max} \mid X_1)|X_0 = x\right) \\ &= p\ \underbrace{\mathbb{P}(X_T = x_{\max} \mid X_1 = x+1)}_{:=P(x+1)} + (1-p)\ \underbrace{\mathbb{P}(X_T = x_{\max} \mid X_1 = x-1)}_{:=P(x-1)}. \end{aligned}$$

On the other hand

$$\begin{aligned} &P(x) = pP(x) + qP(x) = p\ P(x+1) + q\ P(x-1) \\ &\Rightarrow p\,[P(x+1) - P(x)] = q\,[P(x) - P(x-1)] \\ &\Rightarrow [P(x+1) - P(x)] = \tfrac{p}{q}\,[P(x) - P(x-1)]. \end{aligned}$$

Recalling that $P(0) = 0$, this yields

$$[P(2) - P(1)] = \frac{p}{q}\ P(1)\ \Rightarrow [P(3) - P(2)] = \frac{p}{q}\,[P(2) - P(1)] = \left(\frac{p}{q}\right)^2 P(1).$$

By a simple induction w.r.t. x we find that

$$\begin{aligned} &[P(x) - P(x-1)] = \left(\frac{p}{q}\right)^{x-1} P(1) \\ &\Rightarrow [P(x+1) - P(x)] = \frac{p}{q}\,[P(x) - P(x-1)] = \left(\frac{p}{q}\right)^{x} P(1). \end{aligned}$$

On the other hand, we have

$$P(x+1) = [P(x+1) - P(0)] = \sum_{0 \le y \le x} [P(y+1) - P(y)] = P(1) \sum_{0 \le y \le x} \left(\frac{p}{q}\right)^y.$$

We end the proof using the fact that

$$x = x_{\max} - 1 \Rightarrow P(x+1) = P(x_{\max}) = 1 = P(1) \sum_{0 \le y < x_{\max}} \left(\frac{p}{q}\right)^y$$

so that

$$P(1) = 1 / \sum_{0 \le y < x_{\max}} \left(\frac{p}{q}\right)^y.$$

This implies that

$$P(x+1) = \frac{\sum_{0 \le y \le x} \left(\frac{p}{q}\right)^y}{\sum_{0 \le y < x_{\max}} \left(\frac{p}{q}\right)^y} = \begin{cases} \dfrac{1 - \left(\frac{q}{p}\right)^{x+1}}{1 - \left(\frac{q}{p}\right)^{x_{\max}}} & \text{if} \quad p \neq q \\ \dfrac{(x+1)}{x_{\max}} & \text{if} \quad p = q. \end{cases}$$

This ends the proof of the exercise. ■

Chapter 3

Solution to exercise 21:

We have

$$\begin{aligned} x^2+\frac{1}{\epsilon}\ (y-\sqrt{(1-\epsilon)}x)^2 &= x^2+\frac{1}{\epsilon}\ \left(y^2+(1-\epsilon)x^2-2xy\sqrt{(1-\epsilon)}\right) \\ &= \frac{1}{\epsilon}\ (x^2+y^2)-2xy\ \frac{\sqrt{(1-\epsilon)}}{\epsilon}. \end{aligned}$$

By the symmetry property of the last formula w.r.t. the pair (x,y) we readily check that

$$\lambda(x)P(x,y)=\lambda(y)P(y,x).$$

Notice that

$$\frac{\pi(y)P(y,x)}{\pi(x)P(x,y)}=\frac{e^{-\beta V(y)}}{e^{-\beta V(x)}}\times\frac{\lambda(y)P(y,x)}{\lambda(x)P(x,y)}=\ e^{-\beta(V(y)-V(x))}.$$

Thus acceptance ratio of the corresponding Metropolis-Hastings algorithm is given by

$$a(x,y)=\min\left(1,\frac{\pi(y)P(y,x)}{\pi(x)P(x,y)}\right)=\min\left(1,e^{-\beta(V(y)-V(x))}\right).$$

When

$$\lambda(x)=1\quad\text{and}\quad P(x,y)=\frac{1}{\sqrt{2\pi}}\ \exp\left[-\frac{1}{2}\ (y-x)^2\right]=P(y,x)$$

we also have

$$\lambda(x)P(x,y)=\lambda(y)P(y,x)\Rightarrow a(x,y)=\min\left(1,e^{-\beta(V(y)-V(x))}\right).$$

This ends the proof of the exercise. ■

Solution to exercise 22:

The transition probabilities $M(x,y)$ of the Metropolis-Hastings with proposal transition P and target distribution π are given got any $x\in\mathbb{N}$ by

$$\begin{aligned} M(x,x+1) &= P(x,x+1)\ \min\left(1,\frac{\pi(x+1)P(x+1,x)}{\pi(x)P(x,x+1)}\right) \\ &= \frac{1}{2}\ \min\left(1,\frac{e^{-\lambda}\ \frac{\lambda^{x+1}}{(x+1)!}}{e^{-\lambda}\ \frac{\lambda^x}{x!}}\right)=\frac{1}{2}\ \ \min\left(1,\frac{\lambda}{x+1}\right) \end{aligned}$$

and for any $x\geq 1$

$$\begin{aligned} M(x,x-1) &= P(x,x-1)\ \min\left(1,\frac{\pi(x-1)P(x-1,x)}{\pi(x)P(x,x-1)}\right) \\ &= \frac{1}{2}\ \min\left(1,\frac{e^{-\lambda}\ \frac{\lambda^{x-1}}{(x-1)!}}{e^{-\lambda}\ \frac{\lambda^x}{x!}}\right)=\frac{1}{2}\ \ \min\left(1,\frac{x}{\lambda}\right). \end{aligned}$$

Notice that

$$x<\lambda<y \Rightarrow \begin{array}{rcl} M(x,x+1)=\frac{1}{2} & \geq & M(x,x-1)=\frac{1}{2}\,\frac{x}{\lambda} \\ M(y,y+1)=\frac{1}{2}\frac{\lambda}{y+1} & \leq & M(y,y-1)=\frac{1}{2}. \end{array}$$

This ends the proof of the exercise. ■

Solution to exercise 23:

For $1\leq i=j\leq d'$ we clearly have $\pi'(i)M'(i,j)=\pi'(j)M'(j,i)$. For $0\leq i\neq j\leq d'$, recalling that $\pi'(i)\propto\pi(i)$ we have

$$\pi(i)M'(i,j)=\pi(i)M(i,j)=\pi(j)M(j,i)=\pi(j)M'(j,i)\Rightarrow\pi'(i)M'(i,j)=\pi'(j)M'(j,i).$$

The last case is obvious.

This ends the proof of the exercise. ■

Solution to exercise 24:

We clearly have

$$\sum_{x\in S} 1\ M(x,y)=1 \Longleftrightarrow \sum_{x\in S}\frac{1}{\mathrm{card}(S)}\ M(x,y)=\frac{1}{\mathrm{card}(S)}\Longleftrightarrow \pi M=\pi.$$

This ends the proof of the exercise. ■

Solution to exercise 25:

Using the regression formula (3.4), we have

$$\widehat{m}_n=m_n+\frac{\sigma_n^2}{\sigma_n^2+\tau^2}\ (y_n-m_n) \quad \text{and} \quad \widehat{\sigma}_n^{-2}=\tau^{-2}+\sigma_n^{-2}.$$

Using the prediction rule (3.5) we have

$$m_{n+1}=\widehat{m}_n \quad \text{and} \quad \sigma_{n+1}^2=\widehat{\sigma}_n^2+\sigma^2.$$

This implies

$$m_{n+1}=\frac{\tau^2}{\tau^2+\sigma_n^2}\ m_n+\frac{\sigma_n^2}{\tau^2+\sigma_n^2}\ y_n$$

from which we conclude that

$$\left[m_{n+1}-m'_{n+1}\right]=\frac{\tau^2}{\tau^2+\sigma_n^2}\ [m_n-m'_n]=\left\{\prod_{0\leq k\leq n}\frac{\tau^2}{\tau^2+\sigma_n^2}\right\}\ [m_0-m'_0]\,.$$

We end the proof using the fact that $\sigma_{n+1}^2=\widehat{\sigma}_n^2+\sigma^2\geq\sigma^2$. This ends the proof of the exercise. ■

Solution to exercise 26: Notice that the density of the observation variable Y_n given $X_n=x_n$ is given by the Gaussian density

$$p(y_n|x_n)=\frac{1}{\sqrt{2\pi\tau}}\ \exp\left(-\frac{1}{2\tau^2}(y_n-x_n)^2\right).$$

Initially, we start by sampling N i.i.d. random copies $\xi_0 := (\xi_0^i)_{1\leq i\leq N}$ of the signal X_0. Given the observation $Y_0 = y_0$ we sample N random variables $\widehat{\xi}_0 := (\widehat{\xi}_0^i)_{1\leq i\leq N}$ with the discrete distribution

$$\sum_{1\leq i\leq N} \frac{e^{-\frac{1}{2\tau^2}(y_0-\xi_0^i)^2}}{\sum_{1\leq j\leq N} e^{-\frac{1}{2\tau^2}(y_0-\xi_0^j)^2}}\ \delta_{\xi_0^i}.$$

In other words, each random variable $\widehat{\xi}_0^k$ is sampled according to the probability measure

$$\forall i\in\{1,\ldots,N\}\qquad \mathbb{P}(\widehat{\xi}_0^k = \xi_0^i \mid \xi_0) = \frac{e^{-\frac{1}{2\tau^2}(y_0-\xi_0^i)^2}}{\sum_{1\leq j\leq N} e^{-\frac{1}{2\tau^2}(y_0-\xi_0^j)^2}}.$$

During the prediction transition $\widehat{\xi}_0 \rightsquigarrow \xi_1 = (\xi_1^i)_{1\leq i\leq N}$, we sample N i.i.d. copies $(W_1^i)_{1\leq i\leq N}$ of W_1 and we set

$$\forall i\in\{1,\ldots,N\}\qquad \xi_1^i = \widehat{\xi}_0^i + W_1^i.$$

Given the observation $Y_1 = y_1$ we sample N random variables $\widehat{\xi}_1 := (\widehat{\xi}_1^i)_{1\leq i\leq N}$ with the discrete distribution

$$\sum_{1\leq i\leq N} \frac{e^{-\frac{1}{2\tau^2}(y_1-\xi_1^i)^2}}{\sum_{1\leq j\leq N} e^{-\frac{1}{2\tau^2}(y_1-\xi_1^j)^2}}\ \delta_{\xi_1^i}.$$

During the prediction transition $\widehat{\xi}_1 \rightsquigarrow \xi_2 = (\xi_2^i)_{1\leq i\leq N}$, we sample N i.i.d. copies $(W_2^i)_{1\leq i\leq N}$ of W_2 and we set

$$\forall i\in\{1,\ldots,N\}\qquad \xi_2^i = \widehat{\xi}_1^i + W_2^i$$

and so on.

This ends the proof of the exercise. ■

Solution to exercise 27: The solution is discussed in full details on page 55. ■

Solution to exercise 28:

Applying Doeblin-Itō differential formula to the function $f(x) = x^2$ ($\Rightarrow f'(x) = 2x$ and $f''(x) = 2$) we find that

$$df(W_t) = f'(W_t)\ dW_t\ + \frac{1}{2}\ f''(W_t)dt = dW_t^2 = 2\ W_t\ dW_t + dt.$$

This implies that

$$W_t^2 - W_0^2 = \int_0^t dW_s^2 = 2\int_0^t W_s\ dW_s + t \Rightarrow \mathbb{E}(W_t^2) = t.$$

This ends the proof of the exercise. ■

Solution to exercise 29: Applying Doeblin-Itō differential formula to the function $f(x) = x^4$ ($\Rightarrow f'(x) = 4x^3$ and $f''(x) = 12x^2$) we find that

$$df(W_t) = f'(W_t)\ dW_t\ + \frac{1}{2}\ f''(W_t)dt = dW_t^4 = 4\ W_t^3\ dW_t + 6\ W_t^2 dt.$$

This yields

$$W_t^4 = \int_0^t dW_s^4 = 4\int_0^t W_s^3\ dW_s + 6\int_0^t W_s^2\ ds \Rightarrow \mathbb{E}(W_t^4) = 6\int_0^t \underbrace{\mathbb{E}(W_s^2)}_{=s}\ ds = 3t^2.$$

This ends the proof of the exercise. ■

Solution to exercise 30:
Using (3.17), for any $\alpha \in \mathbb{R}$ we have

$$\mathbb{E}\left(X_t^\alpha\right) = \mathbb{E}\left(X_0^\alpha\right) \; \left[\exp\left(\int_0^t \alpha \; \left(b_s - \frac{1}{2}\,\sigma_s^2\right)\; ds\right)\right] \times \mathbb{E}\left(\exp\left(\alpha \; \int_0^t \sigma_s\; dW_s\right)\right).$$

Recalling that $Y_t := \int_0^t \sigma_s \; dW_s$ is a centered Gaussian random variable with variance

$$\mathrm{Var}(Y_t) = \mathbb{E}(Y_t^2) = \int_0^t \sigma_s^2 \; ds$$

we conclude that

$$\begin{aligned}
\mathbb{E}\left(X_t^\alpha\right) &= \mathbb{E}\left(X_0^\alpha\right)\; \exp\left(\alpha \; \int_0^t \; \left(b_s - \frac{1}{2}\,\sigma_s^2\right)\; ds\right) \times \exp\left(\frac{\alpha^2}{2} \; \int_0^t \sigma_s^2 \; ds\right) \\
&= \mathbb{E}\left(X_0^\alpha\right)\; \exp\left(\alpha \; \int_0^t \; b_s\; ds + \frac{\alpha(\alpha-1)}{2} \; \int_0^t \sigma_s^2\; ds\right).
\end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 31:
We set $V_t := \int_0^t \sigma_s(X_s)\; dW_s \Rightarrow dV_t = \sigma_t(X_t)\; dW_t$. Applying the Doeblin-Itō differential formula to the function $f(x) = f'(x) = f''(x) = e^x$ we have

$$dY_t = df(V_t) = f'(V_t)\; dV_t + \frac{1}{2}\; f''(V_t)\; \sigma_t^2(X_t)\; dt = Y_t \; \left(\sigma_t(X_t)\; dW_t + \frac{1}{2}\; \sigma_t^2(X_t)\; dt\right).$$

In the same vein, if we set

$$U_t := \int_0^t \sigma_s(X_s)\; dW_s - \frac{1}{2}\int_0^t \; \sigma_s^2(X_s)\; ds \Rightarrow dU_t = \sigma_t(X_t)\; dW_t - \frac{1}{2}\; \sigma_t^2(X_t)\; dt$$

we have

$$\begin{aligned}
dZ_t &= df(U_t) = f'(U_t)\; dU_t + \frac{1}{2}\; f''(U_t)\; \sigma_t^2(X_t)\; dt \\
&= Z_t\left(\sigma_t(X_t)\; dW_t - \frac{1}{2}\; \sigma_t^2(X_t)\; dt + \frac{1}{2}\; \sigma_t^2(X_t)\; dt\right) = Z_t\; \sigma_t(X_t)\; dW_t.
\end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 32:
Applying Doeblin-Itō differential formula to the function $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ and $f''(x) = n(n-1)x^{n-2}$

$$\begin{aligned}
dX_t^n &= nX_t^{n-1}\; \left[(a_t + b_t\; X_t)\; dt + \; (\tau_t + \sigma_t\; X_t)\; dW_t\right] + \frac{n(n-1)}{2}\; X_t^{n-2}\; \left(\tau_t + \sigma_t\; X_t\right)^2\; dt \\
&= nX_t^n\; \left[\left(b_t + \frac{(n-1)}{2}\; \sigma_t^2\right)\; dt \; + \sigma_t\; dW_t\right] \\
&\qquad + n\; X_t^{n-1}\left[\{a_t + (n-1)\tau_t\sigma_t\}\; dt + \tau_t\; dW_t\right] + \frac{n(n-1)}{2}\; X_t^{n-2}\; \tau_t^2 dt.
\end{aligned}$$

This implies that

$$\begin{aligned} dm_t^n &= n\ m_t^n\ \left(b_t + \frac{(n-1)}{2}\ \sigma_t^2\right)\ dt \\ &\qquad + n\ m_t^{n-1}\left(a_t + (n-1)\tau_t\sigma_t\right)\ dt + \frac{n(n-1)}{2}\ m_t^{n-2}\ \tau_t^2 dt. \end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 33:
Using the Doeblin-Itō formula (3.10), we have

$$df(X_t) = L(f)(X_t)dt + \frac{\partial f}{\partial x}(t, X_t)\ \sigma_t(X_t)\ dW_t.$$

This implies that

$$\begin{aligned} &d\mathbb{E}(f(X_t)) = \int f(x)\ (p_{t+dt}(x) - p_t(x))\ dx = \mathbb{E}(L(f)(X_t))\ dt \\ &\Leftrightarrow \int\ f(x)\ \tfrac{\partial p_t}{\partial t}\ dx = \int\ L(f)(x)\ p_t(x)\ dx. \end{aligned}$$

On the other hand, for any smooth function with compact support, using an integration by parts we have

$$\begin{aligned} \int\ L(f)(x)\ p_t(x)\ dx &= \int\ \left[b_t\ \frac{\partial f}{\partial x}(x)\ + \frac{1}{2}\ \sigma_t^2\ \frac{\partial^2 f}{\partial x^2}(x)\right]\ p_t(x)\ dx \\ &= -\int\ f(x)\ \frac{\partial}{\partial x}(b_t p_t)(x)\ dx + \frac{1}{2}\int\ f(x)\ \frac{\partial^2}{\partial x^2}(\sigma_t p_t)(x) \end{aligned}$$

This result being true for any function f implies that we must have

$$\frac{\partial p_t}{\partial t} = -\frac{\partial}{\partial x}\left(b_t\ p_t\right) + \frac{1}{2}\ \frac{\partial^2}{\partial x^2}\left(\sigma_t^2\ p_t\right).$$

This ends the proof of the exercise. ■

Solution to exercise 34: We have $Y = e^{\mu+\sigma Z}$ with Z being standard normal. We also note that the probabilities of the events $\{Y > K\}$ and $\{Z > \frac{\log(K)-\mu}{\sigma}\}$ coincide. Hence

$$\begin{aligned} \mathbb{E}\left[(Y-K)^+\right] &= \int_{\frac{\log(K)-\mu}{\sigma}}^{\infty} \left(e^{\mu+\sigma t} - K\right)\frac{e^{-t^2/2}}{\sqrt{2\pi}}\ dt \\ &= e^{\mu+\sigma^2/2}\int_{\frac{\log(K)-\mu-\sigma^2}{\sigma}}^{\infty}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx - K\ G\left(\frac{\mu-\log(K)}{\sigma}\right) \\ &= e^{\mu+\sigma^2/2}\ G\left(\frac{\mu-\log(K)+\sigma^2}{\sigma}\right) - K\ G\left(\frac{\mu-\log(K)}{\sigma}\right). \end{aligned}$$

■

Chapter 4

Solution to exercise 35: For any $n\mathbb{N} - \{0\}$, we have

$$\begin{aligned}\mathbb{P}\left(\lfloor X \rfloor = n-1\right) &= \mathbb{P}\left(n-1 \leq X < n\right) = \int_{n-1}^{n} \lambda\; e^{-\lambda t}\; dt \\ &= \left[e^{-\lambda t}\right]_{n}^{n-1} = \left(e^{-\lambda}\right)^{n-1}\; \left(1-e^{-\lambda}\right) = (1-p)^{n-1}\; p.\end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 44: For any bounded function f on $[0,\infty[$, we have

$$\begin{aligned}\mathbb{E}\left(f(X+Y)\right) &= \frac{c^{a+b}}{\Gamma(a)\Gamma(b)} \int_0^\infty x^{a-1}\; \left[\int_0^\infty f(x+y)\quad y^{b-1}\; e^{-c(x+y)} dy\right] dx \\ &= \frac{c^{a+b}}{\Gamma(a)\Gamma(b)} \int_0^\infty x^{a-1}\; \left[\int_x^\infty f(z)\;\; (z-x)^{b-1}\; e^{-cz} dz\right] dx \\ &= \frac{c^{a+b}}{\Gamma(a)\Gamma(b)} \int_{0\leq x\leq z} f(z)\; x^{a-1}\; (z-x)^{b-1}\; e^{-cz} dx dz.\end{aligned}$$

This yields

$$\mathbb{E}\left(f(X+Y)\right) = \frac{c^{a+b}}{\Gamma(a)\Gamma(b)} \int_0^\infty f(z)\; e^{-cz}\; \left[\int_0^z x^{a-1}\; (z-x)^{b-1}\; dx\right] dz.$$

To take the final step, we observe that

$$\begin{aligned}\int_0^z x^{a-1}\; (z-x)^{b-1}\; dx &= z^{a+b-1} \int_0^z \left(\frac{x}{z}\right)^{a-1}\; \left(1-\frac{x}{z}\right)^{b-1}\; \frac{dx}{z} \\ &= z^{(a+b)-1} \times B(a,b)\end{aligned}$$

with

$$B(a,b) = \int_0^1 u^{a-1}\; (1-u)^{b-1}\; du.$$

This implies that

$$\mathbb{E}\left(f(X+Y)\right) = \left[B(a,b)\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right] \int_0^\infty \frac{c^{a+b}}{\Gamma(a+b)}\; f(z)\; z^{(a+b)-1}\; e^{-cz}\; dz$$

from which we conclude that

$$X+Y \sim \mathrm{Gamma}(a+b,c) \quad \text{and} \quad B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

This ends the proof of the exercise. ■

Solution to exercise 45: For any bounded function f on $[0,\infty[$, we have

$$\begin{aligned}&\mathbb{E}\left(f\left(\tfrac{X}{X+Y}\right)\right)\\&=\frac{c^{a+b}}{\Gamma(a)\Gamma(b)}\int_0^\infty x^{a-1}\ \left[\int_0^\infty f\left(\frac{x}{x+y}\right)\quad y^{b-1}\ e^{-c(x+y)}\ dy\right]dx.\end{aligned}$$

For each x, we use the change of variables

$$z=\frac{x}{x+y}\,(\in[0,1])\Rightarrow y=-x+\frac{x}{z}\quad dy=\frac{x}{z^2}\ dz$$

to check that

$$\begin{aligned}&\mathbb{E}\left(f\left(\tfrac{X}{X+Y}\right)\right)\\&=\frac{c^{a+b}}{\Gamma(a)\Gamma(b)}\int_0^\infty x^{a-1}\left[\int_0^1 f(z)\quad\left(-x+\frac{x}{z}\right)^{b-1}\ e^{-cx/z}\,\frac{x}{z^2}\ dz\right]dx\\&=\frac{c^{a+b}}{\Gamma(a)\Gamma(b)}\int_0^1 f(z)\quad(1-z)^{b-1}\ z^{-b-1}\ \left(\frac{z}{c}\right)^{a+b}\ \Gamma(a+b)\\&\qquad\qquad\times\underbrace{\left[\int_0^\infty\frac{\left(\frac{c}{z}\right)^{a+b}}{\Gamma(a+b)}\ x^{(a+b)-1}\ e^{-\frac{c}{z}\ x}\ dx\right]}_{=1}dz.\end{aligned}$$

This yields

$$\mathbb{E}\left(f\left(\tfrac{X}{X+Y}\right)\right)=\int_0^1 f(z)\ \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\ z^{a-1}\ (1-z)^{b-1}\ dz.$$

This ends the proof of the exercise. ∎

Solution to exercise 46: Using exercise 44, we have

$$Z:=\sum_{1\leq i\leq d}X_i\sim\Gamma(\sum_{1\leq i\leq d}a_i,c).$$

Our objective is to prove that for any Dirichlet variable

$$(Y_1,\ldots,Y_d)\sim D(a_1,\ldots,a_d)$$

independent of Z, the collection $(Z\ Y_i)_{1\leq i\leq d}$ forms a sequence of independent r.v. with distribution $(\Gamma(a_i,c))_{1\leq i\leq d}$. In this situation, recalling that $\sum_{1\leq i\leq d}Y_i=1$ we have

$$\forall 1\leq i\leq d\qquad X_i=Z\ Y_i$$

$$\Longrightarrow\left(\frac{X_1}{\sum_{1\leq i\leq d}X_i},\ldots,\frac{X_d}{\sum_{1\leq i\leq d}X_i}\right)=(Y_1,\ldots,Y_d)\sim D(a_1,\ldots,a_d).$$

For any bounded function f on $\mathbb{R}^d$, we have

$$\begin{aligned}
&\mathbb{E}\left(f\left(ZY_1,\ldots,ZY_d\right)\right)\\
&=\int_{\Delta_{d-1}}\int_0^\infty f\left(zy_1,\ldots,zy_d\right)\ \frac{c^{\sum_{1\le i\le d}a_i}}{\Gamma\left(\sum_{1\le i\le d}a_i\right)}\ z^{\sum_{1\le i\le d}a_i-1}\ e^{-cz}\ dz\\
&\qquad\times\frac{\Gamma\left(\sum_{1\le i\le d}a_i\right)}{\prod_{1\le i\le d}\Gamma(a_i)}\left[\prod_{1\le i\le d}y_i^{a_i-1}\right]\ dy_1\ldots dy_{d-1}\\
&=\int_{\Delta_{d-1}}\int_0^\infty f\left(zy_1,\ldots,zy_d\right)\ \frac{c^{\sum_{1\le i\le d}a_i}}{\prod_{1\le i\le d}\Gamma(a_i)}\ z^{d-1}\ e^{-cz}\ dz\\
&\qquad\times\left[\prod_{1\le i\le d}(zy_i)^{a_i-1}\right]\ dy_1\ldots dy_{d-1}.
\end{aligned}$$

For each z we use the change of variables

$$\forall 1\le i\le d\quad z_i=zy_i\quad\text{and we recall that}\quad z_d:=zy_d=z(1-y_1-\ldots-y_{d-1})$$

so that $\sum_{1\le i\le d}z_i=z$, $dy_1\ldots dy_{d-1}=z^{1-d}dz_1\ldots dz_{d-1}$, and finally $dz=dz_d$. This implies that

$$\begin{aligned}
&\mathbb{E}\left(f\left(ZY_1,\ldots,ZY_d\right)\right)\\
&=\int f\left(z_1,\ldots,z_d\right)\ \prod_{1\le i\le d}\left[\frac{c^{a_i}}{\Gamma(a_i)}\ z_i^{a_i-1}\ e^{-cz_i}\right]\ dz_1\ldots dz_d.
\end{aligned}$$

The last assertion is a direct consequence of the additive formula presented in exercise 44. This ends the proof of the exercise. ∎

Solution to exercise 47:

By symmetry, we can assume without loss of generality that $i=1$. Using the fact that $\Gamma(z+1)=z\Gamma(z)$, we prove that

$$\begin{aligned}
&\mathbb{E}\left(U_1\right)\\
&=\frac{\Gamma\left(\sum_{1\le i\le d}a_i\right)}{\prod_{1\le i\le d}\Gamma(a_i)}\ \int_{u_1+\ldots+u_{d-1}<1}u_1^{(a_1+1)-1}\left[\prod_{1<i\le d}u_i^{a_i-1}\right]\ du_1\ldots du_{d-1}\\
&=\frac{\Gamma(a_1+1)}{\Gamma(a_1)}\frac{\Gamma\left(\sum_{1\le i\le d}a_i\right)}{\Gamma\left(1+\sum_{1\le i\le d}a_i\right)}=\frac{a_1}{\sum_{1\le i\le d}a_i}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left(U_1^2\right)\\
&=\frac{\Gamma\left(\sum_{1\le i\le d}a_i\right)}{\prod_{1\le i\le d}\Gamma(a_i)}\ \int_{u_1+\ldots+u_{d-1}<1}u_1^{(a_1+2)-1}\left[\prod_{1<i\le d}u_i^{a_i-1}\right]\ du_1\ldots du_{d-1}\\
&=\frac{\Gamma(a_1+2)}{\Gamma(a_1)}\frac{\Gamma\left(\sum_{1\le i\le d}a_i\right)}{\Gamma\left(2+\sum_{1\le i\le d}a_i\right)}=\frac{1+a_1}{1+\sum_{1\le i\le d}a_i}\times\mathbb{E}\left(U_1\right).
\end{aligned}$$

In the above formulae, we have implicitly used the notation $u_d = 1 - \sum_{1\leq i<d} u_i$. This ends the proof of the exercise. ■

Solution to exercise 48:
For any observed sequence $(x_1, \ldots, x_n) \in \{0,1\}$, we have

$$\mathbb{P}\left((X_1 \ldots, X_n) = (x_1, \ldots, x_n) \mid \ \Theta = \theta\right) = \theta^{\sum_{1\leq i\leq n} x_i} \times (1-\theta)^{\sum_{1\leq i\leq n}(1-x_i)}.$$

Using the Bayes rule, this implies that

$$\begin{aligned}
&\mathbb{P}\left(\Theta \in d\theta \mid (X_1 \ldots, X_n) = (x_1, \ldots, x_n)\right) \\
&\propto \theta^{\left[a+\sum_{1\leq i\leq n} x_i\right]-1} \ (1-\theta)^{\left[b+\sum_{1\leq i\leq n}(1-x_i)\right]-1} \ 1_{[0,1]}(\theta)
\end{aligned}$$

from which we conclude that

$$\begin{aligned}
\mathbb{E}\left(\Theta \mid (X_1 \ldots, X_n)\right) &= \frac{a + \sum_{1\leq i\leq n} X_i}{a + b + \sum_{1\leq i\leq n} X_i + \sum_{1\leq i\leq n}(1-X_i)} \\
&= (1 + (a+b)/n)^{-1} \left(\frac{a}{n} + \frac{1}{n}\sum_{1\leq i\leq n} X_i\right) \\
&\longrightarrow_{n\uparrow\infty} \ \Theta.
\end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 49:
The conditional density of $(Y_1, \ldots, Y_n)$ given X is given by

$$p(y_1, \ldots, y_n \mid x) = p(y_1|x) \ldots p(y_n|x) \propto \prod_{1\leq i\leq n} e^{-\frac{1}{2\tau}(y_i - ax)^2}$$

and

$$\begin{aligned}
p(x \mid y_1, \ldots, y_n) &\propto p(y_1|x) \ldots p(y_n|x) \ p(x) \\
&\propto \prod_{1\leq i\leq n} \underbrace{\exp\left(-\frac{1}{2\tau^2}(y_i - ax)^2 - \frac{1}{2n\sigma^2}(x-m)^2\right)}_{=p_n(x \ \mid \ y_i)}
\end{aligned}$$

with the conditional density

$$p_n(x \mid y_i) \propto \exp\left(-\frac{1}{2\rho_n}\left((x-m) - \beta_n(y_i - am))\right)^2\right)$$

and the parameters

$$\beta_n = a \ \sigma^2 n/(a^2\sigma^2 n + \tau^2) \to a^{-1} \quad \text{and} \quad \rho_n = \left(a^2\tau^{-2} + \sigma^{-2}/n\right)^{-1} \to \tau^2/a^2.$$

We get

$$\begin{aligned}
p(x \mid y_1, \ldots, y_n) &\propto \exp\left(-\frac{1}{2\rho_n}\sum_{1\leq i\leq n}\left((x-m) - \beta_n(y_i - am)\right)^2\right) \\
&\propto \exp\left(-\frac{n}{2\rho_n}\left((x-m) - \beta_n \frac{1}{n}\sum_{1\leq i\leq n}(y_i - am)\right)^2\right).
\end{aligned}$$

This clearly implies that

$$\mathbb{E}\left(X \mid (Y_1,\ldots,Y_n)\right) = (1-a\beta_n)\ m + \beta_n\ \frac{1}{n}\sum_{1\leq i\leq n} Y_i \longrightarrow X.$$

This ends the proof of the exercise. ■

Solution to exercise 50:
We observe that

$$\forall y_k \in \{0,1\} \qquad \mathbb{P}(Y_k = y_k \mid X = x) = x^{y_k}\ (1-x)^{1-y_k} := p(y_k \mid x)$$

and therefore

$$\mathbb{P}(Y_k = y_k,\ 1\leq k\leq n \mid X = x) = \prod_{1\leq k\leq n} p(y_k \mid x) = x^{\sum_{1\leq k\leq n} y_k}\ (1-x)^{n-\sum_{1\leq k\leq n} y_k}.$$

Using Bayes' rule, this implies that

$$\begin{aligned}
&\mathbb{P}(X\in dx \mid Y_k = y_k,\ 1\leq k\leq n)\\
&= \frac{1}{\int_0^1 \mathbb{P}(Y_k = y_k,\ 1\leq k\leq n \mid X = u)\ 1_{[0,1]}(du)\ du}\ \mathbb{P}(Y_k = y_k,\ 1\leq k\leq n \mid X = x)\ 1_{[0,1]}(x)\ dx\\
&= \frac{1}{\int_0^1\ u^{\overline{y}_n}\ (1-u)^{n-\overline{y}_n} du}\ x^{\overline{y}_n}\ (1-x)^{n-\overline{y}_n}\ 1_{[0,1]}(x)\ dx\\
&= \frac{\Gamma(n+2)}{\Gamma(\overline{y}_n+1)\Gamma(n-\overline{y}_n+1)}\ x^{\overline{y}_n}\ (1-x)^{n-\overline{y}_n}\ 1_{[0,1]}(x)\ dx\\
&= \frac{(n+1)!}{\overline{y}_n!(n-\overline{y}_n)!}\ \ x^{\overline{y}_n}\ (1-x)^{n-\overline{y}_n}\ 1_{[0,1]}(x)\ dx
\end{aligned}$$

with $\overline{y}_n = \sum_{1\leq k\leq n} y_k$. We recall that

$$\int_0^1\ u^{\alpha-1}\ (1-u)^{\beta-1} du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \text{and} \quad \Gamma(n+1) = n!$$

where $\alpha\mapsto\Gamma(\alpha)$ is the Gamma function. This ends the proof of the exercise. ■

Solution to exercise 51:
Firstly, we have that

$$\mathbb{P}\left(X\in dx \mid Y = y\right) = x\ e^{-xy} \times x^{a-1}\ e^{-b\ x} = x^{(a+1)-1}\ e^{-(b+y)\ x}\ 1_{]0,\infty[}(x)\ dx$$

so that $\mathrm{Law}(X \mid Y) = \mathrm{Gamma}(a+1, b+Y)$.
In much the same way, we find that

$$\begin{aligned}
\mathbb{P}\left(X\in dx \mid Y = y\right) &\propto x^{y}\ e^{-x} \times x^{a-1}\ e^{-b\ x}\ 1_{]0,\infty[}(x)\ dx\\
&= x^{(a+y)-1}\ e^{-(b+1)\ x}\ 1_{]0,\infty[}(x)\ dx
\end{aligned}$$

so that $\mathrm{Law}(X \mid Y) = \mathrm{Gamma}(a+Y, b+1)$. This ends the proof of the exercise. ■

Solution to exercise 52:
The prior density of the Dirichlet distribution is given by

$$p(x_1,\ldots,x_d) \propto \left[\prod_{1\leq i\leq d} x_i^{a_i-1}\right] 1_{\Delta_{d-1}}(x_1,\ldots,x_d)$$

and the multinomial likelihood is given by

$$\mathbb{P}\left((Y_n^1,\ldots,Y_n^d)=(m_1,\ldots,m_d)\mid X_1=x_1,\ldots,X_d=x_d\right) \propto x_1^{m_1}\ldots x_d^{m_d}.$$

This implies that the posterior density is defined for any $(x_1,\ldots,x_d)\in\Delta_{d-1}$ by

$$\begin{aligned} p((x_1,\ldots,x_d)\mid(y_1,\ldots,y_n)) &\propto \left[\prod_{1\leq i\leq d} x_i^{m_i}\right]\times\left[\prod_{1\leq i\leq d} x_i^{a_i-1}\right] \\ &= \left[\prod_{1\leq i\leq d} x_i^{(a_i+m_i)-1}\right]. \end{aligned}$$

We conclude that

$$\mathrm{Law}((X_1,\ldots,X_d)\mid(Y_1,\ldots,Y_n)) = D(a_1+m_1,\ldots,a_d+m_d).$$

This ends the proof of the exercise. ■

Solution to exercise 53:
We have

$$\begin{aligned} \mathbb{E}\left(\frac{1}{N}\sum_{1\leq i\leq N} X^i \mid \mathcal{Y}\right) &= \sum_{1\leq i\leq N}\widehat{m}(Y^i) \\ &= \left(\frac{\widehat{\sigma}}{\sigma}\right)^2 m + \left(1-\left(\frac{\widehat{\sigma}}{\sigma}\right)^2\right)\frac{1}{N}\sum_{1\leq i\leq N} Y^i. \end{aligned}$$

On the other hand, using the fact that

$$\begin{aligned} \mathbb{E}\left(\mathcal{X}(f)^2\mid\mathcal{Y}\right) &= \sum_{1\leq i,j\leq N}\int f(x^j)f(x^j)\prod_{1\leq k\leq N}\widehat{M}(Y^k,dx^k) \\ &= \sum_{1\leq i\leq N}\widehat{M}(f^2)(Y^i) + 2\sum_{1\leq i<j\leq N}\widehat{M}(f)(Y^i)\widehat{M}(f^2)(Y^i) \\ &= \sum_{1\leq i\leq N}\left[\widehat{M}(f^2)(Y^i)-\widehat{M}(f)(Y^i)^2\right] + \left[\sum_{1\leq i\leq N}\widehat{M}(f)(Y^i)\right]^2 \end{aligned}$$

in the linear Gaussian model discussed here we find that

$$N\,\mathbb{E}\left(\left[\frac{1}{N}\sum_{1\leq i\leq N} X^i - \mathbb{E}\left(\frac{1}{N}\sum_{1\leq i\leq N} X^i\mid\mathcal{Y}\right)\right]^2 \mid \mathcal{Y}\right) = \widehat{\sigma}^2.$$

This ends the proof of the exercise. ■

Solution to exercise 54:

When the function f has compact support, we clearly have the integration by parts formula

$$\int \left(\frac{d}{dw}\ f(w)\right)\ \frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}\ w^2}\ dw = -\int\ \ f(w)\ \left(\frac{d}{dw}\frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}\ w^2}\right)\ dw.$$

Since

$$\frac{d}{dw}\frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}\ w^2} = \frac{w}{\sqrt{2\pi}}\ e^{-\frac{1}{2}\ w^2}$$

we conclude that

$$\int \left(\frac{d}{dw}\ f(w)\right)\ \frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}\ w^2}\ dw = -\int\ \ w\ f(w)\ \frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}\ w^2}\ dw.$$

For more general functions we use an approximation argument. This ends the proof of the exercise. ■

Chapter 5

Solution to exercise 55:
We have

$$\begin{aligned}
V_N(a_1\ f_1 + a_2\ \ f_2) &= \sqrt{N}\ (\eta^N(a_1\ f_1 + a_2\ \ f_2) - \eta(a_1\ f_1 + a_2\ \ f_2)) \\
&= \sqrt{N}\ (a_1\ \eta^N(f_1) + a_2\ \ \eta^N(f_2) - a_1\ \eta(f_1) - a_2\eta(f_2)) \\
&= a_1\ V^N(f_1) + a_2\ V^N(f_2).
\end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 56:
We have

$$\begin{aligned}
V_N(f_1)V_N(f_2) &= \frac{1}{N} \sum_{1\leq i,j\leq N} (f_1(X^i) - \eta(f_1))(f_2(X^j) - \eta(f_2)) \\
&= \frac{1}{N} \sum_{1\leq i\leq N} (f_1(X^i) - \eta(f_1))(f_2(X^i) - \eta(f_2)) \\
&\qquad + \frac{1}{N} \sum_{1\leq i\neq j\leq N} (f_1(X^i) - \eta(f_1))(f_2(X^j) - \eta(f_2)).
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathbb{E}\left(V_N(f_1)V_N(f_2)\right) &= \frac{1}{N} \sum_{1\leq i\leq N} \mathbb{E}\left[(f_1(X^i) - \eta(f_1))(f_2(X^i) - \eta(f_2))\right] \\
&= \mathbb{E}\left[(f_1(X) - \eta(f_1))(f_2(X) - \eta(f_2))\right] = \mathbb{E}(f_1(X)f_2(X)) - \eta(f_1)\eta(f_2).
\end{aligned}$$

The end of the proof of the exercise is now clear. ■

Solution to exercise 57:
We have

$$\begin{aligned}
\mathbb{E}\left(f(Y)\right) &= \int f(y)\ p(y)\ dy = \int f(x)\ \frac{p(x)}{q(x)}\ q(x)\ dx \\
&= \mathbb{E}\left(\overline{f}(X)\right) = \mathbb{E}\left(\frac{1}{N} \sum_{1\leq i\leq N} \overline{f}(X^i)\right)
\end{aligned}$$

with the weight function

$$\overline{f}(x) = f(x)\ \frac{p(x)}{q(x)}.$$

Using simple calculations we find that

$$\begin{aligned} N\ \mathrm{Var}\left(\frac{1}{N}\sum_{1\leq i\leq N}\ \overline{f}(X^i)\right) &= \mathbb{E}\left(\overline{f}(X)^2\right)-\mathbb{E}\left(\overline{f}(X)\right)^2 \\ &= \int\ \left(f(x)\ \frac{p(x)}{q(x)}\right)^2\ q(x)\ dx-\mathbb{E}\left(f(Y)\right)^2 \\ &= \int\ f^2(x)\ \frac{p(x)}{q(x)}\ p(x)\ dx-\mathbb{E}\left(f(Y)\right)^2. \end{aligned}$$

This ends the proof of the exercise. ∎

Solution to exercise 58:
We have

$$\mathbb{P}(U_k=u)=p\ 1_{u=1}+(1-p)\ 1_{u=0}\quad\text{and}\quad\mathbb{P}(\overline{U}_k=u)=\overline{p}\ 1_{u=1}+(1-\overline{p})\ 1_{u=0}.$$

This implies that

$$\begin{aligned} &\mathbb{P}(U_1=u_1,\ldots,U_n=u_n) \\ &=\prod_{1\leq k\leq n}\left(p\ 1_{u_k=1}+(1-p)\ 1_{u_k=0}\right) \\ &=\left[\prod_{1\leq k\leq n}\left(\frac{p\ 1_{u_k=1}+(1-p)\ 1_{u_k=0}}{\overline{p}\ 1_{u_k=1}+(1-\overline{p})\ 1_{u_k=0}}\right)\right]\ \mathbb{P}(\overline{U}_1=u_1,\ldots,\overline{U}_n=u_n). \end{aligned}$$

The end of the proof of the exercise is now clear. ∎

Solution to exercise 59: We have

$$\begin{aligned} \mathbb{E}\left(e^{\lambda U)}\right) &= \frac{1}{2}\ \left(e^{\lambda}+e^{-\lambda}\right) \\ &= \frac{1}{2}\ \left(\sum_{n\geq 0}\frac{\lambda^n}{n!}+\sum_{n\geq 0}\frac{(-\lambda)^n}{n!}\right) \\ &= \sum_{n\geq 0}\frac{\lambda^{2n}}{(2n)!}\leq 1+\sum_{n\geq 1}\frac{\lambda^{2n}}{2^n\ n!}=e^{\lambda^2/2}. \end{aligned}$$

Notice that

$$\mathbb{E}\left(e^{\lambda(X-\mathbb{E}(X))}\right)=\mathbb{E}\left(e^{\lambda\mathbb{E}\left((X-\overline{X})\ \mid\ X\right)}\right)$$

for an independent copy $\overline{X}$ of X. Using Jensen's inequality, we prove that

$$\begin{aligned} \mathbb{E}\left(e^{\lambda(X-\mathbb{E}(X))}\right) &= \mathbb{E}\left(\mathbb{E}\left(e^{\lambda(X-\overline{X})}\mid X\right)\right)=\mathbb{E}\left(e^{\lambda(X-\overline{X})}\right) \\ &= \mathbb{E}\left(\mathbb{E}\left(e^{\lambda U(X-\overline{X})}\mid X,\overline{X}\right)\right)=\mathbb{E}\left(e^{\lambda U(X-\overline{X})}\right). \end{aligned}$$

The last assertion follows from the fact that $(X-\overline{X})$ and $-(X-\overline{X})$ have the same law. Now, we use the fact that

$$\mathbb{E}\left(e^{\lambda U(X-\overline{X})}\mid X,\overline{X}\right)\leq e^{\lambda^2((X-\overline{X}))^2/2}\leq e^{\frac{\lambda^2(b-a)^2}{2}}.$$

Finally, for any $\lambda > 0$ and ρ we have

$$\mathbb{P}\left(X - \mathbb{E}(X) \geq \rho\right) = \mathbb{P}\left(e^{\lambda(X-\mathbb{E}(X))} \geq e^{\lambda\rho}\right) \leq e^{-\lambda\rho}\ \mathbb{E}\left(e^{\lambda(X-\mathbb{E}(X))}\right) \leq e^{-\left(\lambda\rho - \frac{\lambda^2(b-a)^2}{2}\right)}.$$

Choosing

$$\lambda = \rho/(b-a)^2 \Rightarrow \lambda\rho - \frac{\lambda^2(b-a)^2}{2} = \frac{\rho^2}{(b-a)^2} - \frac{\rho^2(b-a)^2}{2(b-a)^4} = \frac{\rho^2}{2(b-a)^2}$$

we find that

$$\mathbb{P}\left(X - \mathbb{E}(X) \geq \rho\right) \leq e^{-\frac{\rho^2}{2(b-a)^2}}.$$

This ends the proof of the exercise. ■

Solution to exercise 60:

For any $a \geq 0$ we have

$$\mathbb{P}(X_n \geq a) = \sum_{A\subset\{1,\ldots,n\},\ |A|\geq a} \left(\prod_{k\in A} p_k\right) \left(\prod_{k\notin A} (1-p_k)\right).$$

We set $m_n = \mathbb{E}(X_n) = \sum_{1\leq i\leq n} p_i$. For any $\epsilon \geq 0$ and $\lambda > 0$ we have

$$\mathbb{P}(X_n \geq (1+\epsilon)m_n) = \mathbb{P}\left(e^{\lambda X_n} \geq e^{-\lambda(1+\epsilon)m_n}\right) \leq e^{-\lambda(1+\epsilon)m_n}\ \mathbb{E}\left(e^{\lambda X_n}\right).$$

On the other hand, we have

$$\mathbb{E}\left(e^{\lambda X_n}\right) = \prod_{1\leq k\leq n} \mathbb{E}\left(e^{\lambda U_k}\right) = \prod_{1\leq k\leq n} \left((1-p_k) + p_k e^{\lambda}\right) = \prod_{1\leq k\leq n} \left(1 + p_k\left(e^{\lambda} - 1\right)\right).$$

Using the elementary bound $1 + x \leq e^x$, for any $x \geq 0$, we prove that

$$\mathbb{E}\left(e^{\lambda X_n}\right) \leq \prod_{1\leq k\leq n} e^{p_k\left(e^{\lambda}-1\right)} = e^{m_n\left(e^{\lambda}-1\right)}$$

and therefore

$$\mathbb{P}(X_n \geq (1+\epsilon)m_n) \leq e^{\left[\left(e^{\lambda}-1\right)-\lambda(1+\epsilon)\right]m_n}.$$

To minimize the function $f(\lambda) = \left(e^{\lambda} - 1\right) - \lambda(1+\epsilon)$ we check that

$$\lambda = \log(1+\epsilon) \Longrightarrow f'(\lambda) = e^{\lambda} - (1+\epsilon) = 0.$$

In this situation, we find that

$$\mathbb{P}(X_n \geq (1+\epsilon)m_n) \leq e^{[\epsilon - \log(1+\epsilon)(1+\epsilon)]m_n} = \left(\frac{e^{\epsilon}}{(1+\epsilon)^{1+\epsilon}}\right)^{m_n} = \left(e\ \left(1 - \frac{\epsilon}{1+\epsilon}\right)^{\frac{1+\epsilon}{\epsilon}}\right)^{\epsilon\ m_n}.$$

Notice that $(1 - 1/x)^x < e^{-1}$ for any $x > 0$ (and $(1-1/x)^x$ increases to e^{-1}, as $x \uparrow \infty$), so that $e\ \left(1 - \frac{\epsilon}{1+\epsilon}\right)^{\frac{1+\epsilon}{\epsilon}} < 1$. This ends the proof of the exercise. ■

Solution to exercise 61:

Firstly, we provide a simple inductive proof of (5.8). We assume without loss of generality that the functions f_i are centered, that is we have that $\eta(f_i) = 0$ for any $1 \leq i \leq d$.

The result is immediate for $n = 1$ (with the conventions $\sum_\emptyset = 0$ and $\prod_\emptyset = 1$). We further assume that the formula has been checked at rank n. For any $(\lambda_j)_{1\leq j\leq n+1} \in \mathbb{R}^{n+1}$ we have

$$\begin{aligned}
&\mathbb{E}\left(e^{\sum_{1\leq j\leq n+1}\lambda_j\ V^N(f_j)}\right)\\
&= \mathbb{E}\left(e^{V^N\left(\sum_{1\leq j\leq n+1}\lambda_j\ f_j\right)}\right) = e^{2^{-1}\ \eta\left(\left[\sum_{1\leq j\leq n+1}\lambda_j\ f_j\right]^2\right)}\\
&= e^{2^{-1}\ \eta\left(\left[\sum_{1\leq j\leq n}\lambda_j\ f_j\right]^2\right)} \times e^{2^{-1}\ \lambda_{n+1}^2\eta(f_{n+1}^2)} \times e^{\lambda_{n+1}\sum_{1\leq j\leq n}\lambda_j\ \eta(f_j f_{n+1})}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\mathbb{E}\left(e^{\sum_{1\leq j\leq n+1}\lambda_j\ V^N(f_j)}\right)\\
&= \mathbb{E}\left(e^{\sum_{1\leq j\leq n}\lambda_j\ V^N(f_j)}\right) \times e^{2^{-1}\ \lambda_{n+1}^2\eta(f_{n+1}^2)} \times e^{\lambda_{n+1}\sum_{1\leq j\leq n}\lambda_j\ \mathbb{E}(V(f_j)V(f_{n+1}))}.
\end{aligned} \tag{30.23}$$

For any $1 \leq i \leq n$ the term $\prod_{1\leq j\neq i\leq n}\lambda_j$ in the series expansion of $\mathbb{E}\left(e^{\sum_{1\leq j\leq n}\lambda_j\ V^N(f_j)}\right)$ is given by $\mathbb{E}\left(\prod_{1\leq j\neq i\leq n} V_N(f_j)\right)$. Thus, the desired formula (5.8) at rank $(n+1)$ results from a simple identification of the terms

$$\lambda_1\ldots\lambda_{n+1} = \left[\prod_{1\leq j\neq i\leq n}\lambda_j\right] \times \lambda_{n+1}\ \lambda_i$$

in the series expansion of (30.23).

We notice that formula (5.8) is also true by a permutation of the indexes. This shows that for any fixed index $j_0 \in \{1,\ldots,n\}$ we have

$$\mathbb{E}\left(V(f_1)\ldots V(f_n)\right) = \frac{1}{2}\sum_{1\leq i\neq j_0\leq n}\mathbb{E}\left(\prod_{k\notin\{i,j_0\}} V_N(f_k)\right) \times \mathbb{E}(V(f_i)V(f_{j_0})).$$

This yields that

$$\begin{aligned}
\mathbb{E}\left(V(f_1)\ldots V(f_n)\right) &= \frac{1}{2n}\sum_{1\leq i\neq j\leq n}\mathbb{E}\left(\prod_{k\notin\{i,j\}} V_N(f_k)\right) \times \mathbb{E}(V(f_i)V(f_j))\\
&= \frac{1}{n}\sum_{1\leq i<j\leq n}\mathbb{E}\left(\prod_{k\notin\{i,j\}} V_N(f_k)\right) \times \mathbb{E}(V(f_i)V(f_j)).
\end{aligned} \tag{30.24}$$

Now we use the inductive proof of (5.9). The result is immediate for $n = 1$. Assuming that it is true at rank n, we have

$$\mathbb{E}\left(\prod_{k\notin\{i,j\}} V_N(f_k)\right) = \sum_{P\in\mathcal{P}_{2n}^{(i,j)}-\{\{i,j\}\}}\ \prod_{\{k,l\}\in P}\mathbb{E}\left(V(f_k)V(f_l)\right)$$

where $\mathcal{P}_{2n}^{(i,j)}$ stands for the set of pairings of $\{1,\ldots,2n\}$ containing $\{i,j\}$. Using (30.24), this implies that

$$\begin{aligned}
&\mathbb{E}\left(V(f_1)\ldots V(f_{2n})\right)\\
&= \tfrac{1}{n}\textstyle\sum_{1\leq i<j\leq n}\sum_{P\in\mathcal{P}_{2n}^{(i,j)}-\{\{i,j\}\}}\quad\prod_{\{k,l\}\in P\cup\{\{i,j\}\}}\mathbb{E}\left(V(f_k)V(f_l)\right)\\
&= \tfrac{1}{n}\textstyle\sum_{1\leq i<j\leq n}\sum_{P\in\mathcal{P}_{2n}^{(i,j)}}\quad\prod_{\{k,l\}\in P}\mathbb{E}\left(V(f_k)V(f_l)\right).
\end{aligned}$$

The end of the proof follows from the fact that each partition $P\in\mathcal{P}_{2n}$ is counted n times in the above display.

This ends the proof of the exercise. ■

Solution to exercise 62:

We have

$$\begin{aligned}
\partial_{x_k}\left(x'R^{-1}x\right) &= \partial_{x_k}\left(\sum_{1\leq i,j\leq r} x_iR^{-1}_{i,j}x_j\right)\\
&= \sum_{1\leq i,j\leq r}\left(1_{i=k}R^{-1}_{i,j}x_j+x_iR^{-1}_{i,j}1_{j=k}\right)=\sum_{1\leq j\leq r}R^{-1}_{k,j}x_j+\sum_{1\leq i\leq r}x_iR^{-1}_{i,k}
\end{aligned}$$

By symmetry arguments we conclude that

$$\frac{1}{2}\partial_{x_k}\left(x'R^{-1}x\right)=\sum_{1\leq i\leq r}R^{-1}_{k,i}\ x_i$$

Using the fact that

$$\sum_{1\leq k\leq r}R_{j,k}\sum_{1\leq i\leq r}R^{-1}_{k,i}\ x_i \;=\; \sum_{1\leq i\leq r}\left(\sum_{1\leq k\leq r}R_{j,k}\ R^{-1}_{k,i}\right)\ x_i=\sum_{1\leq i\leq r}1_{i=j}\ x_i=x_j$$

we conclude that

$$\begin{aligned}
x_j &= \frac{1}{2}\sum_{1\leq k\leq r}R_{j,k}\ \partial_{x_k}\left(x'R^{-1}x\right)\\
&= -\sum_{1\leq k\leq r}R_{j,k}\ e^{\frac{1}{2}\ x'R^{-1}x}\partial_{x_k}\left(e^{-\frac{1}{2}\ x'R^{-1}x}\right)
\end{aligned}$$

The last assertion is checked using the integration by parts formula

$$\begin{aligned}
\mathbb{E}\left(X_i\ f(X)\right) &\propto \int x_i\ f(x)\ e^{-\frac{1}{2}\ x'R^{-1}x}\ dx\\
&= -\sum_{1\leq k\leq r}R_{j,k}\int f(x)\partial_{x_k}\left(e^{-\frac{1}{2}\ x'R^{-1}x}\right)\ dx\\
&= \sum_{1\leq k\leq r}R_{j,k}\int \partial_{x_k}f(x)\ e^{-\frac{1}{2}\ x'R^{-1}x}\ dx=\sum_{1\leq k\leq r}R_{j,k}\ \mathbb{E}\left(\partial_{x_k}f(X)\right)
\end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 63:
By the Borel Cantelli lemma we have $\mathbb{P}(\{X_n = a_n\}\text{infinitely often}) = 0$. Therefore for any $\omega \in \Omega$ there exists some $N(\omega) \in \mathbb{N} - \{0\}$ such that $X_n(\omega) = b_n$, for any $n \geq N(\omega)$. In this case, for any such $n \geq N(\omega)$ we have

$$\frac{1}{n} \sum_{1\leq k\leq n} X_k(\omega) \quad = \quad \frac{A(\omega)}{n} + \frac{1}{n} \sum_{N(\omega)\leq k\leq n} b_k$$

with some finite constant $A(\omega) = \sum_{1\leq k<N(\omega)} X_k(\omega) < \infty$. This implies that

$$\frac{1}{n} \sum_{1\leq k\leq n} X_k(\omega) \quad = \quad \frac{A(\omega)+B(\omega)}{n} + \frac{1}{n} \sum_{1\leq k\leq n} b_k \ \to_{n\uparrow\infty} \ b$$

with $B(\omega) = \sum_{1\leq k<N(\omega)} b_k < \infty$.
This ends the proof of the exercise.
Solution to exercise 64: We check the upper bound by using the fact that

$$\begin{aligned}
\mathbb{P}(X > \delta) \quad &= \quad \frac{1}{\sqrt{2\pi}} \int_\delta^{+\infty} \frac{1}{x} \frac{\partial}{\partial x}\left(-e^{-\frac{x^2}{2}}\right) dx \\
&= \quad \frac{1}{\sqrt{2\pi}} \left[\frac{1}{x}\, e^{-\frac{x^2}{2}}\right]_{+\infty}^{\delta} - \frac{1}{\sqrt{2\pi}} \int_\delta^{+\infty} \frac{1}{x^2}\, e^{-\frac{x^2}{2}}\, dx \leq \frac{1}{\delta} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\delta^2}{2}}.
\end{aligned}$$

To prove the lower bound we use the upper bound we just obtained to check that

$$\mathbb{P}(X > \delta) = \int_\delta^{+\infty} y \left(\frac{1}{y} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\right) dy \geq \delta \int_\delta^{+\infty} \left(\int_y^{+\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx\right) dy.$$

To continue, we use integration by parts to obtain

$$\begin{aligned}
&\int_\delta^{+\infty} \left(\int_y^{+\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx\right) dy \\
&= \left[y \left(\int_y^{+\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx\right)\right]_\delta^{+\infty} + \int_\delta^{+\infty} y\, \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy \\
&= -\delta \int_\delta^{+\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx + \int_\delta^{+\infty} \frac{\partial}{\partial y}\left(-\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\right) dy \\
&= -\delta \int_\delta^{+\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx + \frac{e^{-\frac{\delta^2}{2}}}{\sqrt{2\pi}}.
\end{aligned}$$

We conclude that

$$\mathbb{P}(X > \delta) \geq -\delta^2\, \mathbb{P}(X > \delta) + \delta\, \frac{e^{-\frac{\delta^2}{2}}}{\sqrt{2\pi}}.$$

In other words, we have proved that

$$\mathbb{P}(X > \delta) \geq \frac{\delta}{1+\delta^2} \frac{e^{-\frac{\delta^2}{2}}}{\sqrt{2\pi}} = \frac{1}{\delta+\delta^{-1}} \frac{e^{-\frac{\delta^2}{2}}}{\sqrt{2\pi}}.$$

This ends the proof of Mill's inequalities.

Now, using the fact that $X \stackrel{law}{=} -X$ we have

$$\begin{aligned}
\mathbb{P}(|X| \geq a) &= \mathbb{P}(X \geq a \text{ or } \quad X \leq -a) \\
&\leq \mathbb{P}(X \geq a) + \mathbb{P}(X \leq -a) \\
&= \mathbb{P}(X \geq a) + \mathbb{P}(-X \geq a) = 2\, \mathbb{P}(X \geq a) \\
&\leq \frac{2}{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}}.
\end{aligned}$$

If we choose $a = \sqrt{2(1+\alpha)\ \log n}$, then we find that

$$\mathbb{P}\left(|X| \geq \sqrt{2(1+\alpha)\ \log n}\right) \quad \leq \quad \frac{1}{\sqrt{\pi\alpha}} \frac{1}{n^{1+\alpha}} \frac{1}{\sqrt{\log(n)}}.$$

The last assertion is a direct consequence of the Borel-Cantelli lemma. This ends the proof of the exercise. ∎

Solution to exercise 65: Jensen's inequality states that for any convex function and a random variable W s.t. $f(\mathbb{E}W)$ and $\mathbb{E}(f(W))$ exist, the inequality

$$f(\mathbb{E}W) \leq \mathbb{E}(f(W))$$

holds. Consider a random variable W defined on a interval (a, ∞) and with a density that is proportional to $te^{-t^2/2}, t \in (a, \infty)$. Take $f(t) := 1/t$. Then by applying Jensen's inequality we get:

$$\int_a^\infty t\ e^{-\frac{1}{2}t^2} dt \ / \int_a^\infty t^2\ e^{-\frac{1}{2}t^2} dt \leq \int_a^\infty e^{-\frac{1}{2}t^2} dt \ / \int_a^\infty t\ e^{-\frac{1}{2}t^2} dt.$$

However we readily check that $\int_a^\infty t\ e^{-\frac{1}{2}t^2} dt = e^{-\frac{1}{2}a^2}$ and $\int_a^\infty t^2\ e^{-\frac{1}{2}t^2} dt = a\ e^{-\frac{1}{2}a^2} + \int_a^\infty e^{-\frac{1}{2}t^2} dt$ holds. Substituting back, we get

$$\left(e^{-\frac{1}{2}a^2}\right)^2 \leq a e^{-\frac{1}{2}a^2} \int_a^\infty e^{-\frac{1}{2}t^2} dt + \left(\int_a^\infty e^{-\frac{1}{2}t^2} dt\right)^2.$$

If we set $z := \int_a^\infty e^{-\frac{1}{2}t^2} dt$ we are dealing with a quadratic inequality with respect to z. Hence

$$\frac{\sqrt{4+a^2} - a}{2}\ e^{-\frac{1}{2}a^2} \leq \int_a^\infty e^{-\frac{1}{2}t^2} dt.$$

This ends the proof of the exercise.

∎

Solution to exercise 66:

We consider a pair of standard normal random variables $X = (X_1, X_2)$. Suppose we want to evaluate the quantity

$$\mathbb{P}(X \in A(a, \epsilon)) = \mathbb{E}\left(1_{A(a,\epsilon)}(X)\right) = \eta(1_{A(a,\epsilon)})$$

with the Gaussian distribution η of X, and the indicator function $1_{A(a,\epsilon)}$ of the set

$$A(a, \epsilon) = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1^2 + x_2^2 \geq a \quad \text{and} \quad 0 \leq \arctan(x_2/x_1) \leq 2\pi e^{-b}\}.$$

Using the Box-Muller transformation $(X_1, X_2) = \varphi(U_1, U_2)$ presented in (4.5), we have

$$\varphi^{-1}(A(a,\epsilon)) = [0, e^{-a}] \times [0, e^{-b}] \subset [0,1]^2.$$

If we set $f = 1_{A(a,\epsilon)} \circ \varphi$ then we find that

$$\begin{aligned}\mathbb{P}(X \in A(a,\epsilon)) &= \mathbb{E}(f(U_1, U_2)) \\ &= \int_{[0,1]^2} 1_{[0,e^{-a}]\times[0,e^{-b}]}(u_1, u_2)\, du_1 du_2 = e^{-(a+b)}.\end{aligned}$$

The above equation already shows that a very small part of the cell $[0,1]^2$ is used to compute the desired integral, as soon as a, b are too large. This shows that any fixed grid approximation technique will fail. We let $(U_1^i, U_2^i)_{i\geq 1}$ be a sequence of independent copies of the variable (U_1, U_2). If we set $X^i = (X_1^i, X_2^i) = \varphi(U_1^i, U_2^i)$, then we have

$$\eta^N(1_{A(a,\epsilon)}) = \frac{1}{N} \sum_{1\leq i\leq N} 1_{[0,e^{-a}]\times[0,e^{-b}]}(U_1^i, U_2^i).$$

The chance for a sample (U_1^i, U_2^i) to hit the set $[0, e^{-a}] \times [0, e^{-b}]$ can be extremely small whenever a, b are too large, so that the number of samples needed to get an accurate estimate can be too large to get a feasible solution. In this case we notice that the relative variance is given by

$$\mathbb{E}\left(\left[\frac{\eta^N(1_{A(a,\epsilon)})}{\eta(1_{A(a,\epsilon)})} - 1\right]^2\right) = \frac{1}{N\eta\left(1_{A(a,\epsilon)}\right)}\ \left(1 - \eta\left(1_{A(a,\epsilon)}\right)\right).$$

This ends the proof of the exercise.

$\blacksquare$

Chapter 6

Solution to exercise 68: In this case, we have

$$\mu_\beta(\sigma) \propto e^{-\beta V(\sigma)}.$$

We let $\delta := \inf_{\sigma \in S - V^\star} (V(\sigma) - V_\star)$, with $V_\star := \inf_S V$. Since the state space is finite we have $\delta > 0$. We use the formulae

$$\mu_\beta(\sigma) = \frac{e^{-\beta(V(\sigma)-V_\star)}}{\sum_{\tau \in S} e^{-\beta(V(\tau)-V_\star)}}$$

and

$$\begin{aligned} \mathrm{Card}(V^\star) \leq \sum_{\tau \in S} e^{-\beta(V(\tau)-V_\star)} &= \mathrm{Card}(V^\star) + \sum_{\tau \in S - V_\star} e^{-\beta(V(\tau)-V_\star)} \\ &\leq \mathrm{Card}(V^\star) + \mathrm{Card}(S - V^\star) \times e^{-\beta\delta} \end{aligned}$$

to check that for any $\sigma \in V^\star$

$$\mu_\infty(\sigma) \stackrel{\beta\uparrow\infty}{\longleftarrow} \frac{1}{1 + ce^{-\beta\delta}} \times \mu_\infty(\sigma) \ \leq \mu_\beta(\sigma) \leq \mu_\infty(\sigma)$$

with $c := \mathrm{Card}(S - V^\star)/\mathrm{Card}(V^\star)$. In much the same way, we prove that for any $\sigma \in S - V^\star$ we have that

$$0 = \mu_\infty(\sigma) \leq \mu_\beta(\sigma) \leq e^{-\beta\delta} \downarrow_{\beta\uparrow\infty} 0 = \mu_\infty(\sigma).$$

This ends the proof of the exercise.

■

Solution to exercise 69:
The Metropolis-Hastings ratio is given by the formula

$$\frac{p(y)\epsilon^{-d} q((x-y)/\epsilon)}{p(x)\epsilon^{-d} q((y-x)/\epsilon)} = \frac{p(y)}{p(y)}.$$

This ends the proof of the exercise.

■

Solution to exercise 70:
We have

$$\pi(x)\ M(x,y) \propto w(x,y) = w(y,x) \propto \pi(y)\ M(x,y)$$

and

$$\sum_{x \in \mathcal{V}} \pi(x)\ M(x,y) = \pi(y) \underbrace{\sum_{y \in \mathcal{V}} M(x,y)}_{=1}.$$

This implies that

$$\mathbb{P}(X_1 = y) = \sum_{x\in\mathcal{V}} \mathbb{P}(X_0 = x)\ \mathbb{P}(X_1 = y \mid X_0 = x) = \sum_{x\in\mathcal{V}} \pi(x)M(x,y) = \pi(y).$$

By a simple induction w.r.t. the time parameter we prove that $\mathbb{P}(X_n = x) = \pi(x)$ for any $n \geq 0$. This ends the proof of the exercise.

■

Solution to exercise 71:
We have

$$p(x_{n+1} \mid y_0,\ldots,y_n) = \int\ p(x_{n+1}, x_n \mid y_0,\ldots,y_n)\ dx_n$$

and

$$p(x_{n+1}, x_n \mid y_0,\ldots,y_n) = \underbrace{p(x_{n+1} \mid y_0,\ldots,y_n,x_n)}_{=p(x_{n+1}|x_n)}\ p(x_n \mid y_0,\ldots,y_n).$$

Using Bayes' rule, we also have that

$$p(x_{n+1} \mid y_0,\ldots,y_n,y_{n+1}) = \frac{p(y_{n+1}|x_{n+1})\ p(x_{n+1} \mid y_0,\ldots,y_n)}{\int p(y_{n+1}|x'_{n+1})\ p(x'_{n+1} \mid y_0,\ldots,y_n)\ dx'_{n+1}}$$

and

$$\int p(y_{n+1}|x'_{n+1})\ p(x'_{n+1} \mid y_0,\ldots,y_n)\ dx'_{n+1} = p(y_{n+1}|y_0\ldots,y_n).$$

The last assertion follows from the fact that

$$p(y_0\ldots,y_n) = \prod_{0\leq k\leq n} p(y_k|y_0,\ldots,y_{k-1})$$

and

$$p(y_k|y_0,\ldots,y_{k-1}) = \int p(y_k|x_k)\ p(x_k \mid y_0,\ldots,y_{k-1})\ dx_k$$

and this ends the proof of the exercise.

■

Solution to exercise 72:
We have

$$\mathcal{Z}_\beta = \sum_{x\in S}\ \exp\left(-h\beta\sum_{i\in E}\ x(i) + K\ \sum_{i\sim j} x(i)x(j)\right).$$

By symmetry, this yields

$$\begin{aligned}\mathcal{Z}_\beta &= \sum_{x\in S}\exp\left[J_\beta\sum_{i=1}^{L} x(i)x(i+1) + \frac{h_\beta}{2}\sum_{i=1}^{L}(x(i)+x(i+1))\right]\\ &= \sum_{x\in S}\prod_{i=1}^{L}\ T(x(i),x(i+1))\end{aligned}$$

with

$$T(x(i),x(i+1)) = \exp\left[J_\beta x(i)x(i+1) + \frac{h_\beta}{2}\left(x(i)+x(i+1)\right)\right].$$

We conclude that

$$\begin{aligned} \mathcal{Z}_\beta &= \sum_{x(1)\in\{-1,+1\}} \ldots \sum_{x(L)\in\{-1,+1\}} T(x(1),x(2))\ldots T(x(L-1),x(L))T(X(L),x(1)) \\ &= \mathrm{Trace}(T^L). \end{aligned}$$

To diagonalize the symmetric matrix L we need to compute the eigenvalues. To this end, we check that

$$\begin{aligned} \det(L-\lambda Id) &= \left(e^{J_\beta+h_\beta}-\lambda\right)\left(e^{J_\beta-h_\beta}-\lambda\right)-e^{-2J_\beta} \\ &= \lambda^2-2\lambda e^{J_\beta}\frac{e^{-h_\beta}++h_\beta}{2}+2\frac{e^{2J_\beta}-e^{-2J_\beta}}{2} \\ &= \lambda^2-2e^{J_\beta}\ \lambda\cosh(h_\beta)+2\sinh(2J_\beta) \\ &= \left(\lambda-e^{J_\beta}\cosh(h_\beta)\right)^2-\left(e^{2J_\beta}\cosh(h_\beta)^2-2\sinh(2J_\beta)\right). \end{aligned}$$

Therefore the two eigenvalues are given by

$$\begin{aligned} \lambda_{+,\beta} &= e^{J_\beta}\cosh(h_\beta)+\sqrt{e^{2J_\beta}\cosh(h_\beta)^2-2\sinh(2J_\beta)} \\ \lambda_{-,\beta} &= e^{J_\beta}\cosh(h_\beta)-\sqrt{e^{2J_\beta}\cosh(h_\beta)^2-2\sinh(2J_\beta)}. \end{aligned}$$

The last assertion of the exercise follows from the spectral decomposition

$$D_\beta := \begin{pmatrix} \lambda_{+,\beta} & 0 \\ 0 & \lambda_{-,\beta} \end{pmatrix} = UTU^{-1} \quad \text{with} \quad U^{-1}=U'$$

for some unitary transformation matrix U. Therefore

$$T^L=(U^{-1}D_\beta U)^L \Rightarrow \mathrm{Trace}(T^L)=\mathrm{Trace}(D_\beta^L)=\lambda_{+,\beta}^L+\lambda_{-,\beta}^L.$$

We have

$$\frac{1}{\beta L}\log\mathcal{Z}_\beta=\frac{1}{\beta}\log\lambda_{+,\beta}+\underbrace{\frac{1}{\beta L}\log\left(1+\left[\frac{\lambda_{-,\beta}}{\lambda_{+,\beta}}\right]^L\right)}_{\to_{L\uparrow\infty}0}.$$

This ends the proof of the exercise.

■

Solution to exercise 73:

We follow the solution of exercise 26, so we only describe the first selection-mutation transitions. In this context, the density of the observation variable Y_n given $X_n=x_n=\begin{pmatrix} x_n^{(1)} \\ x_n^{(2)} \\ x_n^{(3)} \end{pmatrix}\in S=\mathbb{R}^3$ is given by the Gaussian density

$$p(y_n|x_n)=\frac{1}{\sqrt{2\pi}\Delta}\ \exp\left(-\frac{1}{2\Delta^2}(y_n-x_n^{(3)})^2\right).$$

Initially, we start by sampling N i.i.d. random copies $\xi_0:=(\xi_0^i)_{1\leq i\leq N}$ of the signal X_0. Notice that in this situation each particle has three components

$$\xi_0^i=\begin{pmatrix} \xi_0^{i,(1)} \\ \xi_0^{i,(2)} \\ \xi_0^{i,(3)} \end{pmatrix}\in S=\mathbb{R}^3.$$

Given the observation $Y_0 = y_0$ we sample N random variables $\widehat{\xi}_0 := (\widehat{\xi}_0^i)_{1\leq i\leq N}$ with the discrete distribution

$$\sum_{1\leq i\leq N} \frac{\exp\left(-\frac{1}{2\Delta^2}(y_0-\xi_0^{i,(3)})^2\right)}{\sum_{1\leq j\leq N}\exp\left(-\frac{1}{2\Delta^2}(y_0-\xi_0^{j,(3)})^2\right)}\ \delta_{\xi_0^i}.$$

Notice that the N selected particles have again three components

$$\widehat{\xi}_0^i = \left(\begin{array}{c} \widehat{\xi}_0^{i,(1)} \\ \widehat{\xi}_0^{i,(2)} \\ \widehat{\xi}_0^{i,(3)} \end{array}\right) \in S = \mathbb{R}^3.$$

During the prediction transition $\widehat{\xi}_0 \rightsquigarrow \xi_1 = (\xi_1^i)_{1\leq i\leq N}$, we sample N i.i.d. copies $(\epsilon_1^i)_{1\leq i\leq N}$ and $(W_1^i)_{1\leq i\leq N}$ of ϵ_1 and W_1 and we set

$$\forall i\in\{1,\ldots,N\} \qquad \xi_1^i = \left(\begin{array}{c} \xi_1^{i,(1)} \\ \xi_1^{i,(2)} \\ \xi_1^{i,(3)} \end{array}\right) \in S = \mathbb{R}^3$$

with

$$\left\{\begin{array}{rcl} \xi_1^{i,(1)} - \widehat{\xi}_0^{i,(1)} &=& \epsilon_1^i\ W_1^i \\ \xi_1^{i,(2)} - \widehat{\xi}_0^{i,(2)} &=& -\alpha\ \widehat{\xi}_0^{i,(2)}\ \Delta + \beta\ \Delta\ \xi_1^{i,(1)} \\ \xi_1^{i,(3)} - \widehat{\xi}_0^{i,(3)} &=& \xi_1^{i,(2)}\ \Delta \end{array}\right.$$

and so on.

This ends the proof of the exercise.

■

Solution to exercise 74:

- Random polynomials

$$V(x) = \sum_{0\leq n\leq d} U_n\ x^n \Rightarrow C(x,y) = \mathbb{E}(V(x)V(y)) = \mathbb{E}(U_1^2)\sum_{0\leq n\leq d}(xy)^n.$$

- Cosine random field

$$V(x) = U_1\ \cos(ax) + U_2\ \sin(ax)$$

$$\begin{array}{rcl} \Rightarrow\ C(x,y) &=& \mathbb{E}(V(x)V(y)) = \mathbb{E}(U_1^2)\,(\cos(ax)\cos(ay)+\sin(ax)\sin(ay)] \\ &=& \mathbb{E}(U_1^2)\ \cos(a(x-y)). \end{array}$$

This ends the proof of the exercise.

■

Solution to exercise 75: We have

$$\mathbb{E}\left(V(x)V(y)\right) = \mathbb{E}(U_1^2)\sum_{n\geq 1}\lambda_n\ \varphi_n(x)\varphi_n(y) = \mathbb{E}(U_1^2)\ C(x,y).$$

This ends the proof of the exercise.

■

Solution to exercise 76: We have

$$\frac{\partial}{\partial a_i}\mathbb{E}\left[\left((V(x)-\sum_{1\leq i\leq n} a_i\ V(x_i)\right)^2\right]=2\mathbb{E}\left[\left((V(x)-\sum_{1\leq j\leq n} a_j\ V(x_j)\right)\ V(x_i)\right]=0$$

for $\sum_{1\leq j\leq n}\ a_j\ C(x_i,x_j)=C(x,x_i)$. This implies that

$$\begin{pmatrix} w_1(x)\\ \vdots \\ w_n(x)\end{pmatrix}=\begin{pmatrix} C(x_1,x_1) & \ldots & C(x_1,x_n)\\ \vdots & \vdots & \vdots\\ C(x_n,x_1) & \ldots & C(x_n,x_n)\end{pmatrix}^{-1}\begin{pmatrix} C(x,x_1)\\ \vdots \\ C(x,x_n)\end{pmatrix}.$$

We set $\mathcal{V}=[V(x_1),\ldots,V(x_n)]$, $\mathcal{C}=\begin{pmatrix} C(x,x_1)\\ \vdots \\ C(x,x_n)\end{pmatrix}$, and

$$\mathcal{Q}=\begin{pmatrix} C(x_1,x_1) & \ldots & C(x_1,x_n)\\ \vdots & \vdots & \vdots\\ C(x_n,x_1) & \ldots & C(x_n,x_n)\end{pmatrix}=\mathbb{E}\left(\mathcal{V}'\mathcal{V}\right).$$

In this notation, we have

$$\begin{aligned}\mathbb{E}\left[(V(x)-\widehat{V}(x))^2\right] &= \mathbb{E}(V(x)^2)-2\mathbb{E}\left(\mathcal{V}\mathcal{Q}^{-1}\mathcal{C}\right)+\mathbb{E}\left(\mathcal{C}'\mathcal{Q}^{-1}\mathcal{V}'\mathcal{V}\mathcal{Q}^{-1}\mathcal{C}\right)\\ &= C(x,x)-\mathcal{C}'\mathcal{Q}^{-1}\mathcal{C}.\end{aligned}$$

This ends the proof of the exercise.

■

Chapter 7

Solution to exercise 77:

By construction, we have

$$
\begin{aligned}
&\mathbb{E}\left(f\left(X_{n+1}\right) \mid X_n\right) \\
&= \int f\left(X_n + b_n(X_n) + \sigma_n(X_n) w\right)\ 1_A\left(X_n + b_n(X_n) + \sigma_n(X_n)\ W_{n+1}\right)\ \mu_{n+1}(dw) \\
&\qquad\qquad + \int f\left(X_n\right)\ 1_A\left(X_n + b_n(X_n) + \sigma_n(X_n) w\right)\ \mu_{n+1}(dw) \\
&= \int f\left(x_{n+1}\right)\ 1_A\left(x_{n+1}\right)\ K_{n+1}(x_n, dx_{n+1}) + f\left(X_n\right) \int 1_{S-A}\left(x_{n+1}\right)\ K_{n+1}(x_n, dx_{n+1})
\end{aligned}
$$

This shows that

$$
\begin{aligned}
M_{n+1}(x_n, dx_{n+1}) &= \mathbb{P}\left(X_{n+1} \in dx_{n+1} \mid X_n = x_n\right) \\
&= K_{n+1}(x_n, dx_{n+1})\ 1_A(x_{n+1}) + (1 - K_{n+1}(x_n, A))\ \delta_{x_n}(dx_{n+1})
\end{aligned}
$$

This ends the proof of the exercise.

■

Solution to exercise 78:

We have $X_n = X_0 + \sum_{1 \le k \le n} W_k$

$$\mathbb{P}(W = +1 \mid \Theta) = 1 - \mathbb{P}(W = -1 \mid \Theta) = \Theta$$

Given Θ, X_n is a simple random walk

$$\mathbb{P}\left(X_{n+1} = X_n + 1 \mid X_n\ \Theta\right) = \Theta$$

For any $\epsilon \in \{-1, +1\} \mapsto \frac{1+\epsilon}{2} \in \{0, 1\}$ we readily check that

$$\mathbb{P}(W = \epsilon \mid \Theta) = \Theta^{\frac{1+\epsilon}{2}}\ (1 - \Theta)^{1 - \frac{1+\epsilon}{2}}$$

This implies that

$$\mathbb{P}\left(W_1 = \epsilon_1, \ldots, W_n = \epsilon_n \mid \Theta\right) = \Theta^{\sum_{1 \le i \le n} \frac{1+\epsilon_i}{2}}\ (1 - \Theta)^{n - \sum_{1 \le i \le n} \frac{1+\epsilon_i}{2}}$$

Applying Bayes' rule, this yields the formula

$$\mathbb{P}\left(\Theta \in d\theta \mid W_1, \ldots, W_n\right) \propto \theta^{\sum_{1 \le i \le n} \frac{1+W_i}{2}}\ (1 - \theta)^{n - \sum_{1 \le i \le n} \frac{1+W_i}{2}}\ \mu(d\theta)$$

If we set $\overline{W}_n = \sum_{1\leq i\leq n} \frac{1+W_i}{2} \in [0,n]$ $\left(\Rightarrow n-\overline{W}_n\right) \in [0,n]$ we also have

$$\mathbb{P}\left(\Theta \in d\theta \mid W_1,\ldots,W_n\right) = \mathbb{P}\left(\Theta \in d\theta \mid \overline{W}_n\right) \propto \theta^{\overline{W}_n}\ (1-\theta)^{n-\overline{W}_n}\ \mu(d\theta)$$

When Θ is uniform on $[0,1]$ we have the conditional Beta distribution

$$\begin{aligned}&\mathbb{P}\left(\Theta \in d\theta \mid \overline{W}_n\right)\\ &= \frac{\Gamma\left((\overline{W}_n+1)+((n-\overline{W}_n)+1)\right)}{\Gamma\left(\overline{W}_n+1\right)\Gamma\left((n-\overline{W}_n)+1\right)}\ \theta^{(\overline{W}_n+1)-1}\ (1-\theta)^{((n-\overline{W}_n)+1)-1}\ 1_{[0,1]}(\theta)\ d\theta\end{aligned}$$

In other words, we have

$$\mathrm{Law}\left(\Theta \in d\theta \mid \overline{W}_n\right) = \mathrm{Beta}\left(\overline{W}_n+1,(n-\overline{W}_n)+1\right)$$

This implies that

$$\mathbb{E}\left(\Theta \mid \overline{W}_n\right) = \frac{\overline{W}_n+1}{\overline{W}_n+1+(n-\overline{W}_n)+1} = \frac{1}{n+2}\ \left(\overline{W}_n+1\right)$$

The mean and the variance of Beta distributions are discussed in exercises 45-47 in terms of Dirichlet distributions. By the Law of Large Numbers, given the value of Θ we have

$$\begin{aligned}&\overline{W}_n/n \rightarrow_{n\rightarrow\infty} \tfrac{\mathbb{E}(W\ \mid\ \Theta)+1}{2} = \tfrac{(2\Theta-1)+1}{2} = \Theta\\ &\Longrightarrow\ \mathbb{E}\left(\Theta \mid \overline{W}_n\right) \rightarrow_{n\rightarrow\infty}\ \Theta\end{aligned}$$

More generally, when Θ is itself a Beta(a,b) distribution on $[0,1]$ we have the conditional Beta distribution

$$\begin{aligned}&\mathbb{P}\left(\Theta \in d\theta \mid \overline{W}_n\right)\\ &= \frac{\Gamma\left((\overline{W}_n+a)+((n-\overline{W}_n)+b)\right)}{\Gamma\left(\overline{W}_n+a\right)\Gamma\left((n-\overline{W}_n)+b\right)}\ \theta^{(\overline{W}_n+a)-1}\ (1-\theta)^{((n-\overline{W}_n)+b)-1}\ 1_{[0,1]}(\theta)\ d\theta\end{aligned}$$

In this situation, we have

$$\begin{aligned}\mathbb{E}\left(\Theta \mid \overline{W}_n\right) &= \frac{\overline{W}_n+a}{\overline{W}_n+a+(n-\overline{W}_n)+b}\\ &= \frac{1}{a+b+n}\ \left(\overline{W}_n+a\right) = \frac{n}{a+b+n}\ \frac{\overline{W}_n}{n} + \frac{a+b}{a+b+n}\ \underbrace{\frac{a}{a+b}}_{=\mathbb{E}(\Theta)}\end{aligned}$$

Here again, we notice that

$$\mathbb{E}\left(\Theta \mid \overline{W}_n\right) \rightarrow_{n\rightarrow\infty}\ \Theta$$

In all the situations, we have

$$\begin{aligned}X_n &= x_0 + \sum_{1\leq k\leq n} W_k\\ &= x_0 + 2\left(\sum_{1\leq k\leq n}\frac{1+W_k}{2}\right) - n = x_0 + 2\overline{W}_n - n\end{aligned}$$

This implies that

$$\overline{W}_n = [(X_n - x_0) + n]/2$$

so that

$$\mathbb{E}\left(\mathbb{P}\left(X_{n+1} = X_n + 1 \mid X_n\ \Theta\right)\ \mid X_n\right) = \mathbb{E}\left(\mathbb{P}\left(X_{n+1} = X_n + 1 \mid X_n\ \Theta\right)\ \mid \overline{W}_n\right) = \mathbb{E}\left(\Theta \mid \overline{W}_n\right)$$

when Θ is itself a $\mathrm{Beta}(a, b)$ distribution on $[0, 1]$ we have

$$\mathbb{P}\left(X_{n+1} = X_n + 1 \mid X_n\right) = \frac{n}{a+b+n}\ \frac{(X_n - x_0) + n}{2n} + \frac{a+b}{a+b+n}\ \frac{a}{a+b} \simeq_{n\uparrow\infty} \Theta$$

This ends the proof of the exercise.

■

Solution to exercise 79:
Observe that

$$Y_q = a + \sum_{1\leq p\leq q} b_p\ Y_{q-p} + V_q = a + b_1\ Y_{q-1} + b_2\ Y_{q-2} + \ldots + b_q\ Y_0 + V_q$$

$$X_0 = \begin{pmatrix} Y_0 \\ \vdots \\ Y_{q-1} \end{pmatrix}$$

and

$$\begin{aligned} X_1 &= \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{q-1} \\ Y_q \end{pmatrix} \\ &= \underbrace{a \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{:=c} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ b_q & b_{q-1} & b_{q-2} & b_{q-3} & \ldots & b_2 & b_1 \end{bmatrix}}_{:=B} \begin{pmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{q-2} \\ Y_{q-1} \end{pmatrix} + \underbrace{V_q \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{:=W_1} \end{aligned}$$

This shows that

$$X_1 = c + BX_0 + W_1$$

In the same vein, we have

$$\begin{aligned} Y_{n+q} &= a + \sum_{1\leq p\leq q} b_p\ Y_{n+q-p} + V_{n+q} \\ &= a + b_1\ Y_{n+q-1} + b_2\ Y_{n+q-2} + \ldots + b_{q-1}\ Y_{n+1} + b_q\ Y_n + V_{n+q} \end{aligned}$$

$$
\begin{aligned}
X_{n+1} &= \begin{pmatrix} Y_{n+1} \\ Y_{n+2} \\ \vdots \\ Y_{n+q-1} \\ Y_{n+q} \end{pmatrix} \\
&= \underbrace{a \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{:=c} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ b_q & b_{q-1} & b_{q-2} & b_{q-3} & \dots & b_2 & b_1 \end{bmatrix}}_{:=B} \begin{pmatrix} Y_n \\ Y_{n+1} \\ \vdots \\ Y_{n+q-2} \\ Y_{n+q-1} \end{pmatrix} + \underbrace{V_{n+q} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{:=W_{n+1}}
\end{aligned}
$$

This shows that

$$X_{n+1} = c + BX_n + W_{n+1}$$

This ends the proof of the exercise.

■

Solution to exercise 80:

By construction, initially G fires at U, U fires at G, and B fires at G. Since nobody fires at B the transition $(GUB) \rightsquigarrow (GU)$ is not possible. Similarly, $(GUB) \rightsquigarrow (G)$ and $(GUB) \rightsquigarrow (U)$ are impossible.

Initially the chain starts at the state (GUB). The next states can be

- (GUB) when all shooters missed their opponents.
- (GB) when G kills U, U misses G, and B misses G.
- (UB) when
  - the U hits G, G misses U, and B hits G.
  - the U hits G, G misses U, and B misses G.
  - the U misses G, G misses U, and B hits G.
- (B) when
  - G hits U, U hits G and B misses G
  - G hits U, U hits G, and B hits G
  - G hits U, U misses G, and B hits G

From (UB) the next states can be

- (UB) when both shooters missed their opponents.
- (U) when U kills B and B misses U.
- (B) when U misses B and B kills U.
- $when(\emptyset)$ U kills B and B kills U.

From (GB) the next states can be

- (GB) when both shooters missed their opponents.

- (G) when G kills B and B missed G.
- (B) when G missed B and B kills G.
- $(\emptyset)$ when G kills B and B kills G.

$$\begin{array}{rcl}\mathbb{P}((GUB) \rightsquigarrow (GUB)) & = & (1-g)(1-u)(1-b)\\ \mathbb{P}((GUB) \rightsquigarrow (GB)) & = & g(1-u)(1-b)\\ \mathbb{P}((GUB) \rightsquigarrow (UB)) & = & (1-g)ub + (1-g)u(1-b) + (1-g)(1-u)b\\ \mathbb{P}((GUB) \rightsquigarrow (B)) & = & gu(1-b) + gub + g(1-u)b.\end{array}$$

In the same way, we have

$$\begin{array}{rcl}\mathbb{P}((UB) \rightsquigarrow (UB)) & = & (1-u)(1-b)\\ \mathbb{P}((UB) \rightsquigarrow (B)) & = & (1-u)b\\ \mathbb{P}((UB) \rightsquigarrow (U)) & = & u(1-b)\\ \mathbb{P}((UB) \rightsquigarrow (\emptyset)) & = & ub\end{array}$$

and

$$\begin{array}{rcl}\mathbb{P}((GB) \rightsquigarrow (GB)) & = & (1-g)(1-b)\\ \mathbb{P}((GB) \rightsquigarrow (G)) & = & g(1-b)\\ \mathbb{P}((GB) \rightsquigarrow (B)) & = & (1-g)b\\ \mathbb{P}((GB) \rightsquigarrow (\emptyset)) & = & gb\end{array}$$

and, of course

$$\mathbb{P}((\emptyset) \rightsquigarrow (\emptyset)) = \mathbb{P}((G) \rightsquigarrow (G) = \mathbb{P}((U) \rightsquigarrow (U) = \mathbb{P}((B) \rightsquigarrow (B)) = 1.$$

This ends the proof of the exercise.

■

Solution to exercise 81:
There are 3 cases to have $(\emptyset)$ at the $(n+1)$-th round:

- have $(\emptyset)$ at the n-th round.
- (GB) at round n and the $(n+1)$-th transition $(GB) \rightsquigarrow (\emptyset)$.
- (UB) at round n and the $(n+1)$-th transition $(UB) \rightsquigarrow (\emptyset)$

Recall that $q_1(n)$, $q_2(n)$ are the probabilities that the states (GB), (UB) are the result of the n-th round. This yields

$$p_1(n+1) = p_1(n) + gb\; q_1(n) + ub\; q_2(n).$$

There are 2 cases to have (G) at the $(n+1)$-th round:

- have (G) at the n-th round.

- (G) at round n and the $(n+1)$-th transition $(GB) \rightsquigarrow (\emptyset)$.
- (GB) at round n and the $(n+1)$-th transition $(GB) \rightsquigarrow (G)$

This yields
$$p_2(n+1) = p_2(n) + g(1-b)\ q_1(n).$$

There are 2 cases to have (U) at the $(n+1)$-th round:

- have (U) at the n-th round.
- (G) at round n and the $(n+1)$-th transition $(GB) \rightsquigarrow (\emptyset)$.
- (UB) at round n and the $(n+1)$-th transition $(UB) \rightsquigarrow (U)$

This yields
$$p_3(n+1) = p_3(n) + u(1-b)\ q_2(n).$$

There are 4 cases to have (B) at the $(n+1)$-th round:

- have (B) at the n-th round.
- (GB) at round n and the $(n+1)$-th transition $(GB) \rightsquigarrow (B)$.
- (UB) at round n and the $(n+1)$-th transition $(UB) \rightsquigarrow (B)$.
- (GUB) at round n and the $(n+1)$-th transition $(GUB) \rightsquigarrow (B)$.

This yields
$$\begin{aligned} p_4(n+1) &= p_4(n) + (1-g)b\ q_1(n) + (1-u)b\ q_2(n) + [gu(1-b) + gub + g(1-u)b]\, q_3(n) \\ &= p_4(n) + (1-g)b\ q_1(n) + (1-u)b\ q_2(n) + [gu + g(1-u)b]\, q_3(n). \end{aligned}$$

In much the same way, we have 2 cases to have (GB) at the $(n+1)$-th round:

- (GB) at round n and the $(n+1)$-th transition $(GB) \rightsquigarrow (GB)$.
- (GUB) at round n and the $(n+1)$-th transition $(GUB) \rightsquigarrow (GB)$.

This yields
$$q_1(n+1) = (1-g)(1-b)\ q_1(n) + g(1-u)(1-b)\ q_3(n).$$

We also have 2 cases to have (UB) at the $(n+1)$-th round:

- (UB) at round n and the $(n+1)$-th transition $(UB) \rightsquigarrow (UB)$.
- (GUB) at round n and the $(n+1)$-th transition $(GUB) \rightsquigarrow (UB)$.

The first of these two cases yields
$$\begin{aligned} q_2(n+1) &= (1-u)(1-b)\ q_2(n) + [(1-g)ub + (1-g)u(1-b) + (1-g)(1-u)b]\ \ q_3(n) \\ &= (1-u)(1-b)\ q_2(n) + [(1-g)u + (1-g)(1-u)b]\ \ q_3(n). \end{aligned}$$

The second one clearly yields
$$q_3(n+1) = (1-g)(1-u)(1-b)\ q_3(n).$$

Notice that $q_1(0) = 0 = q_2(0)$, and $q_3(0) = 1$.

We conclude that

$$q(n+1) = \begin{pmatrix} q_1(n+1) \\ q_2(n+1) \\ q_3(n+1) \end{pmatrix}$$

$$= \begin{pmatrix} (1-g)(1-b) & 0 & g(1-u)(1-b) \\ 0 & (1-u)(1-b) & (1-g)u+(1-g)(1-u)b \\ 0 & 0 & (1-g)(1-u)(1-b) \end{pmatrix} \begin{pmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{pmatrix}$$

and

$$p(n+1) = \begin{pmatrix} p_1(n+1) \\ p_2(n+1) \\ p_3(n+1) \\ p_4(n+1) \end{pmatrix}$$
$$= \begin{pmatrix} p_1(n) \\ p_2(n) \\ p_3(n) \\ p_4(n) \end{pmatrix} + \begin{pmatrix} gb & ub & 0 \\ g(1-b) & 0 & 0 \\ 0 & u(1-b) & 0 \\ (1-g)b & (1-u)b & gu+g(1-u)b \end{pmatrix} \begin{pmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{pmatrix}.$$

This ends the proof of the exercise.

■

Solution to exercise 82:
For $n=1$, we have

$$\frac{1}{2}\begin{pmatrix} \frac{3}{2}+\left(\frac{1}{2}-1\right) & 2^0 & \frac{1}{2}+\left(\frac{1}{2}-1\right) \\ \frac{1}{2} & 2^0 & \frac{1}{2} \\ \frac{1}{2}+\left(\frac{1}{2}-1\right) & 2^0 & \frac{3}{2}+\left(\frac{1}{2}-1\right) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{3}{2}-\frac{1}{2} & 1 & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} & 1 & \frac{3}{2}-\frac{1}{2} \end{pmatrix} = M.$$

We suppose the result is true at rank n. In this case, we have

$$\begin{aligned} M^{n+1} &= \frac{1}{2^{n+1}}\begin{pmatrix} \frac{3}{2}+\left(2^{n-2}-1\right) & 2^{n-1} & \frac{1}{2}+\left(2^{n-2}-1\right) \\ 2^{n-2} & 2^{n-1} & 2^{n-2} \\ \frac{1}{2}+\left(2^{n-2}-1\right) & 2^{n-1} & \frac{3}{2}+\left(2^{n-2}-1\right) \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 1 \end{pmatrix} \\ &= \frac{1}{2^{n+1}}\begin{pmatrix} \frac{3}{2}+\left(2^{n-1}-1\right) & 2^{n} & \frac{1}{2}+\left(2^{n-1}-1\right) \\ 2^{n-1} & 2^{n} & 2^{n-1} \\ \frac{1}{2}+\left(2^{n-1}-1\right) & 2^{n} & \frac{3}{2}+\left(2^{n-1}-1\right) \end{pmatrix}. \end{aligned}$$

In the first row of the above display we have used the fact that

$$\begin{aligned} \frac{3}{2}+\left(2^{n-2}-1\right)+2^{n-2} &= \frac{3}{2}+\left(2\times 2^{n-2}-1\right) = \frac{3}{2}+\left(2^{n-1}-1\right) \\ \frac{3}{2}+\left(2^{n-2}-1\right)+2^{n-1}+\frac{1}{2}+\left(2^{n-2}-1\right) &= 2^{n-2}+2^{n-1}+2^{n-2} = 2^2\ 2^{n-2} = 2^n \\ 2^{n-2}+\frac{1}{2}+\left(2^{n-2}-1\right) &= 2^{n-1}-\frac{1}{2} = \frac{1}{2}+\left(2^{n-1}-1\right). \end{aligned}$$

In the second row, we have used the fact that

$$\begin{gathered} 2^{n-2}+\frac{1}{2}\ 2^{n-1} = 2\ 2^{n-2} = 2^{n-1} \\ 2^{n-2}+2^{n-1}+2^{n-2} = 2\ 2^{n-1} = 2^n. \end{gathered}$$

The last row follows the same computations as the first one. This shows that

$$M^n = \begin{pmatrix} \frac{1}{2^{n+1}} + \frac{1}{4} & \frac{1}{2} & -\frac{1}{2^{n+1}} + \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2^{n+1}} + \frac{1}{4} & \frac{1}{2} & \frac{1}{2^{n+1}} + \frac{1}{4} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} + \frac{1}{2^{n+1}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \to_{n\uparrow\infty} \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}.$$

This ends the proof of the exercise.

■

Solution to exercise 83:

Consider a path $(k, X_k)_{0\leq k\leq n}$ of the simple random walk (1.1) starting in $(0, X_0) = (0, 0)$ (at time $n = 0$) and ending at $(n, X_n) = (n, x)$ (at some terminal time n). We recall that $X_k = X_{k-1} + U_k$ with a sequence of i.i.d. copies of a Bernoulli random variable $U \in \{-1, +1\}$ with $\mathbb{P}(U = +1) = \mathbb{P}(U = -1) = 2^{-1}$. We let a the number of $U_k = +1$ and b the number of $U_k = -1$. By construction, we have

$$a + b = n \quad \text{and} \quad a - b = x \Rightarrow a = \frac{n+x}{2} \quad \text{and} \quad b = \frac{n-x}{2}.$$

Furthermore there are $\begin{pmatrix} n \\ a \end{pmatrix}$ ways of choosing the time-location of the $+1$ in the path. Observe that $(n+x)$ must be even otherwise there is no admissible path going from $(0, X_0) = (0, 0)$ to $(n, X_n) = (n, x)$. It remains to notice that each admissible path has the same probability 2^{-n}. We conclude that

$$\mathbb{P}(X_n = x \mid X_0 = 0) = 2^{-n} \begin{pmatrix} n \\ \frac{n+x}{2} \end{pmatrix}.$$

On the other hand, we have

$$\mathbb{P}(X_n = y \mid X_m = x) = \mathbb{P}(X_{n-m} = (x - y) \mid X_0 = 0).$$

It remains to compute the number of paths from (m, x) to (n, y) that remain in the positive axis, when $x, y > 0$. Using the reflection principle proved in exercise 6, every path that hits the horizontal axis $\mathbb{H} = \{(n, 0)\ , n \in \mathbb{N}\}$ can be reflected into a path going from (m, x) to $(n, -y)$. This reflection transformation is an one-to-one mapping from the paths from (m, x) to $(n, -y)$ into the paths from (m, x) to (n, y) that hit $\mathbb{H}$ at some time between m and n. Since there are $\begin{pmatrix} n-m \\ \frac{(n-m)-(x+y)}{2} \end{pmatrix}$ such paths, we conclude that the number of paths going from (m, x) to (n, y) that remains in the positive axis is given by

$$\begin{pmatrix} n-m \\ \frac{(n-m)+(y-x)}{2} \end{pmatrix} - \begin{pmatrix} n-m \\ \frac{(n-m)-(x+y)}{2} \end{pmatrix}$$

This ends the proof of the second assertion.

For any $z < x \vee y$ the number of paths from (m, x) to (n, y) that remain above the axis $\mathbb{H}_z := \{(n, z),\ n \in \mathbb{N}\}$ coincides with the number of paths from $(m, x - z)$ to $(n, y - z)$ that remain above $\mathbb{H}$ (use the fact that only these paths only depend on the relative positions of x and y w.r.t. z). Using the fact that

$$\frac{(n-m) - ((x-z) + (y-z))}{2} = \frac{(n-m) - (x+y)}{2} + z$$

we end the proof of the third assertion.

The number of paths $(X_k)_{0\leq k\leq n}$ from $(0,0)$ to (n,x) s.t. $X_k \geq 0$ for any $0 \leq k \leq n$ coincides with the number of paths from $(0,0)$ to (n,x) s.t. $X_k > z = -1$ for any $0 \leq k \leq n$, which is the same as the number of paths from $(0,1)$ to $(n,x+1)$ s.t. $X_k > 0$ for any $0 \leq k \leq n$. From previous calculations, this number is given by

$$\begin{aligned}\left(\begin{array}{c} n \\ \frac{n+((x+1)-1)}{2} \end{array}\right) - \left(\begin{array}{c} n \\ \frac{n-(1+(x+1))}{2} \end{array}\right) &= \left(\begin{array}{c} n \\ \frac{n+x}{2} \end{array}\right) - \left(\begin{array}{c} n \\ \frac{n-x}{2}-1 \end{array}\right) \\ &= \left(\begin{array}{c} n \\ \frac{n+x}{2} \end{array}\right) - \left(\begin{array}{c} n \\ \frac{n+x}{2}+1 \end{array}\right).\end{aligned}$$

In the last assertion, we used the fact that

$$n - \left(\frac{n-x}{2} - 1\right) = \frac{n+x}{2} + 1 \Rightarrow \left(\begin{array}{c} n \\ \frac{n-x}{2}-1 \end{array}\right) = \frac{n!}{\left(\frac{n-x}{2}-1\right)!\left(\frac{n+x}{2}+1\right)!} = \left(\begin{array}{c} n \\ \frac{n+x}{2}+1 \end{array}\right).$$

When $x = 0$ and $n = 2m$ we have

$$\begin{aligned}&\left(\begin{array}{c} 2m \\ m \end{array}\right) - \left(\begin{array}{c} 2m \\ m-1 \end{array}\right) = \left(\begin{array}{c} 2m \\ m \end{array}\right) - \left(\begin{array}{c} 2m \\ m+1 \end{array}\right) \\ &= \frac{(2m)!}{m!^2} - \frac{(2m)!}{(m-1)!(m+1)!} \\ &= \frac{(2m+1)!}{m!(m+1)!}\,\frac{1}{2m+1}\,((m+1)-m) = \frac{1}{2m+1}\left(\begin{array}{c} 2m+1 \\ m \end{array}\right).\end{aligned}$$

To prove the last assertion, we recall that

$$\begin{aligned}&\mathbb{P}(X_{2m} = 2y\ ,\ X_k \geq 0,\ 0 \leq k \leq 2m \mid X_0 = 0) \\ &= 2^{-2m}\left[\left(\begin{array}{c} 2m \\ m+y \end{array}\right) - \left(\begin{array}{c} 2m \\ m+(y+1) \end{array}\right)\right] \\ &\Rightarrow \mathbb{P}(X_k \geq 0,\ 0 \leq k \leq 2m \mid X_0 = 0) \\ &= 2^{-2m}\ \sum_{y=0}^{m}\left[\left(\begin{array}{c} 2m \\ m+y \end{array}\right) - \left(\begin{array}{c} 2m \\ m-(y+1) \end{array}\right)\right] \\ &= 2^{-2m}\left(\left[\left(\begin{array}{c} 2m \\ m \end{array}\right) - \left(\begin{array}{c} 2m \\ m+1 \end{array}\right)\right] + \left[\left(\begin{array}{c} 2m \\ m+1 \end{array}\right) - \left(\begin{array}{c} 2m \\ m+2 \end{array}\right)\right]\right. \\ &\qquad\qquad \left. + \ldots + \left[\left(\begin{array}{c} 2m \\ m+m \end{array}\right) - 0\right]\right).\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 84:

The nonlinear Markov chain has the same form as the one defined in (7.27). The corresponding mean field particle model is defined by the system of N interacting equations

$$\xi_n^i - \xi_{n-1}^i = \frac{1}{N}\sum_{1\leq j\leq N} \log\left(1 + \left(\xi_{n-1}^j\right)^2\right) \ + \ W_n^i$$

with $1 \leq i \leq N$. In the above displayed formulae, W_n^i stands for N independent copies of W_n.

This ends the proof of the exercise.

■

Solution to exercise 85:

$$\forall n \geq 1 \qquad X_n = A_n X_{n-1} + B_n W_n$$

where X_0, W_n are $\mathbb{R}^d$-valued independent random variables such that W_n is centered.

$$\begin{aligned} X_n &= A_n A_{n-1} X_{n-2} + A_n B_{n-1} W_{n-1} + B_n W_n \\ &= (A_n A_{n-1} A_{n-2}) X_{n-3} \\ &\qquad\qquad +(A_n A_{n-1}) B_{n-2} W_{n-2} + A_n B_{n-1} W_{n-1} + B_n W_n \\ &= (A_n \ldots A_1)\, X_0 + \sum_{1 \leq p \leq n} (A_n \ldots A_{p+1})\, B_p W_p \end{aligned}$$

This implies that

$$\mathbb{E}(X_n) = A_n \ldots A_1 \; \mathbb{E}(X_0).$$

We use the decompositions

$$X_n - \mathbb{E}\,(X_n) = (A_n \ldots A_1)\,(X_0 - \mathbb{E}(X_0)) + \sum_{1 \leq p \leq n} (A_n \ldots A_{p+1})\, B_p W_p$$

and

$$\begin{aligned} &[X_n - \mathbb{E}\,(X_n)][X_n - \mathbb{E}\,(X_n)]' \\ &= \left[(A_n \ldots A_1)\,(X_0 - \mathbb{E}(X_0)) + \textstyle\sum_{1 \leq p \leq n} (A_n \ldots A_{p+1})\, B_p W_p\right] \\ &\times \left[(X_0 - \mathbb{E}(X_0))'\,(A_n \ldots A_1)' + \textstyle\sum_{1 \leq p \leq n} W_p' B_p'\,(A_n \ldots A_{p+1})'\right] \\ &= (A_n \ldots A_1)\,(X_0 - \mathbb{E}(X_0))\,(X_0 - \mathbb{E}(X_0))'\,(A_n \ldots A_1)' \\ &\qquad\qquad + \textstyle\sum_{1 \leq p,q \leq n} (A_n \ldots A_{p+1})\, B_p W_p W_q' B_q'\,(A_n \ldots A_{q+1})' + R_n \end{aligned}$$

with

$$\begin{aligned} R_n &= \textstyle\sum_{1 \leq p \leq n} (A_n \ldots A_1)\,(X_0 - \mathbb{E}(X_0))\, W_p' B_p'\,(A_n \ldots A_{p+1})' \\ &\qquad\qquad + \textstyle\sum_{1 \leq p \leq n} (A_n \ldots A_{p+1})\, B_p W_p\,(X_0 - \mathbb{E}(X_0))'\,(A_n \ldots A_1)' \end{aligned}$$

Using the fact that

$$\mathbb{E}\left((X_0 - \mathbb{E}(X_0))\, W_p'\right) = 0 = \mathbb{E}\left(W_p\,(X_0 - \mathbb{E}(X_0))'\right) \Rightarrow \mathbb{E}(R_n) = 0$$

we prove that the covariance matrix is given by

$$\begin{aligned} &\mathrm{Cov}(X_n, X_n) \\ &= (A_n \ldots A_1)\,\mathrm{Cov}(X_0, X_0)\,(A_n \ldots A_1)' \\ &\qquad\qquad + \textstyle\sum_{1 \leq p \leq n} (A_n \ldots A_{p+1})\, B_p \mathrm{Cov}(W_p, W_p)\; B_p'\,(A_n \ldots A_{p+1})'. \end{aligned}$$

For one dimensional and time homogeneous models $(A_n, B_n, \mathrm{Cov}(W_n, W_n)) = (a, b, \sigma^2)$ we find that

$$\begin{aligned}\mathbb{E}(X_n) &= a^n\ \mathbb{E}(X_0)\\ \mathrm{Var}(X_n) &= a^{2n}\ \mathrm{Var}(X_0)\ + (\sigma b)^2 \sum_{0\leq p<n} a^{2p}.\end{aligned}$$

For instance, when $a < 1$ and $\mathbb{E}(X_0) = 0$ we have $\mathbb{E}(X_n) = 0$ and

$$\mathrm{Var}(X_n) = \sigma^2\ \frac{b^2}{1-a^2}\ \left(1 - a^{2n}\right).$$

This ends the proof of the exercise.

■

Solution to exercise 86:
We check the reversible property using the fact that

$$\begin{aligned}&\tfrac{1}{\epsilon}\ (y - \sqrt{1-\epsilon}\ x)^2 + x^2\\ &= \tfrac{1}{\epsilon}\ (y^2 - 2\sqrt{1-\epsilon}\ x\ y + (1-\epsilon)x^2) +\ x^2\\ &= \tfrac{1}{\epsilon}\ (y^2 - 2\sqrt{1-\epsilon}\ x\ y + x^2).\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 87:
For any $i \geq 1$, using the fixed point equation $\pi = \pi M$ we have

$$\begin{aligned}\pi(i) &= \sum_{j\geq 0} \pi(j) M(j,i)\\ &= \pi(0)\ p(i) + \pi(i+1)\\ &= \pi(0)\ [p(i) + p(i+1)] + \pi(i+2) = \ldots = \pi(0)\ \mathbb{P}(I_1 \geq i).\end{aligned}$$

On the other hand, we have

$$\sum_{i\geq 1} \mathbb{P}(I_1 \geq i) = \sum_{j\geq i\geq 1} \mathbb{P}(I_1 = j) = \sum_{j\geq 1} \underbrace{\sum_{1\leq i\leq j} 1}_{=j} \times \mathbb{P}(I_1 = j) = \mathbb{E}(I_1).$$

This implies that

$$1 - \pi(0) = \sum_{j\geq 1} \pi(i) = \pi(0)\ \mathbb{E}(I_1) \Rightarrow \pi(0) = \frac{1}{1+\mathbb{E}(I_1)}$$

and

$$\forall i \geq 1 \qquad \pi(i) = \frac{\mathbb{P}(I_1 \geq i)}{1+\mathbb{E}(I_1)}.$$

This ends the proof of the exercise.

■

Chapter 8

Solution to exercise 90:

The transition matrix is doubly stochastic so that the invariant measure of the chain is given by the uniform measure

$$\pi = \frac{1}{4}[1,1,1,1] = \frac{1}{4}[1,1,1,1]\begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}.$$

This ends the proof of the exercise.

■

Solution to exercise 91:

The invariant measure of the chain is given by

$$\pi = \frac{1}{6}\ [1,2,2,1] = \frac{1}{6}\ [1,2,2,1]\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This ends the proof of the exercise.

■

Solution to exercise 92:

The invariant measure of the chain is given by

$$\pi = \frac{1}{8}\ [1,3,3,1] = \frac{1}{8}[1,3,3,1]\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This ends the proof of the exercise.

■

Solution to exercise 93:

The invariant measure of the chain is given by

$$\pi = \frac{1}{2(1+v)}\ [u,2v,2v,u] = \frac{1}{2(1+v)}\ [u,2v,2v,u]\begin{pmatrix} u & v & 0 & 0 \\ u/2 & v & u/2 & 0 \\ 0 & u/2 & v & u/2 \\ 0 & 0 & v & u \end{pmatrix}.$$

This ends the proof of the exercise.

■

Solution to exercise 94:
When $p < 1/2$, the invariant measure $\pi(x)$ of the chain is given by

$$\forall x \in \mathbb{N} \qquad \pi(x) = \frac{1-2p}{1-p}\left(\frac{p}{1-p}\right)^x \qquad \left(\sum_{x\geq 0}\left(\frac{p}{1-p}\right)^x = \frac{1-p}{1-2p}\right).$$

For instance, we check this claim by using the fact that

$$\pi(x-1)\, M(x-1,x) \propto \left(\frac{p}{1-p}\right)^{x-1}\frac{p}{1-p}\,(1-p) = \left(\frac{p}{1-p}\right)^x\,(1-p)\ \propto\ \pi(x)\, M(x,x-1).$$

for any $x \geq 1$. When $p \geq 1/2$ the chain has no invariant measure. This ends the proof of the exercise.

■

Solution to exercise 95:
The invariant measure $\pi(x)$ of the chain is given by

$$\forall x \in \{0,\ldots,d\} \qquad \pi(x) = \frac{1-\left(\frac{p}{1-p}\right)}{1-\left(\frac{p}{1-p}\right)^{d+1}}\left(\frac{p}{1-p}\right)^x \qquad \left(\sum_{0\leq y\leq d}\left(\frac{p}{1-p}\right)^y = \frac{1-\left(\frac{p}{1-p}\right)^{d+1}}{1-\left(\frac{p}{1-p}\right)}\right).$$

For instance, we check this claim by using the fact that

$$\pi(x-1)\, M(x-1,x) \propto \left(\frac{p}{1-p}\right)^{x-1}\frac{p}{1-p}\,(1-p) = \left(\frac{p}{1-p}\right)^x\,(1-p)\ \propto\ \pi(x)\, M(x,x-1)$$

for any $1 \leq x \leq d$.

■

Solution to exercise 96:
We have

$$\begin{aligned}
\mathrm{Det}(M-\lambda Id) &= \mathrm{Det}\begin{pmatrix} (1-p)-\lambda & p \\ q & (1-q)-\lambda \end{pmatrix} \\
&= [(1-p)-\lambda]\,[(1-q)-\lambda] - pq \\
&= \lambda^2 - \lambda\ \left((1-p)+(1-q)\right) + (1-p)(1-q) - pq \\
&= \lambda^2 - 2\lambda\left(1-\frac{p+q}{2}\right) + 1-(p+q).
\end{aligned}$$

This yields

$$\mathrm{Det}(M-\lambda Id) = \left(\lambda - \left(1-\frac{p+q}{2}\right)\right)^2 - \underbrace{\left[\left(1-\frac{p+q}{2}\right)^2 - (1-(p+q))\right]}_{=((p+q)/2)^2}.$$

Therefore

$$\begin{aligned}\mathrm{Det}(M-\lambda Id) &= \left(\lambda-\left(1-\frac{p+q}{2}\right)\right)^2-\left(\frac{p+q}{2}\right)^2\\ &= (\lambda-1)\ (\lambda-(1-(p+q)))\,.\end{aligned}$$

This shows that M has two real eigenvalues

$$\lambda_1=1 \quad \text{and} \quad \lambda_2=1-(p+q) \quad (=(1-p)-q\leq 1 \Leftarrow (1-p),q\in[0,1]).$$

The corresponding eigenvectors are $\varphi_i := \begin{pmatrix}\varphi_i(0)\\ \varphi_i(1)\end{pmatrix}$, with $i=1,2$, and are obtained by solving the linear equations

$$M(\varphi_i)=\lambda_i\ \varphi_i.$$

We observe that

$$\begin{aligned}M(\varphi_i) &:= \begin{pmatrix}M(\varphi_i)(0)\\ M(\varphi_i)(1)\end{pmatrix}\\ &= \begin{pmatrix}M(0,0) & M(0,1)\\ M(1,0) & M(1,1)\end{pmatrix}\begin{pmatrix}\varphi_i(0)\\ \varphi_i(1)\end{pmatrix}\\ &= \begin{pmatrix}1-p & p\\ q & 1-q\end{pmatrix}\begin{pmatrix}\varphi_i(0)\\ \varphi_i(1)\end{pmatrix}\\ &= \begin{pmatrix}(1-p)\varphi_i(0)+p\varphi_i(1)\\ q\varphi_i(0)+(1-q)\varphi_i(1)\end{pmatrix}=\begin{pmatrix}\lambda_i\varphi_i(0)\\ \lambda_i\varphi_i(1)\end{pmatrix}\end{aligned}$$

if and only if we have

$$\left\{\begin{array}{rcl}(1-p)\varphi_i(0)+p\varphi_i(1) &=& \lambda_i\varphi_i(0)\\ q\varphi_i(0)+(1-q)\varphi_i(1) &=& \lambda_i\varphi_i(1)\end{array}\right.$$

For $i=1$, we easily check that $\varphi_1=\begin{pmatrix}\varphi_1(0)\\ \varphi_1(1)\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}$ is a solution of the system.

For $i=2$, the linear system takes the following form

$$\left\{\begin{array}{rcl}(1-p)\varphi_2(0)+p\varphi_2(1) &=& (1-(p+q))\varphi_2(0)\,(=(1-p)\varphi_2(0)-q\varphi_2(0))\\ q\varphi_2(0)+(1-q)\varphi_2(1) &=& (1-(p+q))\varphi_2(1)\,(=(1-q)\varphi_2(1)-p\varphi_2(1))\,.\end{array}\right.$$

These equations are equivalent to the fact that

$$p\varphi_2(1)=-q\varphi_2(0)$$

from which we conclude that $\varphi_2=\begin{pmatrix}\varphi_2(0)\\ \varphi_2(1)\end{pmatrix}=\begin{pmatrix}p\\ -q\end{pmatrix}$ is a solution of the system.

We can check immediately that

$$M(\varphi_1) = \begin{pmatrix}1-p & p\\ q & 1-q\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}=1\times\begin{pmatrix}1\\ 1\end{pmatrix}=\lambda_1\times\varphi_1$$

and

$$\begin{aligned}M(\varphi_2) &= \begin{pmatrix}1-p & p\\ q & 1-q\end{pmatrix}\begin{pmatrix}p\\ -q\end{pmatrix}\\ &= \begin{pmatrix}(1-p)p-pq\\ qp-q(1-q)\end{pmatrix}=\begin{pmatrix}[1-(p+q)]\ p\\ [-p+(1-q)]\ (-q)\end{pmatrix}\\ &= [1-(p+q)]\ \begin{pmatrix}p\\ -q\end{pmatrix}=\lambda_2\times\varphi_2.\end{aligned}$$

We equip $\mathcal{B}(\{0,1\}) = \mathbb{R}^{\{0,1\}}$ with the scalar product

$$\langle f_1, f_2 \rangle := \sum_{x\in\{0,1\}} f_1(x) f_2(x) \quad \text{for any} \quad f_i = \begin{pmatrix} f_i(0) \\ f_i(1) \end{pmatrix} \quad i = 1, 2.$$

We consider the normalized eigenvectors $\overline{\varphi}_i$. They are defined by

$$\overline{\varphi}_i = \frac{\varphi_i}{|\varphi_i|} = \frac{1}{\sqrt{\varphi_i(0)^2 + \varphi_i(1)^2}} \begin{pmatrix} \varphi_i(0) \\ \varphi_i(1) \end{pmatrix}$$

for any $i = 1, 2$. This implies that

$$\overline{\varphi}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad \text{and} \quad \overline{\varphi}_2 = \begin{pmatrix} p/\sqrt{p^2+q^2} \\ -q/\sqrt{p^2+q^2} \end{pmatrix}.$$

Notice that

$$\begin{aligned} \langle \overline{\varphi}_1, \overline{\varphi}_2 \rangle = \frac{p-q}{\sqrt{2(p^2+q^2)}} = 0 \quad &\Longleftrightarrow \quad p = q \\ &\Longleftrightarrow \quad M \ \text{symmetric} \\ &\Longrightarrow \quad \overline{\varphi}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

We introduce the change of variable formula

$$P := (\overline{\varphi}_1, \overline{\varphi}_2) = \begin{pmatrix} \overline{\varphi}_1(0) & \overline{\varphi}_2(0) \\ \overline{\varphi}_1(1) & \overline{\varphi}_2(1) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & p/\sqrt{p^2+q^2} \\ 1/\sqrt{2} & -q/\sqrt{p^2+q^2} \end{pmatrix}.$$

To compute its inverse, we observe that

$$P \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

if and only if

$$\begin{cases} \frac{1}{\sqrt{2}}\, x + \frac{p}{\sqrt{p^2+q^2}}\, y &= u \\ \frac{1}{\sqrt{2}}\, x - \frac{q}{\sqrt{p^2+q^2}}\, y &= v \end{cases}$$

This shows that

$$\frac{p+q}{\sqrt{p^2+q^2}}\, y = (u-v) \Rightarrow y = (u-v)\, \frac{\sqrt{p^2+q^2}}{p+q}.$$

This also implies that

$$\frac{x}{\sqrt{2}} - \frac{q}{p+q} \underbrace{\left(\frac{p+q}{\sqrt{p^2+q^2}}\, y \right)}_{u-v} = v = \frac{x}{\sqrt{2}} - \frac{q}{p+q}\, (u-v).$$

Hence we prove that

$$\begin{aligned} x &= \sqrt{2} \left(\frac{q}{p+q}\, (u-v) + v \right) \\ &= \sqrt{2} \left(\frac{q}{p+q}\, u + v \left(1 - \frac{q}{p+q} \right) \right) \\ &= \frac{\sqrt{2}}{p+q}\, (q\, u + p\, v). \end{aligned}$$

In summary, we get

$$P^{-1} = \frac{1}{p+q} \begin{pmatrix} q\sqrt{2} & p\sqrt{2} \\ \sqrt{p^2+q^2} & -\sqrt{p^2+q^2} \end{pmatrix}.$$

We can also check that

$$\frac{1}{p+q} \begin{pmatrix} 1/\sqrt{2} & p/\sqrt{p^2+q^2} \\ 1/\sqrt{2} & -q/\sqrt{p^2+q^2} \end{pmatrix} \begin{pmatrix} q\sqrt{2} & p\sqrt{2} \\ \sqrt{p^2+q^2} & -\sqrt{p^2+q^2} \end{pmatrix} = Id.$$

We let $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. By construction, we also have that

$$\begin{aligned} PD &= \begin{pmatrix} \overline{\varphi}_1(0) & \overline{\varphi}_2(0) \\ \overline{\varphi}_1(1) & \overline{\varphi}_2(1) \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1\, \overline{\varphi}_1(0) & \lambda_2\, \overline{\varphi}_2(0) \\ \lambda_1\, \overline{\varphi}_1(1) & \lambda_2\, \overline{\varphi}_2(1) \end{pmatrix} = M \begin{pmatrix} \overline{\varphi}_1(0) & \overline{\varphi}_2(0) \\ \overline{\varphi}_1(1) & \overline{\varphi}_2(1) \end{pmatrix} = MP \end{aligned}$$

from which we conclude that

$$M = PDP^{-1}$$

and therefore

$$M^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1} \Rightarrow \ldots \Rightarrow M^n = PD^nP^{-1}.$$

$$\begin{aligned} & D^nP^{-1} \\ &= \frac{1}{p+q} \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} q\sqrt{2} & p\sqrt{2} \\ \sqrt{p^2+q^2} & -\sqrt{p^2+q^2} \end{pmatrix} \\ &= \frac{1}{p+q} \begin{pmatrix} q\sqrt{2} & p\sqrt{2} \\ \lambda_2^n\sqrt{p^2+q^2} & -\lambda_2^n\sqrt{p^2+q^2} \end{pmatrix}. \end{aligned}$$

This shows that

$$\begin{aligned} M^n &= PD^nP^{-1} \\ &= \frac{1}{p+q} \begin{pmatrix} 1/\sqrt{2} & p/\sqrt{p^2+q^2} \\ 1/\sqrt{2} & -q/\sqrt{p^2+q^2} \end{pmatrix} \begin{pmatrix} q\sqrt{2} & p\sqrt{2} \\ \lambda_2^n\sqrt{p^2+q^2} & -\lambda_2^n\sqrt{p^2+q^2} \end{pmatrix} \\ &= \frac{1}{p+q} \begin{pmatrix} q+\lambda_2^n p & p-\lambda_2^n p \\ q-\lambda_2^n q & p+\lambda_2^n q \end{pmatrix}. \end{aligned}$$

This implies that

$$\begin{aligned} M^n &= \frac{1}{p+q} \begin{pmatrix} q & p \\ q & p \end{pmatrix} + \frac{\lambda_2^n}{p+q} \begin{pmatrix} p & -p \\ -q & q \end{pmatrix} \\ &= \pi \times Id + \frac{\lambda_2^n}{p+q} \begin{pmatrix} p & -p \\ -q & q \end{pmatrix} \rightarrow_{n\uparrow\infty} \pi \times Id \end{aligned}$$

with the measure

$$\pi = \left[\frac{q}{p+q}, \frac{p}{p+q}\right].$$

When M is symmetric (i.e. $p = q$), we notice that

$$\pi = \left[\frac{1}{2}, \frac{1}{2}\right]$$

as well as

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = P' = P.$$

In this case we also have that

$$M^n = \pi \times Id + \lambda_2^n \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

This implies that

$$\begin{aligned} M^n(f)(0) &= \pi(f) + \lambda_2^n \left(\frac{1}{\sqrt{2}} f(0) - \frac{1}{\sqrt{2}} f(1) \right) \frac{1}{\sqrt{2}} \\ &= \pi(f) + \lambda_2^n \langle f, \overline{\varphi}_2 \rangle \overline{\varphi}_2(0). \end{aligned}$$

$$\begin{aligned} M^n(f)(1) &= \pi(f) - \lambda_2^n \left(\frac{1}{\sqrt{2}} f(0) - \frac{1}{\sqrt{2}} f(1) \right) \frac{1}{\sqrt{2}} \\ &= \pi(f) + \lambda_2^n \langle f, \overline{\varphi}_2 \rangle \overline{\varphi}_2(1). \end{aligned}$$

This clearly implies that

$$\forall n \geq 1 \quad \forall x \in S \qquad M^n(f)(x) = \pi(f) + \lambda_2^n \langle f, \overline{\varphi}_2 \rangle \overline{\varphi}_2(x).$$

Now, we turn to the proof of (8.9). Using the fact that

$$\langle f_1, f_2 \rangle_\pi = \frac{1}{2} \langle f_1, f_2 \rangle$$

and

$$|\sqrt{2}\, \overline{\varphi}_i|_\pi^2 = |\overline{\varphi}_i|^2 = 1$$

for any $i = 1, 2$, we show that the functions $\psi_i := \sqrt{2}\overline{\varphi}_i$ form an orthogonal basis of $l_2(\pi)$. This also shows that

$$\begin{aligned} M^n(f)(x) &= \pi(f) \ \times 1 + \lambda_2^n \frac{1}{2} \langle f, \sqrt{2}\, \overline{\varphi}_2 \rangle \sqrt{2}\, \overline{\varphi}_2(x) \\ &= \pi(\psi_1)\psi_1(x) + \lambda_2^n \langle f, \psi_2 \rangle_\pi \psi_2(x). \end{aligned}$$

■

Solution to exercise 97:
Choosing $R = R_\epsilon := 2/(1-\epsilon)$, we find that

$$\left(1 - \left(\epsilon + \frac{1}{R}\right)\right) = \left(1 - \left(\epsilon + \frac{1-\epsilon}{2}\right)\right) = 1 - \frac{1+\epsilon}{2} = \frac{1-\epsilon}{2}$$

and

$$1 - \frac{1}{1+2\rho R} = 1 - \frac{(1-\epsilon)}{(1-\epsilon)+4\rho} = \frac{4\rho}{(1-\epsilon)+4\rho}.$$

This implies that

$$W(x) \wedge W(y) \geq R_\epsilon$$

$$\Downarrow$$

$$\forall \rho \in]0,1] \qquad \frac{\|M(x,.)-M(y,.)\|_{V_\rho}}{1+V_\rho(x)+V_\rho(y)} \leq 1-\frac{1}{2}\,\frac{4\rho(1-\epsilon)}{(1-\epsilon)+4\rho} < 1.$$

In much the same way, when $W(x)\vee W(y)\leq 2/(1-\epsilon)$ we have

$$\frac{\|M(x,.)-M(y,.)\|_{V_\rho}}{1+V_\rho(x)+V_\rho(y)} \leq 1-\left(\alpha_\epsilon-\frac{8\rho}{1-\epsilon}\right) < 1$$

with $\alpha_\epsilon := 1-\beta^{(R_\epsilon)}(M)$, as soon as

$$\rho < \alpha_\epsilon\delta/8 \quad \text{with} \quad \delta := (1-\epsilon).$$

If we set $u := 4\rho/\delta$ then we have

$$\frac{1}{2}\,\frac{4\rho(1-\epsilon)}{(1-\epsilon)+4\rho} = \frac{\delta}{2}\,\left(1-\frac{1}{1+u}\right) := g(u)$$

and

$$\left(\alpha_\epsilon-\frac{8\rho}{1-\epsilon}\right) = (\alpha_\epsilon-2u) := h(u).$$

On the interval $u\in[0,\alpha_\epsilon/2]$ (so that $\rho<\alpha_\epsilon\delta/8$) the function g is increasing from $g(0)=0$ to $g(\alpha_\epsilon/2)=\frac{\delta\alpha_\epsilon}{2+\alpha_\epsilon}<1$, while the function h is decreasing from $h(0)=\alpha_\epsilon$ to $h(\alpha_\epsilon/2)=0$. These two functions intersect at a point u such that

$$(1+u)\ (2u-\alpha_\epsilon)+u\delta/2=0.$$

In other words, if we set

$$a := \frac{1}{2}\left(1-b+\frac{\delta}{4}\right) \leq \frac{1}{2} \quad \text{with} \quad b := \frac{\alpha_\epsilon}{2}$$

we need to solve the equation

$$u^2+2ua-b=(u-a)^2-\left[a^2+b\right]=0$$

with $u\in[0,b]$. This implies that

$$0<u=\sqrt{a^2+b}-a\leq b.$$

The r.h.s. inequality is checked using the fact that

$$\begin{aligned}\sqrt{a^2+b}-a\leq b \quad &\Leftrightarrow \quad a^2+b\leq a^2+b^2+2ab\\ &\Leftrightarrow \quad b\leq b\ (b+2a)=b\ \left(1+\frac{\delta}{4}\right).\end{aligned}$$

Using Taylor expansion, for any $v\geq 0$ we have

$$\sqrt{1+v}=1+\frac{v}{2\sqrt{1+v\tau_v}}\geq 1+\frac{v}{2\sqrt{1+v}}$$

for some $\tau_v\in[0,1]$. If we set $v=b/a^2$ we find that

$$\begin{aligned} u=\sqrt{a^2+b}-a\geq \tfrac{b}{2\sqrt{a^2+b}} & \\ \Rightarrow\ g(u)=h(u) \quad &\geq \quad g\left(\tfrac{b}{2\sqrt{a^2+b}}\right)\\ &= \quad \tfrac{\delta}{2}\ \tfrac{b}{2\sqrt{a^2+b}}\ \tfrac{1}{1+\frac{b}{2\sqrt{a^2+b}}}\\ &= \quad \tfrac{\delta b}{2}\ \tfrac{1}{b+2\sqrt{a^2+b}}\geq \tfrac{\delta b}{1+2\sqrt{3}}\ .\end{aligned}$$

The r.h.s. estimate follows from the fact that

$$a \vee b \leq 1/2 \;\Rightarrow\; b + \sqrt{4a^2 + 4b} \leq 1/2 + \sqrt{1+2} = (1 + 2\sqrt{3})/2.$$

Choosing $\rho = u\delta/4$, we conclude that

$$\beta_{V_\rho}(M) = \sup_{x,y} \frac{\|M(x,.) - M(y,.)\|_{V_\rho}}{1 + V_\rho(x) + V_\rho(y)} \leq 1 - g(u) \leq 1 - \frac{(1-\epsilon)(1 - \beta^{(R_\epsilon)}(M))}{2(1+2\sqrt{3})}.$$

This ends the proof of the exercise.

■

Solution to exercise 98:

Since M has positive entries, by theorem 8.1.2 all the entries of γ are positive. We let $\gamma = [\gamma(1), \gamma(2), \gamma(3)]$. We want to solve the equation

$$[\gamma(1), \gamma(2), \gamma(3)] \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} = [\gamma(1), \gamma(2), \gamma(3)].$$

In other words, we have

$$\left\{ \begin{array}{rcl} \gamma(1)\ [1 - (p_{12} + p_{13})] + \gamma(2)\ p_{21} + \gamma(3) p_{31} & = & \gamma(1) \\ \gamma(1)\ p_{12} + \gamma(2)\ [1 - (p_{21} + p_{23})] + \gamma(3) p_{32} & = & \gamma(2) \\ \gamma(1)\ p_{13} + \gamma(2)\ p_{23} + \gamma(3)\ [1 - (p_{31} + p_{32})] & = & \gamma(3) \end{array} \right.$$

which is equivalent to

$$\left\{ \begin{array}{rcl} \gamma(2)\ p_{21} + \gamma(3)\ p_{31} & = & \gamma(1)\ [p_{12} + p_{13}] \\ \gamma(1)\ p_{12} + \gamma(3)\ p_{32} & = & \gamma(2)\ [p_{21} + p_{23}] \\ \gamma(1)\ p_{13} + \gamma(2)\ p_{23} & = & \gamma(3)\ [p_{31} + p_{32}]\,. \end{array} \right.$$

This yields the system

$$\left\{ \begin{array}{rcl} \frac{\gamma(2)}{\gamma(1)}\ p_{21} + \frac{\gamma(3)}{\gamma(1)}\ p_{31} & = & [p_{12} + p_{13}] \\ p_{12} + \frac{\gamma(3)}{\gamma(1)}\ p_{32} & = & \frac{\gamma(2)}{\gamma(1)}\ [p_{21} + p_{23}]\,. \end{array} \right.$$

This shows that

$$\left\{ \begin{array}{rcl} \frac{\gamma(2)}{\gamma(1)}\ p_{21} + \frac{\gamma(3)}{\gamma(1)}\ p_{31} & = & [p_{12} + p_{13}] \\ \frac{\gamma(2)}{\gamma(1)}\ [p_{21} + p_{23}] - \frac{\gamma(3)}{\gamma(1)}\ p_{32} & = & p_{12}. \end{array} \right.$$

Multiplying the first line by $[p_{21} + p_{23}]$ and the second one by p_{21}, we find that

$$\left\{ \begin{array}{rcl} \frac{\gamma(2)}{\gamma(1)}\ p_{21}\, [p_{21} + p_{23}] \quad + \frac{\gamma(3)}{\gamma(1)}\ p_{31}\, [p_{21} + p_{23}] & = & [p_{21} + p_{23}]\ [p_{12} + p_{13}] \\ \frac{\gamma(2)}{\gamma(1)}\ p_{21}\, [p_{21} + p_{23}] \quad - \frac{\gamma(3)}{\gamma(1)}\ p_{21} p_{32} & = & p_{12} p_{21}. \end{array} \right.$$

Then we subtract the two lines to check that

$$\begin{array}{rcl} \frac{\gamma(3)}{\gamma(1)}\ (p_{31}\, [p_{21} + p_{23}] + p_{21} p_{32}) & = & [p_{21} + p_{23}]\ [p_{12} + p_{13}] - p_{12} p_{21} \\ & = & p_{21} p_{13} + p_{23}\, [p_{12} + p_{13}]\,. \end{array}$$

This implies that

$$\frac{\gamma(3)}{\gamma(1)} = \frac{p_{21} p_{13} + p_{23}\, [p_{12} + p_{13}]}{p_{31}\, [p_{21} + p_{23}] + p_{21} p_{32}}.$$

In a similar way, by multiplying the first line by p_{32} and the second one by p_{31}, we find that

$$\left\{\begin{array}{rrcl} \frac{\gamma(2)}{\gamma(1)}\ p_{21}p_{32} & +\frac{\gamma(3)}{\gamma(1)}\ p_{31}p_{32} & = & p_{32}\left[p_{12}+p_{13}\right] \\ \frac{\gamma(2)}{\gamma(1)}\ p_{31}\left[p_{21}+p_{23}\right] & -\frac{\gamma(3)}{\gamma(1)}\ p_{31}p_{32} & = & p_{12}p_{31}. \end{array}\right.$$

Adding the two lines we find that

$$\frac{\gamma(2)}{\gamma(1)}\left(p_{31}\left[p_{21}+p_{23}\right]+p_{21}p_{32}\right)=p_{32}\left[p_{12}+p_{13}\right]+p_{12}p_{31}$$

from which we conclude that

$$\frac{\gamma(2)}{\gamma(1)}=\frac{p_{32}\left[p_{12}+p_{13}\right]+p_{12}p_{31}}{p_{31}\left[p_{21}+p_{23}\right]+p_{21}p_{32}}$$

and

$$\frac{\gamma(3)}{\gamma(2)}=\frac{\gamma(3)}{\gamma(1)}\times\frac{\gamma(1)}{\gamma(2)}=\frac{p_{21}p_{13}+p_{23}\left[p_{12}+p_{13}\right]}{p_{32}\left[p_{12}+p_{13}\right]+p_{12}p_{31}}.$$

We conclude that

$$\begin{aligned} \gamma(1) & \propto p_{31}\left[p_{21}+p_{23}\right]+p_{21}p_{32} \\ & = p_{31}p_{21}+p_{23}p_{31}+p_{32}p_{21}=\prod_{(i,j)\in g_1}p_{i,j}+\prod_{(i,j)\in g_3}p_{i,j}+\prod_{(i,j)\in g_3}p_{i,j} \end{aligned}$$

with the 1-graphs $\{g_1, g_2, g_3\}$ defined on page 214 and

$$\begin{aligned} \gamma(2) & \propto p_{32}\left[p_{12}+p_{13}\right]+p_{12}p_{31} \\ \gamma(3) & \propto p_{21}p_{13}+p_{23}\left[p_{12}+p_{13}\right]. \end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 99:

For any $g\in\mathcal{G}(x)$ and $x'\not x$, the set $h=g\cup\{(x,x')\}$ is a directed graph on S with a single loop at the state x'. We let $\mathcal{L}(x')$ be the set of these graphs. We clearly have that

$$\mathcal{L}(x')=\cup_{x\neq x'}\ \left(\mathcal{G}(x)\cup\{(x,x')\}\right)=\cup_{x\neq x'}\ \left(\mathcal{G}(x')\cup\{(x',x')\}\right).$$

We set

$$M(g):=\prod_{(u,v)\in g}M(u,v).$$

In this notation, we have that

$$\begin{aligned} \sum_{x\ :\ x\neq x'}\gamma(x)M(x,x') & = \sum_{x\ :\ x\neq x'}\left[\sum_{g\in\mathcal{G}(x)}M(g)\right]M(x,x') \\ & = \sum_{x\ :\ x\neq x'}\sum_{g\in\mathcal{G}(x)}M(g\cup\{(x,x')\}) \\ & = \sum_{h\in\mathcal{L}(x')}M(h) \\ & = \sum_{x\ :\ x\neq x'}\sum_{g\in\mathcal{G}(x')}M(g\cup\{(x',x)\}) \\ & = \sum_{x\ :\ x\neq x'}\gamma(x')M(x',x)=\gamma(x')\ \left(1-M(x',x')\right). \end{aligned}$$

The end of the proof of the exercise is now clear.

■

Solution to exercise 100:

$$P(\lambda) = \mathrm{Det}\begin{pmatrix} p_{11}-\lambda & p_{12} & p_{13} \\ p_{21} & p_{22}-\lambda & p_{23} \\ p_{31} & p_{32} & p_{33}-\lambda \end{pmatrix}$$

$$= (p_{11}-\lambda)\,\mathrm{Det}\begin{pmatrix} p_{22}-\lambda & p_{23} \\ p_{32} & p_{33}-\lambda \end{pmatrix}$$
$$-p_{12}\ \mathrm{Det}\begin{pmatrix} p_{21} & p_{23} \\ p_{31} & p_{33}-\lambda \end{pmatrix} + p_{1,3}\ \mathrm{Det}\begin{pmatrix} p_{21} & p_{22}-\lambda \\ p_{31} & p_{32} \end{pmatrix}.$$

$$\begin{aligned} P(\lambda) &= (p_{11}-\lambda)\left[(p_{22}-\lambda)(p_{33}-\lambda)-p_{23}p_{32}\right] \\ &\qquad -p_{12}\left[p_{21}(p_{33}-\lambda)-p_{23}p_{31}\right] \\ &\qquad\qquad +p_{1,3}\left[p_{21}p_{32}-p_{31}(p_{22}-\lambda)\right] \\ &= -\lambda^3+\lambda^2 A+\lambda B+C \end{aligned}$$

with

$$\begin{aligned} A &= p_{11}+p_{22}+p_{33} \\ B &= p_{23}p_{32}+p_{12}p_{21}+p_{13}p_{31}-(p_{11}p_{22}+p_{11}p_{33}+p_{22}p_{33}) \\ C &= 1-(A+B). \end{aligned}$$

The last assertion follows from the fact that $P(1)=0$ so that $A+B+C=1$. We also have that

$$P(\lambda) = (1-\lambda)\ \left(\lambda^2+(1-A)\lambda+C\right)$$

and

$$\lambda^2+(1-A)\lambda+C = \left(\lambda+\left(\frac{1-A}{2}\right)\right)^2-\left(\left(\frac{1-A}{2}\right)^2-C\right).$$

We also notice that

$$\begin{aligned} 1-A &= 1-((1-p_{12}-p_{13})+(1-p_{21}-p_{23})+(1-p_{31}-p_{32})) \\ &= p_{12}+p_{21}+p_{13}+p_{31}+p_{32}+p_{23}-2. \end{aligned}$$

This yields

$$-\frac{1-A}{2} = 1-(q_{12}+q_{13}+q_{23})$$

with

$$q_{i,j} = (p_{ij}+p_{ji})/2.$$

Therefore

$$\begin{aligned} \left(\frac{1-A}{2}\right)^2 &= 1+(q_{12}+q_{13}+q_{23})^2-2(q_{12}+q_{13}+q_{23}) \\ &= 1+(q_{12}^2+q_{13}^2+q_{23}^2)+2\,(q_{12}q_{13}+q_{12}q_{23}+q_{13}q_{23}) \\ &\qquad -2(q_{12}+q_{13}+q_{23}). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
p_{11}p_{22} &= (1-p_{12}-p_{13})(1-p_{21}-p_{23}) \\
&= 1-(p_{12}+p_{13}+p_{21}+p_{23})+(p_{12}p_{21}+p_{12}p_{23}+p_{13}p_{21}+p_{13}p_{23}) \\
p_{11}p_{33} &= (1-p_{12}-p_{13})(1-p_{31}-p_{32}) \\
&= 1-(p_{12}+p_{13}+p_{31}+p_{32})+(p_{12}p_{31}+p_{12}p_{32}+p_{13}p_{31}+p_{13}p_{32}) \\
p_{22}p_{33} &= (1-p_{21}-p_{23})(1-p_{31}-p_{32}) \\
&= 1-(p_{21}+p_{23}+p_{31}+p_{32})+(p_{21}p_{31}+p_{21}p_{32}+p_{23}p_{31}+p_{23}p_{32})
\end{aligned}$$

from which we conclude that

$$\begin{aligned}
B &= p_{23}p_{32}+p_{12}p_{21}+p_{13}p_{31}-[p_{11}p_{22}+p_{11}p_{33}+p_{22}p_{33}] \\
&= -3+4\ (q_{12}+q_{13}+q_{23})-D
\end{aligned}$$

with

$$D=(p_{12}p_{23}+p_{21}p_{32}+p_{12}p_{32})+(p_{13}p_{21}+p_{21}p_{31}+p_{12}p_{31})$$
$$+(p_{13}p_{23}+p_{23}p_{31}+p_{13}p_{32})\,.$$

We also have that

$$\begin{aligned}
4q_{12}q_{23} &= (p_{12}p_{23}+p_{21}p_{32}+p_{12}p_{32})+p_{21}p_{23} \\
4q_{12}q_{13} &= (p_{21}p_{13}+p_{21}p_{31}+p_{12}p_{31})+p_{12}p_{13} \\
4q_{13}q_{23} &= (p_{13}p_{23}+p_{23}p_{31}+p_{13}p_{32})+p_{31}p_{32}
\end{aligned}$$

whence

$$D=4\,(q_{12}q_{13}+q_{12}q_{23}+q_{13}q_{23})-(p_{21}p_{23}+p_{12}p_{13}+p_{31}p_{32})$$

$$\left(\tfrac{1-A}{2}\right)^2-C$$

$$=\left(\tfrac{1-A}{2}\right)^2-1+(A+B)$$

$$=(q_{12}^2+q_{13}^2+q_{23}^2)+2\,(q_{12}q_{13}+q_{12}q_{23}+q_{13}q_{23})-2(q_{12}+q_{13}+q_{23})$$
$$+(3-(p_{12}+p_{21}+p_{13}+p_{31}+p_{32}+p_{23}))$$
$$-3+4\ (q_{12}+q_{13}+q_{23})-D$$

$$=(q_{12}^2+q_{13}^2+q_{23}^2)-2\,(q_{12}q_{13}+q_{12}q_{23}+q_{13}q_{23})$$
$$+[p_{21}p_{23}+p_{12}p_{13}+p_{31}p_{32}]\,.$$

This implies that

$$\left(\frac{1-A}{2}\right)^2-C=\Delta(q)+\delta(p)$$

with the parameters

$$\begin{aligned}
\Delta(q) &= \frac{1}{2}\left[(q_{12}-q_{13})^2+(q_{12}-q_{23})^2+(q_{13}-q_{23})^2\right] \\
\delta(p) &= [p_{12}p_{13}-q_{12}q_{13}]+[p_{21}p_{23}-q_{21}q_{23}]+[p_{31}p_{32}-q_{31}q_{32}]\,.
\end{aligned}$$

This implies that

$$\lambda_2 = (1-(q_{12}+q_{13}+q_{23})) + \sqrt{\Delta(q)+\delta(p)}$$
$$\lambda_3 = (1-(q_{12}+q_{13}+q_{23})) - \sqrt{\Delta(q)+\delta(p)}$$

with the convention $\sqrt{-a} = i\sqrt{a}$, for any $a \geq 0$. In the reversible case, we have $\delta(p) = 0$ and

$$\lambda_2 = (1-(p_{12}+p_{13}+p_{23})) + \sqrt{\Delta(p)}$$
$$\lambda_3 = (1-(p_{12}+p_{13}+_{23})) - \sqrt{\Delta(p)}.$$

We also check that

$$\lambda_2 \leq 1 \Leftrightarrow \frac{1}{2}\left[(p_{12}-p_{13})^2+(p_{12}-p_{23})^2+(p_{13}-p_{23})^2\right] \leq (p_{12}+p_{13}+p_{23}).$$

Since

$$\tfrac{1}{2}\left[(p_{12}-p_{13})^2+(p_{12}-p_{23})^2+(p_{13}-p_{23})^2\right] - (p_{12}+p_{13}+p_{23})$$
$$= -\left(p_{12}p_{13}+p_{12}p_{23}+p_{13}p_{23}\right)$$

we conclude that $\lambda_3 \leq \lambda_2 \leq \lambda_1$.

This ends the proof of the exercise.

■

Solution to exercise 101:

We use mathematical induction to prove the claim.

- When $k = 1$, the assertion is obvious .
- Suppose there are k cards below the bottom card, and that all $k!$ arrangements of these cards are equally likely.

  The next $(k+1)$-th card, to be inserted below the original bottom card, is equally likely to land in any of the $(k+1)$ possible positions among these k cards (between the original bottom card and the first of these k cards, below the first, or the second, and finally below the k-th one). By induction, these remaining k cards are in any of the possible $k!$ random orders, so that $(k+1)\times k! = (k+1)!$ of the arrangements are equally likely. This ends the proof of the induction. ■

Once the bottom card reaches the top, all possible $51!$ permutations of the cards below are equally likely. Therefore, when we are inserting it back at a random position, all $52 \times 51!$ permutations of the deck are equally likely.

This ends the proof of the exercise.

■

Solution to exercise 102:

We use the decomposition

$$M(x,dy) = (1-\epsilon)\ M_\epsilon(x,dy) + \epsilon\nu(dy) \quad \text{with} \quad M_\epsilon(x,dy) := \frac{M(x,dy)-\epsilon\nu(dy)}{1-\epsilon}$$

to check that

$$\operatorname{osc}(M(f)) = (1-\epsilon)\times\operatorname{osc}(M_\epsilon) \leq (1-\epsilon)\operatorname{osc}(f).$$

This implies that $\beta(M) \leq 1$.

This ends the proof of the exercise.

■

Solution to exercise 103:

- Since $X_0^1 = 0 = X_0^2$, we have

$$(T = 1) = \emptyset \quad \text{and} \quad (T = 2) = \left(X_1^1 = 1 = X_1^2\right).$$

This implies that

$$\mathbb{P}(T = 1) = 0 \quad \text{and} \quad \mathbb{P}(T = 2) = \mu(1)^2.$$

In much the same way, we have

$$(T = 3) = \left(X_1^1 = 2 = X_1^2\right) \quad \Longrightarrow \quad \mathbb{P}(T = 3) = \mu(2)^2.$$

- The chain X_n may return to the origin after 4 steps using only two possible random paths. More precisely, we have that

$$(X_0 = 0 \mapsto X_1 = 3 \mapsto X_2 = 2 \mapsto X_3 = 1 \mapsto X_4 = 0) = (X_1 = 3)$$

and

$$(X_0 = 0 \mapsto X_1 = 1 \mapsto X_2 = 0 \mapsto X_3 = 1 \mapsto X_4 = 0) = (X_1 = 1 = X_3).$$

This implies that

$$\begin{aligned}(T = 4) &= \{X_1^1 = 3 = X_1^2\} \\ &\cup \{X_1^1 = 3 \ \& \ X_1^2 = 1 = X_3^2\} \ \cup \ \{X_1^1 = 1 = X_3^1 \ \& \ X_1^2 = 3\}\end{aligned}$$

from which we conclude that

$$\mathbb{P}(T = 4) = \mu(3)^2 + 2\mu(3)\mu(1)^2.$$

- The chain X_n may return to the origin after 5 steps using only two possible random paths. More precisely, we have that

$$\begin{aligned}&(X_0 = 0 \mapsto X_1 = 2 \mapsto X_2 = 1 \mapsto X_3 = 0 \mapsto X_4 = 1 \mapsto X_5 = 0) \\ &= (X_1 = 2, \ X_4 = 1)\end{aligned}$$

and

$$\begin{aligned}&(X_0 = 0 \mapsto X_1 = 1 \mapsto X_2 = 0 \mapsto X_3 = 2 \mapsto X_4 = 1 \mapsto X_5 = 0) \\ &= (X_1 = 1, \ X_3 = 2).\end{aligned}$$

This implies that

$$\begin{aligned}(T = 5) &= \left\{\left(X_1^1 = 2, \ X_4^1 = 1\right) \ \& \ \left(X_1^2 = 1, \ X_3^2 = 2\right)\right\} \\ &\cup \ \left\{\left(X_2^1 = 2, \ X_4^2 = 1\right) \ \& \ \left(X_1^1 = 1, \ X_3^1 = 2\right)\right\}\end{aligned}$$

from which we conclude that

$$\mathbb{P}(T = 5) = 2 \times (\mu(2)\mu(1) \times \mu(1)\mu(2)) = 2\mu(1)^2\mu(2)^2.$$

- The chain X_n may return to the origin after 6 steps using only four possible random paths A, B, C, D described below

$$\begin{aligned}
A &= (X_0 = 0 \mapsto X_1 = 1 \mapsto X_2 = 0 \mapsto X_3 = 1 \mapsto X_4 = 0 \\
&\qquad\qquad\qquad\qquad \mapsto X_5 = 1 \mapsto X_6 = 0) \\
&= (X_1 = X_3 = X_5 = 1) \\
B &= (X_0 = 0 \mapsto X_1 = 1 \mapsto X_2 = 0 \mapsto X_3 = 3 \mapsto X_4 = 2 \\
&\qquad\qquad\qquad\qquad \mapsto X_5 = 1 \mapsto X_6 = 0) \\
&= (X_1 = 1,\ X_3 = 3) \\
C &= (X_0 = 0 \mapsto X_1 = 3 \mapsto X_2 = 2 \mapsto X_3 = 1 \mapsto X_4 = 0 \\
&\qquad\qquad\qquad\qquad \mapsto X_5 = 1 \mapsto X_6 = 0) \\
&= (X_1 = 3,\ X_3 = 1) \\
D &= (X_0 = 0 \mapsto X_1 = 2 \mapsto X_2 = 1 \mapsto X_3 = 0 \mapsto X_4 = 2 \\
&\qquad\qquad\qquad\qquad \mapsto X_5 = 1 \mapsto X_6 = 0) \\
&= (X_1 = 2 = X_4)\,.
\end{aligned}$$

Eliminating the combinations of paths that meet the origin strictly before time 6, and with some obvious abuse of notation we find that

$$\begin{aligned}
(T = 6) = &\left\{(X^1, X^2) = (A, D)\right\} \ \cup\ \left\{(X^1, X^2) = (D, A)\right\} \\
&\cup \left\{(X^1, X^2) = (B, C)\right\} \ \cup\ \left\{(X^1, X^2) = (C, B)\right\} \\
&\cup \left\{(X^1, X^2) = (B, D)\right\} \ \cup\ \left\{(X^1, X^2) = (B, D)\right\} \\
&\cup \left\{(X^1, X^2) = (C, D)\right\} \ \cup\ \left\{(X^1, X^2) = (D, C)\right\}
\end{aligned}$$

from which we conclude that

$$\begin{aligned}
\mathbb{P}(T = 6) &= 2 \times \left(\mu(1)^3 \times \mu(2)^2 + \mu(1)^2 \mu(3)^2 + 2\ \mu(1) \times \mu(3) \times \mu(2)^2\right) \\
&= 2\mu(1)\left(\mu(1)^2 \mu(2)^2 + \mu(1)\mu(3)^2 + 2\mu(3)\mu(2)^2\right).
\end{aligned}$$

- For the geometric distribution with success parameter $p \in]0, 1[$, we have

$$\mu(i) = (1 - p)^{i-1} p \Rightarrow \mu(1) = p, \quad \mu(2) = p(1 - p), \quad \mu(3) = p(1 - p)^2$$

In this situation, we find that

$$\begin{array}{rcl}
\mathbb{P}(T=1) & = & 0 \\
\mathbb{P}(T=2) & = & p^2 \\
\mathbb{P}(T=3) & = & p^2(1-p)^2 \\
\mathbb{P}(T=4) & = & p^2(1-p)^4+2p^3(1-p)^2 \\
\mathbb{P}(T=5) & = & 2p^4(1-p)^2 \\
\mathbb{P}(T=6) & = & 2p^5(1-p)^2+6p^4(1-p)^4
\end{array}$$

This ends the proof of the exercise.

■

Solution to exercise 104:
We use the decomposition

$$\sum_{n=1}^{T} X_n = \sum_{n\geq 1} X_n \ 1_{T\geq n}.$$

Notice that

$$[\forall k\geq 1 \quad \{T=k\}\in\sigma(X_1,\ldots,X_k)] \Rightarrow \{T\geq n\}=\Omega-\{T<n\}\in\sigma(X_0,\ldots,X_{n-1}).$$

This implies that

$$\begin{array}{rcl}
\mathbb{E}\left[\sum_{1\leq n\leq T} X_n\right] & = & \mathbb{E}\left(\sum_{n\geq 1} \overbrace{\mathbb{E}\left(\ X_n \ 1_{T\geq n} \mid (X_0,\ldots,X_{n-1})\right)}^{=\mathbb{E}(X)\mathbb{P}(T\geq n)}\right) \\
& = & \mathbb{E}(X)\mathbb{E}(T).
\end{array}$$

This ends the proof of the first assertion. When $\mathbb{E}(X)=0$, we have

$$\mathrm{Var}\left(\sum_{1\leq n\leq T} X_n\right) = \mathbb{E}\left(\left[\sum_{1\leq n\leq T} X_n\right]^2\right).$$

On the other hand, is we set $\overline{X}_n=\sum_{1\leq k\leq n} X_k$ we have

$$\left[\sum_{1\leq n\leq T} X_n\right]^2 = \overline{X}_T^2 = \sum_{n\geq 1} \underbrace{\left(\overline{X}_n^2-\overline{X}_{n-1}^2\right)}_{=X_n^2+2\overline{X}_{n-1}X_n} \ 1_{T\geq n}$$

with the convention $\overline{X}_0=0$. Arguing as in the proof of the first assertion, we prove the desired result. This ends the proof of the exercise.

■

Solution to exercise 105:

When $\epsilon^i = 0$, for each $1 \leq i \leq r$, the Markov chain reduces to the deterministic evolution equation

$$X_{n+1} = X_n + \tau_n \ [V(X_n) - X_n] = (1-\tau_n) \ X_n \ + \ \tau_n \ V(X_n).$$

In this situation, we have

$$\begin{array}{rcl} X_{n+1} - x^\star & = & (1-\tau_n) \ X_n \ + \ \tau_n \ V(X_n) - (\tau_n x^\star + (1-\tau_n) x^\star) \\ & = & (1-\tau_n) \ (X_n - x^\star) \ + \ \tau_n \ (V(X_n) - V(x^\star)). \end{array}$$

This implies that

$$\begin{array}{rcl} \|X_{n+1} - x^\star\| & \leq & (1-\tau_n) \ \|X_n - x^\star\| \ + \ \tau_n \ (1-\rho) \ \|X_n - x^\star\| \\ & \leq & (1-\tau_n \rho) \ \|X_n - x^\star\| \leq \ldots \leq \left\{ \prod_{0 \leq k \leq n} (1-\tau_k \rho) \right\} \ \|X_0 - x^\star\|. \end{array}$$

Recalling that $\log(1-x) \leq -x$, for any $x \in [0,1[$, we conclude that

$$\|X_n - x^\star\| \leq \exp\left(-\rho \sum_{0 \leq k \leq n} \tau_k \right) \ \|X_0 - x^\star\| \longrightarrow_{n \to \infty} 0.$$

We have

$$(\partial W)(x) = \left(\begin{array}{c} \partial_{x_1} W(x) \\ \vdots \\ \partial_{x_r} W(x) \end{array} \right) = \left(\begin{array}{c} x_1 - x_1^\star \\ \vdots \\ x_r - x_r^\star \end{array} \right)$$

and therefore

$$\begin{array}{rcl} ((\partial W)(x))^T (V(x) - x) & = & \langle x - x^\star, V(x) - x \rangle \\ & = & \langle x - x^\star, V(x) - V(x^\star) \rangle - \langle x - x^\star, x - x^\star \rangle \\ & \leq & \|x - x^\star\| \, \|V(x) - V(x^\star)\| - \|x - x^\star\|^2 \\ & = & (1-\rho) \|x - x^\star\|^2 - \|x - x^\star\|^2 = -\rho \ \|x - x^\star\|^2 < 0 \end{array}$$

for any $x \neq x^\star$. Clearly, we also have that

$$\partial W(x^\star) = 0 \quad \text{and} \quad \langle x^\star - x, V(x) - x \rangle \geq \rho \ \|x^\star - x\|^2.$$

We also have

$$\begin{array}{rcl} \mathbb{E}\left[\|(V(x) + \epsilon) - x\|^2 \right] & = & \mathbb{E}\left[\|(V(x) - V(x^\star)) + (x^\star - x) + \epsilon\|^2 \right] \\ & \leq & \mathbb{E}\left[(\|V(x) - V(x^\star)\| + \|x^\star - x\| + \|\epsilon\|)^2 \right] \\ & \leq & 3 \ \left(\|V(x) - V(x^\star)\|^2 + \|x^\star - x\|^2 + \mathbb{E}\left[\|\epsilon\|^2 \right] \right). \end{array}$$

In the last assertion we have used the variance estimate $((a+b+c)/3)^2 \leq (a^2+b^2+c^2)/3$, which is valid for any $a, b, c \in \mathbb{R}$. This shows that

$$\mathbb{E}\left[\|(V(x) + \epsilon) - x\|^2 \right] \leq \underbrace{3 \ \max\left(((1-\rho)^2 + 1), \mathbb{E}\left[\|\epsilon\|^2 \right] \right)}_{:=c} \ \left(1 + \|x^\star - x\|^2 \right).$$

We have

$$\begin{aligned}
I_{n+1} &= \|X_{n+1} - x^\star\|^2 \\
&= \langle X_n - x^\star + \tau_n \ [(V(X_n) + \epsilon_n) - X_n], X_n - x^\star + \tau_n \ [(V(X_n) + \epsilon_n) - X_n] \rangle \\
&= I_n + 2\tau_n \ \langle [(V(X_n) + \epsilon_n) - X_n], X_n - x^\star \rangle + \tau_n^2 \|[(V(X_n) + \epsilon_n) - X_n]\|^2 .
\end{aligned}$$

We have

$$\begin{aligned}
\mathbb{E}(I_{n+1} \mid \mathcal{F}_n) &= I_n + 2\tau_n \ \langle [V(X_n) - X_n], X_n - x^\star \rangle + \tau_n^2 \mathbb{E}\left(\|[(V(X_n) + \epsilon_n) - X_n]\|^2 \mid X_n \right) \\
&\leq I_n - 2\tau_n \rho \ \|X_n - x^\star\|^2 + \tau_n^2 \ c \ \left(1 + \|X_n - x^\star\|^2\right) \\
&= I_n - \tau_n (2\rho - \tau_n \ c) \ I_n + \tau_n^2 \ c .
\end{aligned}$$

We have

$$M_n := I_n + c\,\tau^2 - c \sum_{0 \leq k < n} \tau_k^2 \geq 0$$

and

$$\begin{aligned}
\mathbb{E}(M_{n+1} \mid \mathcal{F}_n) &= M_n + (\mathbb{E}(I_{n+1} \mid \mathcal{F}_n) - I_n) - c\,\tau_n^2 \\
&= M_n - \tau_n (2\rho - \tau_n \ c) \ I_n \leq M_n.
\end{aligned}$$

This shows that M_n is a non negative super-martingale such that

$$\sup_{n \geq 0} \mathbb{E}(M_n) \leq \mathbb{E}(M_0) = \mathbb{E}\left(\|X_0 - x^\star\|^2\right) + c\,\tau^2.$$

By Doob's convergence theorem (theorem 8.4.23) we conclude that $\lim_{n\to\infty} M_n = M_\infty$ exists. On the other hand, we have

$$\begin{aligned}
\mathbb{E}(M_{n+1}) &= \mathbb{E}(M_n) - \tau_n (2\rho - \tau_n \ c) \ \mathbb{E}(I_n) \\
&= \mathbb{E}(M_0) - \sum_{0 \leq k \leq n} \tau_k (2\rho - \tau_k \ c) \ \mathbb{E}(I_k) \geq 0.
\end{aligned}$$

This implies that

$$0 \leq \sum_{0 \leq k \leq n} \tau_k (2\rho - \tau_k \ c) \ \mathbb{E}(I_k) \leq \mathbb{E}(M_0) = \mathbb{E}\left(\|X_0 - x^\star\|^2\right) + c\,\tau^2.$$

Notice that

$$\tau_k \to_{k\uparrow\infty} 0 \Longrightarrow \exists k_0 \geq 1 \quad \text{s.t.} \quad 2\rho - \tau_k \ c \geq \rho.$$

This yields

$$0 \leq \sum_{0 \leq k < k_0} \tau_k (2\rho - \tau_k \ c) \ \mathbb{E}(I_k) + \rho \ \mathbb{E}\left(\sum_{k_0 \leq k} \tau_k \ I_k \right) \leq \mathbb{E}(M_0) .$$

We conclude that

$$\sum_{k_0 \leq k} \tau_k = \infty \implies \lim_{n\to\infty} I_n = 0 \implies M_\infty = 0.$$

This ends the proof of the exercise.

∎

Solution to exercise 106:
We follow the proof of the exercise 105 but replace the function $V(x)+\epsilon-x$ by $\mathcal{U}(x,\epsilon)$ and $V(x)-x$ by $U(x)$.
When $\mathcal{U}(x,\epsilon)=U(x)$, the Markov chain reduces to the deterministic evolution equation

$$X_{n+1}=X_n+\tau_n\, U(X_n).$$

In this situation, we have

$$\begin{array}{rcl}\langle X_{n+1}-x^\star,X_{n+1}-x^\star\rangle &=& \langle X_n-x^\star,X_n-x^\star\rangle+2\,\tau_n\langle X_n-x^\star,U(X_n)\rangle+\tau_n^2\langle U(X_n),U(X_n)\rangle\\ &\leq& (1-(2\rho-c\tau_n)\tau_n)\ \langle X_n-x^\star,X_n-x^\star\rangle+c\tau_n^2\\ &=& (1-a_n)\ \langle X_n-x^\star,X_n-x^\star\rangle+b_n\end{array}$$

with

$$a_n=(2\rho-c\tau_n)\tau_n\quad\text{and}\quad b_n=c\tau_n^2.$$

Notice that $2\rho-c\tau_n\geq\rho$ for $n\geq n_0$ and some sufficiently large $n_0\geq 1$. This implies that

$$\begin{array}{rcl}\|X_{n+1}-x^\star\|^2 &\leq& (1-(2\rho-c\tau_n)\tau_n)\ \|X_n-x^\star\|^2+c\tau_n^2\\ &=& (1-a_n')\ \langle X_n-x^\star,X_n-x^\star\rangle+b_n\end{array}$$

for any $n\geq n_0$ with $a_n'=\rho\tau_n$. Observe that

$$\frac{b_n}{a_n'}=\tau_n\,\frac{c}{2\rho}\to_{n\uparrow\infty}\ 0\quad\text{and}\quad\prod_{n\geq 0}(1-a_n')\leq e^{-\rho\sum_{n\geq 0}\tau_n}\to_{n\uparrow\infty}\ 0.$$

This implies that $\|X_n-x^\star\|\to_{n\uparrow\infty}\ 0$. We check this claim using the fact that

$$\forall\epsilon>0\quad\exists n(\epsilon)\geq 1\quad\text{s.t.}\quad\forall n\geq n(\epsilon)\qquad b_n\leq\epsilon\ a_n'\quad\text{and}\quad\prod_{n\geq 0}(1-a_n')\leq\epsilon.$$

In this case, for any $n\geq n(\epsilon)$ we have

$$\begin{array}{rcl}\|X_{n+1}-x^\star\|^2-\epsilon &\leq& (1-a_n')\ \|X_n-x^\star\|^2+\epsilon\ a_n'-\epsilon\\ &=& (1-a_n')\ \|X_n-x^\star\|^2-\epsilon(1-a_n')=(1-a_n')\ \left(\|X_n-x^\star\|^2-\epsilon\right).\end{array}$$

The end of the proof is now clear.
Arguing as in the proof of exercise 105 we prove that

$$\begin{array}{rcl}I_{n+1} &=& \langle X_n-x^\star+\tau_n\,\mathcal{U}(X_n,\epsilon_n),X_n-x^\star+\tau_n\,\mathcal{U}(X_n,\epsilon_n)\rangle\\ &=& I_n+2\tau_n\ \langle\mathcal{U}(X_n,\epsilon_n),X_n-x^\star\rangle+\tau_n^2\,\|\mathcal{U}(X_n,\epsilon_n)\|^2\end{array}$$

and therefore

$$\begin{array}{rcl}\mathbb{E}\left(I_{n+1}\mid\mathcal{F}_n\right) &=& I_n+2\tau_n\ \langle U(X_n),X_n-x^\star\rangle+\tau_n^2\mathbb{E}\left(\|\mathcal{U}(X_n,\epsilon_n)\|^2\ \mid X_n\right)\\ &\leq& I_n-2\tau_n\rho\ \|X_n-x^\star\|^2+\tau_n^2\ c\ \left(1+\|X_n-x^\star\|^2\right)\\ &=& I_n-\tau_n\,(2\rho-\tau_n\ c)\ I_n+\tau_n^2\ c\ .\end{array}$$

The end of the proof follows the one of exercise 105, thus it is omitted.
The final assertion is a direct application of the above results to the functions $U(x)=a-W(x)$, $\mathcal{U}(x,\epsilon)=a-\mathcal{W}(x,\epsilon)$, with $x^\star=x_a$ and $U(x_a)=a-W(x_a)=0$.

This ends the proof of the exercise.

■

Solution to exercise 107:
We have

$$X_n = X_{n-1} \times \exp(aU_n + b) \Rightarrow \mathbb{E}(X_n \mid \mathcal{F}_{n-1}) = X_{n-1}\, \mathbb{E}(\exp(aU_n + b)).$$

On the other hand, we have

$$\mathbb{E}(\exp(aU_n + b)) = e^b\, \frac{e^a + e^{-a}}{2} = 1 \Longleftrightarrow \ b \ = -\log\cosh(a).$$

This ends the proof of the exercise.

■

Solution to exercise 108:
After $2a$ consecutive jumps in the same direction the chain X_n exits the set $[-a, a]$. Thus, we have

$$\begin{aligned}
\mathbb{P}(T > 2a + n \mid T > n) &\leq \mathbb{P}(|X_{2a+n}| < a \mid |X_n| < a) \\
&= 1 - \underbrace{\mathbb{P}(|X_{2a+n}| \geq a \mid |X_n| < a)}_{\geq 1/2^{2a}} \leq 1 - \frac{1}{2^{2a}}.
\end{aligned}$$

This implies that

$$\mathbb{P}(T > 2a + n \mid T > n) = \frac{\mathbb{P}(T > 2a + n)}{\mathbb{P}(T > n)} \leq 1 - \frac{1}{2^{2a}}$$

and by induction

$$\mathbb{P}(T > 2an) \leq \left(1 - \frac{1}{2^{2a}}\right)^n.$$

By Borel Cantelli lemma we conclude that $\mathbb{P}(T < \infty) = 1$. Finally, we observe that

$$\begin{aligned}
\mathbb{E}(T) &= \sum_{k \geq 0} \mathbb{P}(T > k) \\
&= \sum_{n \geq 0} \sum_{2na \leq k < 2(n+1)a} \mathbb{P}(T > k) \\
&\leq 2a \sum_{n \geq 0} \mathbb{P}(T > 2an) \leq 2a \sum_{n \geq 0} \left(1 - \frac{1}{2^{2a}}\right)^n = a\ 2^{2a+1}.
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 109:
For any $m \leq n$ we have

$$\mathcal{F}_m \subset \mathcal{F}_n \Rightarrow \mathbb{E}(Z_n \mid \mathcal{F}_m) = \mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_n) \mid \mathcal{F}_m) = \mathbb{E}(X \mid \mathcal{F}_m) = Z_m.$$

This ends the proof of the exercise.

■

Solution to exercise 110:
We have

$$\mathbb{E}(Y_n \mid \mathcal{F}_{n-1}) = \left\{ \prod_{1\leq p\leq (n-1)} X_p \right\} \times \mathbb{E}(X_n \mid \mathcal{F}_{n-1}) = Y_{n-1}\ \mathbb{E}(X_n) = Y_{n-1}.$$

This ends the proof of the exercise.

■

Solution to exercise 111:
Using the Markov property, we have

$$\mathbb{E}(Y_{k+1} \mid \mathcal{F}_k) = \mathbb{E}(\mathbb{E}(f_n(X_n)|X_{k+1}) \mid X_k) = \mathbb{E}(f_n(X_n) \mid \ X_k) = Y_k.$$

This ends the proof of the exercise.

■

Solution to exercise 112:
We have

$$\begin{aligned} M(V)(x) - V(x) &= \mathbb{E}\left(\|x + b(x) + \sigma(x)W\|^2 - \|x\|^2 \right) \\ &= \|x + b(x)\|^2 - \|x\|^2 + \mathrm{tr}\left(\sigma(x)'\sigma(x)\right). \end{aligned}$$

The last assertion comes from the fact that

$$\begin{aligned} \|x + b(x) + \sigma(x)W\|^2 &= (x + b(x) + \sigma(x)W)'\,(x + b(x) + \sigma(x)W) \\ &= (x + b(x))'\,(x + b(x)) + 2\,(x + b(x))'\,\sigma(x)W + W'\sigma(x)'\sigma(x)W \end{aligned}$$

and for any square $(r \times r)$-matrix $A = (A_{i,j})_{1\leq i,j\leq r}$ we have

$$\mathbb{E}\left(W'AW\right) = \sum_{1\leq i,j\leq r} \underbrace{\mathbb{E}\left(W^i A_{i,j} W^j\right)}_{=A_{i,i}1_{i=j}} = \sum_{1\leq i\leq r} A_{i,i} = \mathrm{tr}(A).$$

This yields the formula

$$M(V)(x) - V(x) = 2\ \langle x, b(x)\rangle + \|b(x)\|^2 + \mathrm{tr}\left(\sigma(x)'\sigma(x)\right).$$

This implies that

$$\limsup_{\|x\|\to\infty} 2\ \langle x, b(x)\rangle + \|b(x)\|^2 + \mathrm{tr}\left(\sigma(x)'\sigma(x)\right) < 0$$

$$\Longrightarrow (\exists R > 0 \text{ s.t. } \forall \|x\| \geq R \ \text{ we have } \ M(V)(x) - V(x) \leq 0)$$

(otherwise we arrive at a contradiction). In much the same way, we have

$$\limsup_{\|x\|\to\infty} 2\ \langle x, b(x)\rangle + \|b(x)\|^2 + \mathrm{tr}\left(\sigma(x)'\sigma(x)\right) < -1$$

$$\Longrightarrow (\exists R > 0 \text{ s.t. } \forall \|x\| \geq R \ \text{ we have } \ M(V)(x) - V(x) \leq -1).$$

This ends the proof of the exercise.

∎

Solution to exercise 113: The r.h.s. condition is clearly met for constant diffusion matrices $\sigma(x) = \sigma$ for any

$$\rho_2 \geq \mathrm{tr}\,(\sigma'\sigma) \quad \text{and for any} \quad R \geq 1.$$

We can obviously choose $\rho_2 = \epsilon\ r$, where ϵ stands for the maximal eigenvalue of the symmetric matrix $\sigma'\sigma$.

The l.h.s. conditions are also met for linear drift functions $b(x) = Ax$ associated with a symmetric matrix A with a maximal eigenvalue $-a$, for some $a > 0$, with $\rho_0 = a = \sqrt{\rho_1}$. In this situation, we clearly have

$$\|b(x)\|^2 = x'A'A\ x \leq a^2\ x'x \quad \text{and} \quad \langle x, b(x)\rangle = x'A\ x \leq -a\ x'x.$$

For any $x \notin B(0,R)$ we have

$$\begin{array}{rcl} V(x) = \|x\|^2 \geq R^2 & & \\ \Rightarrow\ [M(V)(x) - V(x)]\,/V(x) & = & \frac{2}{\|x\|^2}\ \langle x, b(x)\rangle + \frac{1}{\|x\|^2}\,\|b(x)\|^2 + \frac{1}{\|x\|^2}\mathrm{tr}\,(\sigma(x)'\sigma(x)) \\ & \leq & \rho_1 + \rho_2 - 2\rho_0. \end{array}$$

We conclude that

$$\begin{array}{l} \rho_0 > (\rho_1+\rho_2)/2 \\ \Longrightarrow [M(V)(x) - V(x)] \leq -\left(\rho_0 - \dfrac{\rho_1+\rho_2}{2}\right)\ V(x) \leq -\left(\rho_0 - \dfrac{\rho_1+\rho_2}{2}\right)\ R^2. \end{array}$$

We return to the example discussed above. We further assume that $\sigma = \epsilon^{1/2}\ Id$ for some $\epsilon > 0$ s.t. $r\epsilon < 1$. In this situation we have $\rho_2 = r\ \epsilon$ and

$$\rho_0 > \frac{\rho_1+\rho_2}{2} \Longleftrightarrow 1 - (1-a)^2 > r\epsilon \Longleftrightarrow 1 - \sqrt{1-r\epsilon} < a < 1 + \sqrt{1-r\epsilon}.$$

This ends the proof of the exercise.

∎

Solution to exercise 114:

By (8.47) the stochastic process

$$\mathcal{M}_n := \mathcal{M}_n(V) = V(X_n) - V(X_0) - \sum_{1\leq p<n} (M(V) - V)\,(X_p)$$

is a martingale. By the optional stopping theorem (theorem 8.4.12) the stopped process

$$\mathcal{M}_{T_A\wedge n} \quad := \quad V(X_{T_A\wedge n}) - V(X_0) + \sum_{1\leq p<T_A\wedge n} \underbrace{(V - M(V))\,(X_p)}_{\geq 1}$$

remains a martingale. On the other hand, we have

$$\begin{array}{rcl} \mathcal{M}_{T_A\wedge n} - \mathcal{M}_{T_A\wedge(n-1)} & = & V(X_{T_A\wedge n}) - V(X_{T_A\wedge(n-1)}) + (V - M(V))\,(X_{(T_A\wedge n)-1}) \\ & \geq & V(X_{T_A\wedge n}) - V(X_{T_A\wedge(n-1)}) + 1 \end{array}$$

and

$$
\begin{aligned}
&\mathcal{N}_n := V(X_{T_A\wedge n}) - V(X_0) + (T_A \wedge n)\\
&\Rightarrow \mathcal{N}_n - \mathcal{N}_{n-1} \quad = \quad V(X_{T_A\wedge n}) - V(X_{T_A\wedge(n-1)}) + \underbrace{(T_A\wedge n) - (T_A\wedge(n-1))}_{=1_{T_A=n}}\\
&\qquad\qquad\qquad\quad \leq \quad V(X_{T_A\wedge n}) - V(X_{T_A\wedge(n-1)}) + 1 \leq \mathcal{M}_{T_A\wedge n} - \mathcal{M}_{T_A\wedge(n-1)}.
\end{aligned}
$$

We readily conclude that

$$\mathbb{E}\left(\mathcal{N}_n \mid \mathcal{F}_{n-1}\right) - \mathcal{N}_{n-1} \leq \mathbb{E}\left(\mathcal{M}_{T_A\wedge n} \mid \mathcal{F}_{n-1}\right) - \mathcal{M}_{T_A\wedge(n-1)} = 0.$$

This ends the proof of the exercise.

■

Solution to exercise 115:

Using exercise 114, the stochastic process $\mathcal{N}_n := V(X_{T_A\wedge n}) - V(X_0) + (T_A\wedge n)$ is a supermartingale. Using Fatou's lemma, this implies that

$$
\begin{aligned}
\mathbb{E}\left(V(X_{T_A}) + T_A \mid X_0 = x_0\right) - V(x_0) \quad &\leq \quad \mathbb{E}\left(\mathcal{N}_n \mid X_0 = x_0\right)\\
&\leq \quad \mathbb{E}\left(\mathcal{N}_{n-1} \mid X_0 = x_0\right) \leq \ldots \leq \mathbb{E}\left(\mathcal{N}_0 \mid X_0 = x_0\right) = 0.
\end{aligned}
$$

On the other hand, we have

$$\mathbb{E}\left(V(X_{T_A}) + T_A \mid X_0 = x_0\right) \geq \inf_{x\in A} V(x) + \mathbb{E}\left(T_A \mid X_0 = x_0\right).$$

We conclude that

$$\mathbb{E}\left(T_A \mid X_0 = x_0\right) \leq V(x_0)/\inf_{x\in A} V(x) \leq 1.$$

This ends the proof of the exercise.

■

Solution to exercise 116:

Arguing as in exercise 114 we check that the stochastic process $\mathcal{N}_n := V(X_{T_A\wedge n})$ is a super-martingale. By the optional stopping theorem (theorem 8.4.12) the stopped process $\mathcal{N}_{T_C\wedge n} := V(X_{T_C\wedge T_A\wedge n})$ is also a super-martingale. This implies that

$$V(x) = \mathbb{E}\left(\mathcal{N}_0 \mid X_0 = x\right) \geq \mathbb{E}\left(\mathcal{N}_{n\wedge T_C} \mid X_0 = x\right) = \mathbb{E}\left(V(X_{T_C\wedge T_A\wedge n}) \mid X_0 = x\right).$$

Applying Fatou's lemma, and recalling that V is non negative we check that

$$
\begin{aligned}
V(x) \quad &= \quad \mathbb{E}\left(\mathcal{N}_0 \mid X_0 = x\right)\\
&\geq \quad \mathbb{E}\left(\mathcal{N}_{T_C} \mid X_0 = x\right) = \mathbb{E}\left(V(X_{T_C\wedge T_A}) \mid X_0 = x\right)\\
&\geq \quad \mathbb{E}\left(V(X_{T_C\wedge T_A})\, 1_{T_A=\infty} \mid X_0 = x\right) \geq c\, \mathbb{P}\left(T_A = \infty \mid X_0 = x\right).
\end{aligned}
$$

This implies that

$$\mathbb{P}\left(T_A = \infty\right) \leq V(x)/c \longrightarrow_{n\to\infty} 0$$

from which we conclude that $\mathbb{P}\left(T_A < \infty \mid X_0 = x\right) = 1$.

This ends the proof of the exercise.

■

Solution to exercise 117:

Using exercise 114, the stochastic process $\mathcal{N}_n := V(X_{T_A \wedge n}) - V(X_0) + (T_A \wedge n)$ is a supermartingale.

In addition, following the proof of exercise 115 and recalling that V is a non negative function, for any $x \in S$ we have

$$\mathbb{E}\left(T_A \mid X_0 = x\right) - V(x) \leq \mathbb{E}\left(V(X_{T_A}) + T_A \mid X_0 = x\right) - V(x) \leq \mathbb{E}\left(\mathcal{N}_0 \mid X_0 = x\right) = 0.$$

We conclude that

$$\mathbb{E}\left(T_A \mid X_0 = x\right) \leq V(x).$$

We have $T'_A = \inf\{n \geq 1 \ : \ X_n \in A\}$, thus for any $x \in A$ we have

$$\mathbb{E}\left(T'_A \mid X_0 = x\right) = \mathbb{E}\left(\mathbb{E}\left(T'_A \mid X_1\right) \mid X_0 = x\right) \leq \mathbb{E}\left(V(X_1) \mid X_0 = x\right) = M(V)(x).$$

This ends the proof of the exercise.

∎

Chapter 9

Solution to exercise 119:

For any $z \in]a,b[$, we have

$$\begin{aligned}
\mathbb{P}\left(Z_{a,b} \leq z\right) &= \mathbb{P}\left(U \leq \frac{F(z)-F(a)}{F(b)-F(a)}\right) \\
&= \frac{F(z)-F(a)}{F(b)-F(a)} = \frac{\int_a^z \lambda(dz)}{\int_a^b \lambda(dz)} = \frac{\mathbb{E}\left(1_{Z\leq z} 1_{]a,b[}(Z)\right)}{\mathbb{E}\left(1_{]a,b[}(Z)\right)} \\
&= \mathbb{E}\left(1_{Z\leq z} \mid Z \in]a,b[\right) = \mathbb{P}\left(Z \leq z \mid Z \in]a,b[\right).
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 120:

The elementary transitions of the chain are given by

$$M_h(f)(v,x) = f(v, x+vh)\ e^{-(U(x+hv)-U(x))_+} + f(-v,x)\ \left(1 - e^{-(U(x+hv)-U(x))_+}\right).$$

Using a change of variable and recalling that μ is symmetric we have

$$\begin{aligned}
&\int \pi(d(v,x))\ \ e^{-(U(x+hv)-U(x))_+}\ f(v,x+vh) \\
&\propto \int\ e^{-U(x)}\ dx\ \mu(dv)\ \ e^{-(U(x+hv)-U(x))_+}\ f(v,x+vh) \\
&\propto \int\ e^{-U(y-vh)}\ dy\ \mu(dv)\ \ e^{-(U(y)-U(y-vh))_+}\ f(v,y) \quad (y = x+vh \Rightarrow x = y - vh) \\
&\propto \int\ e^{-U(y+vh)}\ dy\ \mu(dv)\ \ e^{-(U(y)-U(y+vh))_+}\ f(-v,y) \quad (\text{since } \mu \text{ is symmetric}) \\
&= \int\ e^{-U(y)}\ \mu(dv)\ e^{-(U(y+vh)-U(y))}\ dy\ \ e^{-(U(y)-U(y+vh))_+}\ f(-v,y) \\
&= \int\ e^{-U(y)}\ \mu(dv)\ e^{-(U(y+vh)-U(y))}\ dy\ \ e^{-(U(y+vh)-U(y))_-}\ f(-v,y).
\end{aligned}$$

The last assertion follows from the property $(-a)_+ = a_-$. Using the fact that

$$(U(y+vh)-U(y)) = (U(y+vh)-U(y))_+ - (U(y+vh)-U(y))_-$$

we conclude that

$$\begin{aligned}
&\int \pi(d(v,x))\ e^{-(U(x+hv)-U(x))_+}\ f(v,x+vh)\\
&\propto \int e^{-U(y)}\ dy\ \mu(dv)\ e^{-(U(y+vh)-U(y))_+}\ f(-v,y)\\
&\propto \int \pi(d(v,x))\ e^{-(U(x+vh)-U(x))_+}\ f(-v,x).
\end{aligned}$$

This clearly implies that

$$\pi M_h(f) = \int\int \pi(d(v,x))\ f(-v,x) = \pi(f).$$

Hence we see that any probability distribution $\pi(d(v,x)) \propto e^{-U(x)}\ dx\ \mu(dx)$ is an invariant measure, as soon as μ is a symmetric distribution. For instance, μ can be a centered Gaussian, a Laplace distribution $\mu(dv) \propto e^{-|v|}\ dv$, or the discrete measure $\mu \propto (\delta_{-1}+\delta_{+1})$. This ends the proof of the first part of the exercise.

Notice that

$$\begin{aligned}
&\overline{M}_h(f)(v,x)\\
&= \int f(w,x+vh)\ e^{-(U(x+hv)-U(x))_+}\mu(dw)\ + \int f(w,x)\ \left(1-e^{-(U(x+hv)-U(x))_+}\right)\ \mu(dw)\\
&= K(f)(v,x+vh)\ e^{-(U(x+hv)-U(x))_+} + K(f)(-v,x)\ \left(1-e^{-(U(x+hv)-U(x))_+}\right).
\end{aligned}$$

This clearly implies that $\overline{M}_h = M_h K$.

The last assertion stems from the fact that

$$W \overset{law}{=} -W \implies dy\ P(y,dx) = dx\ P(x,dy).$$

The r.h.s. can be checked using the fact that

$$\begin{aligned}
\int dy\ f(y)\ P(y,dx)\ g(x) &= \int f(y)\ dy\ \mu(dw)\ g(y+w)\\
&= \int f(x-w)\ dx\ \mu(dw)\ g(x)\\
&= \int g(x)\ dx\ \mu(dw)\ f(x+w) = \int dx\ f(x)\ P(x,dy)\ g(y).
\end{aligned}$$

This completes the proof of the exercise. ■

Solution to exercise 121:

The chain being defined as a Gibbs sampler, the chain Z_n is reversible w.r.t. the probability measure π. Next we provide a direct proof without using this property. We further assume that $Z_0 = (X_0, Y_0)$ is a random variable with distribution π.

$$\mathbb{E}\left(f(X_1,Y_1) \mid (X_1,Y_0)\right) = \frac{1}{2\sqrt{1-X_1^2}} \int f(X_1,y)\ 1_{[-\sqrt{1-X_1^2},+\sqrt{1-X_1^2}]}(y)\ dy.$$

This implies that

$$\begin{aligned}
&\mathbb{E}\left(f(X_1,Y_1)\mid (X_0,Y_0)\right)\\
&=\mathbb{E}\left(\tfrac{1}{2\sqrt{1-X_1^2}}\ \int\ f(X_1,y)\ 1_{[-\sqrt{1-X_1^2},+\sqrt{1-X_1^2}]}(y)\ dy\mid (X_0,Y_0)\right)\\
&=\int\ f(x,y)\ \tfrac{1}{2\sqrt{1-x^2}}\ 1_{[-\sqrt{1-x^2},+\sqrt{1-x^2}]}(y)\\
&\qquad\qquad\times\tfrac{1}{2\sqrt{1-Y_0^2}}1_{[-\sqrt{1-Y_0^2},+\sqrt{1-Y_0^2}]}(x)\ dxdy.
\end{aligned}$$

Using the fact that

$$\mathbb{P}\left(Y_0\in dy_0\right)=2\sqrt{1-y_0^2}\ 1_{[0,1]}(y_0)\ dy_0$$

we conclude that

$$\begin{aligned}
&\mathbb{E}\left(f(X_1,Y_1)\right)\\
&=\mathbb{E}\left(\tfrac{1}{2\sqrt{1-X_1^2}}\ \int\ f(X_1,y)\ 1_{[-\sqrt{1-X_1^2},+\sqrt{1-X_1^2}]}(y)\ dy\mid (X_0,Y_0)\right)\\
&=\int\ f(x,y)\ \frac{1}{2\sqrt{1-x^2}}\ 1_{[-\sqrt{1-x^2},+\sqrt{1-x^2}]}(y)\\
&\qquad\qquad\times\left[\int\underbrace{1_{[-\sqrt{1-y_0^2},+\sqrt{1-y_0^2}]}(x)}_{=1_{[-\sqrt{1-x^2},+\sqrt{1-x^2}]}(y_0)}\ 1_{[0,1]}(y_0)\ dy_0\right]\ dxdy.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\mathbb{E}\left(f(X_1,Y_1)\right)\\
&=\int\ f(x,y)\ 1_{[-\sqrt{1-x^2},+\sqrt{1-x^2}]}(y)\ dxdy=\pi(f).
\end{aligned}$$

Let (X,Y) be a uniform random variable the circle $\{(x,y)\ :\ x^2+y^2=1\}$. Observe that

$$\begin{aligned}
\mathbb{P}\left(X\in dx,Y\in dy\right)\ &=\ \overbrace{\frac{1}{\pi}\ \frac{1}{\sqrt{1-x^2}}\ 1_{]-1,1[}(x)\ dx}^{=\mathbb{P}(X\in dx)}\ \times\ \overbrace{\frac{1}{2}\left[\delta_{-\sqrt{1-x^2}}+\delta_{\sqrt{1-x^2}}\right](dy)}^{=\mathbb{P}(Y\in dy\ \mid\ X=x)}\\
&=\ \underbrace{\frac{1}{\pi}\ \frac{1}{\sqrt{1-y^2}}\ 1_{]-1,1[}(y)\ dy}_{=\mathbb{P}(Y\in dy)}\ \times\ \underbrace{\frac{1}{2}\left[\delta_{-\sqrt{1-y^2}}+\delta_{\sqrt{1-y^2}}\right](dx)}_{=\mathbb{P}(X\in dx\ \mid\ Y=y)}
\end{aligned}$$

The Gibbs samplers are based on sampling the second coordinate Y given the first X, and the first given the second. Starting from $\left(\begin{array}{c}x\\y\end{array}\right)$ it is immediate to check that any of these Gibbs samplers gets stuck on the four states

$$\left\{\left(\begin{array}{c}x\\y\end{array}\right),\left(\begin{array}{c}x\\-y\end{array}\right),\left(\begin{array}{c}-x\\-y\end{array}\right),\left(\begin{array}{c}-x\\y\end{array}\right)\right\}.$$

Of course the random states $\begin{pmatrix} (+/\text{-})\ X \\ (+/\text{-})\ Y \end{pmatrix}$ are uniform random variables the circle $\{(x,y)\ :\ x^2+y^2=1\}$, but starting from a given state $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ the resulting Gibbs samplers fail to converge to the desired uniform target measure.

This ends the proof of the exercise.

■

Solution to exercise 122:

$$\forall x \in \mathbb{N}-\{0,1\} \qquad \frac{\pi(x-1)K(x-1,x)}{\pi(x)K(x,x-1)} = \frac{\frac{\lambda^{x-1}}{(x-1)!}}{\frac{\lambda^x}{x!}} = \frac{x}{\lambda} \Rightarrow a(x,x-1) = 1 \wedge \frac{x}{\lambda}$$

and

$$\forall x \in \mathbb{N}-\{0\} \qquad \frac{\pi(x+1)K(x+1,x)}{\pi(x)K(x,x+1)} = \frac{\frac{\lambda^{x+1}}{(x+1)!}}{\frac{\lambda^x}{x!}} = \frac{\lambda}{x+1} \Rightarrow a(x,x+1) = 1 \wedge \frac{\lambda}{x+1}.$$

In addition

$$\frac{\pi(0)K(0,1)}{\pi(1)K(1,0)} = a(1,0) = 2\ \frac{\pi(0)}{\pi(1)} = 2/\lambda \Rightarrow a(1,0) = 1 \wedge \frac{2}{\lambda}$$

and

$$\frac{\pi(1)K(1,0)}{\pi(0)K(0,1)} = \frac{1}{2}\ \frac{\pi(1)}{\pi(0)} = \frac{\lambda}{2} \Rightarrow a(0,1) = 1 \wedge \frac{\lambda}{2}.$$

This ends the proof of the exercise.

■

Solution to exercise 123:

We have

$$K(x,y) = K(y,y) \quad \Rightarrow \quad \frac{\pi(y)K(y,x)}{\pi(x)K(x,y)} = \frac{\pi(y)}{\pi(x)} = \exp\left(-\frac{1}{2\sigma^2}((y-m)^2 - (x-m)^2)\right)$$
$$\Rightarrow \quad a(x,y) = 1 \wedge \exp\left[\frac{1}{\sigma^2}\,(x-y)\left(\frac{x+y}{2} - m\right)\right].$$

This ends the proof of the exercise.

■

Solution to exercise 124

The proposal transition $K_\epsilon(x,dy)$ is symmetric in the sense that

$$dx\ K_\epsilon(x,dy) = dy\ K_\epsilon(y,dx).$$

The Metropolis-Hastings ratio resumes to

$$\frac{p(y)dy\ K_\epsilon(y,dx)}{p(x)\ dx\ K_\epsilon(x,dy)} = \frac{p(y)}{p(x)}$$

as soon as $p(x) > 0$. The resulting transition of MCMC chain is defined as follows: Given some x_0 s.t. $p(x_0) > 0$, we pick randomly a state y_0 in $B(x_0, \epsilon)$. We move to $x_1 = y_0$ with probability $a(x_0, y_0) := 1 \wedge \frac{p(y_0)}{p(x_0)}$. Otherwise we stay in x_0; that is we set $x_1 = x_0$.

This ends the proof of the exercise.

■

Solution to exercise 125:
For any bounded function f we have

$$\eta M(f) = \int \nu(du) \int dt \left[\int p(x) \frac{p(x+tu)}{\int p(x+su)\, ds} f(x+tu)\, dx \right].$$

The change of variable $y = x + tu$ ($\Rightarrow dx = dy$) in the x-integral yields

$$\begin{aligned}
\eta M(f) &= \int \nu(du) \int dt \left[\int p(y-tu) \frac{p(y)}{\int p(y+(s-t)u)\, ds} f(y)\, dy \right] \\
&= \int p(y)\, dy \left[\int dt \frac{p(y-tu)}{\int p(y+(s-t)u)\, ds} \right] f(y)\, dy.
\end{aligned}$$

On the other hand, another change of variable $\tau = t - s$ in the s-integral shows that

$$\int dt \frac{p(y-tu)}{\int p(y+(s-t)u)\, ds} = \int dt \frac{p(y-tu)}{\int p(y-\tau u)\, d\tau} = 1.$$

We conclude that $\eta M = \eta$. This above formulae are also valid if we replace ν by $\nu(du) = \delta_{u_0}(du)$, for some $u_0 \in \mathbb{S}^{r-1}$. In this situation we have

$$M(f)(x) = \eta_{x,u_0}(f) \Rightarrow \int \eta(dx)\, \eta_{x,u_0}(dz) = (\eta M)(dz) = \eta(dz).$$

When the density $p(x)$ is supported by an open bounded subset $S \subset \mathbb{R}^r$ we replace $\eta_{x,u}(dz)$, with $x \in S$ by the distribution

$$\eta_{x,u}(dz) = \frac{\int_{\mathcal{S}(x,u)} p(x+tu)\, dt}{\int_{\mathcal{S}(x,u)} p(x+su)\, ds} \delta_{x+tu}(dz)$$

with

$$\mathcal{S}(x,u) = \{t \in \mathbb{R} \ : \ x + tu \in S\}.$$

The above hit-and-run sampler will always have the desired target measure for any choice of the measure ν, as soon as the change of variable and the restriction of η to the line $A(x,u)$ are well defined. For instance we can choose $\nu(du)$ with a positive density and η with a positive and bounded density. Some clear drawbacks of these samplers are strong correlations and jams around the corner of the set S.

When $r = 2$, using the change of variables

$$\psi_x : (\theta, t) \in (]0, 2\pi] \times]0, \infty[) \mapsto y = \psi_x(\theta, t) \in \mathbb{R}^2 - \{y \ : \ y_2 = x_2\}$$

given by

$$y := \psi_x(\theta, t) := \begin{pmatrix} y_1 & = & x_1 + t\ \cos(\theta) \\ y_2 & = & x_2 + t\ \sin(\theta) \end{pmatrix}$$

$$\Rightarrow t = t_x(y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} \quad \text{and} \quad \theta = \theta_x(y) = \arctan \tfrac{y_2 - x_2}{y_1 - x_1}$$

$$\Rightarrow \left| \begin{pmatrix} \partial_{y_1} t_x(y) & \partial_{y_2} t_x(y) \\ \partial_{y_1} \theta_x(y) & \partial_{y_2} \theta_x(y) \end{pmatrix} \right| = \tfrac{1}{\|x-y\|}$$

and cutting the $[0,2\pi]$-integral w.r.t. the angles $]0,\pi/2[\cup]3\pi/2,2\pi[$ and $]\pi/2,3\pi/2[$ we find that

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{[0,\infty[} f\left(\begin{array}{c} x_1+t\ \cos(\theta) \\ x_2+t\ \sin(\theta)\end{array}\right)\ dt\ d\theta = \frac{1}{\pi}\int\ f(y)\ \|x-y\|^{-1}\ dy.$$

We set $u_\theta = \left(\begin{array}{c}\cos(\theta)\\ \sin(\theta)\end{array}\right)$ and $v_{x,y} = \frac{y-x}{\|y-x\|}$.

In this notation, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{[0,\infty[}\frac{p(x+tu_\theta)f(x+tu_\theta)}{\int_{\mathcal{S}(x,u_\theta)}p(x+su_\theta)\ ds}\ dt\ d\theta = \frac{1}{\pi}\int\ f(y)\ \frac{p(y)\ \|x-y\|^{-1}}{\int_{\mathcal{S}(x,v_{x,y})}p(x+tv_{x,y})\ dt}\ dy.$$

We conclude that

$$\overline{M}(x,dy) = \overline{m}(x,y)\ dy$$

with the probability density

$$\overline{m}(x,y) = \frac{1}{\pi}\ \frac{p(y)}{\|x-y\|}\ \Big/\int_{\mathcal{S}(x,v_{x,y})}p(x+sv_{x,y})\ ds.$$

When p is bounded, we have

$$\overline{m}(x,y)\geq(\pi\mathrm{diam}(S)\|p\|\delta)^{-1}\ p(y)$$

with

$$\mathrm{diam}(S) := \sup_{(x,y)\in S^2}\|x-y\| \quad \text{and} \quad \sup_{x,y}\ \lambda(\mathcal{S}(x,v_{x,y})) = \delta.$$

Here $\lambda(\mathcal{S}(x,v_{x,y}))$ is the Lebesgue measure of the set $\mathcal{S}(x,v_{x,y})$. Using (8.15) and theorem 8.2.13 we conclude that

$$\left\|\mu\overline{M}^n-\eta\right\|_{tv}\leq(1-\epsilon)^n\ \|\mu-\eta\|_{tv}$$

for any initial distribution μ on S.

We further assume that $\nu = \frac{1}{r}\sum_{1\leq i\leq r}\delta_{e_i}$, where $e_i = (1_i(j))_{1\leq j\leq r}$ stands for the r unit vectors of $\mathbb{R}^r$. In this situation, for each selected $1\leq i\leq r$ we have

$$\eta_{x,e_i}(dz) = \frac{\int\ p(x+te_i)\ dt}{\int p(x+se_i)\ ds}\ \delta_{x+te_i}(dz).$$

On the other hand, we have

$$\int p(x+se_i)\ ds = \int p(x_1,\ldots,x_{i-1},x_i+s,x_{i+1},\ldots,x_r)\ ds = p_{-i}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_r)$$

where p_{-i} stands for the density of X_{-i}. In much the same way, we have

$$\eta_{x,e_i}(f)\propto\int\ p(x+te_i)\ f(x+te_i)\ dt = \int p(x_1,\ldots,x_{i-1},y_i,x_i+s,x_{i+1},\ldots,x_r)f(y_i)\ dy_i.$$

This shows that

$$\eta_{X,e_i} = \mathrm{Law}\left(X_i\mid X_{-i}\right).$$

The resulting Markov transition coincides with the one discussed in (9.13). This ends the proof of the exercise. ■

Solution to exercise 126:
In terms of Bayes' formula we have

$$p(d(x,y)) = \underbrace{\eta(dx)\ K(x,dy)}_{p(dx)p(dy|x)} = \underbrace{(\eta K)(dy)\ M(y,dx)}_{p(dy)p(dx|y)}.$$

In this notation a Gibbs sampler wit target measure $p(d(x,y))$ is defined by the following synthetic diagram:

$$\left(\begin{array}{c} X_n = x \\ Y_n = y \end{array}\right) \to \left(\begin{array}{ccl} X_{n+\frac{1}{2}} & = & x' \sim (X \mid Y = y) \\ Y_{n+\frac{1}{2}} & = & y \end{array}\right) \to \left(\begin{array}{ccl} X_{n+1} & = & x' \\ Y_{n+1} & = & y' \sim (Y \mid X = x') \end{array}\right).$$

This ends the proof of the exercise.

■

Solution to exercise 127:
We have

$$\mathbb{E}\left(f(Y) \mid X\right) = \frac{1}{2}\ [f(X+1)+f(X-1)] = \int\ f(y)\ \frac{1}{2}\ [\delta_{X-1}+\delta_{X+1}]\,(dy)$$

and by a change of variables

$$\begin{aligned} \mathbb{E}\left(f(Y)\right) &= \frac{1}{2}\int\ [f(x+1)+f(x-1)]\ p(x)\ dx \\ &= \int\ f(y)\ \frac{1}{2}\,[p(y+1)+p(y-1)]\ dy. \end{aligned}$$

This shows that

$$\mathbb{P}(Y \in dy) = \frac{1}{2}\,[p(y+1)+p(y-1)]\ dy \quad \text{and} \quad \mathbb{P}\left(Y \in dy \mid X\right) = \frac{1}{2}\ [\delta_{X-1}+\delta_{X+1}]\,(dy).$$

We set

$$h(Y) = \frac{p(Y-1)}{p(Y+1)+p(Y-1)}\ f(Y-1) + \frac{p(Y+1)}{p(Y+1)+p(Y-1)}\ f(Y+1).$$

In this notation, we have

$$\begin{aligned} \mathbb{E}\left(g(Y)h(Y)\right) &= \mathbb{E}\left(g(Y)\left[\frac{p(Y-1)}{p(Y+1)+p(Y-1)}\ f(Y-1)+\right.\right. \\ &\qquad\qquad \left.\left.\frac{p(Y+1)}{p(Y+1)+p(Y-1)}\ f(Y+1)\right]\right) \\ &= \frac{1}{2}\int\ g(y)\ p(y-1)\ f(y-1)\ dy + \frac{1}{2}\int\ g(y)\ p(y+1)\ f(y+1)\ dy. \end{aligned}$$

This implies that

$$\begin{aligned} \mathbb{E}\left(g(Y)h(Y)\right) &= \frac{1}{2}\int\ g(x+1)\ p(x)\ f(x)\ dx + \frac{1}{2}\int\ g(x-1)\ p(x)\ f(x)\ dx \\ &= \int\ \frac{1}{2}\,[g(x+1)+g(x-1)]\ p(x)\ f(x)\ dx \\ &= \int\ \mathbb{E}(g(Y) \mid X = x)\ f(x)\ p(x)\ dx = \mathbb{E}(f(X)g(Y)). \end{aligned}$$

We conclude that

$$h(Y) = \mathbb{E}\left(f(X) \mid Y\right).$$

The target distribution $\pi = \text{Law}(X, Y)$ can be written as follows

$$\begin{aligned}\mathbb{P}((X,Y) \in d(x,y)) &= \mathbb{P}\left(Y \in dy \mid X = x\right)\ \mathbb{P}(X \in dx) \\ &= \mathbb{P}\left(X \in dx \mid Y = y\right)\ \mathbb{P}(Y \in dy)\end{aligned}$$

with the couple of conditional distributions

$$\begin{aligned}\mathbb{P}\left(Y \in dy \mid X\right) &= \frac{1}{2}\ \left[\delta_{X-1} + \delta_{X+1}\right](dy) \\ \mathbb{P}\left(X \in dx \mid Y\right) &= \frac{p(Y-1)}{p(Y+1)+p(Y-1)}\ \delta_{Y-1}(dx) + \frac{p(Y+1)}{p(Y+1)+p(Y-1)}\ \delta_{Y+1}(dx).\end{aligned}$$

A Gibbs sampler with target measure π is defined by the following synthetic diagram:

$$\left(\begin{array}{c} X_n = x \\ Y_n = y \end{array}\right) \rightarrow \left(\begin{array}{ccl} X_{n+\frac{1}{2}} &=& x' \sim (X \mid Y = y) \\ Y_{n+\frac{1}{2}} &=& y \end{array}\right) \rightarrow \left(\begin{array}{ccl} X_{n+1} &=& x' \\ Y_{n+1} &=& y' \sim (Y \mid X = x') \end{array}\right).$$

When U is an uniform random variable on $\{-h, +h\}$ the same analysis applies. In this case we have

$$\mathbb{P}(Y \in dy) = \frac{1}{2}\left[p(y+h) + p(y-h)\right]\ dy \quad \text{and} \quad \mathbb{P}\left(Y \in dy \mid X\right) = \frac{1}{2}\ \left[\delta_{X-h} + \delta_{X+h}\right](dy)$$

as well as

$$\mathbb{P}\left(X \in dx \mid Y\right) = \frac{p(Y-h)}{p(Y+h)+p(Y-h)}\ \delta_{Y-h}(dx) + \frac{p(Y+h)}{p(Y+h)+p(Y-h)}\ \delta_{Y+h}(dx).$$

This ends the proof of the exercise.

■

Solution to exercise 128:
We have the conditional distributions

$$\mathbb{P}\left((U,Z) \in d(u,z) \mid X\right) = \nu(du) \int \mu(dt)\ \delta_{X+tu}(dz)$$

and

$$\mathbb{P}\left(Z \in dz \mid U\right) = \left[\int\ p(z - tU)\ \mu(dt)\right]\ dz.$$

We check these formula using the fact that

$$\begin{aligned}\mathbb{E}\left(f(U,Z) \mid X\right) &= \int \nu(du)\ \mu(dt)\ f\left(u, X + tu\right). \\ \mathbb{E}\left(f(U,Z)\right) &= \int p(x)\ dx\ \nu(du)\ \mu(dt)\ f\left(u, x + tu\right) \\ &= \int\ \nu(du)\ \left[\int \mu(dt)\ p(z - tu)\right]\ dz\ f\left(u, z\right). \\ \mathbb{E}\left(g(Z) \mid U\right) &= \int p(x)\ dx\ \mu(dt)\ g\left(x + tU\right) = \int\ \left[\int p(z - tU)\ \mu(dt)\right]\ g\left(z\right)\ dz.\end{aligned}$$

We set
$$h(U,Z) = \frac{\int\ p(Z-sU)\ f(Z-sU)\ \mu(ds)}{\int\ p(Z-s'U)\ \mu(ds')}.$$
In this notation, we have
$$\begin{aligned}
&\mathbb{E}\left(g(U,Z)\ \ h(U,Z)\right)\\
&= \int\ g(u,z)\ \ h(u,z)\ \ \nu(du)\ \left[\int\ \ p(z-t'U)\ \mu(dt')\right]\ dz\\
&= \int\ g(u,z)\ \frac{\int\ p(z-su)\ \ f(z-su)\ \ \mu(ds)}{\int\ p(z-s'u)\ \ \mu(ds')}\ \nu(du)\ \left[\int\ \ p(z-t'u)\ \mu(dt')\right]\ dz\\
&= \int\ \left[\int\ g(u,z)\ \ p(z-su)\ \ f(z-su)\ \ dz\right]\ \nu(du)\ \mu(ds).
\end{aligned}$$
This implies that
$$\begin{aligned}
&\mathbb{E}\left(g(U,Z)\ \ h(U,Z)\right)\\
&= \int\ \left[\int\ g(u,x+su)\ \ p(x)\ \ f(x)\ \ dx\right]\ \nu(du)\ \mu(ds) = \mathbb{E}\left(g(U,Z)\ \ f(X)\right)
\end{aligned}$$
from which we conclude that
$$\mathbb{E}\left(f(X)\mid U,Z\right) = h(U,Z).$$
Equivalently, we have
$$\mathbb{P}\left(X\in dx\mid (U,Z)\right) = \int\ \frac{p(Z-tU)\ \ \mu(dt)}{\int\ p(Z-sU)\ \ \mu(ds)}\ \delta_{Z-tU}(dx).$$

A Gibbs sampler wit target measure $\pi = \text{Law}(X,Y)$ is defined by the following synthetic diagram:
$$\begin{aligned}
&\left(\begin{array}{ccl} X_n &=& x\\ Y_n &=& (U_n,Z_n)=(u,z)\end{array}\right)\\
&\to \left(\begin{array}{ccl} X_{n+\frac{1}{2}} &=& x'\sim (X\mid (U,Z)=(u,z))\\ Y_{n+\frac{1}{2}} &=& \left(U_{n+\frac{1}{2}},Z_{n+\frac{1}{2}}\right)=(u,z)\end{array}\right)\\
&\to \left(\begin{array}{ccl} X_{n+1} &=& x'\\ Y_{n+1} &=& =(U_{n+1},Z_{n+1})=(u',z')\sim ((U,Z)\mid X=x')\end{array}\right).
\end{aligned}$$
This ends the proof of the exercise.

■

Solution to exercise 129:
Using the change of variables $g(x)=y$ (recalling that the Lebesgue measure is invariant w.r.t. rotations) we have
$$\mathbb{E}\left(f(g(X))\right)\ \ \propto\ \ \int_S\ f(g(x))\ dx = \int_{g(S)}\ f(y)\ dy = \int\ f(y)\ \eta_g(dy).$$

This shows that $g(X)$ has the uniform distribution η_g on $g(S)$; that is, we have that

$$\eta_g = \mathrm{Law}(g(X)).$$

For any $x \in g(S)$ we let $T_g.x$ be a random variable with distribution $M_g(x,dy)$ on $g(S)$. In this notation we have

$$\overline{M}_g(x,dy) = \mathbb{P}(g^{-1}T_g.g(x) \in dy).$$

For any $g \in G$ and any function f on S we have

$$\begin{aligned}
\mathbb{E}\left(f\left(g^{-1}T_g.g(X)\right)\right) &= \int_S \eta(dx)\ f\left(g^{-1}T_g.g(x)\right) \\
&= \int_S \eta(dx) \int_{g(S)} \delta_{g(x)}(dx') \int_{g(S)} M_g(x',dy') \int_S \delta_{g^{-1}(y')}(dy)\ f(y) \\
&= \int_{g(S)} \underbrace{\int_S \eta(dx)\ M_g(g(x),dy')}_{=\int_{g(S)} \eta_g(dx')\ M_g(x',dy')=\eta_g(dy')} f\left(g^{-1}(y')\right)
\end{aligned}$$

which yields the fixed point formula

$$\mathbb{E}\left(f\left(g^{-1}T_g.g(X)\right)\right) \propto \int_{g(S)} f\left(g^{-1}(y)\right)\ dy = \int_S f(x)\ dx = \mathbb{E}(f(X)).$$

We conclude that

$$\eta = \eta\overline{M}_g \Leftrightarrow g^{-1}T_g.g(X) \stackrel{law}{=} X.$$

By Fubini's theorem we check immediately that

$$\forall g \in G \quad \eta = \eta\overline{M}_g \Longrightarrow \eta = \eta\overline{M} \quad \text{with} \quad \overline{M}(x,dy) = \int_G \mu(dg)\ \overline{M}_g(x,dy).$$

This ends the proof of the first part of the exercise. We further assume that

$$\eta(dx) \propto p(x)\ 1_S(x)\ dx$$

for some density function $p(x)$ w.r.t. the Lebesgue measure dx. We also set

$$\eta_g = \mathrm{Law}(g(X)) \Leftrightarrow \mathbb{P}(g(X) \in dx) := \eta_g(dx) \propto 1_{g(S)}(x)\ p(g^{-1}(x))\ dx.$$

Notice that

$$\int_{g(S)} p(g^{-1}(x))\ dx = \int_S p(x)\ dx.$$

In this situation, we have

$$\begin{aligned}
\mathbb{E}\left(f\left(g(X)\right)\right) &\propto \int_S p(x)\ f(g(x))\ dx \\
&= \int_{g(S)} p(g^{-1}(x))\ f(x)\ dx \propto \int_{g(S)} \eta_g(dx)\ f(x).
\end{aligned}$$

Assume that $\eta_g = \eta_g M_g$. In this situation, by Fubini's theorem we also have

$$\begin{aligned}
\eta\overline{M}_g(f) &= \int_S \eta(dx) \int_{g(S)} M_g(g(x),dy) f(g^{-1}(y)) \\
&= \int_{g(S)} \eta_g(dz) M_g(z,dy) f(g^{-1}(y)) = \int_{g(S)} \eta_g(dy) f(g^{-1}(y)) = \eta(f).
\end{aligned}$$

Arguing as above, we conclude that $\eta = \eta\overline{M}$.

The extension of these formulae to any transformation group G and any target distribution η follows the same lines of arguments. Let η_g be the distribution of $g(X)$ with a random sample X with distribution η. In this case, we also have

$$\begin{aligned}
\mathbb{E}\left(f\left(g^{-1}T_g.g(X)\right)\right) &= \int_S \eta(dx)\ f\left(g^{-1}T_g.g(x)\right) \\
&= \int_{g(S)} \underbrace{\int_S \eta(dx) \int_{g(S)} \delta_{g(x)}(dx')\quad M_g(x', dy')}_{=\eta_g(dy')} \int_S \delta_{g^{-1}(y')}(dy)\ f(y).
\end{aligned}$$

This implies that

$$\mathbb{E}\left(f\left(g^{-1}T_g.g(X)\right)\right) = \mathbb{E}\left(f\left(g^{-1}(g(X))\right)\right) = \mathbb{E}\left(f\left(X\right)\right).$$

This ends the proof of the exercise.

∎

Solution to exercise 130:

Observe that

$$\begin{aligned}
&\int_S p(x)\ dx\ K(x, dg)\ f(g(x)) \\
&= \int_G \int_S p(x)\ dx\ \frac{p(g(x))\ \ |\partial g(x)/\partial x|}{\int\ p(h(x))\ \ |\partial h(x)/\partial x|\ \ \nu(dh)}\ f(g(x))\ \nu(dg).
\end{aligned}$$

Using the change of variables

$$x = g^{-1}(y) \quad \Rightarrow \quad dx = \left|\partial g^{-1}(y)/\partial y\right|\ dy$$

and recalling that

$$|\partial g/\partial x|_{h(y)}\, |\partial h(y)/\partial y| = |\partial(g\circ h)/\partial x|_y$$

we check that

$$\begin{aligned}
&\int_S p(x)\ dx\ K(x, dg)\ f(g(x)) \\
&= \int_G \int_S p(g^{-1}(y))\ \ dy \frac{p(y)\ \ |\partial g/\partial x|_{g^{-1}(y)} \left|\partial g^{-1}(y)/\partial y\right|}{\int\ p(h(g^{-1}(y)))\ \ |\partial h/\partial x|_{g^{-1}(y)}\ \ \nu(dh)}\ f(y)\ \nu(dg) \\
&= \int_S p(y)\ f(y)\ dy \int_G \frac{p(g^{-1}(y)) \left|\partial g^{-1}(y)/\partial y\right|}{\int\ p(h(g^{-1}(y)))\ \ |\partial h/\partial x|_{g^{-1}(y)} \left|\partial g^{-1}(y)/\partial y\right|\ \ \nu(dh)}\ \nu(dg) \\
&= \int_S p(y)\ f(y)\ dy \underbrace{\int_G \frac{p(g^{-1}(y)) \left|\partial g^{-1}(y)/\partial y\right|\ \ \nu(dg)}{\int\ p(h(g^{-1}(y)))\ \ |\partial (h\circ g^{-1})(y)/\partial y|\ \ \nu(dh)}}_{=1}.
\end{aligned}$$

In the last assertion we have used the fact that

$$\begin{aligned}
&\left(H\ \sim\ \nu \Rightarrow \forall h\in G\quad H\circ h^{-1}\ \sim \nu\right) \\
&\Rightarrow \int\ p(h(g^{-1}(y)))\ \ \left|\partial(h\circ g^{-1})(y)/\partial y\right|\ \ \nu(dh) = \int\ p(g(y))\ \ |\partial g(y)/\partial y|\ \ \nu(dg)
\end{aligned}$$

and

$$\left(H \sim \nu \Rightarrow H^{-1} \sim \nu\right) \Rightarrow \int p(g(y))\ |\partial g(y)/\partial y|\ \nu(dg) = \int p(g^{-1}(y))\ \left|\partial g^{-1}(y)/\partial y\right|\ \nu(dg).$$

This ends the proof of the exercise.

■

Solution to exercise 131:
Notice that

$$\eta_{g_0}(dx) = \eta(dx) \propto 1_{[-1,1]}(x_1) \times 1_{[-1,1]}(x_2)\ dx_1 dx_2.$$

A Gibbs sampler with target uniform measure on S starts with a given state $(x_1, x_2) \in S$. Then we change x_2 by sampling a random sample x_2' in the set $[-1, 1]$. Given that state we change x_1 by a random sample x_1' in the set $[-1, 1]$, and so on. Notice that after two steps the Gibbs sampler is at equilibrium; that is the random states have the desired distribution η after two steps.

We let g_θ the rotation with angle $\theta \in [0, 2\pi]$. In this notation, we notice that

$$\begin{aligned} g_\theta(S) &= \{g_\theta(x)\ :\ x \in S\} \\ &= \{(\cos(\theta)x_1 - \sin(\theta)x_2, \sin(\theta)x_1 + \cos(\theta)x_2)\ :\ x \in S\}. \end{aligned}$$

We also have

$$\begin{aligned} g_\theta(S) &= \cup_{x_1 \in [-(\cos\theta + \sin(\theta)), \cos\theta + \sin(\theta)]} \left(\{x_1\} \times A_{2,\theta}(x_1)\right) \\ &= \cup_{x_2 \in [-(\cos\theta + \sin(\theta)), \cos\theta + \sin(\theta)]} \left(A_{1,\theta}(x_2) \times \{x_2\}\right) \end{aligned}$$

for any $\theta \in [0, \pi/2[$ with the sections

$$\begin{aligned} A_{1,\theta}(x_1) &:= \{x_2 \in \mathbb{R}\ :\ x = (x_1, x_2) \in g_\theta(S)\}, \\ A_{2,\theta}(x_2) &:= \{x_1 \in \mathbb{R}\ :\ x = (x_1, x_2) \in g_\theta(S)\}. \end{aligned}$$

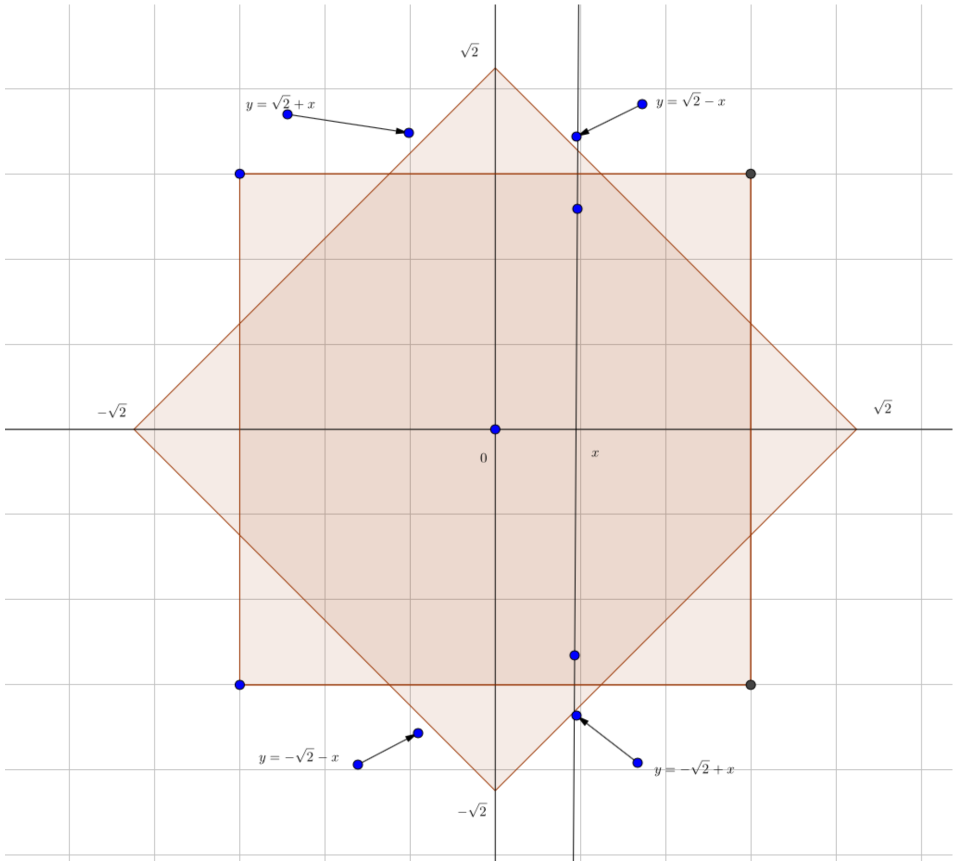
This implies that

$$\eta_{g_\theta}(dx) = \rho_{1,\theta}(dx_1)\ K_\theta^{(2)}(x_1, dx_2) = \rho_{2,\theta}(dx_2)\ K_\theta^{(1)}(x_2, dx_1)$$

for some probability measures $\rho_{i,\theta}$ on $[-(\cos\theta + \sin(\theta)), \cos\theta + \sin(\theta)]$ (the i-th marginal of η_{g_θ}), some Markov transitions $K_{1,\theta}$ from $x_2 \in [-(\cos\theta + \sin(\theta)), \cos\theta + \sin(\theta)]$ into $A_{1,\theta}(x_1)$ (the η_{g_θ}-conditional distribution of the first coordinate given the second one) and $K_{2,\theta}$ from $x_1 \in [-(\cos\theta + \sin(\theta)), \cos\theta + \sin(\theta)]$ into $A_{2,\theta}(x_2)$ (the η_{g_θ}-conditional distribution of the second coordinate given the first one) . For instance, for $\theta = \pi/4$ we have

$$\begin{aligned} &\eta_{g_{\pi/4}}(dx) \\ &\propto \left(1_{[0,\sqrt{2}]}(x_2)\ 1_{\left[-(\sqrt{2}-x_2),(\sqrt{2}-x_2)\right]}(x_1)\ + 1_{[-\sqrt{2},0]}(x_2)\ 1_{\left[-(\sqrt{2}+x_2),(\sqrt{2}+x_2)\right]}(x_1)\right)\ dx_1 dx_2 \\ &= \left(1_{[0,\sqrt{2}]}(x_1)\ 1_{\left[-(\sqrt{2}-x_1),(\sqrt{2}-x_1)\right]}(x_2)\ + 1_{[-\sqrt{2},0]}(x_1)\ 1_{\left[-(\sqrt{2}+x_1),(\sqrt{2}+x_1)\right]}(x_2)\right)\ dx_1 dx_2. \end{aligned}$$

The following diagram illustrates the rotation of the cell and the new direction axis of the Gibbs sampler.



In this case, we have

$$
\begin{aligned}
& K^{(2)}_{\pi/4}(x_1, dx_2) \\
& \propto 1_{[0,\sqrt{2}]}(x_1)\ \tfrac{1}{2(\sqrt{2}-x_1)}\ 1_{\left[-(\sqrt{2}-x_1),(\sqrt{2}-x_1)\right]}(x_2)dx_2 \\
& \qquad\qquad\qquad +1_{[-\sqrt{2},0]}(x_1)\ \tfrac{1}{2(\sqrt{2}+x_1)}\ 1_{\left[-(\sqrt{2}+x_1),(\sqrt{2}+x_1)\right]}(x_2)dx_2
\end{aligned}
$$

and by symmetry arguments we also have

$$
\begin{aligned}
& K^{(1)}_{\pi/4}(x_2, dx_1) \\
& \propto 1_{[0,\sqrt{2}]}(x_2)\ \tfrac{1}{2(\sqrt{2}-x_2)}\ 1_{\left[-(\sqrt{2}-x_2),(\sqrt{2}-x_2)\right]}(x_1)dx_1 \\
& \qquad\qquad\qquad +1_{[-\sqrt{2},0]}(x_2)\ \tfrac{1}{2(\sqrt{2}+x_2)}\ 1_{\left[-(\sqrt{2}+x_2),(\sqrt{2}+x_2)\right]}(x_1)dx_1.
\end{aligned}
$$

A Gibbs sampler with target uniform measure on $g_{\pi/4}(S)$ starts with a given state $(x_1, x_2) \in g_{\pi/4}(S)$, with say $x_2 \in [0, \sqrt{2}]$. Then we change x_1 by a random sample x'_1 in the set $\left[-(\sqrt{2} - x_2), (\sqrt{2} - x_2)\right]$. Given that state x'_1, say in $[0, \sqrt{2}]$ we sample a state x'_2 uniformly on $\left[-(\sqrt{2} - x_1), (\sqrt{2} - x_1)\right]$, and so on. The resulting Gibbs sampler transition from $g_{\pi/4}(S)$ into itself is given by $M_{g_{\pi/4}} = M^{(1)}_{g_{\pi/4}} M^{(2)}_{g_{\pi/4}}$ with

$$\begin{array}{rcl} M^{(1)}_{g_{\pi/4}}((x_1,x_2),d(y_1,y_2)) & = & \delta_{x_2}(dy_2)\ K^{(1)}_{\pi/4}(x_2,dy_1), \\ M^{(2)}_{g_{\pi/4}}((x_1,x_2),d(y_1,y_2)) & = & \delta_{x_1}(dy_1)\ K^{(2)}_{\pi/4}(x_1,dy_2). \end{array}$$

We can alternatively use the transition

$$M_{g_{\pi/4}} = \frac{1}{2}\ M^{(1)}_{g_{\pi/4}} + \frac{1}{2}\ M^{(2)}_{g_{\pi/4}}.$$

Notice that $M^{(i)}_g(x',dy')$ coincides with the distribution of a random state $x' + T_i(x')\ e_i$ where $T_i(x')$ is a uniform random variable on $\mathcal{T}_i(x')$. For instance for $i=1$, $g = g_{\pi/4}$, and $x' \in g(S)$ s.t. $x'_2 \in [0,\sqrt{2}]$ we have

$$\begin{array}{rcl} \mathcal{T}_{1,g}(x') & := & \left\{ t \in \mathbb{R}\ :\ \left(\begin{array}{c} x'_1 \\ x'_2 \end{array}\right) + t \left(\begin{array}{c} 1 \\ 0 \end{array}\right) \in g(S) \right\} \\ & = & \left\{ t \in \mathbb{R}\ :\ x'_1 + t\ \in \left[-(\sqrt{2}-x'_2), (\sqrt{2}-x'_2) \right] \right\}. \end{array}$$

In this situation, we have

$$\begin{array}{rcl} M^{(1)}_g(f)(x'_1,x'_2) & \propto & \displaystyle\int_{\mathcal{T}_{i,g}(x')} dt\ f\left(x' + te_1\right) \\ & = & \displaystyle\int_{-(\sqrt{2}-x'_2)}^{(\sqrt{2}-x'_2)} f\left(t,x'_2\right)\ dt \propto \int\ K^{(1)}_g(x'_2,dy'_1)\ f\left(y'_1,x'_2\right). \end{array}$$

More generally, we can show that the Markov transitions $M^{(i)}_g$ defined in the exercise statement coincide with the transitions defined by

$$\begin{array}{rcl} M^{(1)}_{g_\theta}((x_1,x_2),d(y_1,y_2)) & = & \delta_{x_2}(dy_2)\ K^{(1)}_{\theta}(x_2,dy_1), \\ M^{(2)}_{g_\theta}((x_1,x_2),d(y_1,y_2)) & = & \delta_{x_1}(dy_1)\ K^{(2)}_{\theta}(x_1,dy_2). \end{array}$$

For any $x \in S$, $g \in SO(2)$ and any $i = 1,2$ we have

$$T_g(x) = x + te_i \implies g^{-1}T_g.g(x) = g^{-1}\left(g(x) + te_i\right) = x +\ t\ g^{-1}(e_i).$$

This shows that

$$M_g := \frac{1}{2}\left(M^{(1)}_g + M^{(2)}_g\right) \Rightarrow \overline{M} = \frac{1}{2}\left(\overline{M}^{(1)} + \overline{M}^{(2)}\right)$$

with

$$\overline{M}^{(i)}(x,dy) \propto \int_G \mu(dg)\ \int_{\mathcal{T}_{i,g}(g(x))} dt\ \delta_{x+tg^{-1}(e_i)}(dy).$$

Also observe that

$$\mathcal{T}_{i,g}(g(x))\ \ :=\ \ \{t \in \mathbb{R}\ :\ g(x) + te_i \in g(S)\}$$

and

$$\begin{array}{rcl} \left\{x + tg^{-1}(e_i)\ :\ t \in \mathcal{T}_{i,g}(g(x))\right\} & = & \left\{x + tg^{-1}(e_i)\ :\ \ t \text{ s.t. } g(x) + te_i \in g(S)\right\} \\ & = & \left\{x + tg^{-1}(e_i)\ :\ t \text{ s.t. } x + tg^{-1}(e_i) \in S\right\} \\ & =: & \mathcal{L}(x, g^{-1}(e_i)), \end{array}$$

where $\mathcal{L}(x, g^{-1}(e_i))$ stands for the line segment in S passing through x with a direction vector $g^{-1}(e_i)$. This yields the formula

$$\overline{M}^{(i)}(x, dy) = \int_G \mu(dg)\ P_{\mathcal{L}(x,g^{-1}(e_i))}(dy),$$

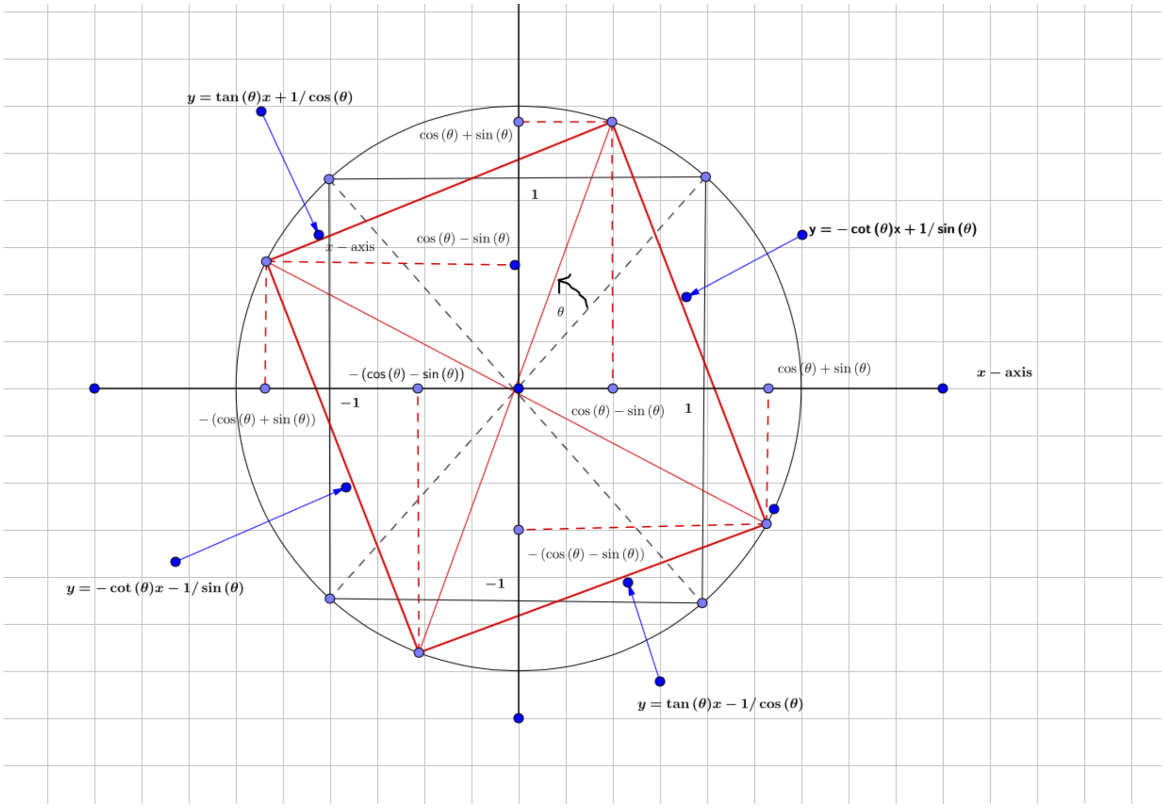
where $P_{\mathcal{L}(x,g^{-1}(e_i))}(dy)$ stands for the uniform distribution on $\mathcal{L}(x, g^{-1}(e_i))$. To be more precise, let us assume that we are given a line segment $\mathcal{L}$ of he following form

$$\begin{aligned} \mathcal{L} \ &:= \ \left\{(x,y) \in \mathbb{R}^2 \ : \ ax + b = y \quad \text{with } x \in [x_\star, x^\star]\right\} \\ &= \ \left\{(x,y) \in \mathbb{R}^2 \ : \ ax + b = y \quad \text{with } y \in [y_\star, y^\star]\right\} \end{aligned}$$

for some parameters $(a, b) \in \mathbb{R}^2$, $a \neq 0$, $x_\star < x^\star$ and $y_\star < y^\star$. In this situation, the uniform mesure $P_{\mathcal{L}}$ on $\mathcal{L}$ is given by

$$\begin{aligned} P_{\mathcal{L}}(d(x,y)) \ &:= \ \frac{1}{x^\star - x_\star}\ 1_{[x_\star,x^\star]}(x)\ dx\ \delta_{ax+b}(dy) \\ &= \ \frac{1}{y^\star - y_\star}\ 1_{[y_\star,y^\star]}(y)\ dy\ \delta_{(y-b)/a}(dx). \end{aligned}$$

The following diagram illustrates the rotation of the cell and the new direction axis of the Gibbs sampler for any angle θ.



The main difference between this sampler and the one discussed above arises from the random directions explored by the sampler. More precisely, mapping back and forth the samples of the Gibbs sampler with target $\eta_{g_{\pi/4}}$ to the original set S we define a Gibbs sampler on S that differs from the one discussed above by only rotating the random direction of the samples. Equivalently, the sampling according to the transition $\overline{M}_\theta$ amounts to replacing the coordinate exploration axis $(0, x_1)$ and $(0, x_2)$ of the usual Gibbs sampler by rotating these directions by an angle θ.

Therefore, sampling according to $\overline{M}$ first amounts to choosing randomly an angle θ. Then, we explore the space coordinate by coordinate, according to the couple directions defined by the θ-rotation of the coordinate exploration axis $(0, x_1)$ and $(0, x_2)$.

This ends the proof of the exercise.

■

Solution to exercise 132:

We consider the Rotation group MCMC sampler discussed in exercise 129 when $r = 2$. Assume that S is given by the boundary of the cell discussed in exercise 131; that is, we have

$$S = \partial([-1,1]\times[-1,1]) = (\{-1,1\}\times[-1,1]) \cup ([-1,1]\times\{-1,1\}).$$

Let η be the uniform probability measure on S is given in cartesian coordinates by

$$\begin{aligned}\eta(d(x_1,x_2)) \quad &\propto \quad (\delta_{-1}(dx_1)+\delta_1(dx_1))\ 1_{[-1,1]}(x_2)\ dx_2 \\ &\qquad + 1_{[-1,1]}(x_1)\ dx_1\ (\delta_{-1}(dx_2)+\delta_1(dx_2)).\end{aligned}$$

One way to sample a random variable X with probability η on S is given below: Firstly we choose randomly one of the states

$$Y := \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}.$$

Given $Y \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$ we sample a random variable Z uniformly on $[-1,1]$ and we set $X = \begin{pmatrix} Y_1 \\ Z \end{pmatrix}$. Given $Y \in \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$ we sample a random variable Z uniformly on $[-1,1]$ and we set $X = \begin{pmatrix} Z \\ Y_2 \end{pmatrix}$. Two equivalent ways of sampling X are given below: First we sample a random variable X_1 on $[-1,1]$, then given X_1 we choose X_2 in $\{-1,1\}$. By symmetry, we can also also start by sampling a random variable X_2 on $[-1,1]$, and then given X_2 we choose X_1 in $\{-1,1\}$.

In all cases, for any subset $C \subset S$ of length c we have

$$\mathbb{P}(X \in C) = c/8.$$

For instance, the chance to have $X \in ([a,b]\times\{1\})$ with the first sampling technique is the same as the chance $(1/4)$ for Y to select $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the chance $(b-a)/2$ for Z (uniformly on $[-1,1]$) to hit the set $[a,b]$. This yields

$$\mathbb{P}(X \in [a,b]\times\{1\}) = \mathbb{P}\left(Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \times \mathbb{P}\left(Z \in [a,b]\ \middle|\ Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \frac{1}{4}\ \frac{b-a}{2}.$$

In much the same way the uniform measure on $g_{\pi/4}(S)$ is given in cartesian coordinates by

$$\begin{aligned}\eta_{g_{\pi/4}}(d(x_1,x_2)) \quad &\propto \quad 1_{[0,\sqrt{2}]}(x_1)\ dx_1\ \left(\delta_{-(\sqrt{2}-x_1)}(dx_2) + \delta_{\sqrt{2}-x_1}(dx_2)\right) \\ &\qquad + 1_{[-\sqrt{2},0]}(x_1)\ dx_1\ \left(\delta_{-(\sqrt{2}+x_1)}(dx_2) + \delta_{\sqrt{2}+x_1}(dx_2)\right) \\ &= \quad 1_{[0,\sqrt{2}]}(x_2)\ dx_2\ \left(\delta_{-(\sqrt{2}-x_2)}(dx_1) + \delta_{\sqrt{2}-x_2}(dx_1)\right) \\ &\qquad + 1_{[-\sqrt{2},0]}(x_2)\ dx_2\ \left(\delta_{-(\sqrt{2}+x_2)}(dx_1) + \delta_{\sqrt{2}+x_2}(dx_1)\right).\end{aligned}$$

The sampling of these distributions follows the same arguments as the one above.

The Gibbs samplers with the target measures η_{g_θ} are defined as the ones discussed in exercise 131. First we notice that the conditional distribution of one coordinate given the other one resumes to a discrete measure. For instance, given x_1 say in $[0,\sqrt{2}]$, the second coordinate is randomly chosen in the set $\{-(\sqrt{2}-x_1), +(\sqrt{2}-x_1)\}$. It is readily checked that the resulting Gibbs sampler discussed above evolves between the 4 states

$$\left\{ \begin{pmatrix} x_1 \\ -(\sqrt{2}-x_1) \end{pmatrix}, \begin{pmatrix} x_1 \\ -(\sqrt{2}-x_1), \end{pmatrix}, \begin{pmatrix} -x_1 \\ -(\sqrt{2}-x_1) \end{pmatrix}, \begin{pmatrix} -x_1 \\ -(\sqrt{2}-x_1) \end{pmatrix} \right\}.$$

In cartesian coordinates, the Gibbs sampler evolves on a boundary of some bounded regular domain embedded in $\mathbb{R}^r$ using the conditional distributions of one coordinate given the other ones. These distributions are discrete probability measures, thus they are easy to sample (as soon as we identify the possible states).

This ends the proof of the exercise.

■

Solution to exercise 133:

Let $S = \partial D$ be the boundary of some smooth convex surface D. It is readily checked that

$$r_x = r_x' \quad \text{and} \quad r_x^2 = r_x.$$

In addition, we have

$$\begin{aligned}
\langle u, n(x)\rangle \geq 0 \Rightarrow \langle r_x(u), n(x)\rangle &= n(x)' r_x(u) = n(x)' \left(Id - 2\; n(x)n(x)'\right)(u) \\
&= -n(x)'u = -\langle u, n(x)\rangle \leq 0.
\end{aligned}$$

We conclude that the mapping r_x is the reflection w.r.t. the tangent line $T_x(D)$ at the surface at $x \in \partial D$.

For any $(x,u) \in \left(S \times \mathbb{S}_x^1\right)$ we set

$$t(x,u) := \inf\left\{t \geq 0 \;:\; x + tu \in \partial D\right\}.$$

By construction, we have

$$y = x + t(x,u)u \Rightarrow t(x,u) = \|y-x\| \Rightarrow y = x + \|y-x\|\; u \Rightarrow u = \frac{y-x}{\|y-x\|}$$

as soon as $x \neq y$ (otherwise we set $t(x,u) = 0$ for any $u \in \mathbb{S}_x^1$).

The uniform measure on the $\mathbb{S}^1$ is given by

$$\nu(d(u_1,u_2)) = \frac{1}{\pi}\; \frac{1}{\sqrt{1-u_1^2}}\; 1_{]-1,1[}(u_1)\; du_1\; \frac{1}{2}\left(\delta_{-\sqrt{1-u_1^2}} + \delta_{-\sqrt{1-u_1^2}}\right)(du_2).$$

We let ν_x be the restriction of ν to the hemisphere $\mathbb{S}_x^1$ and U_x be a random variable with distribution ν_x. Now

$$\mathbb{E}\left(f(x + t(x,U_x)U_x)\right) = \frac{1}{\nu(\mathbb{S}_x^1)} \int_{\mathbb{S}_x^1} f(x + t(x,u)u)\; 1_{\mathbb{S}_x^1}(u)\; \nu(du).$$

We further assume that ∂D is the null level set $\partial D = \varphi^{-1}(\{0\})$ of a continuously differentiable function s.t. $\partial_{y_2}\varphi(y) \neq 0$ on ∂D. By `the implicit function theorem`, for any given $y \in \partial D$ (s.t. $\partial_{y_2}\varphi(y) \neq 0$) there exists a product of open sets $y \in O := (O_1 \times O_2) \subset \mathbb{R}^2$ and some height function $h \;:\; z_1 \in O_1 \mapsto h(z_1) = z_2 \in O_2$ such that

$$\{z = (z_1,z_2) \in O \;:\; \varphi(z) = 0\} = \{(z_1, h(z_1)) \;:\; z_1 \in O_1\}.$$

For a given $x \in \partial D$ and a given direction $u \in \mathbb{S}_x^1$ we set $y = x + t(x,u)u$. We let h be the height function defined above. We assume that O_1 is chosen sufficiently small so that we can find some open subset $\mathcal{U} \in \mathbb{S}_x^1$ s.t.

$$O_1 = \{x + t(x,u)u \;:\; u \in \mathcal{U}\}.$$

We use the change of variables from $u \in \mathcal{U}$ into $z_1 \in O_1$ given by

$$z_1 = x_1 + t(x,u)u_1 = x_1 + \|z - x\| \ u_1 \quad \text{with} \quad z = (z_1, z_2) = (z_1, h(z_1)).$$

We have

$$u_1 = \frac{z_1 - x_1}{\sqrt{(z_1 - x_1)^2 + (h(z_1) - x_2)^2}}$$
$$\Longrightarrow \partial_{z_1} u_1 = \frac{1}{\sqrt{(z_1 - x_1)^2 + (h(z_1) - x_2)^2}} \left(1 - \frac{(z_1 - x_1)^2 + (z_1 - x_1)(h(z_1) - x_2)\partial h(z_1)}{(z_1 - x_1)^2 + (h(z_1) - x_2)^2}\right).$$

This implies that

$$\begin{aligned}
\partial_{z_1} u_1 &= \frac{(h(z_1) - x_2)}{[(z_1 - x_1)^2 + (h(z_1) - x_2)^2]^{3/2}} \left((h(z_1) - x_2) - (z_1 - x_1)\partial h(z_1)\right) \\
&= \frac{(h(z_1) - x_2)}{[(z_1 - x_1)^2 + (h(z_1) - x_2)^2]^{3/2}} \left\langle \begin{pmatrix} (z_1 - x_1) \\ (h(z_1) - x_2) \end{pmatrix}, \begin{pmatrix} -\partial h(z_1) \\ 1 \end{pmatrix} \right\rangle \\
&= \frac{(h(z_1) - x_2)}{\|z - x\|^2} \left\langle \frac{z - x}{\|z - x\|}, n(z) \right\rangle \sqrt{1 + (\partial h(z_1))^2}
\end{aligned}$$

with the outward pointing unit normal vector $n(z)$ at $z \in \partial D$ given by

$$n(z) = \frac{1}{\sqrt{1 + (\partial h(z_1))^2}} \begin{pmatrix} -\partial h(z_1) \\ 1 \end{pmatrix}.$$

On the other hand, we have

$$\sqrt{1 - u_1^2} = \sqrt{1 - \frac{(z_1 - x_1)^2}{(z_1 - x_1)^2 + (h(z_1) - x_2)^2}} = \frac{|h(z_1) - x_2|}{\|z - x\|}.$$

This yields the change of variable formula

$$1_{]-1,1[}(u_1) \ \frac{du_1}{\sqrt{1 - u_1^2}} = \frac{1}{\|z - x\|} \left\langle \frac{z - x}{\|z - x\|}, n(z) \right\rangle \ \mu_h(dz_1)$$

with the surface measure in the coordinate system associated with the height function h given by

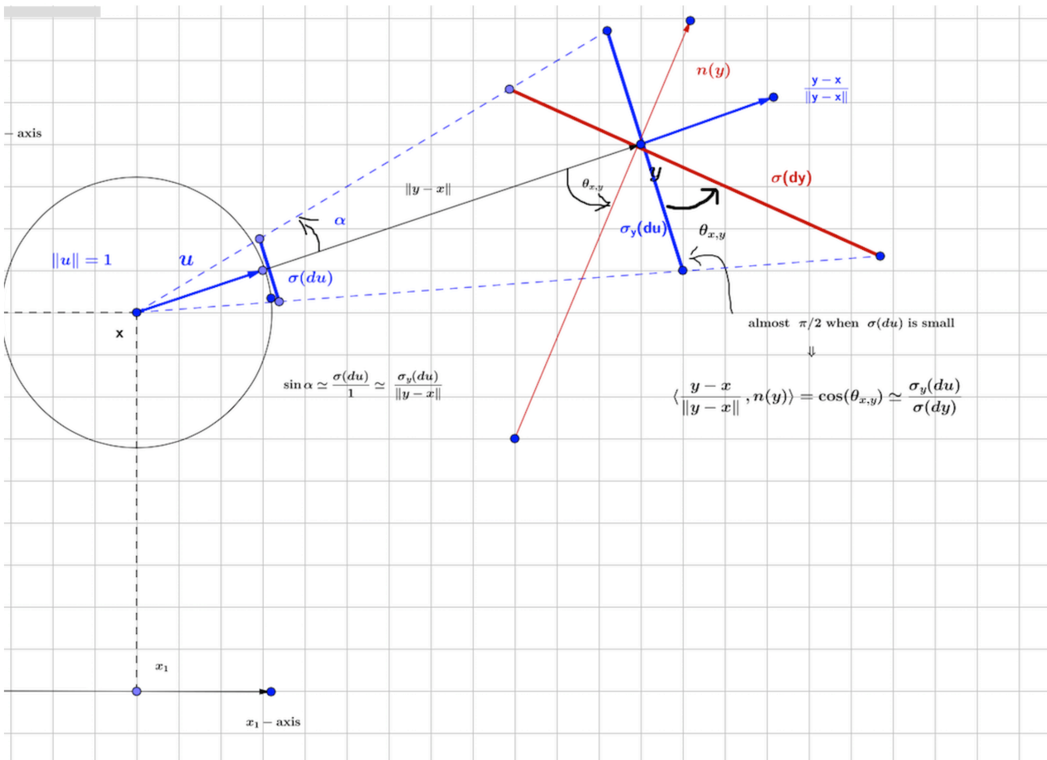
$$\mu_h(dz_1) = \sqrt{1 + (\partial h(z_1))^2} \ dz_1.$$

We conclude that

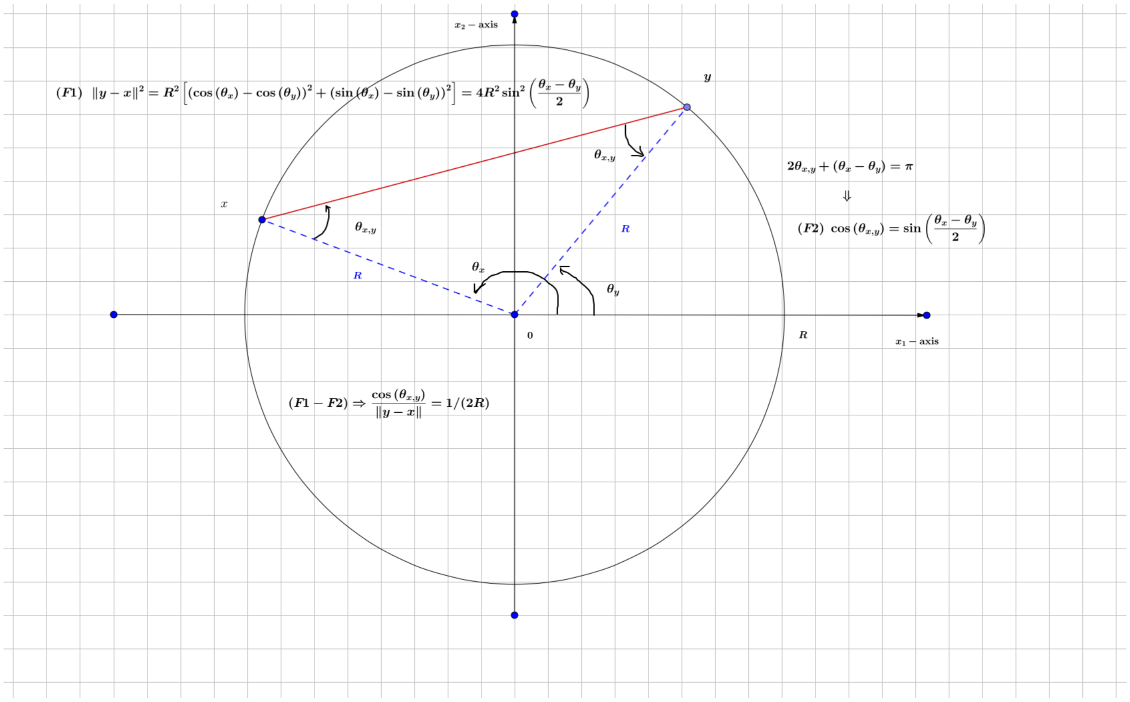
$$\mathbb{E}\left(f(x + t(x, U_x)U_x)\right) \propto \int_{\partial D} f(z) \ \frac{1}{\|z - x\|} \left\langle \frac{z - x}{\|z - x\|}, n(z) \right\rangle \ \sigma(dz)$$

with the surface measure $\sigma(dz)$ (to be more rigorous, we can use the patching techniques discussed on pages 595 and 648 to perform the change of variable discussed above on the whole state space).

The following diagram illustrates this change of variable formula.



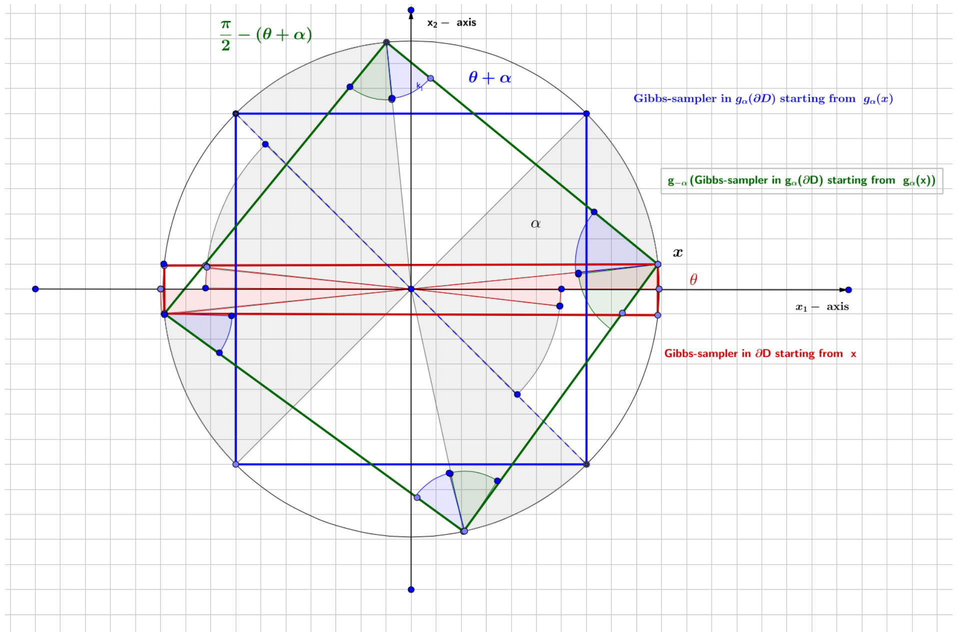
The following diagram illustrates the stochastic billiard when $S = \partial D = \{x \in \mathbb{R}^2 \ : \ \|x\| = R\}$.



A list of trigonometric formulae can be found in the wikipedia website. In this situation, we have

$$\overline{M}(x,dy) \quad\propto\quad \frac{1}{\|y-x\|}\left\langle\frac{y-x}{\|y-x\|}, n(y)\right\rangle\ \sigma(dy) = \frac{1}{\|y-x\|}\ \cos(\theta_{x,y})\ \sigma(dy) \propto \sigma(dy).$$

The following diagram illustrates the Gibbs sampler on the boundary and the one associated with the α-rotation of the state.



This ends the proof of the exercise.

■

Solution to exercise 134:
When ν_x is replaced by the measure

$$\nu_{x,\kappa}(du) \propto \kappa_x(u)\nu_x(du) \quad \text{with} \quad \kappa_x(u) = -\langle u, n(x)\rangle = |\langle u, n(x)\rangle|$$

using the same arguments as above we have

$$\begin{aligned}
\mathbb{E}\left(f(x + t(x, U_x)U_x)\right) &\propto \int_{\partial D} f(z)\ \kappa_x\left(\frac{z-x}{\|z-x\|}\right) \frac{1}{\|z-x\|}\ \left\langle \frac{z-x}{\|z-x\|}, n(z)\right\rangle\ \sigma(dz) \\
&= \int_{\partial D} f(z)\ \frac{1}{\|z-x\|}\ \left\langle \frac{x-z}{\|x-z\|}, n(x)\right\rangle\ \left\langle \frac{z-x}{\|z-x\|}, n(z)\right\rangle\ \sigma(dz).
\end{aligned}$$

When $S = \partial D = \{x \in \mathbb{R}^2\ :\ \|x\| = R\}$, using the diagram presented in exercise 133 we have

$$\overline{M}(x, dy) \quad \propto \quad \left|\sin\left(\frac{\theta_x - \theta_y}{2}\right)\right|\ \sigma(dy).$$

This ends the proof of the exercise.

■

Solution to exercise 135:

By construction, we have

$$\begin{aligned}
\mathbb{P}\left((X,Y)\in d(x,y)\right) &= p(x)\ dx\ \frac{1}{p(x)}\ 1_{[0,p(x)]}(y)\ dy,\\
&= 1_{[0,p(x)]}(y)\ \ dx\ dy = 1_{p^{-1}([y,\infty[)}(x)\ \ dx\ dy\\
\mathbb{P}\left(Y\in dy\right) &= \left[\int\ 1_{p^{-1}([y,\infty[)}(x')\right]\ dy.
\end{aligned}$$

Observe that

$$\mathbb{P}\left((X,Y)\in d(x,y)\right) = \underbrace{\frac{1}{\int\ 1_{p^{-1}([y,\infty[)}(x')\ dx'}\ 1_{p^{-1}([y,\infty[)}(x)\ dx}_{=\mathbb{P}(X\in dx\ \mid\ Y=y)} \times \underbrace{\left[\int\ 1_{p^{-1}([y,\infty[)}(x')\right]\ dy}_{=\mathbb{P}(Y\in dy)}$$

This implies that

$$\mathbb{P}\left(X\in dx\mid Y\right) = \frac{1}{\int\ 1_{p^{-1}([y,\infty[)}(x')\ dx'}\ 1_{p^{-1}([y,\infty[)}(x)\ dx.$$

A Gibbs sampler wit target measure $\pi = \mathrm{Law}(X,Y)$ is defined by the following synthetic diagram:

$$\left(\begin{array}{ccc} X_n & = & x\\ Y_n & = & y\end{array}\right) \rightarrow \left(\begin{array}{ccc} X_{n+\frac{1}{2}} & = & x'\sim (X\mid Y=y)\\ Y_{n+\frac{1}{2}} & = & y\end{array}\right) \rightarrow \left(\begin{array}{ccc} X_{n+1} & = & x'\\ Y_{n+1} & = & y'\sim (Y\mid x=x)\end{array}\right).$$

This ends the proof of the exercise.

■

Solution to exercise 136:
We have

$$\begin{aligned}
\left(\eta K'\right)(dx') &= \int_{S^X\times S^Y} \overbrace{\eta(dx)\ K(x,dy)}^{(\eta K)(dy)\times M(y,dx)}\ M'_y(x,dx')\\
&= \int_{S^Y} (\eta K)(dy)\times \underbrace{\int_{S^X}\ M(y,dx)M'_y(x,dx')}_{M(y,dx)}.
\end{aligned}$$

This implies that

$$\left(\eta K'\right)(dx') = \int_{S^Y} \underbrace{(\eta K)(dy)M(y,dx')}_{=\eta(dx')\ K(x',dy)} = \int_{S^Y} \eta(dx')\ K(x',dy) = \eta(dx').$$

This ends the proof of the exercise.

■

Solution to exercise 137:
The first assertion is a direct consequence of the formula

$$\forall a>0 \qquad \mathbb{E}(f(aU_1)) = \int_0^1\ f(au)\ du = \frac{1}{a}\int_0^a\ f(u)\ du = \int\ f(u)\ \frac{1}{a}\ 1_{[0,a]}(u)\ du$$

which is valid for any bounded function f. This shows that (aU_1) is an uniform random variable on $[0,a]$.

This implies that

$$\begin{aligned} K(x,dy) &:= \mathbb{P}(Y \in dy \mid X = x) \\ &= \left\{ \prod_{1\leq i\leq r} e^{-V_i(x)}\ 1_{[0,e^{V_i(x)}]}(y_i) \right\}\ dy. \end{aligned}$$

In the above display, $dy = dy_1 \times \ldots \times dy_r$ stands for an infinitesimal neighborhood of the state $y = (y_1, \ldots, y_r)$.

We conclude that

$$\begin{aligned} \mathbb{P}((X,Y) \in d(x,y)) &= \mathbb{P}(Y \in dy \mid X = x) \times \mathbb{P}(X \in dx) \\ &= \left\{ \prod_{1\leq i\leq r} 1_{[0,e^{V_i(x)}]}(y_i) \right\}\ \lambda(dx)\ dy. \end{aligned}$$

This shows that

$$\mathbb{P}(X \in dx \mid Y = y) = \frac{1}{\mathcal{Z}(y)} \left\{ \prod_{1\leq i\leq r} 1_{[0,e^{V_i(x)}]}(y_i) \right\}\ \lambda(dx)$$

for some normalizing constant (here the density of the sequence Y w.r.t. dy). The last assertion is easily completed using exercise 136 (and running a Markov chain with transition K').

This ends the proof of the exercise.

■

Solution to exercise 138:

The conditional density of the observation sequence $(Y_{i,j})_{(i,j)\in(I\times J)}$ given $(X_i)_{i\in I} = x$, $Z = z$, $(V_1, V_2) = (v_1, v_2)$ is given for any $y = (y_{i,j})_{(i,j)\in(I\times J)}$ by

$$p(y \mid x,\ v_1,\ v_2,\ z) = \prod_{i\in I} \frac{1}{\sqrt{2\pi V_2}}\ \exp\left(-\frac{1}{2v_2} \sum_{j\in J} (y_{i,j} - x_i)^2 \right)$$

and

$$p(x \mid v_1,\ v_2,\ z) = \prod_{i\in I} \frac{1}{\sqrt{2\pi V_1}}\ \exp\left(-\frac{1}{2v_1}\ (x_i - z)^2 \right).$$

On the other hand we have

$$p(x \mid y,\ v_1,\ v_2,\ z) \propto p(y \mid x,\ v_1,\ v_2,\ z)\ p(x \mid v_1,\ v_2,\ z)$$

$$\propto \prod_{i\in I} \frac{1}{\sqrt{2\pi v_2}}\ \exp\left(-\frac{1}{2v_2}\ \sum_{j\in J} (y_{i,j} - x_i)^2 \right) \frac{1}{\sqrt{2\pi V_1}}\ \exp\left(-\frac{1}{2v_1}\ (x_i - z)^2 \right).$$

Using the fact that

$$\begin{aligned}
&\frac{1}{v_2}\ \sum_{j\in J}(y_{i,j}-x_i)^2+\frac{1}{v_1}\ (x_i-z)^2\\
&=\frac{1}{v_2}\ \sum_{j\in J}\left[y_{i,j}^2+x_i^2-2x_iy_{i,j}\right]+\frac{1}{v_1}\ (x_i^2+z^2-2x_iz)\\
&=x_i^2\left(\frac{1}{v_1}+\frac{|J|}{v_2}\right)-2x\ \left(\frac{z}{v_1}+\frac{1}{v_2}\sum_{j\in J}y_{i,j}\right)+\left[\frac{1}{v_2}\ \sum_{j\in J}y_{i,j}^2+\frac{z}{v_1}\right]\\
&=\left(\frac{1}{v_1}+\frac{|J|}{v_2}\right)\ \left[x_i-\frac{\frac{z}{v_1}+\frac{1}{v_2}\sum_{j\in J}y_{i,j}}{\frac{1}{v_1}+\frac{|J|}{v_2}}\right]^2-\left[\frac{\left(\frac{z}{v_1}+\frac{1}{v_2}\sum_{j\in J}y_{i,j}\right)^2}{\frac{1}{v_1}+\frac{|J|}{v_2}}-\left[\frac{1}{v_2}\ \sum_{j\in J}y_{i,j}^2+\frac{z}{v_1}\right]\right]
\end{aligned}$$

we prove that

$$p(x\mid y,\ v_1,\ v_2,\ z)=\prod_{i\in I}\frac{1}{\sqrt{2\pi\sigma^2(v)}}\exp\left(-\frac{1}{2\sigma^2(v)}\ \left[x_i-\alpha\left((y_{i,j})_{j\in J},v_1,v_2\right)\right]^2\right),\quad (30.25)$$

with

$$\sigma^{-2}(v_1,v_2)=\left(\frac{1}{v_1}+\frac{|J|}{v_2}\right)\quad\text{and}\quad\alpha\left((y_{i,j})_{j\in J},v_1,v_2\right)=\frac{\frac{z}{v_1}+\frac{1}{v_2}\sum_{j\in J}y_{i,j}}{\frac{1}{v_1}+\frac{|J|}{v_2}}.$$

We also notice that

$$p(v_1\mid x,\ y,\ z,\ v_2)=p(v_1\mid x,\ z)\propto p(x\mid z,\ v_1)\ p(v_1\mid z)=p(x\mid z,\ v_1)\ p(v_1).\quad (30.26)$$

Now we use the fact that

$$\begin{aligned}
&p(x\mid z,\ v_1)\ p(v_1)\\
&\propto\frac{1}{(\sqrt{2\pi v_1})^{|I|}}\ \exp\left(-\frac{1}{2v_1}\sum_{i\in I}(x_i-z)^2\right)\ \frac{1}{v_1^{a_1+1}}\ \exp\left(-\frac{b_1}{v_1}\right)\ 1_{]0,\infty[}(v_1)\\
&\propto\frac{1}{v_1^{\frac{|I|}{2}+a_1+1}}\ \exp\left(-\frac{1}{v_1}\left(\frac{1}{2}\sum_{i\in I}(x_i-z)^2+b_1\right)\right)\ 1_{]0,\infty[}(v_1).
\end{aligned}$$

This shows that the conditional distribution of V_1 given $X=(X_i)_{i\in I}=x$, $Y=(Y_{i,j})_{(i,j)\in(I\times J)}=y$ and $(Z,V_2)=(z,v_2)$ coincides with the conditional distribution of V_1 given $X=x$ and $Z=z$ and it is given by an inverse Gamma distribution with shape parameter $A_1(a_1)$ and scale parameter $B_1(z)$ given by

$$A_1=\frac{|I|}{2}+a_1\quad\text{and}\quad B_1(z)=b_1+\frac{1}{2}\sum_{i\in I}(x_i-z)^2.$$

In much the same way, we have

$$p(v_2\mid x,\ y,\ z,\ v_1)=p(v_2\mid y,\ x)\propto p(y\mid x,\ v_2)\ p(v_2\mid x)=p(y\mid x,\ v_2)\ p(v_2).\quad (30.27)$$

Notice that

$$\begin{aligned}
&p(y \mid x,\ v_2)\ p(v_2)\\
&\propto \frac{1}{v_2^{|I|/2}}\ \exp\left(-\frac{1}{2v_2}\sum_{(i,j)\in(I\times J)}(y_{i,j}-x_i)^2\right)\ \frac{1}{v_2^{a_2+1}}\ \exp\left(-\frac{b_2}{v_2}\right)\ 1_{]0,\infty[}(v_2)\\
&\propto \frac{1}{v_2^{|I|/2+a_2+1}}\ \exp\left(-\frac{1}{v_2}\left(b_2+\frac{1}{2}\sum_{(i,j)\in(I\times J)}(y_{i,j}-x_i)^2\right)\right)\ 1_{]0,\infty[}(v_2).
\end{aligned}$$

This shows that the conditional distribution of V_2 given $X = (X_i)_{i\in I} = x$, $Y = (Y_{i,j})_{(i,j)\in(I\times J)} = y$ and $(Z, V_1) = (z, v_1)$ coincides with the conditional distribution of V_2 given $Y = y$ and $X = x$ and it is given by an inverse Gamma distribution with shape parameter $A_2(a_2)$ and scale parameter $B_2(x, y)$ given by

$$A_2 = \frac{|I|}{2} + a_2 \quad \text{and} \quad B_2(x,y) = b_2 + \frac{1}{2}\sum_{(i,j)\in(I\times J)}(y_{i,j}-x_i)^2.$$

Finally, we have

$$p(z,x \mid y,\ v_1,\ v_2) = p(z,x \mid v_1) \propto p(x \mid z,\ v_1)\ p(z \mid v_1) = p(x \mid z,\ v_1)\ p(z). \tag{30.28}$$

To take the final step, we notice that

$$\begin{aligned}
&p(x \mid z,\ v_1)\ p(z)\\
&\propto \frac{1}{\sqrt{2\pi v_1}^{|I|}}\ \exp\left(-\frac{1}{2v_1}\sum_{i\in I}(x_i-z)^2\right)\ \frac{1}{\sqrt{2\pi v}}\ \exp\left(-\frac{1}{2v}(z-m)^2\right).
\end{aligned}$$

Using the fact that

$$\begin{aligned}
&\frac{1}{v_1}\sum_{i\in I}(x_i-z)^2 + \frac{1}{v}(z-m)^2\\
&= \frac{1}{v_1}\sum_{i\in I}x_i^2 + \frac{|I|}{v_1}\ z^2 - 2z\ \frac{1}{v_1}\sum_{i\in I}x_i + \frac{1}{v}\ z^2 + \frac{1}{v}\ m^2 - 2z\frac{m}{v}\\
&= z^2\left(\frac{|I|}{v_1}+\frac{1}{v}\right) - 2z\left(\frac{m}{v}+\frac{1}{v_1}\sum_{i\in I}x_i\right) + \frac{1}{v}m^2 + \frac{1}{v_1}\sum_{i\in I}x_i^2\\
&= \left(\frac{|I|}{v_1}+\frac{1}{v}\right)\left[z - \frac{\left(\frac{m}{v}+\frac{1}{v_1}\sum_{i\in I}x_i\right)}{\left(\frac{|I|}{v_1}+\frac{1}{v}\right)}\right]^2 - \frac{\left(\frac{m}{v}+\frac{1}{v_1}\sum_{i\in I}x_i\right)^2}{\left(\frac{|I|}{v_1}+\frac{1}{v}\right)} + \frac{1}{v}m^2 + \frac{1}{v_1}\sum_{i\in I}x_i^2
\end{aligned}$$

we conclude that

$$p(z \mid x,\ y,\ v_1,\ v_2) \propto \exp\left(-\frac{1}{2\tau^2(v,v_1)}\ [z-\beta(x,v_1)]^2\right)$$

with

$$\tau^{-2}(v_1) = \left(\frac{|I|}{v_1} + \frac{1}{v}\right) \quad \text{and} \quad \beta(x, v_1) = \frac{\left(\frac{m}{v} + \frac{1}{v_1}\sum_{i\in I} x_i\right)}{\left(\frac{|I|}{v_1} + \frac{1}{v}\right)}.$$

Now we can design a Gibbs sampler using the conditional distributions computed above as follows.

Suppose we are given the random state $\mathcal{X}_k = (X^{(k)}, Z^{(k)}, V_1^{(k)}, V_2^{(k)}) = (x^{(k)}, z^{(k)}, v_1^{(k)}, v_2^{(k)})$ at the k-th iteration. The transition

$$\mathcal{X}_k \rightsquigarrow \mathcal{X}_{k+1} = (X^{(k+1)}, Z^{(k+1)}, V_1^{(k+1)}, V_2^{(k+1)}) = (x^{(k+1)}, z^{(k+1)}, v_1^{(k+1)}, v_2^{(k+1)})$$

is defined in 4 steps:

- Firstly, we sample $X^{(k+1)} = x^{(k+1)}$ with the conditional Gaussian distribution (30.25) given $(Y, Z, V_1, V_2) = (y, z^{(k)}, v_1^{(k)}, v_2^{(k)})$.
- Then we sample $V_1^{(k+1)} = v_1^{(k+1)}$ with the inverse Gamma distribution (30.26) with shape parameter A_1 and scale parameter $B_1(z^{(k)})$.
- At the third step, we sample $V_2^{(k+1)} = v_2^{(k+1)}$ with the inverse Gamma distribution (30.27) with shape parameter A_2 and scale parameter $B_2(x^{(k+1)}, y)$.
- Finally, we sample $Z^{(k+1)} = z^{(k+1)}$ with the Gaussian distribution (30.28) with mean and variance $(\beta(x^{(k+1)}, v_1^{(k+1)}), \tau^2(v_1^{(k+1)}))$

$$\tau^{-2}(v_1^{(k+1)}) = \left(\frac{|I|}{v_1^{(k+1)}} + \frac{1}{v}\right) \quad \text{and} \quad \beta(x, v_1^{(k+1)}) = \frac{\left(\frac{m}{v} + \frac{1}{v_1}\sum_{i\in I} x_i^{(k+1)}\right)}{\left(\frac{|I|}{v_1^{(k+1)}} + \frac{1}{v}\right)}.$$

This ends the proof of the exercise.

Solution to exercise 139:

The filtering problem (9.103) has the same form as the filtering problem discussed in (6.7) and in section 9.9.2.

- Using some abusive Bayesian notation, we set

$$\mathbb{P}_n(d(x_0, \ldots, x_n)) := \mathbb{P}((X_0, \ldots, X_n) \in d(x_0, \ldots, x_n)) = p(x_0, \ldots, x_n)\ dx_0 \ldots dx_n.$$

Notice that

$$p(x_0, \ldots, x_n)\ dx_0 \ldots dx_n = \underbrace{p(x_0)dx_0}_{:=\eta_0(dx_0)} \prod_{1\le k\le n} \underbrace{p(x_k \mid x_{k-1})dx_k}_{=M_k(x_{k-1}, dx_k)}$$

with $\eta_0 = \mathrm{Law}(X_0)$ and for any $k \geq 1$

$$M_k(x_{k-1}, dx_k) = \mathbb{P}(X_k \in dx_k \mid X_{k-1} = x_{k-1}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_k - a_k(x_{k-1}))^2\right) dx_k.$$

We let $(y_n)_{n\geq 0}$ be a given sequence of observations. Consider the sequence of likelihood functions

$$\forall n \geq 0 \qquad G_n(x_n) := p(y_n|x_n) := \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\ (y_n - b_n(x_n))^2\right).$$

Using Bayes' rule, we have

$$
\begin{aligned}
&\mathbb{P}((X_0,\ldots,X_n)\in d(x_0,\ldots,x_n)\mid Y_k=y_k,\ 0\le k<n)\\
&=p((x_0,\ldots,x_n)|(y_0,\ldots,y_{n-1}))\ dx_0\ldots dx_n\\
&=\frac{1}{p(y_0,\ldots,y_{n-1})}\ \left\{\prod_{1\le k<n}p(y_k|x_k)\right\}\ p(x_0,\ldots,x_n)\ dx_0\ldots dx_n\\
&=\frac{1}{\mathcal{Z}_n}\ \left\{\prod_{0\le k<n}G_k(x_k)\right\}\ \mathbb{P}_n(d(x_0,\ldots,x_n)):=\mathbb{Q}_n(d(x_0,\ldots,x_n))
\end{aligned}
$$

with the normalizing constant

$$
\mathcal{Z}_n=p(y_0,\ldots,y_{n-1})=\prod_{0\le k<n}p(y_k|y_0,\ldots,y_{k-1})
$$

and

$$
p(y_k|y_0,\ldots,y_{k-1})=\int\underbrace{p(y_k|x_k)}_{:=G_k(x_k)}\ p(x_k|y_0,\ldots,y_{k-1})\ dx_k.
$$

We use the convention $p(y_0|y_0,\ldots,y_{-1})=p(y_0)$, for $k=0$.

We let η_n be the n-th time marginal of the path space measure $\mathbb{Q}_n$ defined above. From previous calculations, we have

$$
\begin{aligned}
\eta_n &= \mathrm{Law}(X_n\mid Y_k=y_k,\ 0\le k<n)\\
\mathbb{Q}_n &= \mathrm{Law}((X_0,\ldots,X_n)\mid Y_k=y_k,\ 0\le k<n)\\
\mathcal{Z}_n &= p(y_0,\ldots,y_{n-1})=\prod_{0\le k<n}\eta_k(G_k)=\gamma_n(1)
\end{aligned}
$$

with the unnormalized Feynman-Kac measures γ_n defined in (9.23).

- Using (9.30) the sequence of distributions η_n satisfies the nonlinear updating-prediction equation

$$
\eta_n=\Psi_{G_{n-1}}(\eta_{n-1})M_n.
$$

This ends the proof of the exercise.

Solution to exercise 140:

- We use the mean field particle models described in section 9.6.1. We let ϵ_n any positive number s.t. $\epsilon_n G_n(x_n)\le 1$ for any $x_n\in\mathbb{R}$. For instance we can choose $\epsilon_n=0$, or

$$
\epsilon_n=\sqrt{2\pi}\Rightarrow\epsilon_n G_n(x_n)=\exp\left(-\frac{1}{2}\ (y_n-b_n(x_n))^2\right)\in[0,1].
$$

The particle filter starts with N i.i.d. copies $\xi_0 = (\xi_0^i)_{1\leq i\leq N} \in \mathbb{R}$ of the initial state X_0. The evolution of the particles is decomposed into 2 transitions

$$\xi_n = (\xi_n^i)_{1\leq i\leq N} \xrightarrow{\text{updating}} \widehat{\xi}_n = (\widehat{\xi}_n^i)_{1\leq i\leq N} \xrightarrow{\text{prediction}} \xi_{n+1} = (\xi_{n+1}^i)_{1\leq i\leq N}.$$

During the updating-selection transition, for each $1 \leq i \leq N$ we set $\widehat{\xi}_n^i = \xi_n^i$ with a probability $\epsilon\ G_n(\xi_n^i)$; otherwise we set $\widehat{\xi}_n^i = \widetilde{\xi}_n^i$, where $\widetilde{\xi}_n^i$ denotes a random variable with distribution

$$\Psi_{G_n}(\eta_n^N) = \sum_{1\leq i\leq N} \frac{G_n(\xi_n^i)}{\sum_{1\leq j\leq N} G_n(\xi_n^j)}\ \delta_{\xi_n^i}.$$

Here η_n^N is the occupation measure $\eta_n^N = \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi_n^i}$ of the particles $\xi_n = (\xi_n^i)_{1\leq N}$. During the prediction-mutation transition, we sample N independent copies (W_{n+1}^i) of W_{n+1} and we set

$$\forall 1 \leq i \leq N \qquad \xi_{n+1}^i = a_n(\widehat{\xi}_n^i) + W_n^i.$$

In other words, during the prediction transition we sample N independent random variables ξ_{n+1}^i with distribution $M_{k+1}\left(\widehat{\xi}_k^i, dx_{k+1}\right)$, with $1 \leq i \leq N$.

By (9.49) we have

$$\eta_n^N = \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi_n^i} \ \longrightarrow_{N\uparrow\infty}\ \eta_n.$$

- Using the product formula (9.50) an unbiased particle approximation of the normalizing constant $\gamma_n(1)$ is given by the formula

$$\gamma_n^N(1) := \prod_{0\leq k<n}\eta_k^N(G_k) = \prod_{0\leq k<n}\frac{1}{\sqrt{2\pi}}\ \frac{1}{N}\sum_{1\leq i\leq N}\exp\left(-\frac{1}{2}\ (y_k - b_k(\xi_k^i))^2\right).$$

By (9.50), for any function f on $\mathbb{R}$ we also have

$$\gamma_n^N(f) := \gamma_n^N(1)\times\eta_n^N(f) \longrightarrow_{N\uparrow\infty}\ \gamma_n(f).$$

- Using the Feynman-Kac models on path space discussed in (9.6.2), the ancestral lines

$$\forall 1 \leq i \leq N \qquad \left(\xi_{0,n}^i, \xi_{1,n}^i, \ldots, \xi_{n-1,n}^i, \xi_{n,n}^i\right)$$

of the individuals $\xi_{n,n}^i = \xi_n^i$ can be interpreted as a mean field particle approximations of the Feynman-Kac models on path space. Using (9.55), we have

$$\frac{1}{N}\sum_{1\leq i\leq N}\delta_{\left(\xi_{0,n}^i,\xi_{1,n}^i,\ldots,\xi_{n,n}^i\right)} \longrightarrow_{N\to\infty} \mathbb{Q}_n = \mathrm{Law}((X_0,\ldots,X_n)\mid Y_k = y_k,\ 0\leq k<n).$$

- Notice that the likelihood functions G_n and the Markov transitions M_n satisfy the regularity property (9.56)

$$\begin{aligned}
&G_{n-1}(x_{n-1})\ M_n(x_{n-1}, dx_n)\\
&= \exp\left(-\frac{1}{2}\ (y_n - b_n(x_n))^2\right)\times\exp\left(-\frac{1}{2}\,(x_n - a_n(x_{n-1}))^2\right)\ \frac{dx_n}{2\pi}\\
&= H_n(x_{n-1}, x_n)\ \lambda_n(dx_n)
\end{aligned}$$

with $\lambda_n(dx_n) = \dfrac{dx_n}{2\pi}$ and

$$H_n(x_{n-1}, x_n) = \exp\left(-\frac{1}{2}\left[(y_n - b_n(x_n))^2 + (x_n - a_n(x_{n-1}))^2\right]\right).$$

The backward particle model (9.58) is given by

$$\mathbb{Q}_n^N(d(x_0,\ldots,x_n)) = \eta_n^N(dx_n) \prod_{0\leq k<n} \mathbb{M}_{k+1,\eta_k^N}(x_{k+1}, dx_k) \longrightarrow_{N\uparrow\infty} \mathbb{Q}_n$$

with the backward Markov transitions

$$\mathbb{M}_{k+1,\eta_k^N}(x_{k+1}, dx_k) = \sum_{1\leq i\leq N} \frac{H_{k+1}(\xi_k^i, x_{k+1})}{\sum_{1\leq j\leq N} H_{k+1}(\xi_k^j, x_{k+1})}\ \delta_{\xi_k^i}(dx_k).$$

This ends the proof of the exercise.

Solution to exercise 141:

- By construction, we have

$$\begin{aligned}\frac{\partial}{\partial\theta} p_\theta(y_0,\ldots,y_n) &= \frac{\partial}{\partial\theta}\mathbb{E}\left(\prod_{0\leq k\leq n} G_{\theta,k}(X_k)\right)\\ &= \mathbb{E}\left(\frac{\partial}{\partial\theta}\left[\prod_{0\leq k\leq n} G_{\theta,k}(X_k)\right]\right)\end{aligned}$$

with

$$G_{\theta,k}(X_k) = \frac{1}{\sqrt{2\pi}}\ \exp\left(-\frac{1}{2}\ (y_k - c_k(X_k) - \theta d_k(X_k))^2\right).$$

Using the fact that

$$\frac{\partial}{\partial\theta}\left[\prod_{0\leq k\leq n} G_{\theta,k}(X_k)\right] = \left[\sum_{0\leq k\leq n} \frac{\partial}{\partial\theta}\log G_{\theta,k}(X_k)\right]\ \left[\prod_{0\leq k\leq n} G_{\theta,k}(X_k)\right]$$

and

$$\frac{\partial}{\partial\theta}\log G_{\theta,k}(X_k) = (y_k - c_k(X_k) - \theta d_k(X_k))\ d_k(X_k) := l_{\theta,k}(X_k)$$

we prove that

$$\frac{\partial}{\partial\theta} p_\theta(y_0,\ldots,y_n) = \mathbb{E}\left(L_{n,\theta}(X_0,\ldots,X_n)\ \prod_{0\leq k\leq n} G_{\theta,k}(X_k)\right)$$

with the additive functional

$$L_{n,\theta}(X_0,\ldots,X_n) = \sum_{0\leq k\leq n} l_{\theta,k}(X_k).$$

We conclude that

$$\frac{\partial}{\partial\theta}\log p_\theta(y_0,\ldots,y_n)=\mathbb{Q}_{n+1,\theta}(\mathcal{L}_{n+1,\theta})$$

with

$$\mathcal{L}_{n+1,\theta}(X_0,\ldots,X_n,X_{n+1})=L_{n,\theta}(X_0,\ldots,X_n).$$

- We fix the parameter θ. With a slight abuse of notation we let $\eta_n^N=\frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi_n^i}$ and $(\xi_{0,n}^i,\ldots,\xi_{n,n}^i)_{1\leq i\leq N}$ be the occupation measures and the ancestral lines of the genetic type N-particle model associated with the likelihood-selection fitness functions $(G_{k,\theta})_{k\geq 0}$ defined in (9.104).

  Using the Feynman-Kac particle approximation on path-space in terms of ancestral lines we have

$$\frac{1}{N}\sum_{1\leq i\leq N}\mathcal{L}_{n+1,\theta}(\xi_{0,n+1}^i,\xi_{1,n+1}^i,,\ldots,\xi_{n+1,n+1}^i)$$

$$\longrightarrow_{N\uparrow\infty}\ \frac{\mathbb{E}\left(\mathcal{L}_{n+1,\theta}(X_0,\ldots,X_{n+1})\ \prod_{0\leq k\leq n}G_{\theta,k}(X_k)\right)}{\mathbb{E}\left(\prod_{0\leq k\leq n}G_{\theta,k}(X_k)\right)}=\mathbb{Q}_{n+1,\theta}(\mathcal{L}_{n+1,\theta})=\frac{\partial}{\partial\theta}\log p_\theta(y_0,\ldots,y_n).$$

  On the other hand, using the backward particle approximation

$$\mathbb{Q}_{n+1,\theta}^N(d(x_0,\ldots,x_{n+1}))=\eta_{n+1}^N(dx_{n+1})\prod_{0\leq k\leq n}\mathbb{M}_{k+1,\eta_k^N}^{(\theta)}(x_{k+1},dx_k)\longrightarrow_{N\uparrow\infty}\ \mathbb{Q}_{n+1,\theta}$$

  with the collection of transitions $\mathbb{M}_{k+1,\eta}^{(\theta)}$ defined as $\mathbb{M}_{k+1,\eta}$ by replacing G_k by the function $G_{k,\theta}$ defined in (9.104). This yields the particle approximation

$$\mathbb{Q}_{n+1}^N(\mathcal{L}_{n+1,\theta})=\sum_{0\leq k\leq n}\eta_{n+1}^N M_{n+1,\eta_n^N}\ldots M_{k+1,\eta_k^N}(l_{\theta,k}).$$

This ends the proof of the exercise.

■

Solution to exercise 142:

- The Feynman-Kac measures (γ_n,η_n) are described in exercise 139. The one step optimal predictor

$$\eta_n=\text{Law}(X_n\mid Y_k=y_k,\ 0\leq k<n)$$

  is defined by the Feynman-Kac model

$$\eta_n(f)=\gamma_n(f)/\gamma_n(1)\quad\text{with}\quad\gamma_n(f)=\mathbb{E}\left(f(X_n)\prod_{0\leq p<n}G_p(X_p)\right)$$

  for any function f on $\mathbb{R}$, and with the likelihood potential functions

$$G_n(x_n):=p(y_n|x_n):=\frac{1}{\sqrt{2\pi}}\ \exp\left(-\frac{1}{2}\ (y_n-b_n(x_n))^2\right).$$

- We consider the genetic-type particle filter approximation of the $1d$-nonlinear filtering problem defined in (9.103):

$$\xi_n=(\xi_n^i)_{1\leq i\leq N}\xrightarrow{\text{updating}}\widehat{\xi}_n=(\widehat{\xi}_n^i)_{1\leq i\leq N}\xrightarrow{\text{prediction}}\xi_{n+1}=(\xi_{n+1}^i)_{1\leq i\leq N},\tag{30.29}$$

starting with N i.i.d. copies $\xi_0 = (\xi_0^i)_{1\leq i\leq N} \in \mathbb{R}$ of the initial state X_0. During the updating transition, we sample N random variables $(\widehat{\xi}_n^i)_{1\leq i\leq N}$ with distribution

$$\sum_{1\leq i\leq N} \frac{G_n(\xi_n^i)}{\sum_{1\leq j\leq N} G_n(\xi_n^j)}\ \delta_{\xi_n^i}.$$

During the prediction-mutation transition, we sample N independent copies (W_{n+1}^i) of W_{n+1} and we set

$$\forall 1\leq i\leq N \qquad \xi_{n+1}^i = a_n(\widehat{\xi}_n^i) + W_n^i.$$

By (9.49) and (9.50) we have

$$\eta_n^N = \frac{1}{N}\sum_{1\leq i\leq N} \delta_{\xi_n^i} \longrightarrow_{N\uparrow\infty} \eta_n \quad \text{and} \quad \gamma_n^N(f) := \gamma_n^N(1)\times\eta_n^N(f) \longrightarrow_{N\uparrow\infty} \gamma_n(f).$$

- The many-body Feynman-Kac measures $(\overline{\gamma}_n, \overline{\eta}_n)$ associated with (γ_n, η_n) are the Feynman-Kac measures on $\mathbb{R}^N$ defined for any function $\overline{f}$ on $\overline{S} = \mathbb{R}^N$ by

$$\overline{\eta}_n(\overline{f}) = \overline{\gamma}_n(\overline{f})/\overline{\gamma}_n(1) \quad \text{with} \quad \overline{\gamma}_n(\overline{f}) := \mathbb{E}\left(\overline{f}(\overline{X}_n)\prod_{0\leq p<n}\overline{G}_p(\overline{X}_p)\right).$$

In the above display, $\overline{X}_n := \left(\xi_n^i\right)_{1\leq i\leq N} \in S_n^N$ and the collection of particle likelihood functions $\overline{G}_n$ on $\mathbb{R}^N$ are defined by

$$\overline{G}_n(\overline{X}_n) := \eta_n^N(G_n) = \frac{1}{N}\sum_{1\leq i\leq N} G_n(\xi_n^i).$$

We recall that

$$\overline{f}(\overline{X}_n) := \frac{1}{N}\sum_{1\leq i\leq N} f(\xi_n^i) \Rightarrow \overline{\gamma}_n(\overline{f}) = \gamma_n(f).$$

This ends the proof of the exercise. ∎

Solution to exercise 143:

- For path-space models, the conditional distributions

$$\eta_n = \mathrm{Law}((X_0',\ldots,X_n') \mid Y_k = y_k,\ 0\leq k<n) = \mathrm{Law}(X_n \mid Y_k = y_k,\ 0\leq k<n)$$

are defined for any function f_n on $S_n = \mathbb{R}^{n+1}$ by the Feynman-Kac model

$$\eta_n(f_n) = \gamma_n(f_n)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f_n) = \mathbb{E}\left(f(X_n)\prod_{0\leq p<n} G_p(X_p)\right).$$

In the above display, the likelihood potential functions G_n on the path space are defined for any $x_n = (x_0',\ldots,x_n') \in S_n = \mathbb{R}^{n+1}$ by

$$G_n(x_n) := G_n'(x_n') := p(y_n|x_n') := \frac{1}{\sqrt{2\pi}}\ \exp\left(-\frac{1}{2}\ (y_n - b_n(x_n'))^2\right).$$

- The corresponding genetic type particle model (30.29) is defined in terms of path-particles

$$\xi_n^i = \left(\xi_{0,n}^i, \xi_{1,n}^i, \ldots, \xi_{n,n}^i\right) \quad \text{and} \quad \widehat{\xi}_n^i = \left(\widehat{\xi}_{0,n}^i, \widehat{\xi}_{1,n}^i, \ldots, \widehat{\xi}_{n,n}^i\right) \in S_n = \mathbb{R}^{n+1}.$$

We also start with N i.i.d. copies $\xi_0 = (\xi_0^i)_{1\leq i\leq N} \in \mathbb{R}$ of the initial state $X_0 = X_0'$.

During the updating transition, we sample N random paths $(\widehat{\xi}_n^i)_{1\leq i\leq N}$ with the weighted distribution

$$\sum_{1\leq i\leq N} \frac{G_n(\xi_n^i)}{\sum_{1\leq j\leq N} G_n(\xi_n^j)}\, \delta_{\xi_n^i} = \sum_{1\leq i\leq N} \frac{G_n'(\xi_{n,n}^i)}{\sum_{1\leq j\leq N} G_n'(\xi_{n,n}^j)}\, \delta_{(\xi_{0,n}^i, \xi_{1,n}^i, \ldots, \xi_{n,n}^i)}.$$

During the prediction-mutation transition, we sample N independent copies (W_{n+1}^i) of W_{n+1} and for each $1 \leq i \leq N$ we set

$$\begin{aligned} \xi_{n+1}^i &= \left(\left(\xi_{0,n+1}^i, \xi_{1,n+1}^i, \ldots, \xi_{n,n+1}^i\right), \xi_{n+1,n+1}^i\right) \\ &= \left(\left(\widehat{\xi}_{0,n}^i, \widehat{\xi}_{1,n}^i, \widehat{\xi}_{2,n}^i, \ldots, \widehat{\xi}_{n,n}^i\right), \xi_{n+1,n+1}^i\right) \left(= \left(\widehat{\xi}_n^i, \xi_{n+1,n+1}^i\right)\right) \in \mathbb{R}^{n+2} \end{aligned}$$

with

$$\xi_{n+1,n+1}^i = a_n(\widehat{\xi}_{n,n}^i) + W_n^i.$$

By (9.49) and (9.50) we have

$$\eta_n^N = \frac{1}{N} \sum_{1\leq i\leq N} \delta_{\xi_n^i} \longrightarrow_{N\uparrow\infty} \eta_n \quad \text{and} \quad \gamma_n^N(f_n) := \gamma_n^N(1) \times \eta_n^N(f_n) \longrightarrow_{N\uparrow\infty} \gamma_n(f_n).$$

- The many-body Feynman-Kac measures $(\overline{\gamma}_n, \overline{\eta}_n)$ associated with (γ_n, η_n) are the Feynman-Kac measures on $(\mathbb{R}^{n+1})^N$ defined for any function $\overline{f}_n$ on $\overline{S}_n = (\mathbb{R}^{n+1})^N$ by

$$\overline{\eta}_n(\overline{f}_n) = \overline{\gamma}_n(\overline{f}_n)/\overline{\gamma}_n(1) \quad \text{with} \quad \overline{\gamma}_n(\overline{f}_n) := \mathbb{E}\left(\overline{f}(\overline{X}_n) \prod_{0\leq p<n} \overline{G}_p(\overline{X}_p)\right).$$

In the above display, the reference Markov chain $\overline{X}_n$ is the path-space particle model

$$\overline{X}_n := \left(\xi_n^i\right)_{1\leq i\leq N} = \left(\left(\xi_{0,n}^i, \xi_{1,n}^i, \ldots, \xi_{n,n}^i\right)\right)_{1\leq i\leq N} \in S_n^N \tag{30.30}$$

and the collection of particle likelihood functions $\overline{G}_n$ on $(\mathbb{R}^{n+1})^N$ is defined by

$$\overline{G}_n(\overline{X}_n) = \frac{1}{N} \sum_{1\leq i\leq N} G_n(\xi_n^i) = \frac{1}{N} \sum_{1\leq i\leq N} G_n'(\xi_{n,n}^i).$$

We recall that

$$\overline{f}_n(\overline{X}_n) := \frac{1}{N} \sum_{1\leq i\leq N} f_n(\xi_n^i) \Rightarrow \overline{\gamma}_n(\overline{f}_n) = \gamma_n(f_n).$$

- We consider the Feynman-Kac measures on $\boldsymbol{S_n} = (\overline{S}_0 \times \ldots \times \overline{S}_n)$ defined for any bounded function $\boldsymbol{f_n}$ on $\boldsymbol{S_n}$ by

$$\boldsymbol{\overline{\eta}_n}(\boldsymbol{\overline{f}_n}) := \boldsymbol{\overline{\gamma}_n}(\boldsymbol{\overline{f}_n})/\boldsymbol{\overline{\gamma}_n}(\mathbf{1}) \quad \text{with} \quad \boldsymbol{\overline{\gamma}_n}(\boldsymbol{\overline{f}_n}) := \mathbb{E}\left(\boldsymbol{\overline{f}_n}(\boldsymbol{\overline{X}_n}) \prod_{0\leq p<n} \overline{G}_p(\overline{X}_p)\right)$$

In the above display, $\boldsymbol{\overline{X}_n} := (\overline{X}_0, \ldots, \overline{X}_n) \in \boldsymbol{\overline{S}_n} = (\overline{S}_0 \times \ldots \times \overline{S}_n)$ is the historical process associated with the path-particle Markov chain (30.30).

We design a particle Metropolis-Hastings algorithm with a target measure $\boldsymbol{\overline{\eta}_n}$ following the methodology developed in section 9.7.2.

More precisely, we fix the time horizon n and we define the particle Metropolis-Hastings Markov chain $(\mathcal{X}_k)_{k\geq 0}$ on $\boldsymbol{\overline{S}_n}$ as follows.

Given some historical trajectory of the path-particle model

$$\mathcal{X}_k = \boldsymbol{\overline{x}_n} = (\overline{x}_0, \ldots, \overline{x}_n) \in \boldsymbol{\overline{S}_n} = (\overline{S}_0 \times \ldots \times \overline{S}_n)$$

we sample an independent trajectory

$$\mathcal{Y}_k = \boldsymbol{\overline{y}_n} = (\overline{y}_0, \ldots, \overline{y}_n) \in \boldsymbol{\overline{S}_n}$$

of the historical process $\boldsymbol{\overline{X}_n} := (\overline{X}_0, \ldots, \overline{X}_n)$. With a probability

$$a\left(\boldsymbol{\overline{x}_n}, \boldsymbol{\overline{y}_n}\right) = 1 \wedge \frac{\prod_{0\leq k<n} \overline{G}_k(\overline{y}_k)}{\prod_{0\leq k<n} \overline{G}_k(\overline{x}_k)}$$

we set $\mathcal{X}_{k+1} = \mathcal{Y}_k$, otherwise we set $\mathcal{X}_{k+1} = \mathcal{X}_k$. By construction, $\mathcal{X}_k$ is a Markov chain with invariant measure $\boldsymbol{\overline{\eta}_n}$. In addition, for any function of the form

$$\boldsymbol{\overline{f}_n}(\boldsymbol{\overline{X}_n}) = \overline{f}_n(\overline{X}_n) = \frac{1}{N} \sum_{1\leq i\leq N} f_n(\xi_n^i)$$

with

$$\overline{X}_n = \left(\xi_n^i\right)_{1\leq i\leq N} \quad \text{and} \quad \forall 1 \leq i \leq N \quad \xi_n^i = \left(\xi_{0,n}^i, \ldots, \xi_{n,n}^i\right) \in \mathbb{R}^{n+1}$$

we have

$$\boldsymbol{\overline{\eta}_n}(\boldsymbol{\overline{f}_n}) = \eta_n(f_n) = \mathbb{E}(f_n(X'_0, \ldots, X'_n) \mid Y_k = y_k,\ 0 \leq k < n).$$

This ends the proof of the exercise. ∎

Solution to exercise 144:

- The posterior distributions of $X_n = (X'_0, \ldots, X'_n) \in \mathbb{R}^{n+1}$ given the sequence of observations $(Y'_k)_{0\leq k<n} = (y'_k)_{0\leq k<n}$ are defined by the Feynman-Kac measures (γ_n, η_n) defined in (9.23) with

$$G_n(X_n) = G'_n(X'_n) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y'_n - b_n(X'_n))^2\right).$$

- The genealogical tree based particle approximation of the measures η_n are discussed in full details in section 9.6.2.The many-body Feynman-Kac measures associated with the mean field particle interpretation of the measures η_n are defined in section 9.7.1. The dual mean field model with frozen trajectory X_n is described in full details in section 9.7.3, and a couple of particle Gibbs-Glauber algorithms are presented is section 9.7.4.

This ends the proof of the exercise. ∎

Chapter 10

Solution to exercise 145:

Since N_t is a Poisson random variable with parameter λt, we have

$$\mathbb{E}(N_t) = e^{-\lambda t}\sum_{n\geq 1} n\,\frac{(\lambda t)^n}{n!} = \lambda t\; e^{-\lambda t}\sum_{n\geq 1}\frac{(\lambda t)^{n-1}}{(n-1)!} = \lambda t.$$

and

$$\mathbb{E}(N_t^2) = e^{-\lambda t}\sum_{n\geq 1}(n(n-1)+n)\,\frac{(\lambda t)^n}{n!} = \mathbb{E}(N_t) + (\lambda t)^2 e^{-\lambda t}\sum_{n\geq 2}\frac{(\lambda t)^{n-2}}{(n-2)!} = \lambda t + (\lambda t)^2.$$

This ends the proof of the exercise. ■

Solution to exercise 146:

- For any $s \in [0,t]$, we have

$$\begin{aligned}\mathbb{P}(T_1 \leq s \mid N_t = 1) &= \mathbb{P}(N_s = 1 \mid N_t = 1)\\ &= \frac{\mathbb{P}\left(N_s = 1,\ (N_t - N_s) = 0\right)}{\mathbb{P}(N_t = 1)} = \frac{\frac{(\lambda s)^1}{1!}e^{-\lambda s}\,\frac{(\lambda(t-s))^0}{0!}e^{-\lambda(t-s)}}{\frac{(\lambda t)^1}{1!}e^{-\lambda t}} = \frac{s}{t}.\end{aligned}$$

  This shows that the conditional distribution of T_1 given $N_t = 1$ is the uniform distribution on $[0,t]$.

- For any $s \in [0,t]$, we have

$$\begin{aligned}\mathbb{P}(T_1 \leq s \mid N_t = 2) &= \mathbb{P}(N_s \geq 1 \mid N_t = 2)\\ &= \frac{\mathbb{P}\left(N_s = 1,\ (N_t - N_s) = 1\right)}{\mathbb{P}(N_t = 2)} + \frac{\mathbb{P}\left(N_s = 2,\ (N_t - N_s) = 0\right)}{\mathbb{P}(N_t = 2)}\\ &= \frac{\frac{(\lambda s)^1}{1!}e^{-\lambda s}\,\frac{(\lambda(t-s))^1}{1!}e^{-\lambda(t-s)}}{\frac{(\lambda t)^2}{2!}e^{-\lambda t}} + \frac{\frac{(\lambda s)^2}{2!}e^{-\lambda s}\,\frac{(\lambda(t-s))^0}{0!}e^{-\lambda(t-s)}}{\frac{(\lambda t)^2}{2!}e^{-\lambda t}}\\ &= 2\,\frac{s(t-s)}{t^2} + \frac{s^2}{t^2} = \frac{2st - s^2}{t^2} = 2\,\frac{s}{t} - \left(\frac{s}{t}\right)^2 = 1 - \left(1 - \frac{s}{t}\right)^2.\end{aligned}$$

  This shows that the conditional distribution of T_1 given $N_t = 2$ is the distribution on $[0,t]$ with density

$$\frac{\partial}{\partial s}\mathbb{P}(T_1 \leq s \mid N_t = 2) = 2\,\frac{1}{t}\,\left(1 - \frac{s}{t}\right).$$

- For any $s \in [0,t]$, we have

$$\begin{aligned}\mathbb{P}(T_2 \leq s \mid N_t = 2) &= \mathbb{P}(N_s \geq 1 \mid N_t = 2)\\ &= \frac{\mathbb{P}\left(N_s = 2,\ (N_t - N_s) = 0\right)}{\mathbb{P}(N_t = 2)} = \frac{\frac{(\lambda s)^2}{2!}e^{-\lambda s}\,\frac{(\lambda(t-s))^0}{0!}e^{-\lambda(t-s)}}{\frac{(\lambda t)^2}{2!}e^{-\lambda t}} = \left(\frac{s}{t}\right)^2.\end{aligned}$$

This shows that the conditional distribution of T_2 given $N_t = 2$ is the distribution on $[0,t]$ with density

$$\frac{\partial}{\partial s}\mathbb{P}(T_2 \leq s \mid N_t = 2) = 2\ \frac{1}{t}\ \frac{s}{t}.$$

This ends the proof of the exercise. ■

Solution to exercise 147:
We have

$$\begin{aligned}
\mathbb{P}(N(t)=n) &= \sum_{0\leq m\leq n} \mathbb{P}(N_1(t)=m)\mathbb{P}(N_2(t)=n-m)\\
&= e^{-(\lambda_1+\lambda_2)t} \sum_{0\leq m\leq n} \frac{(\lambda_2 t)^m}{m!}\ \frac{(\lambda_1 t)^{n-m}}{(n-m)!}\\
&= e^{-(\lambda_1+\lambda_2)t}\frac{1}{n!} \sum_{0\leq m\leq n} \frac{n!}{m!(n-m)!}\ (\lambda_2 t)^m\ (\lambda_1 t)^{n-m}\\
&= e^{-(\lambda_1+\lambda_2)t}\frac{1}{n!}\ \left([\lambda_1+\lambda_2]t\right)^n.
\end{aligned}$$

This shows that $N(t)$ is a Poisson random variable with parameter $(\lambda_1+\lambda_2)t$. The conditional distribution of $N_1(t)$ given $N(t)$ is given for any $0 \leq m \leq n$ by

$$\begin{aligned}
\mathbb{P}(N_1(t)=m \mid N(t)=n) &= \frac{\mathbb{P}(N_1(t)=m\ N_2(t)=n-m)}{\mathbb{P}(N(t)=n)}\\
&= \frac{\mathbb{P}(N_1(t)=m)\times\mathbb{P}(N_2(t)=n-m)}{\mathbb{P}(N(t)=n)}\\
&= \frac{\frac{1}{m!}\ (\lambda_1 t)^m \times \frac{1}{(n-m)!}\ (\lambda_2 t)^{n-m}}{\frac{1}{n!}\ ([\lambda_1+\lambda_2]t)^n}\\
&= \frac{n!}{m!(n-m)!}\ \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^m\ \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-m}.
\end{aligned}$$

This is clearly a binomial distribution. This ends the proof of the exercise. ■

Solution to exercise 148: Clearly $N_n \sim Bin(n,p)$ is a Binomial random variable with parameters n and p. We have

$$m < n \Rightarrow N_n - N_m = \sum_{m<k\leq n} \mathcal{E}_k \stackrel{\text{law}}{=} N_{n-m} \sim Bin(n-m,p).$$

In addition N_n and $(N_n - N_m)$ are independent. More generally,

$$(N_{n_1}, N_{n_2} - N_{n_1}, \ldots, N_{n_k} - N_{n_{k-1}})$$

are independent Binomal random variables with parameters

$$((n_1,p),(n_2-n_1)p,\ldots,(n_k-n_{k-1})p)$$

for any sequence of parameters $1 \leq n_1 < \ldots < n_k$.
Finally, we observe that

$$(T_1 \leq n) = (N_n \geq 1) = \Omega - (N_n = 0) \Rightarrow \mathbb{P}(T_1 \leq n) = 1 - \mathbb{P}(N_n = 0) = 1-(1-p)^n.$$

Therefore, we have

$$(T_1 = n) = (T_1 \leq n) - (T_1 \leq (n-1))$$

$$\Rightarrow \mathbb{P}(T_1 = n) = (1-(1-p)^n) - \left(1-(1-p)^{n-1}\right) = (1-p)^{n-1}(1-(1-p)) = p(1-p)^{n-1}$$

We conclude that T_1 is a Geometric random variable with (success) parameter p.

Finally, we have

$$\begin{aligned}
(T_n = k) &= (T_n \leq k) \cap (T_n > k-1) \\
&= (N_k \geq n) \cap (N_{k-1} < n) = (N_k \geq n) \cap (N_{k-1} = (n-1)) \\
&= (N_{k-1} = (n-1)) \cap (\mathcal{E}_k = 1)
\end{aligned}$$

This implies that S_n is distributed according to the negative binomial probability

$$\begin{aligned}
\mathbb{P}(T_n = k) &= \binom{k-1}{n-1} \; p^{n-1} \; (1-p)^{(k-1)-(n-1)} \; \times p \\
&= \binom{k-1}{n-1} \; p^n \; (1-p)^{k-n}.
\end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 149:

The key idea is to run a Poisson process $N(t)$ with intensity $\lambda = \lambda_1 + \lambda_2$. At each jump time, $N_1(t)$ jumps with a probability $\frac{\lambda_1}{\lambda_1+\lambda_2}$ (and $N_2(t)$ does not jump), and $N_2(t)$ jumps with a probability $\frac{\lambda_2}{\lambda_1+\lambda_2}$ (and $N_1(t)$ does not jump). Among the first $(n+(m-1))$ jumps of $N(t)$, $N_1(t)$ jumps at least n times. Therefore we have

$$\begin{aligned}
&\mathbb{P}\left(N_1 \text{ jumps } n \text{ times before } N_2 \text{ jumps } m \text{ times}\right) \\
&= \textstyle\sum_{n \leq k < m+n} \binom{n+(m-1)}{k} \; \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \; \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{(n+(m-1))-k}.
\end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 150:

By construction, we have $N_2(T^{(2)}_{n-1}) = N^{(2)}_0 + (n-1)$ so that

$$T^{(2)}_n - T^{(2)}_{n-1} = \frac{T^{(1)}_n - T^{(1)}_{n-1}}{\lambda(N^{(2)}_0 + (n-1))}.$$

This ends the proof of the exercise. ■

Solution to exercise 151:

Combining

$$\begin{aligned}
&\mathbb{P}\left((T_1, \ldots, T_n) \in d(t_1, \ldots, t_n)\right) \\
&= \left[\prod_{1 \leq p < n} 1_{[t_{p-1},\infty[}(t_p) \; \lambda_{t_p} \;\; dt_p\right] \exp\left(-\textstyle\int_0^{t_n} \lambda_s \; ds\right) 1_{[t_{n-1},\infty[}(t_n) \; dt_n
\end{aligned}$$

and

$$\mathbb{P}\left(T_n \in dt\right) = \frac{\left(\int_0^t \lambda_s ds\right)^{n-1}}{(n-1)!}\ \lambda_t\ \exp\left(-\int_0^t \lambda_s\ ds\right)\ dt$$

with the Bayes formula we prove

$$\begin{aligned} &\mathbb{P}\left((T_1,\ldots,T_{n-1}) \in d(t_1,\ldots,t_{n-1}) \mid T_n = t_n\right) \\ &= \ (n-1)!\ 1_{0\leq t_1\leq \ldots \leq t_{n-1}\leq t_n} \prod_{1\leq p<n} \frac{\lambda_{t_p} dt_p}{\int_0^t \lambda_s ds}. \end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 152:
We have

$$X_t - X_{t-} = dX_t = a\ X_{t-}\ dN_t = a\ X_{t-}\ (N_t - N_{t-}).$$

Thus, at the first jump time say T_1 of N_t we have

$$X_{T_1} = X_0 + a\ X_0 = (1+a)^{N_{T_1}}\ X_0.$$

Let T_2 be second jump time of N_t. By construction, we have

$$\forall T_1 \leq t < T_2 \qquad X_t = X_{T_1} = (1+a)^{N_{T_1}} X_0\ = (1+a)^{N_t}\ X_0$$

and

$$X_{T_2} = X_{T_2-} + a\ X_{T_2-} = (1+a)\ (1+a)^{N_{T_2-}}\ X_0 = (1+a)^{N_{T_2-}+1}\ X_0 = (1+a)^{N_{T_2}}\ X_0.$$

Iterating the argument, we prove that

$$X_t = (1+a)^{N_t}\ X_0.$$

This ends the proof of the exercise.

■

Solution to exercise 153:
Using the same arguments as in exercise 152, we find that

$$X_t = X_0 \prod_{0\leq k\leq N_t} (1+a_{T_k}) = X_0 \prod_{0\leq s\leq t\ ,dN_s=1} (1+a_s).$$

This ends the proof of the exercise.

■

Solution to exercise 154:
Between two jump times, say $T_n \leq t \leq T_{n+1}$ the process X_t satisfies the linear equation

$$dX_t = (b_t - \lambda\ a_t)\ X_{t-} dt.$$

This implies that

$$\forall T_n \leq t \leq T_{n+1} \qquad X_t = X_{T_n}\ \exp\left(\int_0^t (b_s - \lambda\ a_s) ds\right).$$

Using the same arguments as in exercise 153, we find that

$$X_t = X_0 \ \exp\left(\int_0^t (b_s - \lambda \ a_s) ds\right) \prod_{0 \le k \le N_t} (1 + a_{T_k}).$$

This ends the proof of the exercise.

■

Chapter 11

Solution to exercise 155:

The process $X_t = Y_{N_t}$ can be interpreted as the embedding of a Markov chain Y_n with transition probabilities K, with a Poisson process N_t with intensity $\lambda >$. We let $(T_n)_{n\geq 0}$ be the jump times of the process N_t defined by

$$T_0 = 0 \qquad \forall n \geq 1 \quad T_{n+1} - T_n = -\frac{1}{\lambda}\ \log U_n$$

where U_n stands for a sequence of independent uniform random variables on $]0,1[$ (independent of the sequence $(Y_n)_{n\geq 0}$). We recall that $E_n = -\frac{1}{\lambda}\ \log U_n$ forms a sequence of independent exponential random variables with parameter λ.

Given $N_t = n$ we have

$$\forall n \geq 0 \quad \forall t \in [T_n, T_{n+1}[\qquad X_t = X_{T_n} = Y_n.$$

This implies that

$$\int_0^{T_n} V(X_t)\ dt \ = \ \sum_{0\leq k<n} \int_{T_k}^{T_{k+1}} V(X_t)\ dt = \sum_{0\leq k<n} V(X_{T_k})\ (T_{k+1} - T_k).$$

Therefore

$$\begin{aligned}
&\mathbb{E}\left(f(X_{T_n})\ \exp\left(\int_0^{T_n} V(X_s)\ ds\right) \mid (T_0,\ldots,T_n)\right) \\
&= \mathbb{E}\left(f(Y_n) \prod_{0\leq k<n} e^{(T_{k+1}-T_k)V(Y_k)} \mid (T_0,\ldots,T_n)\right),
\end{aligned}$$

as well as

$$\mathbb{E}\left(f(X_{T_n})\ \exp\left(\int_0^{T_n} V(X_s)\ ds\right)\right) \ = \ \mathbb{E}\left(f(Y_n) \prod_{0\leq k<n} e^{E_k V(Y_k)}\right).$$

We also find that

$$\begin{aligned}
N_t = n \ \Rightarrow \ \int_0^{T_n} V(X_t)\ dt \ &= \ \int_0^{T_n} V(X_t)\ dt + \int_{T_n}^{t} V(X_t)\ dt \\
&= \sum_{0\leq k<n} V(X_{T_k})\ (T_{k+1} - T_k) + V(T_n)\ (t - T_n).
\end{aligned}$$

This yields

$$\begin{aligned}
&\mathbb{E}\left(f(X_t)\ \exp\left(\int_0^{t} V(X_s)\ ds\right) \mid N_t = n\right) \\
&= \mathbb{E}\left(f(Y_n) e^{(t-\sum_{0\leq k<n} E_k)V(Y_n)} \prod_{0\leq k<n} e^{E_k V(Y_k)}\right)
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 156:
Reversing the integration order, we have

$$\begin{aligned}\partial_t \eta_t(f) &= -\eta_t(f) + \int f(x) \left[\int_{-\infty}^{x} q(x-y)\ p_t(y)\ dy\right] dx \\ &= -\eta_t(f) + \int \left[\int f(y)\ 1_{[x,\infty[}\ q(y-x)\ dy\ \right] p_t(x)\ dx = \eta_t(L(f)),\end{aligned}$$

with the pure jump generator

$$L(f)(x) = \int\ (f(y) - f(x))\ K(x, dy)$$

and the Markov transition

$$K(x, dy) = 1_{[x,\infty[}(y)\ q(y-x)\ dy.$$

Notice that a random sample from $M(x, dy)$ is simply defined by $x + U$ where U stands for a random variable with probability density q on $[0, \infty[$. The embedded Markov model Y_n is defined by

$$Y_n = Y_{n-1} + U_n = \ldots = X_0 + V_n \quad \text{with} \quad V_n := \sum_{1\leq i\leq n} U_i$$

where X_0 is a random variable with probability density p_0 and U_n stands for a sequence of independent copies of U. By (11.12) we have

$$X_t = Y_{N_t} \quad \text{and} \quad P_t(f)(x) = \mathbb{E}\left(f(x + V_{N_t})\right).$$

Also observe that

$$\eta_t(L(f)) = \eta_t(K(f)) - \eta_t(f) = \mathbb{E}(f(X_t + U)) - \mathbb{E}(f(X_t)) = \partial_t\mathbb{E}(f(X_t)). \tag{30.31}$$

We consider (whenever their exist) the Laplace transforms

$$\phi_t(\lambda) := \mathbb{E}(e^{\lambda X_t}) \quad \text{and} \quad \varphi(\lambda) := \mathbb{E}(e^{\lambda U}).$$

Notice that

$$\mathbb{E}(e^{\lambda X_t + U}) = \mathbb{E}(e^{\lambda X_t})\ \mathbb{E}(e^{\lambda U}).$$

Therefore, choosing $f(x) = e^{\lambda x}$ in (30.31) we find that

$$\partial_t\phi_t(\lambda) = \phi_t(\lambda)\ \ (\varphi(\lambda) - 1) \Rightarrow \phi_t(\lambda) = \phi_0(\lambda)\ \exp\left(t\left(\varphi(\lambda) - 1\right)\right).$$

This shows the existence and uniqueness of $\phi_t(\lambda)$ for any λ s.t. $\varphi(\lambda) < \infty$. For instance, for exponential jumps with parameter $\alpha > 0$ we have

$$q(u) = \alpha\ e^{-\alpha u}\ 1_{[0,\infty[}(u) \Rightarrow \varphi(\lambda) := \mathbb{E}(e^{\lambda U}) = \frac{\alpha}{\alpha - \lambda} \int_0^\infty\ (\alpha - \lambda)\alpha\ e^{-(\alpha-\lambda)u}\ = \frac{\alpha}{\alpha - \lambda}$$

for any $\lambda < \alpha$. In this situation

$$\phi_t(\lambda) = \phi_0(\lambda)\ \exp\left(t\left(\frac{\alpha}{\alpha - \lambda} - 1\right)\right) = \phi_0(\lambda)\ \exp\left(\frac{\lambda t}{\alpha - \lambda}\right).$$

For exponential jumps (cf. exercise 41), the random variables V_n are Gamma with parameter (n, α).

$$\mathbb{P}(V_n \in dv) = \frac{v^{n-1}}{(n-1)!} \times \alpha^n\ e^{-\alpha v}\ 1_{[0,\infty[}(v)\ dv.$$

This shows that

$$K^n(f)(x) = \mathbb{E}(f(x+V_n)) = \int\ f(y)\ \frac{(y-x)^{n-1}}{(n-1)!} \times \alpha^n\ e^{-\alpha(y-x)}\ 1_{[x,\infty[}(y)\ dy$$

from which we conclude that

$$P_t(x,dy) = e^{-t} \sum_{n \geq 0} \frac{t^n}{n!}\ \frac{(y-x)^{n-1}}{(n-1)!} \times \alpha^n\ e^{-\alpha(y-x)}\ 1_{[x,\infty[}(y)\ dy.$$

This ends the proof of the exercise. ∎

Solution to exercise 157:
The infinitesimal generator of the process is given by

$$L(f)(1) = \lambda(1)\ (f(2) - f(1)) \quad \text{and} \quad L(f)(2) = \lambda(2)\ (f(1) - f(2)).$$

We have

$$d\eta_t(f) = \eta_t(L(f)).$$

On the other hand, we have

$$f(x) = 1_1(x) \Rightarrow L(1_1)(1) = L(1,1) = -\lambda(1) \quad \text{and} \quad L(1_1)(2) = L(2,1) = \lambda(2).$$

By symmetry arguments, we also have

$$f(x) = 1_2(x) \Rightarrow L(1_2)(1) = L(1,2) = \lambda(1) \quad \text{and} \quad L(1_2)(2) = L(2,2) = -\lambda(2).$$

This implies that

$$\frac{d}{dt}\eta_t(1) = \eta_t(L(1_1)) = \eta_t(1)L(1_1)(1) + \eta_t(2)L(1_1)(2) = -\lambda(1)\eta_t(1) + \lambda(2)\eta_t(2)$$

and

$$\frac{d}{dt}\eta_t(2) = \eta_t(L(1_2)) = \eta_t(1)L(1_2)(1) + \eta_t(2)L(1_2)(2) = \lambda(1)\eta_t(1) - \lambda(2)\eta_t(2).$$

Notice that

$$\begin{aligned}
\frac{d}{dt}\left(\lambda(1)\eta_t(1) - \lambda(2)\eta_t(2)\right) &= \lambda(1)\left[-\lambda(1)\eta_t(1) + \lambda(2)\eta_t(2)\right] - \lambda(2)\left[\lambda(1)\eta_t(1) - \lambda(2)\eta_t(2)\right] \\
&= -\left(\lambda(1) + \lambda(2)\right)\ \left(\lambda(1)\eta_t(1) - \lambda(2)\eta_t(2)\right).
\end{aligned}$$

This implies

$$\lambda(1)\eta_t(1) - \lambda(2)\eta_t(2) = e^{-(\lambda(1)+\lambda(2))t}\ \left(\lambda(1)\eta_0(1) - \lambda(2)\eta_0(2)\right).$$

Recalling that $\eta_t(2) = 1 - \eta_t(1)$ we conclude that

$$\left(\lambda(1) + \lambda(2)\right)\eta_t(1) = \lambda(2) + e^{-(\lambda(1)+\lambda(2))t}\ \left(\lambda(1)\eta_0(1) - \lambda(2)\eta_0(2)\right).$$

Therefore

$$\eta_t(1) = \frac{\lambda(2)}{\lambda(1)+\lambda(2)} + \frac{1}{\lambda(1)+\lambda(2)} e^{-(\lambda(1)+\lambda(2))t} \ (\lambda(1)\eta_0(1) - \lambda(2)\eta_0(2)).$$

By symmetry arguments, we also have

$$\eta_t(2) = \frac{\lambda(1)}{\lambda(1)+\lambda(2)} - \frac{1}{\lambda(1)+\lambda(2)} e^{-(\lambda(1)+\lambda(2))t} \ (\lambda(1)\eta_0(1) - \lambda(2)\eta_0(2)).$$

This ends the proof of the exercise.

■

Solution to exercise 158:

$$L(x,x+1) = \lambda_+(x) \qquad L(x,x-1) = \lambda_-(x) \quad \text{and} \quad L(x,x) = \lambda_+(x) + \lambda_-(x).$$

For any $x \in \mathbb{N} - \{0\}$ we have

$$\begin{array}{rcl} (\pi L)(x) & = & \pi(x-1)\ \lambda_+(x-1) - \pi(x)\left[\lambda_+(x)+\lambda_-(x)\right] + \pi(x+1)\ \lambda_-(x+1) \\ & = & \left[\pi(x+1)\ \lambda_-(x+1) - \pi(x)\lambda_+(x)\right] - \left[\pi(x)\lambda_-(x) - \pi(x-1)\ \lambda_+(x-1)\right]. \end{array}$$

For $x = 0$, we find that

$$(\pi L)(0) = \pi(1)\ \lambda_-(1) - \pi(0)\ \lambda_+(0) = 0.$$

Using induction w.r.t. x we conclude that

$$\pi(x)\lambda_-(x) - \pi(x-1)\ \lambda_+(x-1) = 0 \Rightarrow \ \pi(x) \ = \ \frac{\lambda_+(x-1)}{\lambda_-(x)}\ \pi(x-1) \ = \ \cdots \ = \ \left[\prod_{0\le y<x} \frac{\lambda_+(y)}{\lambda_-(y+1)}\right] \pi(0).$$

Finally, we have

$$\sum_{x\ge 0}\left[\prod_{0\le y<x} \frac{\lambda_+(y)}{\lambda_-(y+1)}\right]\pi(0) = 1 \Rightarrow \pi(0) = \left(\sum_{x\ge 0}\left[\prod_{0\le y<x} \frac{\lambda_+(y)}{\lambda_-(y+1)}\right]\right)^{-1}.$$

This ends the proof of the exercise.

■

Solution to exercise 159:
For any $n \in S := \mathbb{N}$ the infinitesimal generator L of N_t is defined by

$$L(f)(n) = \lambda\ [f(n+1) - f(n)] \quad \Leftrightarrow \quad L(n,m) = \lambda\ [1_{n+1}(m) - 1_n(m)].$$

For any $m \ge 1$

$$\frac{d}{dt}\eta_t(m) = \sum_{n\ge 0} \eta_t(n)\ L(n,m) = \lambda\ [\eta_t(m-1) - \eta_t(m)].$$

Since $N_0 = 1$, we have $\eta_0(m) = 1_0(m)$ and

$$\frac{d}{dt}\eta_t(0) = -\lambda\ \eta_t(0) \Rightarrow \eta_t(0) = e^{-\lambda t}$$

$$\frac{d}{dt}\eta_t(1) = \lambda\ [e^{-\lambda t} - \eta_t(1)] \Rightarrow \eta_t(1) = \int_0^t e^{-\lambda(t-s)}\lambda\ e^{-\lambda s} ds = \lambda t\ e^{-\lambda t}.$$

We use mathematical induction. Assuming that $\eta_t(n) = \frac{(\lambda t)^n}{n!}\ e^{-\lambda t}$, we find that

$$\frac{d}{dt}\eta_t(n+1) = \lambda\ \left[\frac{(\lambda t)^n}{n!}\ e^{-\lambda t} - \eta_t(n+1)\right].$$

Therefore

$$\begin{aligned}\eta_t(n+1) &= \int_0^t e^{-\lambda(t-s)}\left[\lambda\ \frac{(\lambda s)^n}{n!}\ e^{-\lambda s}\right] ds \\ &= e^{-\lambda t}\int_0^t \lambda\ \frac{(\lambda s)^n}{n!}\ ds\ =\ \frac{(\lambda s)^{n+1}}{(n+1)!}\ e^{-\lambda t}.\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 160:

The process jumps up by $+1$ unit at rate λ and jumps down by -1 unit at the same rate. We conclude that the generator of X_t is defined by

$$\begin{aligned}L(f)(x) &= \lambda\,(f(x+1) - f(x)) + \lambda'\ (f(x-1) - f(x)) \\ &= (\lambda+\lambda')\left[\frac{\lambda}{\lambda+\lambda'}\,(f(x+1) - f(x)) + \frac{\lambda'}{\lambda+\lambda'}\,(f(x-1) - f(x))\right] \\ &= 2\lambda\int\ (f(y) - f(x))\ \ K(x,dy),\end{aligned}$$

with the Markov transition

$$K(x,dy) = \frac{\lambda}{\lambda+\lambda'}\ \delta_{x+1}(dy) + \frac{\lambda'}{\lambda+\lambda'}\ \delta_{x-1}(dy).$$

On the other hand, for any bounded function f on $S = \mathbb{Z}$ we have

$$\begin{aligned}\partial_t\eta_t(f) &= \sum_{x\in\mathbb{Z}} f(x)\ \partial_t\eta_t(x) = \eta_t(L(f)) \\ &= \lambda\sum_{x\in\mathbb{Z}} \eta_t(x)\,(f(x+1) - f(x)) + \lambda'\sum_{x\in\mathbb{Z}} \eta_t(x)\,(f(x-1) - f(x)) \\ &= \lambda\sum_{x\in\mathbb{Z}}\ (\eta_t(x-1) - \eta_t(x))\ \ f(x) + \lambda'\sum_{x\in\mathbb{Z}}\ (\eta_t(x+1) - \eta_t(x))\ \ f(x) \\ &= \sum_{x\in\mathbb{Z}}[\lambda\,(\eta_t(x-1) - \eta_t(x)) + \lambda'\,(\eta_t(x+1) - \eta_t(x))]\ \ f(x).\end{aligned}$$

By choosing $f = 1_x$, we conclude that

$$\partial_t\eta_t(x) = \lambda\,(\eta_t(x-1) - \eta_t(x)) + \lambda'\,(\eta_t(x+1) - \eta_t(x))\,.$$

We have

$$\begin{aligned}
\partial_t g_t(z) &= \sum_{x\in\mathbb{Z}} \partial_t \eta_t(x)\ z^x \\
&= \sum_{x\in\mathbb{Z}} z^x\ \left[\lambda\eta_t(x-1) - (\lambda+\lambda')\eta_t(x) + \lambda'\eta_t(x+1)\right] \\
&= \sum_{x\in\mathbb{Z}} \left[z^{x-1}\eta_t(x-1)\ \lambda\ z^{+1} - z^x\eta_t(x)\ (\lambda+\lambda') + z^{x+1}\eta_t(x+1)\ \lambda' z^{-1}\right] \\
&= g_t(z)\ \left[\lambda\ z^{+1} - (\lambda+\lambda') + \lambda' z^{-1}\right].
\end{aligned}$$

This implies that

$$g_t(z) = \exp\left(\left[\lambda\ z - (\lambda+\lambda') + \lambda' z^{-1}\right]\ t\right)\ \underbrace{g_0(z)}_{=z^0=1}.$$

Hence

$$g_t(z) = e^{-(\lambda+\lambda')t}\ e^{\left(\lambda z+\lambda' z^{-1}\right)\ t}.$$

This ends the proof of the exercise.

■

Solution to exercise 161:

By construction, X_t jumps at rate λ with an amplitude Y. Thus, its infinitesimal generator of X_t is given by

$$L(f)(x) = \lambda \int\ (f(x+y) - f(x))\ \mu(dy).$$

We have

$$\mathbb{P}(X_t \leq x) = \mathbb{E}\left(\mathbb{P}(X_t \leq x \mid N_t)\right) = e^{-\lambda t} \sum_{n\geq 0} \frac{(\lambda t)^n}{n!}\ \mathbb{P}(\sum_{1\leq i\leq n} Y_i \leq x).$$

Since $\sum_{1\leq i\leq n} Y_i$ is a centered Gaussian with variance n, we have

$$\sum_{1\leq i\leq n} Y_i \stackrel{\text{in law}}{=} \sqrt{n}\ Y \Rightarrow \mathbb{P}(\sum_{1\leq i\leq n} Y_i \leq x) = \mathbb{P}(Y \leq x/\sqrt{n}).$$

This ends the proof of the exercise.

■

Solution to exercise 162:

- We have n individuals after the $(n-1)$-th split (2 individuals at the first split, $2+1=3$ at the second split, and so on). The splitting time of each individual is an exponential random variable with parameter λ. These variables being independent, the first splitting time T_n after T_{n-1} is the minimum between these n splitting times and it is an exponential random variable with parameter $\lambda\times n$. The splitting time of each offspring being independent of the last splitting times, the random time T_n is independent of $(T_k)_{1\leq k<n}$.

  We conclude that T_n are independent exponential random variables with parameters $\lambda\times n$, with $n\geq 1$.

- $\overline{T}_n = \sum_{1\leq k\leq n} T_k$ is the sum of independent exponential random variables T_k with parameter λk. Suppose n students are starting an exam at time $t=0$. Each of them completes the exam at an exponential rate with parameter λ. Arguing as above, the time the first student leaves is an exponential random variable S_n with parameter λn; the time the second student leaves is an (independent) exponential random variable S_{n-1} with parameter $\lambda(n-1)$, and so on. This shows that $\overline{T}_n$ can be interpreted as the time at which all students have left, so that

$$\mathbb{P}\left(\overline{T}_n \leq t\right) = \mathbb{P}\left(S_1 \leq t, \ldots, S_n \leq t\right) = \prod_{1\leq k\leq n} \mathbb{P}(S_k \leq t) = (1-e^{-\lambda t})^n.$$

- Notice that

$$\{X_t > n\} = \left\{\overline{T}_n \leq t\right\} \Rightarrow \mathbb{P}\left(X_t > n\right) = \left(1-e^{-\lambda t}\right)^n.$$

This implies that

$$\mathbb{P}(X_t = n) = \mathbb{P}(X_t \geq n) - \mathbb{P}(X_t > n) = \left(1-e^{-\lambda t}\right)^{n-1} - \left(1-e^{-\lambda t}\right)^n = e^{-\lambda t} \times \left(1-e^{-\lambda t}\right)^{n-1}$$

for any $n \geq 1$. In addition, we have

$$\mathbb{E}(X_t) = \sum_{n\geq 0} \mathbb{P}(X_t > n) = \sum_{n\geq 0} \left(1-e^{-\lambda t}\right)^n = e^{\lambda t}.$$

This ends the proof of the exercise.

■

Solution to exercise 163:

- For any $s \leq t$, we have

$$M_t^{(1)} - M_s^{(1)} = (N_t - N_s) - \lambda(t-s).$$

Since $(N_t - N_s)$ is independent of $(N_r)_{0\leq r\leq s}$ and $(N_t - N_s)$ is a Poisson random variable with parameter $\lambda(t-s)$ we conclude that

$$\mathbb{E}(N_t - N_s) = \lambda(t-s)$$

and

$$\mathbb{E}\left(M_t^{(1)} - M_s^{(1)} \mid \mathcal{F}_s\right) = \mathbb{E}\left(((N_t - N_s) - \lambda(t-s)) \mid \mathcal{F}_s\right) = \mathbb{E}\left(((N_t - N_s) - \lambda(t-s))\right) = 0.$$

- For any $s \leq t$, $(N_t - N_s)$ is a Poisson random variable with parameter $\lambda(t-s)$. Thus, we have

$$\mathrm{Var}\left(N_t - N_s\right) = \mathbb{E}\left([(N_t - N_s) - \lambda(t-s)]^2\right) = \lambda(t-s).$$

$$\begin{aligned}
M_t^{(2)} - M_s^{(2)} &= \left(M_s^{(1)} + (M_t^{(1)} - M_s^{(1)})\right)^2 - \left(M_s^{(1)}\right)^2 - \lambda(t-s) \\
&= \left(M_t^{(1)} - M_s^{(1)}\right)^2 + 2M_s^{(1)}(M_t^{(1)} - M_s^{(1)}) - \lambda(t-s) \\
&= ((N_t - N_s) - \lambda(t-s))^2 + 2M_s^{(1)}(M_t^{(1)} - M_s^{(1)}) - \lambda(t-s).
\end{aligned}$$

Arguing as above, we find that

$$\mathbb{E}\left(M_s^{(1)}(M_t^{(1)} - M_s^{(1)}) \mid \mathcal{F}_s\right) = M_s^{(1)}\ \mathbb{E}\left((M_t^{(1)} - M_s^{(1)}) \mid \mathcal{F}_s\right) = 0$$

and

$$\mathbb{E}\left(M_t^{(2)} - M_s^{(2)} \mid \mathcal{F}_s\right) = 0 \Leftrightarrow \mathbb{E}\left(M_t^{(2)} \mid \mathcal{F}_s\right) = M_s^{(2)}.$$

- For any $s \leq t$, $(N_t - N_s)$ is a Poisson random variable with parameter $\lambda(t-s)$ and $(N_t - N_s)$ is independent of $(N_r)_{0\leq r\leq s}$. Thus, have

$$\begin{aligned}\mathbb{E}\left(e^{a(N_t-N_s)-\lambda(t-s)(e^a-1)} \mid \mathcal{F}_s\right) &= \mathbb{E}\left(e^{a(N_t-N_s)-\lambda(t-s)(e^a-1)}\right) \\ &= e^{-\lambda(t-s)}\sum_{n\geq 0}\frac{(\lambda(t-s))^n}{n!}\ e^{an-\lambda(t-s)(e^a-1)} \\ &= e^{-\lambda(t-s)e^a}\sum_{n\geq 0}\frac{(e^a\lambda(t-s))^n}{n!} = 1.\end{aligned}$$

  This implies that
$$\mathbb{E}\left(e^{aN_t-\lambda t(e^a-1)} \mid \mathcal{F}_s\right) = e^{aN_s-\lambda s(e^a-1)}.$$

  This shows that $M_t^{(3)}$ is a martingale.

- Notice that

$$M_t^{(4)} = (1+b)^{N_t}\ e^{-\lambda bt} = \exp\left(\log(1+b)N_t - \lambda bt\right) \overset{\text{when } a=\log(1+b)}{=} M_t^{(3)}.$$

  This ends the proof of the exercise.

■

Solution to exercise 164:

- At an arrival time, X_t jumps to X_t+1, whereas at the end of service times it jumps to X_t-1. If $X_t=0$, at rate λ_1 it jumps up by one unit. If $X_t=x>0$, at rate $\lambda_1+\lambda_2$ (we recall that the minimum of a couple of independent exponential random variables with parameters λ_1 and λ_2 is an exponential random variable with parameter $\lambda_1+\lambda_2$). At that time, with a probability $\lambda_1/(\lambda_1+\lambda_2)$ it jumps up by one unit; otherwise it jumps down by one unit (we recall that the probability that E_1 coincides with the minimum of exponential random variables E_1 and E_2 with parameters λ_1 and λ_2 is equal to $\lambda_1/(\lambda_1+\lambda_2)$).

- We have $X_t = Y_{N_t}$ where N_t is a Poisson process with rate $\lambda := \lambda_1+\lambda_2$, and the embedded Markov chain Y_n on $\mathbb{N}$ is given by

$$\mathbb{P}(Y_n = Y_{n-1}+1 \mid Y_{n-1}=y) = \lambda_1/(\lambda_1+\lambda_2) = 1-\mathbb{P}(Y_n = Y_{n-1}-1 \mid Y_{n-1}=y)$$

  for any $y>0$, with $\mathbb{P}(Y_n=1 \mid Y_{n-1}=0)$=1 for $y=0$.

This ends the proof of the exercise.

■

Solution to exercise 165: Firstly, suppose that $a=2$. When $X_t=0$ at rate λ_1 it jumps up by one unit. When $X_t=1$ at rate $\lambda_1+\lambda_2$, with a probability $\lambda_1/(\lambda_1+\lambda_2)$ it jumps up by one unit, otherwise it jumps down by one unit.

When $X_t = x \geq a = 2$, the two customers are served independently at a rate λ_2. Thus, one of them is served at a rate $2\lambda_2$ (here again, we recall that the minimum of a couple of independent exponential random variables with parameters λ_2 is an exponential random variable with parameter $2\lambda_2$). On the other hand (and independently of the random service times), a new arrival occurs at a rate λ_1. Therefore, X_t with jump at a rate $\lambda_1+2\lambda_2$. At the jump time, with a probability $\lambda_1/(\lambda_1+2\lambda_2)$ it jumps up by one unit, otherwise it jumps down by one unit.

We have $X_t = Y_{N_t}$ where N_t is a Poisson process with rate $\lambda := \lambda_1 + 2\lambda_2$, and the embedded Markov chain Y_n on $\mathbb{N}$ given by

$$\mathbb{P}(Y_n = Y_{n-1}+1 \mid Y_{n-1} = y) = \frac{\lambda_1}{\lambda_1 + a\lambda_2} \quad \text{and} \quad \mathbb{P}(Y_n = Y_{n-1}-1 \mid Y_{n-1} = y) = \frac{a\lambda_2}{\lambda_1 + a\lambda_2}$$

for any $y \geq a = 2$. In much the same way, we have

$$\mathbb{P}(Y_n = Y_{n-1}+1 \mid Y_{n-1} = y) = \frac{\lambda_1}{\lambda_1 + y\lambda_2} \quad \text{and} \quad \mathbb{P}(Y_n = Y_{n-1}-1 \mid Y_{n-1} = y) = \frac{y\lambda_2}{\lambda_1 + y\lambda_2}$$

for any $0 \leq y \leq 1 = a-1$, with the convention $\mathbb{P}(Y_n = 1 \mid Y_{n-1} = 0){=}1$ for $y = 0$. The same formulae are valid for any $a \geq 2$. This ends the proof of the exercise.

■

Solution to exercise 166:
By construction, we have

$$\begin{aligned}
&\mathbb{P}\left(A_t = k \ |T'_n \ , n \geq 0\right) \\
&= \mathbb{P}\left(U_1 > \lambda_{T'_1}, \ldots, U_{k-1} > \lambda_{T'_{k-1}}, \ U_k \leq \lambda_{T'_k} \ |T'_n \ , n \geq 0\right) \\
&= \left\{\prod_{1\leq l<k}\left(1 - \frac{\lambda_{T'_l}}{\lambda}\right)\right\} \times \frac{\lambda_{T'_k}}{\lambda}.
\end{aligned}$$

Integrating out the uniform random times T'_l, $l \leq k$, we find that

$$\mathbb{P}\left(A_t = k\right) = \left(1 - \frac{\int_0^t \lambda_s ds}{\lambda t}\right)^{k-1} \frac{\int_0^t \lambda_s ds}{\lambda t}.$$

This shows that A_t is a geometric r.v. with success probability given by the area ratio $\int_0^t \lambda_s ds/(\lambda t)$. In particular, we have

$$\mathbb{E}(A_t) = \lambda \left[\frac{1}{t}\int_0^t \lambda_s ds\right]^{-1}.$$

Now

$$\begin{aligned}
\lambda_t = e^{-t} \leq \lambda_0 = \lambda := 1 \quad &\Rightarrow \quad \frac{1}{t}\int_0^t \lambda_s ds = \frac{1-e^{-t}}{t} \downarrow_{t\uparrow\infty} 0 \\
&\Rightarrow \quad \mathbb{E}(A_t) = \frac{t}{1-e^{-t}} \uparrow_{t\uparrow\infty} \ +\infty.
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 167:
The jump times T_n are defined by

$$T_n = \inf\left\{t \geq T_{n-1} \ : \ \int_{T_{n-1}}^t \lambda_s \ ds \geq -\log U_n\right\}.$$

Notice that

$$\int_{T_{n-1}}^{T_n} \lambda_s \; ds = -\int_{T_{n-1}}^{T_n} \partial_s \log(1-s) \; ds = \log \frac{1-T_{n-1}}{1-T_n} = -\log U_n$$

as soon as

$$\begin{aligned}\frac{1-T_n}{1-T_{n-1}} = U_n \quad &\Longleftrightarrow \quad (1-T_n) = (1-T_{n-1})U_n \\ &\Longleftrightarrow \quad T_n = 1-(1-T_{n-1})U_n = (1-U_n) + T_{n-1}U_n.\end{aligned}$$

This yields

$$\begin{aligned}(1-T_n) &= U_n \; (1-T_{n-1}) \\ &= U_n U_{n-1} \; (1-T_{n-2}) = \cdots = (U_n \ldots U_1) \; (1-T_0) = \prod_{1\leq k\leq n} U_k.\end{aligned}$$

On the other hand

$$\mathbb{E}(U^p) = \frac{1}{p+1} \Rightarrow \mathbb{E}\left((1-T_n)^p\right) = (\mathbb{E}(U^p))^n = (p+1)^{-n}.$$

By Borel-Cantelli lemma, this clearly implies that $T_n \to_{n\to\infty} T_\infty = 1$.

This ends the proof of the exercise.

■

Solution to exercise 168

If we consider the random times defined in (11.19), for some $\lambda_n \geq 1$ then we have that

$$\mathbb{P}\left(T_n \leq \sum_{1\leq p\leq n} \lambda_p^{-1} + t + 2\sqrt{\sum_{1\leq p\leq n} \lambda_p^{-2}} \; \sqrt{t}\right) \geq 1-e^{-t}.$$

In the explosive case, using the fact that $\lambda_n^2 \geq \lambda_n \geq 1$, we find that

$$|\lambda|_2^2 := \sum_{p\geq 1} \lambda_p^{-2} \leq |\lambda|_1 := \sum_{p\geq 1} \lambda_p^{-1} < \infty.$$

In this case we have

$$\mathbb{P}\left(T_n \leq |\lambda|_1 + t + 2|\lambda|_2 \; \sqrt{t}\right) \geq 1-e^{-t}$$

and by the monotone convergence theorem

$$\mathbb{P}\left(T_\infty \leq t + |\lambda|_1(1+2\sqrt{t})\right) \geq \mathbb{P}\left(T_\infty \leq |\lambda|_1 + t + 2|\lambda|_2 \; \sqrt{t}\right) \geq 1-e^{-t}.$$

If we choose $t=3$ then we find that

$$\mathbb{P}\left(T_\infty \leq 3 + |\lambda|_1(1+2\sqrt{3})\right) \geq 1-e^{-3} \geq .95$$

For $\lambda_n = n^2$, we have $|\lambda|_1 = \pi^2/6$

$$\mathbb{P}\left(T_\infty \leq 3 + \pi^2\left(\frac{1}{6}+\frac{1}{\sqrt{3}}\right) \leq 10.35\right) \geq 1-e^{-3} \geq 0.95.$$

Using Markov inequality, we have

$$\mathbb{P}\left(T_\infty \geq t\right) \leq t^{-1}\ \mathbb{E}\left(T_\infty\right) = t^{-1}\ \sum_{n\geq 1} \lambda_n^{-1} = t^{-1}\ \pi^2/6 \leq .005 = 5\times 10^{-2}$$

when $t \geq 10\ \pi^2/3 \simeq 32.9$. This inequality implies that

$$\mathbb{P}\left(T_\infty \leq 32.9\right) \geq 0.95.$$

Theorem 11.3.7 also provides an estimate of the T_n even in the case where $\sum_{p\geq 1}\lambda_p^{-1} = \infty$. For instance, for time homogeneous models $\lambda_n = 1$ we find that

$$\mathbb{P}\left(T_n \leq n + t + 2\sqrt{nt}\right) = \mathbb{P}\left(\sqrt{T_n} \leq \sqrt{n} + \sqrt{t}\right) \geq 1 - e^{-t}.$$

If we choose $t = 3$ then we find that

$$T_n \leq n + 3 + 2\sqrt{3n}$$

with a probability larger than 95%. For instance the seventh jump time occurs before 20 units of time, with a probability 95%; and the 10^3-th time occurs before 1.1113×10^3 units of time, with a probability larger than 95%.

This ends the proof of the exercise.

■

Chapter 12

Solution to exercise 169:

Let T_n be the jump times of the Poisson process N_t. At jump times the process X_{T_n-} jumps to

$$X_{T_n} = X_{T_n-} - 2X_{T_n-} = -X_{T_n-}.$$

In other words, the process changes its sign at a rate λ. Recalling that $X_0 = 1$ we conclude that $X_t = (-1)^{N_t}$. This also shows that the generator of X_t is given by

$$L(f)(x) = \lambda \left(f(-x) - f(x)\right).$$

This ends the proof of the exercise.

■

Solution to exercise 170:

We have

$$L(1_0)(0) = -\lambda(0) \quad \text{and} \quad L(1_0)(1) = \lambda(1).$$

By (12.5) this implies that

$$\begin{aligned}
\partial_t \eta_t(0) &= \eta_t\left(L(1_0)\right) = \eta_t(0)\ L(1_0)(0) + \eta_t(1)\ L(1_0)(1) \\
&= -\eta_t(0)\ \lambda(0) + \eta_t(1)\ \lambda(1) \\
&= -\eta_t(0)\ \lambda(0) + (1-\eta_t(0))\ \ \lambda(1) = \lambda(1) - \eta_t(0)\ \ (\lambda(0)+\lambda(1)).
\end{aligned}$$

In much the same way, we have

$$\eta_t(1) = 1 - \eta_t(0) \Rightarrow \partial_t \eta_t(1) = -\partial_t \eta_t(0) = \eta_t(0)\ \lambda(0) - \eta_t(1)\ \lambda(1).$$

The solution is given by the formula

$$\begin{aligned}
\eta_t(0) &= e^{-(\lambda(0)+\lambda(1))t}\ \left[\eta_0(0) + \int_0^t e^{(\lambda(0)+\lambda(1))s}\ \lambda(1)\ ds\right] \\
&= \frac{\lambda(1)}{\lambda(1)+\lambda(0)} + e^{-(\lambda(0)+\lambda(1))t}\left(\eta_0(0) - \frac{\lambda(1)}{\lambda(0)+\lambda(1)}\right).
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 171: We consider the compound Poisson process discussed in exercise 161. We recall that the infinitesimal generator of X_t is given by

$$L(f)(x) = \lambda \int \left(f(x+y) - f(x)\right)\ \mu(dy)$$

with the carrè du champ

$$\Gamma_L(f,f)(x) = \lambda \int \left[f(x+y) - f(x)\right]^2\ \mu(dy).$$

Notice that

$$f(x) = x \Rightarrow L(f)(x) = \lambda\ \mathbb{E}(Y) \quad \text{and} \quad \Gamma_{L_t}(f,f)(x) = \lambda\ \mathbb{E}(Y^2).$$

Therefore, applying the Doeblin-Itō formula (12.25) to $f(x) = x$ we have

$$dX_t = \lambda\ \mathbb{E}(Y)\ dt + dM_t \Rightarrow X_t = \lambda\ \mathbb{E}(Y)\ t + M_t$$

for the martingale M_t with the angle bracket

$$\langle M\rangle_t = \lambda\ \mathbb{E}(Y^2)\ t.$$

This implies

$$\begin{array}{rcl} \mathbb{E}\left(X_t\right) & = & -\lambda\ \mathbb{E}(Y)\ t \\ \mathrm{Var}\left(X_t\right) & = & \mathbb{E}\left[\left(X_t - \lambda\mathbb{E}(Y)t\right)^2\right] = \mathbb{E}(M_t^2) = \lambda\ \mathbb{E}(Y^2)\ t. \end{array}$$

We assume that $\mathbb{E}(Y) = 0 = \mathbb{E}(Y^3)$. In this situation we have

$$\mathbb{E}\left(X_t\right) = 0 \quad \text{and} \quad \mathbb{E}(X_t^2) = \mathrm{Var}\left(X_t\right) = \lambda\ \mathbb{E}(Y^2)\ t.$$

Also, observe that

$$f(x) = x^2 \Rightarrow L(f)(x) = \lambda\ \mathbb{E}(Y^2) \quad \text{and} \quad \Gamma_{L_t}(f,f)(x) = \lambda\ \left(\mathbb{E}\left(Y^4\right) + 4\ x^2\ \mathbb{E}\left(Y^2\right)\right).$$

By applying the Doeblin-Itō formula (12.25) to $f(x) = x^2$ we have

$$dX_t^2 = \lambda\ \mathbb{E}(Y^2)\ dt + dM_t \Rightarrow X_t^2 = \lambda t\ \mathbb{E}(Y^2) + M_t$$

with a martingale M_t with angle bracket

$$\langle M\rangle_t = \lambda\ \left(\mathbb{E}(Y^4)\ t + 4\ \mathbb{E}\left(Y^2\right)\ \int_0^t X_s^2\ ds\right)$$

$$\Longrightarrow \mathbb{E}\left(\langle M\rangle_t\right) = \lambda\ \left(\mathbb{E}(Y^4)\ t + 2\ \lambda\ t^2\ \mathbb{E}(Y^2)^2\right).$$

This implies that

$$\begin{array}{rcl} \mathbb{E}\left(X_t^2\right) & = & -\lambda\ \mathbb{E}(Y^2)\ t \\ \mathrm{Var}\left(X_t^2\right) & = & \mathbb{E}\left[\left(X_t^2 - \lambda t\ \mathbb{E}(Y^2)\right)^2\right] = \mathbb{E}(M_t^2) = \lambda\ \left(\mathbb{E}(Y^4)\ t + 2\ \lambda\ t^2\ \mathbb{E}(Y^2)^2\right). \end{array}$$

This ends the proof of the exercise.

■

Solution to exercise 172:

Suppose we start from some $X_0 = e_{i_0}$, with $1 \leq i_0 \leq r$. Before the first jump time we have

$$dX_t = \sum_{1\leq i\neq j\leq r} (e_i - e_j)\ \overbrace{\langle e_j, e_{i_0}\rangle}^{=1_{j=i_0}}\ dN_t^{(i,j)} = \sum_{1\leq i\leq r,\ i\neq i_0} (e_i - e_{i_0})\ dN_t^{(i,i_0)}.$$

Let $T_1 := T^{i_1,i_0}$ be the first jump time of some Poisson process $N^{(i_1,i_0)}$, for some index $1 \leq i_1 \leq r$, $i_1 \not{i}_0$. At that time, the process jumps from $X_{T_1-} = e_{i_0}$ to

$$X_{T_1} = X_{T_1-} + (e_{i_1} - e_{i_0}) = e_{i_0} + (e_{i_1} - e_{i_0}) = e_{i_1}.$$

Before the next jump time, we have

$$dX_t = \sum_{1\leq i\neq j\leq r} (e_i - e_j)\ \overbrace{\langle e_j, e_{i_1}\rangle}^{=1_{j=i_1}}\ dN_t^{(i,j)} = \sum_{1\leq i\leq r,\ i\neq i_0} (e_i - e_{i_1})\ dN_t^{(i,i_1)}.$$

Let $T_2 := T^{i_2,i_1}$ be the first jump time of some Poisson process $N^{(i_2,i_1)}$, for some index $1 \leq i_2 \leq r,\ i_2 \not{i}_1$. At that time, the process jumps from $X_{T_2-} = e_{i_1}$ to

$$X_{T_2} = e_{i_1} + (e_{i_2} - e_{i_1}) = e_{i_2}$$

and so on. This shows that X_t is an S-valued Markov process with generator

$$L(f)(e_j) = \sum_{1\leq i\leq r\ :\ i\neq j} \lambda(j,i)\ \ (f(e_i) - f(e_j))\,.$$

For any function $f\ :\ S \mapsto \mathbb{R}$ we have

$$\begin{aligned}
\partial_t\eta_t(f) &= \sum_{1\leq j\leq r} f(e_j)\ \partial_t\eta_t(j) = \eta_t L(f) = \sum_{1\leq i\neq j\leq r} \eta_t(j)\lambda(j,i)\ \ (f(e_i) - f(e_j)) \\
&= \sum_{1\leq i\leq r} f(e_i) \left[\sum_{1\leq j\leq r,\ j\neq i} \eta_t(j)\lambda(j,i)\right] - \sum_{1\leq j\leq r} f(e_j)\ \eta_t(j) \sum_{1\leq i\leq r,\ i\neq j} \lambda(j,i) \\
&= \sum_{1\leq j\leq r} f(e_j) \left[\sum_{1\leq i\leq r,\ i\neq j} \eta_t(i)\lambda(i,j) - \eta_t(j) \sum_{1\leq i\leq r,\ i\neq j} \lambda(j,i)\right].
\end{aligned}$$

This yields

$$\partial_t\eta_t(j) = \sum_{1\leq i\leq r,\ i\neq j} \eta_t(i)\lambda(i,j) - \eta_t(j) \sum_{1\leq i\leq r,\ i\neq j} \lambda(j,i)$$

In vector form:

$$\begin{aligned}
\partial_t\eta_t &= [\partial_t\eta_t(1), \ldots, \partial_t\eta_t(r)] \\
&= \eta_t \begin{bmatrix}
-\sum_{1\leq i\leq r,\ i\neq 1} \lambda(1,i) & \lambda(1,2) & \lambda(1,3) & \cdots & \lambda(1,r) \\
\lambda(2,1) & -\sum_{1\leq i\leq r,\ i\neq 2} \lambda(2,i) & \lambda(2,3) & \cdots & \lambda(2,r) \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\lambda(r,1) & \lambda(r,2) & \lambda(r,3) & \cdots & -\sum_{1\leq i\leq r,\ i\neq r} \lambda(r,i)
\end{bmatrix}.
\end{aligned}$$

Observe that

$$-\sum_{1\leq i\leq r,\ i\neq j} \lambda(j,i) = \lambda(j,j) - \sum_{1\leq i\leq r} \lambda(j,i)$$

for any choice of the diagonal terms $\lambda(i,i)$.

This ends the proof of the exercise.

■

Solution to exercise 173:

For any function $F(x,y)=f(x)$ that depends on the first coordinate, we have

$$\begin{aligned}\mathcal{L}(F)(x,y) &= 1_{x\neq y}\left[\overbrace{L(F(x,.))(y)}^{=L(f(x))(x)=0}+\overbrace{L(F(.,y))(x)}^{=L(f)(x)}\right]+1_{x=y}\int [f(y)-f(x)]\;Q(x,dy)\\ &= 1_{x\neq y}L(f)(x)+1_{x=y}\;L(f)(x)=L(f)(x).\end{aligned}$$

By symmetry arguments we also have $\mathcal{L}(F)(x,y)=L(g)(y)$ for any function $F(x,y)=g(y)$ that depends on the second coordinate. This implies that $\mathcal{X}_t$ and $\mathcal{Y}_t$ have the same law as X_t. This ends the proof of the exercise.

■

Solution to exercise 174: For any function $F(x,y)=f(x)$ that depends on the first coordinate, we have

$$\begin{aligned}\mathcal{L}(F)(x,y) &= \int [f(z)-f(x)]\;(q(x,z)\wedge q(y,z))\;\lambda(dz)\\ &\qquad\qquad +\int [f(z)-f(x)]\;(q(x,z)-q(y,z))_+\;\lambda(dz)\\ &= \int [f(z)-f(x)]\;q(x,z)\;\lambda(dz)=L(f)(z).\end{aligned}$$

The last assertion follows from the fact that

$$\begin{aligned}(q(x,z)\wedge q(y,z))+(q(x,z)-q(y,z))_+ &= 1_{q(x,z)\geq q(y,z)}\;[q(y,z)+(q(x,z)-q(y,z))]\\ &\qquad +1_{q(x,z)<q(y,z)}\;[q(x,z)+0]=q(x,z).\end{aligned}$$

By symmetry arguments we also have $\mathcal{L}(F)(x,y)=L(g)(y)$ for any function $F(x,y)=g(y)$ that depends on the second coordinate. This implies that $\mathcal{X}_t$ and $\mathcal{Y}_t$ have the same law as X_t.

This ends the proof of the exercise.

■

Solution to exercise 175:

The exercise is a direct consequence of theorem 12.7.6. To be more precise, we let P_t be the Markov semigroup of X_t. Combining theorem 12.7.6 with theorem 8.3.2 we find that

$$\|\delta_x P_t-\delta_{x'}P_t\|_{tv}\leq c\;\exp\left(-\rho\epsilon t\right)$$

for some non negative parameters c and $\rho>0$. This implies that the Dobrushin contraction coefficient $\beta(P_t)$ of P_t is less than 1 for t sufficiently large. Thus we conclude by using theorem 8.2.13.

This ends the proof of the exercise.

■

Solution to exercise 176:

We have

$$\lambda K=\lambda\Longrightarrow\lambda\left(e^{-V}\;L(f)\right)\propto\lambda\left((K-Id)f\right))=\lambda(K(f))-\lambda(f)=0.$$

When K is λ-reversible we have

$$\lambda\left(g\ K(f)\right)=\lambda\left(f\ K(g)\right)$$

from which we check that

$$\lambda\left(e^{-V}\ g\ L(f)\right)\propto\lambda\left(g\ (K-Id)(f)\right)=\lambda\left(f\ (K-Id)(g)\right)\propto\lambda\left(e^{-V}\ g\ L(f)\right).$$

This ends the proof of the exercise.

■

Solution to exercise 177:

We have

$$\begin{aligned}\partial_t\gamma_t^{[\beta]}(1) &= \mathbb{E}\left[\partial_t \exp\left(-\beta\ \int_0^t V(X_s)\ ds\right)\right]\\ &= -\beta\ \mathbb{E}\left[V(X_t)\exp\left(-\beta\ \int_0^t V(X_s)\ ds\right)\right]=-\beta\ \gamma_t^{[\beta]}(V)=-\beta\ \eta_t^{[\beta]}(V)\ \gamma_t^{[\beta]}(1).\end{aligned}$$

This implies that

$$\partial_t\log\gamma_t^{[\beta]}(1)=\frac{1}{\gamma_t^{[\beta]}(1)}\ \partial_t\gamma_t^{[\beta]}(1)=-\beta\ \eta_t^{[\beta]}(V)$$

from which we prove that

$$\gamma_t^{[\beta]}(1)=\exp\left(-\beta\ \int_0^t\ \eta_s^{[\beta]}(V)\ ds\right).$$

The end of the proof of the second assertion is now clear.

Under our assumptions we have

$$-t\eta_\infty^{[\beta]}(V)-c_\beta\leq-\int_0^t\eta_s^{[\beta]}(V)\ ds\leq-t\eta_\infty^{[\beta]}(V)+c_\beta.$$

Taking the exponential this implies that

$$e^{-t\eta_\infty^{[\beta]}(V)}\ C_\beta^{-1}\leq e^{-\int_0^t\eta_s^{[\beta]}(V)\ ds}=\left[\gamma_t^{[\beta]}(1)\right]^{1/\beta}\leq e^{-t\eta_\infty^{[\beta]}(V)}C_\beta.$$

This ends the proof of the exercise.

■

Solution to exercise 178:

We clearly have

$$\begin{aligned}L_t(f)(x) &= \int\ (f(x+u)-f(x))\ g_t(x,u)\ du\\ &= \int\ (f(y)-f(x))\ g_t(x,y-x)\ dy=\int\ (f(y)-f(x))\ q_t(x,y)\ dy,\end{aligned}$$

with $q_t(x,y) = g_t(x, y-x)$. By construction, we have the integral evolution equation

$$\begin{aligned}
\partial_t \eta_t(f) &= \int f(x)\ \partial_t p_t(x)\ dx = \eta_t\left(L_t(f)\right) \\
&= \int \left(f(y) - f(x)\right)\ p_t(x)\ q_t(x,y)\ dxdy \\
&= \int f(x)\ p_t(y)\ q_t(y,x)\ dxdy - \int f(x)\ \left[\int q_t(x,y)\ dy\right] p_t(x)\ dx \\
&= \int f(x)\ \left(\int p_t(y)\ q_t(y,x)dy - \left[\int p_t(x)\ q_t(x,y)\ dy\right].\right)\ dx
\end{aligned}$$

This implies that

$$\partial_t p_t(x) = \int \left[p_t(y)\ q_t(y,x) - p_t(x)\ q_t(x,y)\right]\ dy.$$

We also observe that

$$\begin{aligned}
\partial_t p_t(x) &= \int \left[p_t(y)\ g_t(y, x-y) - p_t(x)\ g_t(x, y-x)\right]\ dy \\
&= \int \left[p_t(x-(x-y))\ g_t(x-(x-y), x-y) - p_t(x)\ g_t(x, -(x-y))\right]\ dy \\
&= \int_{-\infty}^{+\infty} p_t(x-(x-y))\ g_t(x-(x-y), x-y)\ dy - p_t(x)\ \int_{-\infty}^{+\infty} g_t(x, -(x-y))\ dy \\
&= -\int_{+\infty}^{-\infty} p_t(x-z)\ g_t(x-z, z)\ dz + p_t(x)\ \int_{+\infty}^{-\infty} g_t(x, -z)\ dz \\
&= \int_{-\infty}^{+\infty} p_t(x-z)\ g_t(x-z, z)\ dz - p_t(x)\ \int_{-\infty}^{+\infty} g_t(x, -z)\ dz.
\end{aligned}$$

This implies that

$$\begin{aligned}
\partial_t p_t(x) &= \int p_t(x-z)\ g_t(x-z, z)\ dz - p_t(x)\ \int g_t(x, -z)\ dz \\
&= \int \left[p_t(x-z)\ g_t(x-z, z) - p_t(x)\ g_t(x, z)\right]\ dz.
\end{aligned}$$

The last assertion is a consequence of the fact that

$$\int_{-\infty}^{+\infty} g_t(x, -z)\ dz = -\int_{+\infty}^{-\infty} g_t(x, z)\ dz = \int_{-\infty}^{+\infty} g_t(x, z)\ dz.$$

Using the Taylor's expansion,

$$p_t(x-z)\ g_t(x-z, z) = p_t(x)\ g_t(x, z) + \sum_{n\geq 1} \frac{(-1)^n}{n!}\ z^n\ \partial_x^n\left(p_t(x) g_t(x, z)\right)$$

we find that the formula

$$\begin{aligned}
\partial_t p_t(x) &= \sum_{n\geq 1} \frac{(-1)^n}{n!} \int z^n\ \partial_x^n\left(p_t(x) g_t(x, z)\right)\ dz \\
&= \sum_{n\geq 1} \frac{(-1)^n}{n!}\ \partial_x^n\left(\left[\int z^n\ g_t(x, z)\ dz\right]\ p_t(x)\right) = \sum_{n\geq 1} \frac{(-1)^n}{n!}\ \partial_x^n\left(\alpha_t^n(x)\ p_t(x)\right).
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 179:
It clearly suffices to check the result for $\lambda = 1$. In this case, we have

$$\begin{aligned}
\eta L_h(f) &= \int p(x) \left[(f(x+h) - f(x)) \frac{p(x+h)}{p(x)+p(x+h)} \right. \\
&\qquad \left. + (f(x-h) - f(x)) \frac{p(x-h)}{p(x)+p(x-h)} \right] dx \\
&= \int p(x)\ f(x+h) \frac{p(x+h)}{p(x)+p(x+h)}\ dx - \int p(x)\ f(x) \frac{p(x+h)}{p(x)+p(x+h)}\ dx \\
&\quad + \int p(x)\ f(x-h) \frac{p(x-h)}{p(x)+p(x-h)}\ dx - \int p(x)\ f(x) \frac{p(x-h)}{p(x)+p(x-h)}\ dx.
\end{aligned}$$

The change of variables $y = x + h$ yields

$$\begin{aligned}
\int p(x)\ f(x+h) \frac{p(x+h)}{p(x)+p(x+h)}\ dx &= \int p(y-h)\ f(y) \frac{p(y)}{p(y-h)+p(y)}\ dy \\
&= \int p(x)\ f(x) \frac{p(x-h)}{p(x)+p(x-h)}\ dx.
\end{aligned}$$

By symmetry $(h \rightsquigarrow -h)$ we also have

$$\int p(x)\ f(x-h) \frac{p(x-h)}{p(x)+p(x-h)}\ dx = \int p(x)\ f(x) \frac{p(x+h)}{p(x)+p(x+h)}\ dx.$$

We conclude that $\eta L_h(f)$. This ends the proof of the exercise.

■

Solution to exercise 180: The proof of first part of the exercise follows the same arguments as the proof of exercise 178, so it is skipped. The last assertion is immediate. This ends the proof of the exercise.

■

Solution to exercise 181:
In exercise 171 we proved that the process X_t and $M_t = X_t^2 - \langle X \rangle_t = X_t^2 - \lambda\ t\ \mathbb{E}(Y^2)$ are both martingales starting at $0 \in [-a, a]$. Therefore we can apply directly (12.23) to the martingale X_t and its angle bracket $\langle X \rangle_t = \lambda\ t\ \mathbb{E}(Y^2)$. Otherwise we reformulate the proof of (12.23). By the optional stopping theorem the stopped process $M_{t\wedge T_D}$ is a martingale so that

$$\mathbb{E}(M_{t\wedge T_D}) = 0 \Rightarrow \lambda\ \mathbb{E}(t \wedge T_D)\ \mathbb{E}(Y^2) = \mathbb{E}(X^2_{t\wedge T_D}) \leq a^2 \Rightarrow \mathbb{E}(t\wedge T_D)\ \leq a^2/(\lambda\ \mathbb{E}(Y^2)).$$

Applying Fatou's lemma, we find that

$$\mathbb{E}(T_D) = \mathbb{E}\left(\lim_{t\uparrow\infty}(t\wedge T_D)\right) \leq \liminf_{t\uparrow\infty} \mathbb{E}\left(t\wedge T_D\right) \leq a^2/(\lambda\ \mathbb{E}(Y^2)).$$

The compound process starting at some $x \in D$ is given by $X_t^x = x + X_t$. In this situation, we have

$$(X_t^x)^2 = x^2 + X_t^2 + 2xX_t$$

$$\overset{\mathbb{E}(X_{t\wedge T_D^x})=0}{\Longrightarrow} \quad \mathbb{E}\left(X_{t\wedge T_D^x}^2\right) = \lambda\ \mathbb{E}(t\wedge T_D^x)\ \mathbb{E}(Y^2) = \mathbb{E}\left(\left(X_{t\wedge T_D^x}^x\right)^2\right) - x^2 \leq a^2 - x^2.$$

Arguing as above, this yields the estimate

$$\mathbb{E}(T_D^x) \leq (a^2 - x^2)/(\lambda\mathbb{E}(Y^2)).$$

This ends the proof of the exercise.

■

Solution to exercise 182:

- Since customers wait in line before entering in the first free server, the number of customers being served X_t, at some given time t, varies between 0 and d. Transitions $x \rightsquigarrow x-1$ depend on the number x of customers being served. The departure rate of customers is related to the minimum of x exponentials with parameter λ_2. Thus, the departure rate happens to be
$$\lambda(x) = \lambda_2 x.$$
The arrival rate is constant and it is equal to λ_1.

- The infinitesimal generator of X_t is given for any $x \in \{1,\ldots,d-1\}$ by
$$L(f)(x) = \lambda_2 x\ (f(x-1) - f(x)) + \lambda_1\ (f(x+1) - f(x))$$
and
$$L(f)(0) = \lambda_1\ (f(1) - f(0)) \quad \text{and} \quad L(f)(d) = \lambda_2 d\ (f(d-1) - f(d)).$$
By choosing $f = 1_{x-1}$:
$$L(f)(x) = L(1_{x-1})(x) = L(x, x-1) = \lambda_2 x.$$
In the same vein, we find that
$$L(x,x+1) = \lambda_1 = L(0,1) \quad L(d,d) = -\lambda_2 d \quad \text{and} \quad L(x,x) = -(\lambda_1 + \lambda_2 x).$$
In the other situations we have $L(x,y) = 0$.

- For any $x \in \{1,\ldots,d-1\}$
$$\begin{array}{rcl} 0 = (\pi L)(x) & = & \pi(x-1)\ L(x-1,x) + \pi(x)L(x,x) + \pi(x+1)\ L(x+1,x) \\ & = & \pi(x-1)\ \lambda_1 - \pi(x)\,(\lambda_1 + \lambda_2 x) + \pi(x+1)\ \lambda_2(x+1) \end{array}$$
and
$$0 = (\pi L)(d) = \pi(d-1)\ L(d-1,d) + \pi(d)\ L(d,d) = \pi(d-1)\ \lambda_1 - \pi(d)\ \lambda_2 d.$$
We conclude that
$$\pi(d-1)\ \lambda_1 - \pi(d)\ \lambda_2 d = 0$$
and
$$[\pi(x-1)\ \lambda_1 - \pi(x)\lambda_2 x] = [\pi(x)\lambda_1 - \pi(x+1)\ \lambda_2(x+1)].$$

By a simple backward induction w.r.t. x we conclude that for any $x \in \{1, \ldots, d\}$

$$\pi(x-1)\ \lambda_1 - \pi(x)\lambda_2 x = 0 \Rightarrow \pi(x) = \frac{\lambda_1}{\lambda_2 x}\ \pi(x-1) = \cdots = \left(\frac{\lambda_1}{\lambda_2}\right)^x\ \frac{1}{x!}\ \pi(0)$$

and

$$\sum_{0\leq x\leq d}\left(\frac{\lambda_1}{\lambda_2}\right)^x\ \frac{1}{x!}\ \pi(0) = 1 \Rightarrow \pi(0) = \frac{1}{\sum_{0\leq x\leq d}\left(\frac{\lambda_1}{\lambda_2}\right)^x\ \frac{1}{x!}}.$$

This ends the proof of the exercise.

■

Solution to exercise 183:

The solution follows the arguments developed in exercise 182 but with a countable number of servers.

In this situation we have

$$0 = (\pi L)(0) = \pi(0)L(0,0) + \pi(1)\ L(1,0) = -\lambda_1\pi(0) + \pi(1)\lambda_2 \Rightarrow \pi(1) = \frac{\lambda_1}{\lambda_2}$$

and

$$[\pi(x)\lambda_1 - \pi(x+1)\ \lambda_2(x+1)] = [\pi(x-1)\ \lambda_1 - \pi(x)\lambda_2 x] = \cdots = \pi(0)\ \lambda_1 - \pi(1)\lambda_2 = 0.$$

This implies

$$\pi(x) = \frac{\lambda_1}{\lambda_2}\ \frac{1}{x}\ \pi(x-1).$$

Thus, the invariant measure is the Poisson distribution given for any $x \in \mathbb{N}$ by

$$\pi(x) = \left(\frac{\lambda_1}{\lambda_2}\right)^x\ \frac{1}{x!}\ \pi(0) = e^{-\frac{\lambda_1}{\lambda_2}}\ \left(\frac{\lambda_1}{\lambda_2}\right)^x\ \frac{1}{x!}.$$

This ends the proof of the exercise.

■

Solution to exercise 184:

Since the queue has m servers, the infinitesimal generator of X_t is defined for any $0 \leq x \leq m$ by

$$L(f)(x) = \lambda_2 x\ (f(x-1) - f(x)) + \lambda_1\ (f(x+1) - f(x))$$

and for any $x > m$

$$L(f)(x) = \lambda_2 m\ (f(x-1) - f(x)) + \lambda_1\ (f(x+1) - f(x)).$$

In this situation, the invariant measure is the Poisson distribution given for any $1 \leq x < m$ by

$$\pi(x) = \left(\frac{\lambda_1}{\lambda_2}\right)^x\ \frac{1}{x!}\ \pi(0).$$

For any $x \geq m$, we have

$$\begin{aligned} 0 = (\pi L)(x) &= \pi(x-1)\ L(x-1,x) + \pi(x)L(x,x) + \pi(x+1)\ L(x+1,x) \\ &= \pi(x-1)\ \lambda_1 - \pi(x)\,(\lambda_1 + \lambda_2 m) + \pi(x+1)\ \lambda_2 m. \end{aligned}$$

For $x = (m-1)$ we find that

$$\begin{aligned} 0 &= \pi(m-2)\ \lambda_1 - \pi(m-1)\left(\lambda_1 + \lambda_2(m-1)\right) + \pi(m)\ \lambda_2 m \\ &= -\left(\frac{\lambda_1}{\lambda_2}\right)^{m-1}\ \frac{1}{(m-1)!}\ \lambda_1\ \pi(0) + \pi(m)\ \lambda_2 m. \end{aligned}$$

This implies that

$$\pi(m) = \left(\frac{\lambda_1}{\lambda_2}\right)^m\ \frac{1}{m!}\ \pi(0).$$

This also yields

$$\begin{aligned} \pi(m+1) &= \pi(m)\ \left(1 + \frac{\lambda_1}{\lambda_2 m}\right) - \left[\left(\frac{\lambda_1}{\lambda_2}\right)^m\ \frac{1}{m!}\ \pi(0)\ \right] \\ &= \pi(m)\ \frac{\lambda_1}{\lambda_2 m} = \frac{1}{m}\ \left(\frac{\lambda_1}{\lambda_2}\right)^{m+1}\ \frac{1}{m!}\ \pi(0). \end{aligned}$$

Let us check by induction w.r.t. x that for any $x \geq m$ we have

$$\pi(x) = \frac{1}{m^{x-m}}\ \left(\frac{\lambda_1}{\lambda_2}\right)^x\ \frac{1}{m!}\ \pi(0).$$

Notice that in this case we have

$$\frac{\lambda_1}{\lambda_2}\ \frac{1}{m}\ \pi(x-1) = \pi(x).$$

This implies that

$$\pi(x+1) = \pi(x)\left(\frac{\lambda_1}{\lambda_2}\ \frac{1}{m} + 1\right) - \pi(x-1)\ \frac{\lambda_1}{\lambda_2}\ \frac{1}{m} = \pi(x)\ \frac{\lambda_1}{\lambda_2}\ \frac{1}{m}.$$

The end of he recursion is now clear. This ends the proof of the exercise.

■

Solution to exercise 185: We denote by $R_\theta^{(i,j)} = \left(R_\theta^{(i,j)}(k,l)\right)_{1\leq k,l\leq N}$ the $(N \times N)$-rotation matrix given for any $1 \leq i < j \leq N$ by

$$\forall k \notin \{i,j\} \qquad R_\theta^{(i,j)}(k,l) = 1_{k=l}$$

with the i-th row given by

$$R_\theta^{(i,j)}(i,i) = \cos(\theta) \qquad R_\theta^{(i,j)}(i,j) = \sin(\theta) \quad \text{and} \quad \forall k \notin \{i,j\} \quad R_\theta^{(i,j)}(i,k) = 0$$

and the j-th row given by

$$R_\theta^{(i,j)}(j,j) = \cos(\theta) \qquad R_\theta^{(i,j)}(j,i) = -\sin\theta \quad \text{and} \quad \forall k \notin \{i,j\} \quad R_\theta^{(i,j)}(j,k) = 0.$$

The embedded Markov chain model $Y_n = (Y_n^i)_{1\leq i\leq N}$ is defined by

$$Y_n = R_{\Theta_n}^{(I_n,J_n)} Y_{n-1}$$

where (I_n, J_n) are i.i.d. uniform random variables on $\{(i,j) \in \{1,\ldots,N\}^2\ :\ i < j\}$ and

Θ_n a sequence of i.i.d. uniform random variables on $[0, 2\pi[$. We also assume that (I_n, J_n) and Θ_n are independent sequences.

We have

$$(Y_n^{I_n}, Y_n^{J_n}) = \left(\cos(\Theta_n)\ Y_{n-1}^{I_n} + \sin(\Theta_n)\ Y_{n-1}^{J_n}, -\sin(\Theta_n)\ Y_{n-1}^{I_n} + \cos(\Theta_n)\ Y_{n-1}^{J_n}\right)$$

$$\Rightarrow \left(Y_n^{I_n}\right)^2 + \left(Y_n^{J_n}\right)^2 = \left(Y_{n-1}^{I_n}\right)^2 + \left(Y_{n-1}^{J_n}\right)^2 \Longrightarrow \forall n \geq 0 \quad \sum_{1\leq i\leq N} \left(Y_n^i\right)^2 = \sum_{1\leq i\leq N} \left(Y_0^i\right)^2.$$

The Markov transition M of the chain Y_n is defined for any bounded function f on $S = \mathbb{R}^2$ by

$$M(f)(x) = \frac{1}{2\pi \left(\begin{array}{c} 2 \\ N \end{array}\right)} \sum_{1\leq i<j<N} \int_0^{2\pi} f\left(R_\theta^{(i,j)} x\right)\, d\theta.$$

The continuous time model is defined as an embedding of the chain Y_n at the jump times of a Poisson process with intensity $N\lambda$. Thus, the infinitesimal generator is given by $L = N\lambda\ (M - Id)$.

This ends the proof of the exercise.

■

Solution to exercise 186: We let $P_t(f)(x) = \mathbb{E}(f(X_t) \mid X_0 = x)$ be the semigroup of the process X_t with infinitesimal generator L. By construction, we have

$$f(x^\star) = \sup_{x\in S} f(x) \Longrightarrow \forall t \geq 0 \quad \forall x \in S \quad P_t(f)(x) \leq f(x^\star).$$

This implies that

$$L(f)(x^\star) = \lim_{t\downarrow 0} \frac{P_t(f)(x^\star) - f(x^\star)}{t} \leq 0.$$

This ends the proof of the exercise.

■

Solution to exercise 187:

Using (12.5) we have

$$\begin{aligned} \frac{d}{dt} P_t(g)_{t=0} &= L(P_t(g))_{|t=0} = L(g) \\ &\Rightarrow \frac{d}{dt}\pi(f P_t(g))_{|t=0} = \pi(f L(g)) = -\mathcal{E}\left(f, g\right). \end{aligned}$$

Notice that

$$\pi P_t = \pi \Rightarrow \mathrm{Var}_\pi(P_t(f)) = \pi((P_t(f) - \pi(f))^2) = \pi((P_t(f))^2) - (\pi(f))^2 = \|P_t(f) - \pi\|^2_{\mathbb{L}_2(\pi)}.$$

Arguing as above, we also have

$$\begin{aligned} \frac{d}{dt}\mathrm{Var}_\pi(P_t(f)) &= \frac{d}{dt}\pi((P_t(f))^2) = 2\pi(P_t(f)\frac{d}{dt}P_t(f)) = 2\pi(P_t(f)L(P_t(f))) \\ &= -2\mathcal{E}\left(P_t(f), P_t(f)\right). \end{aligned}$$

Taking $t = 0$, we also find that the Dirichlet form measures the infinitesimal changes in the variance of the semigroup; that is, we have

$$\mathcal{E}(f, f) = -\frac{1}{2}\frac{d}{dt}\mathrm{Var}_\pi(P_t(f))_{|t=0}.$$

This ends the proof of the exercise.

■

Solution to exercise 188: Arguing as in the proof of exercise 187, we have

$$\frac{d}{dt}\mathrm{Var}_\pi(P_t(f)) = -2\mathcal{E}(P_t(f), P_t(f)).$$

If π satisfies a Poincaré inequality for some parameter $a > 0$, then we have

$$-a\ \mathrm{Var}_\pi(P_t(f)) \geq -2\mathcal{E}(P_t(f), P_t(f)) \Rightarrow \frac{d}{dt}\mathrm{Var}_\pi(P_t(f)) \leq -a\ \mathrm{Var}_\pi(P_t(f)).$$

Recalling thet $P_0(f) = f$, this implies that

$$\mathrm{Var}_\pi(P_t(f)) \leq e^{-at}\ \mathrm{Var}_\pi(f). \tag{30.32}$$

Inversely, for small times $t \sim 0$ we have

$$P_t(f) = f + L(f)\,t + \mathrm{o}(t) \Rightarrow \mathrm{Var}_\pi(P_t(f)) = \pi((P_t(f))^2) - (\pi(f))^2 = \mathrm{Var}_\pi(f) - 2t\,\mathcal{E}(f, f) + \mathrm{o}(t)$$

and

$$e^{-at}\ \mathrm{Var}_\pi(f) = \mathrm{Var}_\pi(f) - a\ t\mathrm{Var}_\pi(f) + \mathrm{o}(t).$$

This shows that

$$(30.32) \Rightarrow a\ t\ \mathrm{Var}_\pi(f) + \mathrm{o}(t) \leq 2t\ \mathcal{E}(f, f) + \mathrm{o}(t) \Rightarrow a\ \mathrm{Var}_\pi(f) \leq 2\mathcal{E}(f, f).$$

This ends the proof of the exercise.

■

Solution to exercise 189:
We have

$$\frac{1}{2}\sum_{(v_1,v_2)\in\mathcal{E}} (x(v_1) - x(v_2))^2 = |E| - \sum_{(v_1,v_2)\in\mathcal{E}} x(v_1)x(v_2)$$

and for any given $v \in \mathcal{V}$

$$\sum_{u\sim v} x(u) = |\{u \sim v,\ x(u) = 1\}| - |\{u \sim v,\ x(u) = -1\}|.$$

Notice that

$$x(v) = -1 \iff x^{v,+1} \neq x \iff x(v) - x^{v,+1}(v) = -1 - 1 = -2.$$

Thus, for any $x^{v,+1} \neq x$ we have

$$\begin{aligned} H(x) - H(x^{v,+1}) &= \sum_{(u_1,u_2)\in\mathcal{E}} \left(x(u_1)x(u_2) - x^{v,+1}(u_1)x^{v,+1}(u_2)\right) \\ &= \sum_{(u_1,u_2)\in\mathcal{E},\ u_1\neq v\neq u_2} (x(u_1)x(u_2) - x(u_1)x(u_2)) \\ &\quad + \sum_{u\sim v} \left(x(v) - x^{v,+1}(v)\right)\ x(u). \end{aligned}$$

This implies that for $x^{v,+1} \neq x$ we have

$$H(x) - H(x^{v,+1}) \quad = \quad -2 \sum_{u \sim v} x(u).$$

In much the same way, we have

$$x(v) = +1 \quad \Longleftrightarrow \quad x^{v,-1} \neq x \quad \Longleftrightarrow \quad x(v) - x^{v,-1}(v) = 1 + 1 = 2,$$

and for $x^{v,-1} \neq x$ we have

$$H(x) - H(x^{v,-1}) = 2 \sum_{u \sim v} x(u).$$

Using the fact that

$$x \neq x^{v,+1} \Leftrightarrow (x^{v,+1})^{v,-1} = x \quad \text{and} \quad x \neq x^{v,-1} \Leftrightarrow (x^{v,-1})^{v,+1} = x$$

we also have

$$\begin{aligned} x \neq x^{v,+1} \Rightarrow \quad H(x) - H\left(x^{v,+1}\right) \quad &= \quad H\left((x^{v,+1})^{v,-1}\right) - H\left(x^{v,+1}\right) \\ &= \quad -\left[H\left(x^{v,+1}\right) - H\left((x^{v,+1})^{v,-1}\right)\right] = -2 \textstyle\sum_{u \sim v} x^{v,+1}(u). \end{aligned}$$

This implies that for $x \neq x^{v,+1}$ we have

$$\begin{aligned} \pi_\beta(x^{v,+1})\ Q(x^{v,+1}, x) \quad &= \quad \pi_\beta(x^{v,+1})\ Q(x^{v,+1}, (x^{v,+1})^{v,-1}) \\ &= \quad \pi_\beta(x^{v,+1})\ q_{\epsilon=-1}(v, x^{v,+1}) \\ &\propto \quad e^{-\beta H(x^{v,+1})} \wedge e^{-\beta\left[H(x^{v,+1}) + (H(x) - H\left(x^{v,+1}\right))\right]} \\ &= \quad e^{-\beta H(x^{v,+1})} \wedge e^{-\beta H(x)}. \end{aligned}$$

By symmetry, this implies

$$\forall x \in S \qquad \pi_\beta(x^{v,+1})\ Q(x^{v,+1}, x) = \pi_\beta(x)\ Q(x, x^{v,+1}).$$

Replacing x by $x^{v,-1}$ we deduce that

$$\begin{aligned} \pi_\beta((x^{v,-1})^{v,+1})\ Q((x^{v,-1})^{v,+1}, x^{v,-1}) \quad &= \quad \pi_\beta(x)\ Q(x, x^{v,-1}) \\ &= \quad e^{-\beta H(x)} \wedge e^{-\beta H(x^{v,-1})} \propto \pi_\beta(x^{v,-1})\ Q(x^{v,-1}, x). \end{aligned}$$

This ends the proof of the second assertion.

On the other hand, we have

$$L = Q - Id \Rightarrow P_t(f)(x) = \mathbb{E}(f(X_t) \mid X_0 = x) = f(x) + \int_0^t L(P_s(f))(x)\ ds.$$

Recalling that $\pi_\beta Q = \pi_\beta \Rightarrow \pi_\beta L = 0$, we conclude that

$$\text{Law}(X_0) = \pi_\beta \Rightarrow \mathbb{E}(f(X_t)) = \pi_\beta(f) + \int_0^t \pi_\beta(L(P_s(f)))\ ds = \pi_\beta(f).$$

This ends the proof of the exercise.

■

Solution to exercise 190
By (12.12) we have

$$P^h_{t_n,t_{n+1}} = \sum_{0\leq p\leq m} \frac{1}{p!}\, h^p\, L^p_{t_n} \ + \mathrm{O}(h^{m+1}) = P_{t_n,t_{n+1}}. \tag{30.33}$$

Rephrasing the proof of the theorem we conclude that

$$P^h_{t_p,t_n} = P_{t_p,t_n} + \mathrm{O}\left(h^m\right).$$

For $m=1$, and $m=2$, the pure jump sg (12.35) takes the form

$$P^h_{t_n,t_{n+1}} = Id + \lambda_{t_n} h\ (M_{t_n} - Id) = \lambda_{t_n} h\ M_{t_n} + (1-\lambda_{t_n} h)\ \ Id$$

and

$$\begin{aligned} P^h_{t_n,t_{n+1}} &= Id + \lambda_{t_n} h\ (M_{t_n} - Id) + 2^{-1}(\lambda_{t_n} h)^2\ (M_{t_n} - Id)^2 \\ &= 2^{-1}(\lambda_{t_n} h)^2\ M^2_{t_n} + \lambda_{t_n} h\,(1-\lambda_{t_n} h)\ \ M_{t_n} \\ &\qquad\qquad + \left[1 - \left(2^{-1}(\lambda_{t_n} h)^2 + \lambda_{t_n} h\,(1-\lambda_{t_n} h)\right)\right]. \end{aligned}$$

More generally, for any order $m\geq 1$, we have

$$\begin{aligned} P^h_{t_n,t_{n+1}} &= \sum_{0\leq p\leq q\leq m} \frac{1}{q!}\ (\lambda_{t_n} h)^q \begin{pmatrix} q \\ p \end{pmatrix} (-1)^{q-p} M^p_{t_n} \\ &= \sum_{0\leq p\leq m} \left(\sum_{p\leq q\leq m} \frac{(-1)^{q-p}}{p!(q-p)!}\ (\lambda_{t_n} h)^{(q-p)+p} \right) M^p_{t_n} \\ &= \sum_{0\leq p\leq m} \frac{(\lambda_{t_n} h)^p}{p!} \left(\sum_{0\leq q\leq m-p} \frac{(-\lambda_{t_n} h)^q}{q!} \right) M^p_{t_n} \end{aligned}$$

with

$$\alpha^h_{t_n}(p) = \frac{(\lambda_{t_n} h)^p}{p!} \left(\sum_{0\leq q\leq m-p} \frac{(-\lambda_{t_n} h)^q}{q!} \right).$$

By construction, we have $L^0_{t_n} = Id$, and

$$\left[\forall p>0 \quad L^p_{t_n}(1) = 0\ \right] \Longrightarrow P^h_{t_n,t_{n+1}}(1) = 1.$$

This ensures that

$$P^h_{t_n,t_{n+1}}(1) = \sum_{0\leq p\leq m} \alpha^h_{t_n}(p) = 1.$$

It remains to notice that the functions $\theta_n(x) = \sum_{0\leq q\leq n} \frac{(-x)^q}{q!}$ map $[0,1]$ into itself. Firstly, we prove that these functions are non negative. For odd parameters the result is immediate since

$$\theta_{2n+1}(x) = (1-x) + \left(\frac{x^2}{2!} - \frac{x^3}{3!}\right) + \ldots + \left(\frac{x^{2n}}{(2n)!} - \frac{x^{2n+1}}{(2n+1)!}\right)$$

and

$$\frac{x^{2n}}{(2n)!} \geq \frac{x^{2n}}{(2n+1)!} \geq \frac{x^{2n+1}}{(2n+1)!}.$$

For even parameters, we conclude that

$$\theta_{2(n+1)}(x) = \theta_{2n+1}(x) + \frac{x^{2n}}{(2n)!} \geq 0.$$

On the other hand, we have $\theta'_n = -\theta_{n-1}$. This implies that θ_n is a non-increasing function from $\theta_n(0) = 1$ to $\theta_n(1) \geq 0$. This clearly implies that θ_n maps $[0,1]$ into itself.

This ends the proof of the exercise.

■

Solution to exercise 191

The martingale property of M_t has been proved in the end of section 12.5.2. Using the fact that

$$\mathbb{E}\left(d\overline{M}_t \mid \mathcal{F}_t\right) = \varphi_{t-}\ \mathbb{E}\left(dM_t \mid \mathcal{F}_t\right) = 0$$

we conclude that $\overline{M}_t$ is a martingale.

In addition, recalling that

$$M_t^2 - \int_0^t \lambda_s ds$$

is a martingale, we have

$$\mathbb{E}\left(dM_t^2 - dN_t \mid \mathcal{F}_t\right) = \mathbb{E}\left(dM_t^2 \mid \mathcal{F}_t\right) - \lambda_t dt = 0.$$

In a similar way, we have

$$\begin{aligned}
\mathbb{E}\left(d\overline{M}_t^2 \mid \mathcal{F}_t\right) &= \mathbb{E}\left((d\overline{M}_t)^2 \mid \mathcal{F}_t\right) \\
&= \mathbb{E}\left(\left(\varphi_{t-} dM_t\right)^2 \mid \mathcal{F}_t\right) \\
&= \varphi_{t-}^2\ \mathbb{E}\left(\left(dM_t\right)^2 \mid \mathcal{F}_t\right) = \varphi_{t-}^2 \lambda_t^2 dt.
\end{aligned}$$

This implies that $\widetilde{M}_t$ is a martingale. Finally, we have

$$\begin{aligned}
\mathcal{E}_{t+dt} &= \mathcal{E}_t \times \exp\left(\int_t^{t+dt} \varphi_{s-} dN_s - \int_t^{t+dt} \lambda_s\ \left[e^{\varphi_{s-}} - 1\right]\ ds\right) \\
&= \mathcal{E}_t \times \exp\left(\varphi_{t-} dN_t\right)\ \exp\left(-\lambda_t\ \left[e^{\varphi_{t-}} - 1\right] dt\right).
\end{aligned}$$

Recalling that $dN_t = N_{t+dt} - N_t$ is a Poisson r.v. with intensity $\lambda_t dt$, we prove that

$$\begin{aligned}
\mathbb{E}\left(e^{\varphi_{t-} dN_t} \mid \mathcal{F}_t\right) &= e^{-\lambda_t dt} \sum_{n \geq 1} \frac{\left(\lambda_t dt\right)^n}{n!}\ e^{n\varphi_{t-}} \\
&= e^{-\lambda_t dt}\ e^{\lambda_t\ e^{\varphi_{t-}} dt} = e^{\lambda_t\ \left(e^{\varphi_{t-}} - 1\right) dt}.
\end{aligned}$$

The martingale property of $\mathcal{E}_t$ is now clear. For the constant function $\varphi_{s-} = \log(1+\epsilon)$ we have

$$\mathcal{E}_t = \exp\left(\log(1+\epsilon)\ N_t - \epsilon\ \int_0^t \lambda_s\ ds\right) = \mathcal{E}_t^\epsilon.$$

This ends the proof of the exercise.

■

Chapter 13

Solution to exercise 192:

By construction, we have

$$dC_t = a\ C_t\ dt\ +\ dX_t.$$

Therefore the infinitesimal generator of C_t is given by

$$L(f)(x) = a\ x\ \partial_x f + \lambda \int (f(x+y) - f(x))\ \mu(dy),$$

where μ stands for the distribution of the random variables Y_n.

This ends the proof of the exercise.

■

Solution to exercise 193:

We let T_n be the jump times of the Poisson process N_t. We start at some $X_0 = x_0$ and we solve the system (13.40) up to time T_1-. We calculate the value $X_{T_1-} := \varphi_{0,T_1-}(X_0)$ using the flow map of the deterministic system, and we set

$$X_{T_1} = X_{T_1-} + b(X_{T_1-}).$$

Given X_{T_1}, we solve the system (13.40) from T_1 up to time T_2-. We calculate the value $X_{T_2-} := \varphi_{T_1,T_2-}(X_{T_1})$ using the flow map of the deterministic system, and we set

$$X_{T_2} = X_{T_2-} + b(X_{T_2-})$$

and so on.

The infinitesimal generator of X_t is given for any differentiable function f by the formula

$$L(f)(x) = a(x)\frac{\partial f}{\partial x}(x) + \lambda\ \left(f(x+b(x)) - f(x)\right).$$

This ends the proof of the exercise.

■

Solution to exercise 194:

The stochastic differential equation (13.38) is a particular case of the one discussed in exercise 193. The infinitesimal generator of X_t is given for any differentiable function f by the formula

$$L(f)(x) = a\ x\ \partial_x f(x) + \lambda\ \left(f(x(1+b)) - f(x)\right).$$

We let T_n be the jump times of the Poisson process N_t, with the convention $(N_0, T_0) = (0, 0)$. For $t \in [T_0, T_1[$ we have

$$X_t = e^{at}\quad X_0$$

and at the jump time

$$X_{T_1} = (1+b)\ X_{T_1-} = (1+b)\ e^{at}\ X_0.$$

For $t \in [T_1, T_2[$ we have
$$X_t = e^{at}\ X_{T_1} = (1+b)\ e^{2at}\ X_0$$
and at the jump time
$$X_{T_2} = (1+b)\ X_{T_2-} = (1+b)^2\ e^{2at}\ X_0.$$
For $t \in [T_2, T_3[$ we have
$$X_t = e^{at}\ X_{T_2} = (1+b)^2\ e^{3at}\ X_0$$
and at the jump time
$$X_{T_3} = (1+b)\ X_{T_3-} = (1+b)^3\ e^{3at}\ X_0.$$
Using a simple induction, we check that $t \in [T_n, T_{n+1}[$ we have
$$X_t = e^{at}\ X_{T_n} = (1+b)^n\ e^{(n+1)\ at}\ X_0$$
and at the jump time
$$X_{T_{n+1}} = (1+b)\ X_{T_{n+1}-} = (1+b)^{n+1}\ e^{(n+1)at}\ X_0.$$
This ends the proof of the exercise.

■

Solution to exercise 195:

We let T_n be the jump times of the Poisson process N_t and $\varphi_{s,t}(x)$ the flow map of the deterministic system (13.40). We start at some $X_0 = x_0$ and we solve the system (13.40) up to time
$$R_1 = \inf\{t \geq 0\ :\ \int_0^t b(\varphi_{0,s}(X_0))\ ds \geq T_1\}.$$
We calculate the value $X_{R_1-} := \varphi_{0,R_1-}(X_0)$ using the flow map of the deterministic system, and we set
$$X_{R_1} = X_{R_1-} + 1.$$
We solve the system (13.40) up to time
$$R_2 = \inf\{t \geq R_1\ :\ \int_{R_1}^t b(\varphi_{R_1,s}(X_{R_1}))\ ds \geq (T_2 - T_1)\}.$$
We calculate the value $X_{R_2-} := \varphi_{R_1,R_2-}(X_{R_1})$ using the flow map of the deterministic system, and we set
$$X_{R_2} = X_{R_2-} + 1$$
and so on.

The infinitesimal generator of X_t is given for any differentiable function f by the formula
$$L(f)(x) = a(x)\frac{\partial f}{\partial x}(x) + b(x)\ \ (f(x+1) - f(x))\,.$$
This ends the proof of the exercise.

■

Solution to exercise 196:

By construction, the infinitesimal generator of X_t is given for any differentiable function f by the formula

$$L(f)(x) = a(x)\frac{\partial f}{\partial x}(x) + \lambda(x) \int \ (f(y) - f(x)) \ K(x, dy).$$

This ends the proof of the exercise.

∎

Solution to exercise 197:

By construction, the infinitesimal generator of X_t is given for any differentiable function f by the formula

$$L(f)(x) = -a(x)\frac{\partial f}{\partial x}(x) + \lambda(x) \int \ (f(x+y) - f(x)) \ K(x, dy).$$

This ends the proof of the exercise.

∎

Solution to exercise 198:

The jump times T_n of the storage process are defined by

$$T_0 = 0 \qquad \text{and} \quad \forall n \geq 1 \quad \Delta T_n = T_n - T_{n-1} = Z_n.$$

Between the jumps the PDMP process is given by

$$\forall t \in [T_n, T_{n+1}[\qquad X_t = e^{-b(t-T_n)} \ X_{T_n}.$$

At the $(n+1)$-th jump time we have

$$X_{T_{n+1}} = e^{-b \ (T_{n+1}-T_n)} \ X_{T_n} + Y_{n+1} = e^{-b \ Z_{n+1}} \ X_{T_n} + Y_{n+1}.$$

This ends the proof of the exercise.

∎

Solution to exercise 199:

By applying the Doeblin-Itō formula to the function $f(x) = x$ we have

$$L(f)(x) = -a \ f + \lambda \int \ y \ \nu(dy).$$

This implies that

$$\partial_t \mathbb{E}_x(X_t) = -a \ \mathbb{E}_x(X_t) + \lambda \ m,$$

from which we conclude that

$$\begin{array}{rcl} \mathbb{E}_x(X_t) & = & e^{-at} \ \left[x + (\lambda m/a) \displaystyle\int_0^t \ ae^{as} \ ds\right] = e^{-at} \ \left[x + (\lambda m/a) \ \left(e^{at} - 1\right)\right] \\ & = & e^{-at} \ x + (\lambda m/a) \ \left(1 - e^{-at}\right) = e^{-at} \ (x - \lambda m/a) + (\lambda m/a). \end{array}$$

This ends the proof of the exercise.

∎

Solution to exercise 200:
Applying the Doeblin-Itō formula to the function

$$f(x) = e^{ux} \Rightarrow \partial_x f(x) = u\ f(x) \quad \text{and} \quad f(x+y) - f(x) = f(x)\ [f(y) - 1]$$

we have

$$L(f)(x) = -aux\ f(x) + \lambda\ f(x) \int\ [f(y) - 1]\ \nu(dy) = -aux\ f(x) + \lambda\ f(x)\ [h(u) - 1].$$

This yields the evolution equation

$$\partial_t g_t(u) = -au\ \underbrace{\mathbb{E}\left(X_t e^{uX_t}\right)}_{=\partial_u g_t(u)}\ dt + \lambda g_t(u)\ \ [h(u) - 1]\,,$$

from which we conclude that

$$\partial_t g_t(u) = -au\ \partial_u g_t(u) + g_t(u)\ V(u) \quad \text{with} \quad V(u) = \lambda\ [h(u) - 1]\,.$$

The solution is given by

$$g_t(u) = \exp\left(\int_0^t V\left[e^{-as}u\right]\ ds\right) = \exp\left(a^{-1}\int_{e^{-at}u}^{u} V(\tau)/\tau\ d\tau\right).$$

The r.h.s. expression is obtained using the change of variable

$$\tau = e^{-as}u \Rightarrow d\tau = -a\tau\ ds.$$

Under our assumptions we also have

$$u \in]-\infty, u_0[\Rightarrow \forall s \geq 0 \qquad e^{-as}u \in]-\infty, u_0[.$$

We conclude that

$$\begin{aligned} g_t(u) &= \exp\left(a^{-1}\int_{e^{-at}u}^{u} \partial_\tau \log \overline{V}(\tau)\ d\tau\right) \\ &= \exp\left(a^{-1}\left[\log \overline{V}(u) - \log \overline{V}\left(e^{-at}u\right)\right]\right) = \left(\frac{\overline{V}(u)}{\overline{V}\left(e^{-at}u\right)}\right)^{1/a}. \end{aligned}$$

We check that g_t satisfies the desired evolution equation using the fact that

$$\partial_t g_t(u) = V\left[e^{-at}u\right]\ g_t(u)$$

and

$$\begin{aligned} -a\ u\ \partial_u g_t(u) &= -\left[\int_0^t a\ u\ e^{-as}\ \partial_v V\left[e^{-as}u\right]\ ds\right]\ g_t(u) \\ &= \left[\int_0^t \partial_s\left(e^{-as}\ u\right)\ \partial_v V\left[e^{-as}u\right]\ ds\right]\ g_t(u) \\ &= \left[\int_0^t \partial_s\left\{V\left[e^{-as}u\right]\right\}\ ds\right]\ g_t(u) = \left[V\left[e^{-at}u\right] - V(u)\right]\ g_t(u). \end{aligned}$$

This implies that

$$\partial_t g_t(u) = -a\ u\ \partial_u g_t(u) + V(u)\ g_t(u).$$

When $\nu(dy) = b\ e^{-by}\ 1_{[0,\infty[}(y)\ dy$ we have

$$h(u) = \frac{b}{b-u} \int_0^\infty (b-u)\ e^{-(b-u)y}\ dy = \frac{b}{b-u} \Rightarrow V(u) = \lambda\ \left[\frac{b}{b-u} - 1\right] = \frac{\lambda}{b} \frac{u}{1-u/b}.$$

In this situation, $V(u)$ is well defined for any $u \neq b$ and

$$V(u)/u = \lambda/(b-u) = -\lambda\ \partial_u \log(b-u)$$

$$\Rightarrow g_t(u) = \exp\left(\lambda a^{-1} \int_u^{e^{-at}u} \partial_\tau \log(b-\tau)\ d\tau\right) = \exp\left(\lambda a^{-1}\ \log \frac{1-e^{-at}u/b}{1-u/b}\right).$$

This implies that

$$g_t(u) = \left(\frac{1-e^{-at}u/b}{1-u/b}\right)^{\lambda/a} \longrightarrow_{t\uparrow\infty} g_\infty(u) := \mathbb{E}(e^{uX_\infty}) = (1-u/b)^{-\lambda/a}$$

for any $u < b$, with $\alpha := \lambda/a$. We conclude that the invariant probability measure is given by the `Laplace distribution with shape` α `and rate` b; that is we have that

$$\pi(dx) = \mathbb{P}(X_\infty \in dx) = \frac{b^\alpha}{\Gamma(\alpha)}\ x^{\alpha-1}\ e^{-\beta x}\ 1_{]0,\infty[}(x)\ dx.$$

This ends the proof of the exercise.

■

Solution to exercise 201:

Let X_t^x and X_t^y be a couple of storage processes starting at x and y and sharing the same sequence of random variables defined in exercise 198. By construction, the jump times of the processes coincide. In addition, when a jump occur the same amount Y_n are added to the process. Thus, we have

$$X_t^x - X_t^y = e^{-at}\ (x-y) \Rightarrow \mathbb{W}\left(\mathrm{Law}(X_t^x), \mathrm{Law}(X_t^y)\right) \leq e^{-at}\ |x-y|.$$

This ends the proof of the exercise.

■

Solution to exercise 202:

Let X_t^x and X_t^y be a couple of storage processes starting at x and y and sharing the same jump times defined in terms of the sequence of random variables $(Z_n)_{n\geq 1}$ defined in exercise 198. The number of jumps at time t is given by a Poisson process N_t. The jump amplitudes are defined using the maximal coupling discussed in example 8.3.5. For each $n \geq 0$ and given $\left(X^x_{T_{n+1}-}, X^y_{T_{n+1}-}\right) = (a_1, a_2)$ we let $\left(X^x_{T_{n+1}}, X^y_{T_{n+1}}\right)$ be a couple of exponential random variables with conditional exponential distributions

$$\begin{aligned}
\mathbb{P}\left(X^x_{T_{n+1}} \in dz \mid \left(X^x_{T_{n+1}-}, X^y_{T_{n+1}-}\right)\right) &= b\ e^{-b(z-a_1)}\ 1_{[a_1,\infty[}(z)\ dz \\
\mathbb{P}\left(X^y_{T_{n+1}} \in dz \mid \left(X^x_{T_{n+1}-}, X^y_{T_{n+1}-}\right)\right) &= b\ e^{-b(z-a_2)}\ 1_{[a_2,\infty[}(z)\ dz.
\end{aligned}$$

Using the maximal coupling described in example 8.3.5 these variables can be coupled in such a way that

$$\begin{aligned}
\mathbb{P}\left(X^x_{T_{n+1}} = X^y_{T_{n+1}} \mid \left(X^x_{T_{n+1}-}, X^y_{T_{n+1}-}\right)\right) &= \exp\left(-b\left|X^x_{T_{n+1}-} - X^y_{T_{n+1}-}\right|\right) \\
&\geq 1 - b\left|X^x_{T_{n+1}-} - X^y_{T_{n+1}-}\right|.
\end{aligned}$$

Observe that on these successful coupling events the process is such that $X^x_{T_n} = X^y_{T_n}$ for any $n \geq 1$, thus $X^x_t = X^y_t$ for any time t as soon as $N_t > 0$. On the other hand, at any time t s.t. $N_t > 0$ the chance of coupling is given by

$$\mathbb{P}\left(X^x_{T_{N_t}} = X^y_{T_{N_t}} \mid \left(X^x_{T_{N_t}-}, X^y_{T_{N_t}-}\right)\right) \geq 1 - b\,|x-y|\ e^{-aT_{N_t}-}.$$

Using theorem 8.3.2 we conclude that

$$\begin{aligned}
\|\mathrm{Law}(X^x_t) - \mathrm{Law}(X^y_t)\|_{tv} &\leq \mathbb{P}\left(X^x_t \neq X^y_t\right)\\
&= 1 - \mathbb{P}\left(X^x_t = X^y_t\right) = 1 - \mathbb{E}\left([1 - b\,|x-y|]\ e^{-aT_{N_t}}\ 1_{N_t>0}\right)\\
&= \mathbb{P}(N_t = 0) + b\,|x-y|\ \mathbb{E}\left(e^{-aT_{N_t}}\ 1_{N_t>0}\right)\\
&= e^{-\lambda t} + b\,|x-y|\ \mathbb{E}\left(e^{-aT_{N_t}}\ 1_{N_t>0}\right).
\end{aligned}$$

Given $N_t = n$ the random variable $(T_1/n, \ldots, T_n/t)$ is an ordered uniform statistic on $[0,1]$ (cf. exercise 41). Given $N_t = n$, this shows that T_n/t has the same law as the maximum $\max_{1\leq i\leq n} U_i$ of n uniform random variables U_i on $[0,1]$. Since

$$\mathbb{P}\left(\max_{1\leq i\leq n} U_i \leq u\right) = u^n \Rightarrow \mathbb{P}\left(\max_{1\leq i\leq n} U_i \in du\right) = nu^{n-1}$$

for any $u \in [0,1]$, we conclude that

$$\mathbb{P}\left(T_{N_t}/t \in du \mid N_t = n\right) = nu^{n-1}\ 1_{[0,1]}(u)\ du \Leftrightarrow \mathbb{P}\left(T_{N_t}/t \in du \mid N_t\right) = N_t\ u^{N_t-1}\ 1_{[0,1]}(u)\ du.$$

This implies that

$$\mathbb{E}\left(e^{-atT_{N_t}/t}\ 1_{N_t>0}\right) = \int_0^1 e^{-atu}\ \mathbb{E}\left(N_t\ u^{N_t-1}\right)\ du.$$

Recalling that

$$\begin{aligned}
\mathbb{E}\left(N_t\ u^{N_t-1}\right) &= e^{-\lambda t}\sum_{n\geq 1} n\ u^{n-1}\ \frac{(\lambda t)^n}{n!}\\
&= \lambda t\ e^{-\lambda t}\sum_{n\geq 1} \frac{(\lambda t)^{n-1}}{(n-1)!} = \lambda t\ e^{-\lambda t(1-u)},
\end{aligned}$$

we find that

$$\begin{aligned}
\mathbb{E}\left(e^{-atT_{N_t}/t}\ 1_{N_t>0}\right) &= \lambda t\ e^{-\lambda t}\int_0^1 e^{u(\lambda-a)t}\ du = \frac{\lambda t}{(\lambda-a)t}\ e^{-\lambda t}\ \left[e^{(\lambda-a)t} - 1\right]\\
&= \frac{\lambda}{(\lambda-a)}\ \left[e^{-at} - e^{-\lambda t}\right].
\end{aligned}$$

We conclude that

$$\|\mathrm{Law}(X^x_t) - \mathrm{Law}(X^y_t)\|_{tv} \leq e^{-\lambda t} + b\,|x-y|\ \frac{\lambda}{(\lambda-a)}\ \left[e^{-at} - e^{-\lambda t}\right].$$

This ends the proof of the exercise.

■

Solution to exercise 203:

Using (13.15), for any non negative and bounded function f on $S=\mathbb{R}^r$ we have

$$P_{s,t}(f)(x) \quad \geq \quad \lambda_\star \epsilon \int_s^t \quad e^{-\lambda^\star(u-s)} \; (\nu P_{u,t})(f) \; du = \epsilon_{s,t} \; \nu_{s,t}(f),$$

with the probability measure

$$\nu_{s,t}(f) := \frac{\int_s^t \quad e^{-\lambda^\star(u-s)} \; (\nu_u P_{u,t})(f) \; du}{\int_s^t \quad e^{-\lambda^\star(u-s)} \; du}$$

and the parameter

$$\epsilon_{s,t} := \frac{\lambda_\star}{\lambda^\star} \; \epsilon \; \left(1-e^{-\lambda^\star(t-s)}\right) > 0.$$

We also observe that

$$\begin{aligned}\left(1-e^{-\lambda^\star(t-s)}\right) \geq 1/2 \quad &\Leftrightarrow \quad e^{-\lambda^\star(t-s)} \leq 1/2\\ &\Leftrightarrow \quad \lambda^\star(t-s) \geq \log 2 \;\Leftrightarrow\; (t-s) \geq \log 2/\lambda^\star.\end{aligned}$$

The last assertion is a direct consequence of (8.18).

This ends the proof of the exercise.

■

Solution to exercise 204:

Following the analysis of switching processes developed in section 13.3.1, the infinitesimal generator of X_t is given for any function

$$f \;:\; (i,x) \in\in (\{i\}\times\mathbb{R}^{r_i}) \mapsto f(i,x) \in \mathbb{R}$$

differentiable w.r.t. the second component by the formula

$$L(f)(i,x) = \sum_{1\leq j\leq r_i} a_i^j(x) \; \partial_{x_j} f(i,x) + \lambda(i,x) \int \quad (f(j,y)-f(i,x)) \;\; K((i,x),d(j,y))$$

for any $i \in J$ and $x=(x_i)_{1\leq i\leq r_i}$.

This ends the proof of the exercise.

■

Solution to exercise 205:

Observe that

$$\begin{aligned}2^{-1}\partial_t \left\|Y_t\right\|^2 \quad &= \quad \sum_{1\leq i\leq r} a^i(X_t,Y_t)\; Y_t^i = \langle a(X_t,Y_t)-a(X_t,0),Y_t-0\rangle + \langle a(X_t,0),Y_t-0\rangle\\ &\leq \quad -V(X_t) \;\; \left\|Y_t\right\|^2 + \;\; \langle a(X_t,0),Y_t-0\rangle\,.\end{aligned}$$

We set $I_t := \left\|Y_t\right\|^2$. Applying Cauchy-Schwartz inequality, we find that

$$2^{-1}\partial_t I_t \leq -V(X_t)\; I_t + \sqrt{I_t} \;\; \|a(X_t,0)\|$$

$$\Rightarrow \partial_t\sqrt{I_t} = \tfrac{1}{2\sqrt{I_t}} \; \partial_t I_t \leq -V(X_t)\; \sqrt{I_t} + \;\; \|a(X_t,0)\|\,.$$

By Gronwall's inequality we find that

$$\|Y_t\| \leq e^{-\int_0^t V(X_s)ds} \;\; \|Y_0\| + \int_0^t \quad e^{-\int_s^t V(X_u)du} \quad \|a(X_s,0)\| \;\; ds.$$

Recalling that Y_0 does not depend on X_t and using the generalized Minkowski inequality we prove the Feynman-Kac formula

$$\begin{aligned}\mathbb{E}\left[\|Y_t\|^\beta\right]^{1/\beta} &\leq \mathbb{E}\left(e^{-\beta\int_0^t V(X_s)ds}\right)^{1/\beta}\ \mathbb{E}\left(\|Y_0\|^\beta\right)^{1/\beta}\\ &\qquad\qquad + \int_0^t \mathbb{E}\left[\ e^{-\beta\int_s^t V(X_u)du}\quad \|a(X_s,0)\|^\beta\ \ ds\right]^{1/\beta}\end{aligned}$$

for any $\beta > 1$. We conclude that

$$\mathbb{E}\left[\|Y_t\|^\beta\right]^{1/\beta} \leq \gamma_t^{[\beta]}(1)^{1/\beta}\ \mathbb{E}\left(\|Y_0\|^\beta\right)^{1/\beta} + \int_0^t \left[\frac{\gamma_t^{[\beta]}(1)}{\gamma_s^{[\beta]}(1)}\right]^{1/\beta} \left(\eta_s^{[\beta]}(f_\beta)\right)^{1/\beta}\ ds.$$

Using the above estimates, we have

$$\mathbb{E}\left[\|Y_t\|^\beta\right]^{1/\beta} \leq C_\beta\left[e^{-t\eta_\infty^{[\beta]}(V)}\ \mathbb{E}\left(\|Y_0\|^\beta\right)^{1/\beta} + C_\beta\int_0^t e^{-(t-s)\eta_\infty^{[\beta]}(V)}\left(\eta_s^{[\beta]}(f^\beta)\right)^{1/\beta}\ ds\right].$$

When f is uniformly bounded we clearly have

$$\mathbb{E}\left[\|Y_t\|^\beta\right]^{1/\beta} \leq C_\beta\ e^{-t\eta_\infty^{[\beta]}(V)}\ \mathbb{E}\left(\|Y_0\|^\beta\right)^{1/\beta} + C_\beta^2\ \|f\|\ \left(1 - e^{-t\eta_\infty^{[\beta]}(V)}\right).$$

We further assume that $a^\star := \sup_{x\in S}\|a(x,0)\| < \infty$ and $\inf_{x\in\mathbb{R}^r} V(x) = V_\star > 0$. In this situation, we have

$$\begin{aligned}\|Y_t\| &\leq e^{-\int_0^t V(X_s)ds}\ \|Y_0\| + \int_0^t e^{-\int_s^t V(X_u)du}\quad \|a(X_s,0)\|\ \ ds\\ &\leq e^{-\int_0^t V(X_s)ds}\left[\|Y_0\| + (a^\star/V_\star)\ \int_0^t V(X_s)\ e^{\int_0^s V(X_u)du}\ ds\right]\\ &= e^{-\int_0^t V(X_s)ds}\ \|Y_0\| + (a^\star/V_\star)\ \left(1 - e^{-\int_0^t V(X_u)du}\right).\end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 206:

We couple the stochastic processes $\mathcal{X}_t = (X_t, Y_t)$ and $\mathcal{X}'_t = (X_t, Y'_t)$ starting at different states $\mathcal{X}_0 = (x,y)$ and $\mathcal{X}'_0 = (x,y')$ (using with the same first coordinate process X_t).

We have

$$\begin{aligned}\partial_t\,\|Y_t - Y'_t\|^2 &= \partial_t \sum_{1\leq i\leq r}\left(Y_t^i - Y_t^{\prime,i}\right)^2\\ &= 2\sum_{1\leq i\leq r}\left(Y_t^i - Y_t^{\prime,i}\right)\ \left(a^i(X_t,Y_t) - a^i(X_t,Y'_t)\right)\\ &= 2\ \langle a\,(X_t,Y_t) - a\,(X_t,Y'_t)), (Y_t - Y'_t)\rangle\,.\end{aligned}$$

Under our assumptions, we have the estimate

$$\partial_t\,\|Y_t - Y'_t\|^2 \leq -2\ V(X_t)\ \ \|Y_t - Y'_t\|^2\,.$$

This implies that

$$\|Y_t - Y'_t\| \leq \exp\left(-\int_0^t V(X_s)\ ds\right)\ \|y - y'\|\,.$$

We conclude that for any parameter $\beta > 0$

$$\mathbb{E}\left(\|Y_t - Y'_t\|^{\beta} \mid (Y_0, Y'_0) = (y, y'), X_0 = x\right) \leq \mathcal{Z}_{0,t}(x) \;\; \|y - y'\|^{\beta}$$

with the Feynman-Kac normalizing constants $\mathcal{Z}_{s,t}(x)$ defined for any $0 \leq s \leq t$ by

$$\mathcal{Z}^{(\beta)}_{s,t}(x) := \mathbb{E}\left(\exp\left(-\beta \int_s^t V(X_s)\, ds\right) \mid X_0 = x\right).$$

This ends the proof of the exercise. ■

Solution to exercise 207:

One natural way to couple the stochastic processes $\mathcal{X}_t = (X_t, Y_t)$ and $\mathcal{X}'_t = (X'_t, Y'_t)$ starting at different states $\mathcal{X}_0 = (x, y)$ and $\mathcal{X}'_0 = (x', y')$ is to use the coupling described in the second statement of theorem 12.7.6. To be more precisely, we couple the first components (X_t, X'_t) until their coupling time T, and we set $X_t = X'_t$ for any $t \geq T$. In this situation we have

$$\begin{aligned}
&\mathbb{E}\left(\|Y_t - Y'_t\|^{\beta} \;\; 1_{T\leq \epsilon t} \mid (Y_0, Y'_0) = (y, y'), (X_0, X'_0) = (x, x')\right)\\
&= \mathbb{E}\left(\mathbb{E}\left(\|Y_t - Y'_t\|^{\beta} \mid (Y_T, Y'_T), X_T = X'_T\right) \;\; 1_{T\leq \epsilon t} \mid (Y_0, Y'_0) = (y, y'), (X_0, X'_0) = (x, x')\right)\\
&\leq \mathbb{E}\left(\mathcal{Z}^{(\beta)}_{T,t}(X_T) \;\; 1_{T\leq \epsilon t} \;\; \|Y_T - Y'_T\|^{\beta} \mid (Y_0, Y'_0) = (y, y'), (X_0, X'_0) = (x, x')\right).
\end{aligned}$$

By Hölder's inequality, for any conjugate parameters $1 \leq \alpha, \alpha' \leq \infty$ (s.t. $1 = 1/\alpha + 1/\alpha'$) we have

$$\begin{aligned}
&\mathbb{E}\left(\|Y_t - Y'_t\|^{\beta} \;\; 1_{T\leq \epsilon t} \mid (Y_0, Y'_0) = (y, y'), (X_0, X'_0) = (x, x')\right)\\
&\leq \underbrace{\mathbb{E}\left(\left(\mathcal{Z}^{(\beta)}_{T,t}(X_T)\right)^{\alpha} \;\; 1_{T\leq \epsilon t} \mid (X_0, X'_0) = (x, x')\right)^{1/\alpha}}_{\leq \exp\left(-\beta\;\; V_\star(1-\epsilon)t\right)}\\
&\qquad\qquad \times \mathbb{E}\left(\|Y_T - Y'_T\|^{\alpha'\beta} \mid (Y_0, Y'_0) = (y, y'), (X_0, X'_0) = (x, x')\right)^{1/\alpha'}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\mathbb{E}\left(\|Y_t - Y'_t\|^{\beta} \;\; 1_{T\leq \epsilon t} \mid (Y_0, Y'_0) = (y, y'), (X_0, X'_0) = (x, x')\right)\\
&\leq 2 \;\; \exp\left(-\beta V_\star(1-\epsilon)t\right) \;\; \left[\|y\|^{\beta} \vee \|y'\|^{\beta} \vee (a^\star/V_\star)^{\beta}\right].
\end{aligned}$$

On the other hand, for any conjugate parameters $1 \leq \alpha, \alpha' \leq \infty$ we also have

$$\begin{aligned}
&\mathbb{E}\left(\|Y_t - Y'_t\|^{\beta} \;\; 1_{T> \epsilon t} \mid (Y_0, Y'_0) = (y, y'), (X_0, X'_0) = (x, x')\right)\\
&= \mathbb{E}\left(\|Y_t - Y'_t\|^{\beta\alpha'} \mid (Y_0, Y'_0) = (y, y'), (X_0, X'_0) = (x, x')\right)^{1/\alpha'}\\
&\qquad\qquad \times \mathbb{P}\left(T > \epsilon t \mid (X_0, X'_0) = (x, x')\right)^{1/\alpha}.
\end{aligned}$$

By theorem theorem 12.7.6 there exists some non negative parameters c and $\rho > 0$ s.t.

$$\mathbb{P}\left(T > \epsilon t \mid (X_0, X_0') = (x, x')\right) \leq c\ \exp\left(-\rho\epsilon t\right).$$

This implies that

$$\mathbb{E}\left(\|Y_t - Y_t'\|^\beta\ 1_{T>\epsilon t} \mid (Y_0, Y_0') = (y, y'), (X_0, X_0') = (x, x')\right)$$

$$= 2c\ \exp\left(-\rho\epsilon t/\alpha\right)\left[\|y\|^\beta \vee \|y'\|^\beta \vee (a^\star/V_\star)^\beta\right].$$

By choosing $\alpha = 1$ we conclude that

$$\mathbb{E}\left(\|Y_t - Y_t'\|^\beta\ \mid (Y_0, Y_0') = (y, y'), (X_0, X_0') = (x, x')\right)$$

$$\leq 2\left[\ \exp\left(-\beta V_\star(1-\epsilon)t\right) + c\ \exp\left(-\rho\epsilon t\right)\right]\left[\|y\|^\beta \vee \|y'\|^\beta \vee (a^\star/V_\star)^\beta\right].$$

Notice that

$$\beta V_\star(1-\epsilon) = \rho\epsilon \Leftrightarrow \epsilon = \delta_\beta := \beta V_\star / \left(\beta V_\star + \rho\right)$$

from which we conclude that

$$\mathbb{E}\left(\|Y_t - Y_t'\|^\beta\ \mid (Y_0, Y_0') = (y, y'), (X_0, X_0') = (x, x')\right)$$

$$\leq 2(1+c)\ \exp\left(-\delta_\beta t\right)\left[\|y\|^\beta \vee \|y'\|^\beta \vee (a^\star/V_\star)^\beta\right].$$

This ends the proof of the exercise.

∎

Solution to exercise 208:

Between the jumps X_t evolves as $\dot{X}_t = 1$ so that the semigroup of the deterministic flow $\varphi_{s,t}(x)$ is given by

$$\forall x \in \mathbb{R} \quad \forall s \leq t \qquad \varphi_{s,t}(x) = x + (t-s).$$

By applying the integral formula (13.15) to $K_u(x, dy) = \delta_0(dy)$ and $\lambda_u(x) = \lambda$ we have

$$P_{s,t}(f)(0) = f(t-s)\ e^{-\lambda(t-s)} + \int_s^t \lambda\ e^{-\lambda\ (u-s)}\ P_{u,t}(f)(0)\ du.$$

The semigroup is time homogenous $P_{s,t}(f)(x) = P_{0,t-s}(f)(x)$, so if we set $P_t = P_{0,t}$ we find that

$$P_t(f)(0) = f(t)\ e^{-\lambda t} + \int_0^t \lambda\ e^{-\lambda(t-s)}\ P_s(f)(0)\ ds.$$

This yields the formula

$$I_t(f) := e^{\lambda t} P_t(f)(0) = f(t) + \int_0^t \lambda\ I_s(f)\ ds$$

from which we conclude that

$$\begin{aligned}
I_t(f) &= f(t) + \lambda \int_0^t I_s(f)\ ds \\
&= f(t) + \lambda \int_0^t f(s)\ ds + \lambda^2 \int_0^t \int_0^s I_r(f)\ dr \\
&= \ldots \\
&= \sum_{n\geq 0} \frac{(\lambda t)^n}{n!}\ U_n(f)(t),
\end{aligned}$$

with

$$U_0(f)(t) = f(t) \quad \text{and} \quad U_{n+1}(f)(t) = \frac{n+1}{t} \int_0^t U_n(f)(s)\, ds.$$

In other words, we have

$$\begin{array}{rcl} U_1(f)(t) & = & \frac{1}{t} \int_0^t f(s)\, ds \\ U_2(f)(t) & = & \frac{2}{t^2} \int_0^t \left(\int_0^{s_2} f(s_1)\, ds_1 \right) ds_2 \\ & \vdots & \\ U_n(f)(t) & = & \frac{n!}{t^n} \int_0^t \int_0^{s_n} \ldots \left[\int_0^{s_2} f(s_1)\, ds_1 \right] ds_2 \ldots ds_n. \end{array}$$

We let $(S_t^{k,n})_{1 \leq k \leq n}$ be an n-ordered uniform statistics on $[0,t]$

$$0 \leq S_t^{1,n} \leq S_t^{2,n} \leq \ldots \leq S_t^{n,n} \leq t \implies U_n(f)(t) = \mathbb{E}(f(S_t^{1,n}))$$

We conclude that

$$P_t(f)(0) = e^{-\lambda t} \sum_{n \geq 0} \frac{(\lambda t)^n}{n!}\, \mathbb{E}(f(S_t^{1,n})) \implies X_t = S_t^{1,N_t}.$$

This ends the proof of the first part of the exercise. Next, we consider the PDMP X_t with generator

$$L_t(f)(x) = \lambda\, (f(0) - f(x)) \; + \; b_t(x)\, f'(x)$$

for some some smooth and bounded drift function b_t. We let $x_t = \varphi_{s,t}(x)$ be the deterministic flow map of the deterministic system starting at $x_s = x$ and defined for any $t \in [s, \infty[$ by the dynamical equations

$$\left\{ \begin{array}{rcl} \dot{x}_t & = & b_t(x_t) \\ x_s & = & x. \end{array} \right.$$

Applying the integral formula (13.15) to $K_u(x, dy) = \delta_0(dy)$ and $\lambda_u(x) = \lambda$ we have

$$I_{s,t}(f) := e^{\lambda(t-s)} P_{s,t}(f)(0) = f(\varphi_{s,t}(0)) \; + \int_s^t \lambda\, I_{u,t}(f)\, du.$$

Arguing as above, we find that

$$I_{s,t}(f) = \sum_{n \geq 0} \frac{(\lambda(t-s))^n}{n!} \underbrace{\frac{n!}{(t-s)^n} \int_s^t \int_{s_1}^t \ldots \left[\int_{s_{n-1}}^t f(\varphi_{s_n,t}(0))\, ds_n \right] ds_{n-1} \ldots ds_1}_{= \mathbb{E}\left(f\left(\varphi_{S_{s,t}^{n,n},t}(0) \right) \right)}$$

with an n-ordered uniform statistics $(S_{s,t}^{k,n})_{1 \leq k \leq n}$ on $[s,t]$

$$s \leq S_{s,t}^{1,n} \leq S_{s,t}^{2,n} \leq \ldots \leq S_{s,t}^{n,n} \leq t.$$

This shows that

$$X_s = 0 \implies X_t = \varphi_{S_{s,t}^{N_{t-s},N_{t-s}},t}(0),$$

where N_t stands for a Poisson random variable with parameter λt. This ends the proof of the exercise.

■

Solution to exercise 209:

At some rate, say $\lambda(X_t)$ the process $X_t \in \{-1,1\}$ changes its sign. Between the jumps the process evolves according to

$$\left\{\begin{array}{rcl} dX_t &=& 0 \\ dY_t &=& a(Y_t) + X_t\ b(Y_t)\ dt. \end{array}\right.$$

We conclude that the generator of $Z_t = (X_t, Y_t)$ is defined by

$$L_t(f)(x,y) = (a(y) + xb(y))\ \partial_y f(x,y) + \lambda(x)\ \ (f(-x,y) - f(x,y))\,.$$

Assume that (X_t, Y_t) has a density given by

$$\forall x \in \{-1,1\} \qquad \mathbb{P}(X_t = x\ ,\ Y_t \in dy) = p_t(x,y)\ dy$$

where dy stands for the Lebesgue measure on $\mathbb{R}$. Notice that

$$\begin{array}{rcl} \displaystyle\sum_{x\in\{-1,1\}} \int_{\mathbb{R}} p_t(x,y)\ L^d(f)(x,y)\ dy &=& \displaystyle\int \lambda(1) \int_{\mathbb{R}} (f(-1,y) - f(1,y))\ \ p_t(1,y)\ dy \\ && \displaystyle +\int \lambda(-1) \int_{\mathbb{R}} (f(1,y) - f(-1,y))\ \ p_t(-1,y)\ dy. \end{array}$$

Choosing $f(x,y) = 1_{x=1}\ g(y)$ we find that

$$\sum_{x\in\{-1,1\}} \int_{\mathbb{R}} p_t(x,y)\ L^d(f)(x,y)\ dy = \int g(y)\ \ (\lambda(-1)\ p_t(-1,y) - \lambda(1)p_t(1,y))\ \ dy.$$

In this situation, for any smooth function g with compact support we have

$$\begin{array}{rcl} \displaystyle\sum_{x\in\{-1,1\}} \int_{\mathbb{R}} p_t(x,y)\ L^c_t(f)(x,y)\ dy &=& \displaystyle\int_{\mathbb{R}} p_t(1,y)\ (a(y) + b(y))\partial_y(g)(y)\ dy \\ &=& \displaystyle -\int_{\mathbb{R}} g(y)\ \partial_y\left(p_t(1,y)\ (a(y)+b(y))\right)\ dy. \end{array}$$

This implies that

$$\partial_t p_t(1,y) = (\lambda(-1)\ p_t(-1,y) - \lambda(1)p_t(1,y)) - \partial_y\left(p_t(1,y)\ (a(y)+b(y))\right).$$

In much the same way, by choosing $f(x,y) = 1_{x=-1}\ g(y)$ we find that

$$\sum_{x\in\{-1,1\}} \int_{\mathbb{R}} p_t(x,y)\ L^d(f)(x,y)\ dy = \int_{\mathbb{R}} g(y)\ \ (\lambda(1)p_t(1,y) - \lambda(-1)\ p_t(-1,y))\ \ dy.$$

In this situation, for any smooth function g with compact support we have

$$\begin{array}{rcl} \displaystyle\sum_{x\in\{-1,1\}} \int_{\mathbb{R}} p_t(x,y)\ L^c_t(f)(x,y)\ dy &=& \displaystyle\int_{\mathbb{R}} p_t(-1,y)\ (a(y) - b(y))\partial_y(g)(y)\ dy \\ &=& \displaystyle -\int_{\mathbb{R}} g(y)\ \partial_y\left(p_t(-1,y)\ (a(y)-b(y))\right)\ dy. \end{array}$$

This implies that

$$\partial_t p_t(-1,y) = (\lambda(1)p_t(1,y) - \lambda(-1)\ p_t(-1,y)) - \partial_y\left(p_t(-1,y)\ (a(y)-b(y))\right).$$

We set

$$\begin{array}{rclcrcl} q_t^+(y) &=& p_t(1,y)+p_t(-1,y) && q_t^-(y) &=& p_t(1,y)-p_t(-1,y)\\ \lambda^+ &=& \lambda(1)+\lambda(-1) && \lambda^- &=& \lambda(1)-\lambda(-1). \end{array}$$

By construction, we have

$$\begin{array}{rcl} \partial_t q_t^+(y) &=& (\lambda(-1)\ p_t(-1,y) - \lambda(1)p_t(1,y)) - \partial_y\left(p_t(1,y)\ (a(y)+b(y))\right)\\ && \quad +(\lambda(1)p_t(1,y) - \lambda(-1)\ p_t(-1,y)) - \partial_y\left(p_t(-1,y)\ (a(y)-b(y))\right)\\ &=& -\partial_y\left(q_t^+(y)\ a(y) + q_t^-(y)\ b(y)\right) \end{array}$$

and

$$\begin{array}{rcl} \partial_t q_t^-(y) &=& 2\left(\lambda(-1)\ p_t(-1,y) - \lambda(1)p_t(1,y)\right)\\ && -\partial_y\left(p_t(1,y)\ (a(y)+b(y))\right) - \partial_y\left(p_t(-1,y)\ (b(y)-a(y))\right)\\ &=& -\left(\lambda(1)-\lambda(-1)\right)\ q_t^+(y) - \left(\lambda(1)+\lambda(-1)\right)\ q_t^-(y)\\ && \qquad\qquad -\partial_y\left(q_t^-(y)\ a(y) + q_t^+(y)\ b(y)\right)\\ &=& -\left(\lambda^-\ q_t^+(y) + \lambda^+\ q_t^-(y)\right) - \partial_y\left(q_t^-(y)\ a(y) + q_t^+(y)\ b(y)\right). \end{array}$$

We further assume that $b > |a|$. The steady state $(q^-(y), q^+(y))$ of these coupled equations satisfies

$$\partial_t q^+(y) = 0 = \partial_t q^-(y).$$

This implies that

$$q^+(y)\ a(y) + q^-(y)\ b(y) = c\ \Rightarrow\ q^-(y) = -\frac{a(y)}{b(y)}\ q^+(y) + \frac{c}{b(y)}$$

for some constant c, and

$$\partial_y\left(q^-(y)\ a(y) + q^+(y)\ b(y)\right) = -\left(\lambda^-\ q^+(y) + \lambda^+\ q^-(y)\right).$$

To solve this system of equations we observe that

$$q^+(y)\ b(y) + q^-(y)\ a(y) = \left[q^+(y)b(y)\ \left(1-\left(\frac{a(y)}{b(y)}\right)^2\right) + c\ \frac{a(y)}{b(y)}\right]$$

and

$$\begin{array}{rcl} \lambda^-\ q^+(y) + \lambda^+\ q^-(y) &=& \lambda^-\ q^+(y) - \lambda^+\ \left(\dfrac{a(y)}{b(y)}\ q^+(y) + \dfrac{c}{b(y)}\right)\\ &=& q^+(y)\ \left(\lambda^- - \lambda^+\dfrac{a(y)}{b(y)}\right) - \dfrac{\lambda^+}{b(y)}\ c. \end{array}$$

This implies that

$$\partial_y\left(q^+(y)\frac{(b^2(y)-a^2(y))}{b(y)} + c\ \frac{a(y)}{b(y)}\right) = q^+(y)\frac{\lambda^+ a(y) - \lambda^- b(y)}{b(y)}\ + \frac{\lambda^+}{b(y)}\ c$$

or equivalently

$$\frac{\left(b^2(y)-a^2(y)\right)}{b(y)}\,\partial_y q^+(y) = q^+(y)\left[\frac{\lambda^+ a(y)-\lambda^- b(y)}{b(y)} - \partial_y\left(\frac{\left(b^2(y)-a^2(y)\right)}{b(y)}\right)\right] + c\left[\frac{\lambda^+}{b(y)} - \partial_y\left(\frac{a(y)}{b(y)}\right)\right].$$

This yields the familiar ordinary differential equation

$$\partial_y q^+(y) = q^+(y)\ A(y) + c\ B(y)$$

with

$$A(y) := \frac{\lambda^+ a(y)-\lambda^- b(y)}{b^2(y)-a^2(y)} + \partial_y \log\frac{b(y)}{b^2(y)-a^2(y)} \quad \text{and} \quad B(y) = \frac{\lambda^+ - b(y)\ \partial_y\left(\frac{a(y)}{b(y)}\right)}{b^2(y)-a^2(y)}.$$

The solution of the above system is given by

$$q^+(y) = e^{\int_{c_1}^y A(z)\ dz}\ c_2 + c\int_{c_1}^y\ e^{\int_z^y A(z)\ dz}\ B(y)\ dy$$

for some constants c_1, c_2. Whenever $c_3 := \int_{c_1}^\infty A(z)\ dz$ we can choose $c = 0$.

For instance, when $\lambda(1) = \lambda(-1) := \lambda$ we have $\lambda^+ = 2\lambda$ and $\lambda^- = 0$. In addition, when $a = 0$ and $0 < \int_{c_1}^\infty b^{-1}(y)dy < \infty$ we have

$$A(y) := -\partial_y \log b(y) \quad \text{and} \quad B(y) = 2\lambda/b^2(y).$$

In this situation, by choosing $c = 0$ we have

$$q^- = 0 \quad \text{and} \quad q^+(y) = \frac{b(c_1)}{b(y)}\ c_2.$$

In this case we have

$$\int q_t^+(y)\ dy = \int\ (p_t(1,y)+p_t(-1,y))\ dy = 1 \Rightarrow c_2 = \frac{1}{b(c_1)}\ \left(\int_{c_1}^\infty b^{-1}(y)dy\right)^{-1}.$$

This ends the proof of the exercise. ■

Solution to exercise 210:

Under appropriate regularity conditions that allow us to interchange the order of differentiation and integration this yields

$$\partial_t p_t^1(x^1) = -\sum_{1\leq i\leq r_1} \partial_{x_i^1}\int a_t^i(x^1,x^2)p_t(x^1,x^2)\ \nu(dx^2).$$

In addition, for compactly supported drift functions a_t^i w.r.t. the first coordinate we also have the equation

$$\partial_t p_t^2(x^2) = \int \lambda_t(x^1,y^2)\ p_t(x^1,y^2)\ dx^1\ \nu(dy^2) - \int\ \lambda_t(x^1,x^2)\ p_t(x^1,x^2)\ dx^1.$$

This ends the proof of the exercise.

■

Solution to exercise 211:

By construction, the evolution equation of the number of molecules X_t^i of the i-th species is given by

$$X_t^i = X_0^i + \sum_{1\leq j\leq n_s} \left(b'_{i,j} - b_{i,j}\right)\ N_j\left(\int_0^t \lambda_j\left(X_s\right) ds\right)$$

where $(N_j)_{1\leq j\leq n_c}$ stands for n_c independent Poisson processes with unit intensity.

This ends the proof of the exercise.

■

Solution to exercise 212:

The following formula describes the process $X_t = (X_t^1, X_t^2)$ in terms of 4 independent Poisson processes $(N_j)_{1\leq j\leq 4}$ with unit intensity

$$\left\{\begin{array}{rcl} X_t^1 &=& X_0^1 + N_1\left(\int_0^t\ 1_{X_s^1=0}\lambda_1\left(X_s^2\right) ds\right) - N_2\left(\int_0^t\ 1_{X_s^1=1}\lambda_2\left(X_s^2\right) ds\right)\\ X_t^2 &=& X_0^2 + N_3\left(\int_0^t\ 1_{X_s^1=0}\lambda_3\left(X_s^2\right) ds\right) - N_4\left(\int_0^t\ 1_{X_s^1=1}\lambda_4\left(X_s^2\right) ds\right). \end{array}\right.$$

This ends the proof of the exercise.

■

Solution to exercise 213:

Following the analysis of switching processes developed in section 13.3.1 we have

$$L(f)(u,v) = a(u,v)\ \partial_u(f)(u,v) + \lambda(u,v)\ \int_{\mathbb{S}}\ (f(u,w) - f(u,v))\ \ K((u,v),dw).$$

This ends the proof of the exercise.

■

Solution to exercise 214: Following the stability analysis of switching processes developed in the end of section 13.3.1 we can choose any intensity function satisfying the following condition:

$$\lambda(u,v)\ q(u) = \lambda^\star(u) - \sum_{1\leq i\leq 2} a^i(v)\ \partial_{u^i}(q)(u) \geq 0$$

for some sufficiently large function $\lambda^\star$ such that $\lambda^\star(u) \geq \sum_{1\leq i\leq 2}\ \|a^i\|\ \ |\partial_{u^i}(q)(u)|$, for any $u = (u^1, u^2)$.

This ends the proof of the exercise.

■

Solution to exercise 215:

We have

$$\sum_{\epsilon\in\{-1,+1\}}\int e^{-(b-a)|y|}L(f)(\epsilon,y)\ dy$$

$$= \sum_{\epsilon\in\{-1,+1\}}\int_0^\infty e^{-(b-a)y}\left(\epsilon\ \partial_y f(\epsilon,y) + (a+(b-a)\ 1_{\epsilon\geq 0})\ \ (f(-\epsilon,y)-f(\epsilon,y))\right)\ dy$$
$$+ \sum_{\epsilon\in\{-1,+1\}}\int_{-\infty}^0 e^{(b-a)y}\left(\epsilon\ \partial_y f(\epsilon,y) + (a+(b-a)\ 1_{\epsilon\leq 0})\ \ (f(-\epsilon,y)-f(\epsilon,y))\right)\ dy.$$

In addition, we have the decompositions

$$\sum_{\epsilon\in\{-1,+1\}}\int_0^\infty e^{-(b-a)y}\left(\epsilon\ \partial_y f(\epsilon,y) + (a+(b-a)\ 1_{\epsilon\geq 0})\ \ (f(-\epsilon,y)-f(\epsilon,y))\right)\ dy$$

$$= \int_0^\infty e^{-(b-a)y}\left(\partial_y f(1,y) + b\ (f(-1,y)-f(1,y))\right)\ dy$$
$$+ \int_0^\infty e^{-(b-a)y}\left(-\partial_y f(-1,y) + a\ (f(1,y)-f(-1,y))\right)\ dy.$$

Observe that

$$\int_0^\infty e^{-(b-a)y}\left[(b\ (f(-1,y)-f(1,y))) + (a\ (f(1,y)-f(-1,y)))\right]\ dy$$

$$= \int_0^\infty (b-a)\ e^{-(b-a)y}\ \ (f(-1,y)-f(1,y))\ dy.$$

A simple integration by parts shows that

$$\int_0^\infty e^{-(b-a)y}(\partial_y(f(1,y)-f(-1,y)))\ dy$$

$$= [f(-1,0)-f(+1,0)] + \int_0^\infty (b-a)\ e^{-(b-a)y}\,(f(1,y)-f(-1,y))\ dy.$$

We conclude that

$$\sum_{\epsilon\in\{-1,+1\}}\int_0^\infty e^{-(b-a)y}\left(\epsilon\ \partial_y f(\epsilon,y) + (a+(b-a)\ 1_{\epsilon\geq 0})\ \ (f(-\epsilon,y)-f(\epsilon,y))\right)\ dy$$

$$= [f(-1,0)-f(1,0)]\,.$$

In much the same way, we have

$$\sum_{\epsilon\in\{-1,+1\}}\int_{-\infty}^0 e^{(b-a)y}\left(\epsilon\ \partial_y f(\epsilon,y) + (a+(b-a)\ 1_{\epsilon\leq 0})\ \ (f(-\epsilon,y)-f(\epsilon,y))\right)\ dy$$

$$= \int_{-\infty}^0 e^{(b-a)y}\left(\partial_y f(1,y) + a\ (f(-1,y)-f(1,y))\right)\ dy$$
$$+ \int_{-\infty}^0 e^{(b-a)y}\left(-\partial_y f(-1,y) + b\ (f(1,y)-f(-1,y))\right)\ dy.$$

Arguing as above we find that

$$\int_{-\infty}^{0} e^{(b-a)y}(\partial_y(f(1,y)-f(-1,y)))\ dy$$

$$= [f(1,0)-f(-1,0)] + \int_{0}^{\infty} (b-a)\ e^{(b-a)y}\,(f(-1,y)-f(1,y))\ dy$$

and

$$\int_{-\infty}^{0} e^{(b-a)y}\,[(b\ (f(1,y)-f(-1,y))) + (a\ (f(-1,y)-f(1,y)))]\ dy$$

$$= \int_{-\infty}^{0} (b-a)\ e^{(b-a)y}\ \ (f(1,y)-f(-1,y))\ dy.$$

This shows that

$$\sum_{\epsilon\in\{-1,+1\}} \int_{-\infty}^{0} e^{(b-a)y}\,(\epsilon\ \partial_y f(\epsilon,y) + (a+(b-a)\ 1_{\epsilon\leq 0})\ \ (f(-\epsilon,y)-f(\epsilon,y)))\ dy$$

$$= [f(1,0)-f(-1,0)]\,.$$

This implies that $\pi L(f)=0$. This ends the proof of the exercise.

■

Solution to exercise 216:
Using a simple integration by parts we have

$$\sum_{\epsilon\in\{-1,+1\}} \int e^{-V(y)}L(f)(\epsilon,y)\ dy$$

$$= \sum_{\epsilon\in\{-1,+1\}} \int e^{-V(y)}\epsilon\ \partial_y f(\epsilon,y)\ dy$$

$$+ \sum_{\epsilon\in\{-1,+1\}} \int e^{-V(y)}\ \ (\epsilon\partial_y V(y))_+\ (f(-\epsilon,y)-f(\epsilon,y))\ dy$$

$$= \sum_{\epsilon\in\{-1,+1\}} \int e^{-V(y)}(\epsilon\ \partial_y V)\ f(\epsilon,y)\ dy$$

$$+ \sum_{\epsilon\in\{-1,+1\}} \int e^{-V(y)}\ \ [(-\epsilon\partial_y V(y))_+ - (\epsilon\partial_y V(y))_+]\ \ f(\epsilon,y))\ dy.$$

Recalling that $(-a)_+ = a_-$, we conclude that

$$\sum_{\epsilon\in\{-1,+1\}} \int e^{-V(y)}L(f)(\epsilon,y)\ dy$$

$$= \sum_{\epsilon\in\{-1,+1\}} \int e^{-V(y)}\ \underbrace{\{(\epsilon\ \partial_y V) - [(\epsilon\partial_y V(y))_+ - (\epsilon\partial_y V(y))_-]\}}_{=0}\ f(\epsilon,y))\ dy.$$

This shows that $\pi L = 0$. This ends the proof of the exercise.

■

Solution to exercise 217:
An integration by part w.r.t. the x-coordinate gives

$$\begin{aligned}
&\int a\ v\ \partial_x f(v,x)\ e^{-U(x)}\ dx\ \mu(dv)\\
&= -\int\ \left[\left(a\ v\ e^{U(x)}\partial_x e^{-U(x)}\right)\right]\ f(v,x)\ e^{-U(x)}\ dx\ \mu(dv)\\
&= -\int\ \left[\left(a\ v\ e^{U(x)}\partial_x e^{-U(x)}\right)\right]_+\ f(v,x)\ e^{-U(x)}\ dx\ \mu(dv)\\
&\qquad\qquad +\int\ \left[\left(a\ v\ e^{U(x)}\partial_x e^{-U(x)}\right)\right]_-\ f(v,x)\ e^{-U(x)}\ dx\ \mu(dv)
\end{aligned}$$

for any smooth function f wit compact support.

We also have

$$\begin{aligned}
&\int\ e^{-U(x)}dx\ \mu(dv)\ \left(a\ v\ e^{U(x)}\partial_x e^{-U(x)}\right)_-\ (f(-v,x)-f(v,x))\\
&= -\int\ \left[\left(a\ v\ e^{U(x)}\partial_x e^{-U(x)}\right)\right]_-\ f(v,x)\ e^{-U(x)}\ dx\ \mu(dv)\\
&+\int\ e^{-U(x)}dx\ \mu(dv)\ \left(a\ v\ e^{U(x)}\partial_x e^{-U(x)}\right)_-\ f(-v,x)\\
&= -\int\ \left[\left(a\ v\ e^{U(x)}\partial_x e^{-U(x)}\right)\right]_-\ f(v,x)\ e^{-U(x)}\ dx\ \mu(dv)\\
&+\int\ e^{-U(x)}dx\ \mu(dv)\ \underbrace{\left(-a\ v\ e^{U(x)}\partial_x e^{-U(x)}\right)_-}_{\left(a\ v\ e^{U(x)}\partial_x e^{-U(x)}\right)_+}\ f(v,x).
\end{aligned}$$

We conclude that π is L-invariant. Recalling that $(-a)_- = a_+$, we check immediately that this stochastic process coincides with the one discussed in exercise 216 when $a = 1$ and $\mu(dv) \propto \delta_{-1} + \delta_{+1}$. When $a = 1$ their generators coincide. The non uniqueness property of the invariant measure is clear since $\mu(dv)$ represents any symmetric probability measure.

This ends the proof of the exercise.

■

Solution to exercise 218:
An integration by part w.r.t. the x-coordinate gives

$$\begin{aligned}
&\int\ a(v)\ \partial_x f(v,x)\ e^{-U(x)}\ dx\ \mu(dv)\\
&= -\int\ \left(a(v)\ e^{U(x)}\partial_x e^{-U(x)}\right)\ f(v,x)\ e^{-U(x)}\ dx\ \mu(dv)
\end{aligned}$$

for any smooth function f wit compact support. On the other hand we have

$$\begin{aligned}
&\int e^{-U(x)}dx\ \mu(dv)\ \lambda(v,x)\ \int\ (f(v,x)-f(v,x))\ \mu(dv)\\
&= -\int\ e^{-U(x)}dx\ \mu(dv)\ \left(a(v)\ e^{U(x)}\partial_x e^{-U(x)}\right)\ \int\ (f(w,x)-f(v,x))\ \mu(dw)\\
&+\int\ e^{-U(x)}dx\ \left[\alpha(x)+\sup_{w\in\mathbb{R}}\left(a(w)\ e^{U(x)}\partial_x e^{-U(x)}\right)\right]\\
&\qquad\qquad\times\underbrace{\left[\int\ \mu(dv)\ \int\ (f(w,x)-f(v,x))\ \mu(dw)\right]}_{=0}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\int\ e^{-U(x)}dx\ \mu(dv)\ \lambda(v,x)\ \int\ (f(v,x)-f(v,x))\ \mu(dv)\\
&=\int\ e^{-U(x)}dx\ \mu(dv)\ \left(a(v)\ e^{U(x)}\partial_x e^{-U(x)}\right)\ f(v,x)\\
&-\int\ e^{-U(x)}dx\ \underbrace{\left[\int \mu(dv)\ a(v)\right]}_{=0}\ \left(e^{U(x)}\ \partial_x e^{-U(x)}\right)\ \int\ (w,x)\ \mu(dw).
\end{aligned}$$

We conclude that π is L-invariant. The function

$$x\mapsto\alpha_-(x):=\sup_{w\in\mathbb{R}}\left(a(w)\ e^{U(x)}\partial_x e^{-U(x)}\right)$$

is difficult to compute in practical situations. If we only have an upper bound

$$\sup_{w\in\mathbb{R}}\left(a(w)\ e^{U(x)}\partial_x e^{-U(x)}\right)\leq\alpha_+(x)$$

we can choose

$$\alpha(x)=\alpha_+(x)-\alpha_-(x).$$

In this case we have

$$\lambda(v,x)=\alpha(x)+\alpha_-(x)-\left(a(v)\ e^{U(x)}\partial_x e^{-U(x)}\right)=\alpha_+(x)-\left(a(v)\ e^{U(x)}\partial_x e^{-U(x)}\right).$$

This ends the proof of the exercise.

■

Solution to exercise 219:
We clearly have

$$A(x)'=(I-2U(x)U(x)')'=I-2\left(U(x)U(x)'\right)'=I-2U(x)U(x)'$$

and

$$A(x)^2=(I-2U(x)U(x)')^2=I-4U(x)U(x)'+4U(x)\underbrace{U(x)'U(x)}_{=\|U(x)\|^2=1}U(x)'=I.$$

This shows that $A(x)$ is an orthogonal matrix. For any $x, y \in \mathbb{R}^r$, we also have

$$\begin{aligned}\langle U(x), A(x)y\rangle &= \langle U(x), (I - 2U(x)U(x)')\, y\rangle \\ &= \langle U(x), y\rangle - 2\langle U(x), U(x)U(x)'y\rangle \\ &= \langle U(x), y\rangle - 2\ \underbrace{U(x)'U(x)}_{=1} U(x)'y = -\langle U(x), y\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}\langle U(x), A(x)y\rangle_+ &:= \max\left(\langle U(x), A(x)y\rangle, 0\right) \\ &= \max\left(-\langle U(x), y\rangle, 0\right) = -\min\left(\langle U(x), y\rangle, 0\right) = \langle U(x), y\rangle_-\end{aligned}$$

from which we conclude that

$$\begin{aligned}\langle \partial V(x), A(x)y\rangle_+ &= \|\partial V(x)\|\ \langle U(x), A(x)y\rangle_+ \\ &= \|\partial V(x)\|\ \langle U(x), y\rangle_- = \langle \partial V(x), y\rangle_-\end{aligned}$$

and

$$\langle \partial V(x), A(x)y\rangle_+ - \langle \partial V(x), y\rangle_+ = -\langle \partial V(x), y\rangle.$$

After dividing by $\|\partial V(x)\|$ we also have

$$\langle U(x), A(x)y\rangle_+ - \langle U(x), y\rangle_+ = -\langle U(x), y\rangle.$$

The infinitesimal generator of the process X_t is given by

$$L(f)(x^1, x^2) = \sum_{1\leq i\leq r} x_i^2\ \partial_{x_i^1} f(x) + \lambda(x)\ \left[f\left(x^1, A(x^1)x^2\right) - f\left(x\right)\right].$$

Observe that

$$\int \pi(dx)\ \lambda(x)\ f\left(x^1, A(x^1)x^2\right) \quad \propto \quad \int e^{-V(x^1)}\ dx^1\ \left[\int \nu(dx^2)\ \langle \partial V(x^1), x^2\rangle_+ f\left(x^1, A(x^1)x^2\right)\right].$$

Using the change of variable

$$y^2 = A(x^1)x^2 \Rightarrow x^2 = A(x^1)y^2$$

and recalling that ν is spherically symmetric we check that

$$\int \nu(dx^2)\ \langle \partial V(x^1), x^2\rangle_+\ f\left(x^1, A(x^1)x^2\right) \quad = \quad \int \nu(dy^2)\ \langle \partial V(x^1), A(x^1)y^2\rangle_+\ f\left(x^1, y^2\right).$$

This implies that

$$\begin{aligned}&\int e^{-V(x^1)}\ dx^1 \int \nu(dx^2)\ \langle \partial V(x^1), x^2\rangle_+\ \left[f\left(x^1, A(x^1)x^2\right) - f(x)\right] \\ &= \int e^{-V(x^1)}\ dx^1 \int \nu(dy^2)\ \left[\langle \partial V(x^1), A(x^1)y^2\rangle_+ - \langle \partial V(x^1), y^2\rangle_+\right]\ f\left(x^1, y^2\right) \\ &= -\int e^{-V(x^1)}\ dx^1 \int \nu(dy^2)\ \langle \partial V(x^1), y^2\rangle\ f(x^1, y^2).\end{aligned}$$

By a simple integration by parts, we also have

$$\int e^{-V(x^1)}\, dx^1 \int \nu(dx^2) \sum_{1\leq i\leq r} x_i^2\, \partial_{x_i^1} f(x)$$

$$= \int e^{-V(x^1)}\, dx^1 \int \nu(dx^2) \left(\sum_{1\leq i\leq r} x_i^2\, \partial_{x_i^1} V(x^1)\right) f(x)$$

$$\propto \int \pi(dx)\, \langle \partial V(x^1), x^2\rangle\, f(x)$$

for any smooth function f with compact support. This clearly implies that $\pi L = 0$

Notice that the invariant distribution is not unique as any distribution π with spherically symmetric distribution ν satisfies $\pi L = 0$. We can choose the centered product Gaussian distribution, student distributions, or Laplace type distributions.

This ends the proof of the exercise. ■

Chapter 14

Solution to exercise 220:

We recall that the increments $(W_t - W_s)$ and $(W_s - W_0) = W_s$ are independent centered Gaussian random variables with variance $(t-s)$ and s. Recalling W_t and W_s are centered Gaussian with variance s and t, we prove that

$$\mathrm{Cov}(W_s, W_t) = \mathbb{E}(W_s W_t) = \mathbb{E}(\ W_s\ [W_s + (W_t - W_s)]\ ) = \mathbb{E}(W_s^2) = s.$$

This ends the proof of the exercise.

■

Solution to exercise 221:

For any $s \leq t$ we have

$$W_t - W_s = \sum_{i \in I} a_i\ (W_t^i - W_s^i).$$

The process W_t is clearly a martingale w.r.t. $\mathcal{F}_t = \sigma\left(W_s^i,\ i \in I,\ s \leq t\right)$, with Gaussian independent increments. Suppose that $\sum_{i \in I} a_i^2 = 1$. In this case, we have

$$\mathbb{E}\left(W_t^2\right) = \sum_{i \in I} a_i^2\ t = t.$$

Inversely, we have

$$\mathbb{E}\left(W_t^2\right) = \sum_{i \in I} a_i^2\ t = t \Rightarrow \sum_{i \in I} a_i^2 = 1.$$

This ends the proof of the exercise.

■

Solution to exercise 222:

For any $s \leq t$, we have

$$\begin{aligned}
\mathrm{Cov}(W_s^1, W_t) &= \mathbb{E}\left(W_s^1 W_t\right) = \mathbb{E}\left(W_s^1 W_s\right) \\
&= \mathbb{E}\left(W_s^1 \left[\epsilon W_s^1 + \sqrt{1-\epsilon^2}\ W_s^2\right]\right) = \epsilon\ \mathbb{E}\left(W_s^1 W_s^1\right) = \epsilon\ s.
\end{aligned}$$

In much the same way, we have

$$\mathrm{Cov}(W_t^1, W_s) \quad = \quad \mathbb{E}\left(W_s^1 W_s\right) = \epsilon\ s.$$

This implies that

$$\mathrm{Cov}(W_t^1, W_s) = \epsilon\ \ (t \wedge s)\,.$$

This ends the proof of the exercise.

■

Solution to exercise 223:
We have

$$X_{t_k} - X_{t_{k-1}} = b\,\epsilon \;+\; \sigma\,\sqrt{\epsilon}\;\times\;\left[\sqrt{\epsilon}^{-1}\;\left(W_{t_k} - W_{t_{k-1}}\right)\right].$$

The random variables $\left(W_{t_k} - W_{t_{k-1}}\right)/\sqrt{\epsilon}$ are independent centered and Gaussian with unit variance. Thus the conditional densities of the sequence of state $(X_{t_1}, \ldots, X_{t_n})$ given $X_0 = x_0$ are given by

$$\begin{aligned}
p_{t_1,\ldots,t_n}(x_1,\ldots,x_n \mid x_0) \quad &\propto \quad \exp\left[-\frac{1}{2\sigma^2\epsilon}\sum_{1\leq k\leq n}\left((x_k - x_{k-1}) - b\epsilon\right)^2\right] \\
&= \quad \exp\left[-\frac{1}{2\sigma^2\epsilon}\sum_{1\leq k\leq n}\left(x_k - x_{k-1}\right)^2\right] \\
&\qquad \times \exp\left[\frac{b}{\sigma^2}\sum_{1\leq k\leq n}(x_k - x_{k-1})\right]\;\exp\left[-\frac{b^2t_n}{2\sigma^2}\right].
\end{aligned}$$

In the last assertion we have used the fact that $n\epsilon = t_n$. We also notice that

$$\sum_{1\leq k\leq n}(x_k - x_{k-1}) = x_n - x_0.$$

On the other hand, we have

$$-2bt_n(x_n - x_0) + b^2t_n^2 = \left[(x_n - x_0) - bt_n\right]^2 - (x_n - x_0)^2.$$

This yields

$$\begin{aligned}
p_{t_1,\ldots,t_n}(x_1,\ldots,x_n \mid x_0) \quad &\propto \quad \exp\left[-\frac{1}{2\sigma^2\epsilon}\sum_{1\leq k\leq n}\left(x_k - x_{k-1}\right)^2\right] \\
&\qquad \times \exp\left[-\frac{-2bt_n}{2\sigma^2t_n}\;(x_n - x_0)\right]\;\exp\left[-\frac{b^2t_n^2}{2\sigma^2t_n}\right] \\
&= \quad \exp\left[-\frac{1}{2\sigma^2\epsilon}\sum_{1\leq k\leq n}\left(x_k - x_{k-1}\right)^2\right] \\
&\qquad \times \exp\left[-\frac{1}{2\sigma^2t_n}\;\left(\left[(x_n - x_0) - bt_n\right]^2 - (x_n - x_0)^2\right)\right].
\end{aligned}$$

In much the same way, the conditional densities of the terminal state X_{t_n} given $X_0 = x_0$ is given by

$$p_{t_n}(x_n \mid x_0) \;\propto\; \exp\left[-\frac{1}{2\sigma^2t_n}\;\left((x_n - x_0) - bt_n\right)^2\right].$$

We conclude that the conditional densities of the sequence of state $(X_{t_1}, \ldots, X_{t_{n-1}})$ given $(X_0, X_{t_n}) = (x_0, x_n)$ are given by

$$p_{t_1,\ldots,t_{n-1}}(x_1,\ldots,x_{n-1} \mid x_0, x_n) \propto \exp\left[-\frac{1}{2\sigma^2\epsilon}\sum_{1\leq k\leq n}\left(x_k - x_{k-1}\right)^2 + \frac{1}{2\sigma^2t_n}\;(x_n - x_0)^2\right].$$

The last assertion is a consequence of the above formula.

This ends the proof of the exercise.

■

Solution to exercise 224:
The generator of the process is given by

$$L(f)(x) = \lambda \left(f\left(x + \frac{1}{\sqrt{2\lambda}}\right) - f(x)\right) + \lambda\left(f\left(x - \frac{1}{\sqrt{2\lambda}}\right) - f(x)\right).$$

A simple Taylor expansion implies that

$$\begin{aligned} f\left(x+\frac{1}{\sqrt{2\lambda}}\right) - f(x) &= +\frac{1}{\sqrt{2\lambda}} f'(x) + \frac{1}{2}\frac{1}{2\lambda} f''(x) + \mathrm{O}\left(\frac{1}{\lambda^{3/2}}\right), \\ f\left(x-\frac{1}{\sqrt{2\lambda}}\right) - f(x) &= -\frac{1}{\sqrt{2\lambda}} f'(x) + \frac{1}{2}\frac{1}{2\lambda} f''(x) + \mathrm{O}\left(\frac{1}{\lambda^{3/2}}\right). \end{aligned}$$

Summing the two terms we find that

$$L(f)(x) = \frac{1}{2} f''(x) + \mathrm{O}\left(\frac{1}{\lambda^{1/2}}\right).$$

Using the decomposition

$$\forall 0 \le s \le t \qquad X_t - X_s = \frac{1}{\sqrt{2\lambda}} \left((N_t - N_s) - (N'_t - N'_s)\right)$$

we check that $(X_t - X_s)$ is independent of X_s. In addition, we have

$$\mathbb{E}(X_t - X_s) = \frac{1}{\sqrt{2\lambda}} \left(\mathbb{E}(N_t - N_s) - \mathbb{E}(N'_t - N'_s)\right) = 0$$

and

$$\begin{aligned} \mathbb{E}\left((X_t - X_s)^2\right) &= \frac{1}{2\lambda}\, \mathbb{E}\left(\left((N_t - N_s) - (N'_t - N'_s)\right)^2\right) \\ &= \frac{1}{\lambda}\, \mathbb{E}\left((N_t - N_s)^2\right) = (t-s). \end{aligned}$$

Finally, we have

$$\mathbb{E}(e^{\alpha X_t}) = \mathbb{E}(e^{\frac{\alpha}{\sqrt{2\lambda}} N_t})\, \mathbb{E}(e^{-\frac{\alpha}{\sqrt{2\lambda}} N_t}).$$

Recalling that

$$\mathbb{E}(e^{\beta N_t}) = e^{-\lambda t} \sum_{n\ge 0} \frac{\left(e^{\beta}\lambda t\right)^n}{n!} = e^{\lambda t\ (e^{\beta}-1)}$$

we find that

$$\begin{aligned} \mathbb{E}(e^{\alpha X_t}) &= e^{\lambda t\ (e^{\frac{\alpha}{\sqrt{2\lambda}}}-1)}\ e^{\lambda t\ (e^{-\frac{\alpha}{\sqrt{2\lambda}}}-1)} \\ &= \exp\left[\left(\lambda(e^{\frac{\alpha}{\sqrt{2\lambda}}} - 1) + \lambda(e^{-\frac{\alpha}{\sqrt{2\lambda}}} - 1)\right) t\right]. \end{aligned}$$

Using the fact that

$$\lambda\left(e^{\frac{\alpha}{\sqrt{2\lambda}}} - 1\right) = \alpha\sqrt{\frac{\lambda}{2}} + \frac{1}{2}\frac{\alpha^2}{2} + \mathrm{O}\left(\lambda^{-1/2}\right)$$

we conclude that

$$\log \mathbb{E}(e^{\alpha X_t}) = \frac{\alpha^2 t}{2} + \mathrm{O}\left(t\lambda^{-1/2}\right).$$

This ends the proof of the exercise.

■

Solution to exercise 225:
Applying the integration by parts formula (14.20), we have

$$d(tW_t) = W_t\ dt + t\ dW_t + \underbrace{dtdW_t}_{=0} = W_t\ dt + t\ dW_t.$$

This implies that

$$tW_t = \int_0^t d(sW_s) = \int_0^t W_s ds + \int_0^t\ s\ dW_s.$$

This ends the proof of the exercise.

■

Solution to exercise 226:
Applying the integration by parts formula (14.20), we have

$$d(t^2 W_t) = 2t\ W_t\ dt + t^2\ dW_t + \underbrace{dt^2 dW_t}_{=0} = 2t\ W_t\ dt + t^2\ dW_t.$$

This implies that

$$M_t = t^2\ W_t - 2\int_0^t\ s\ W_s\ ds \Rightarrow dM_t = t^2\ dW_t.$$

This clearly shows that M_t and $M_t/2$ are martingale w.r.t. $\mathcal{F}_t = \sigma(W_s,\ s \leq t)$. In addition, we have

$$dM_t dM_t = t^4\ dt \Longrightarrow\ \langle M\rangle_t = \int_0^t\ s^4\ ds = t^5/5.$$

This ends the proof of the exercise.

■

Solution to exercise 227:
Applying the integration by parts formula (14.20), we have

$$d(f(t)W_t) = f'(t)\ W_t\ dt + f(t)\ dW_t + \underbrace{df(t)dW_t}_{=0} = f'(t)\ W_t\ dt + f(t)\ dW_t.$$

This implies that

$$M_t := f(t)\ W_t - \int_0^t\ f'(s)\ W_s\ ds\ \ \Rightarrow\ dM_t = f(t)\ dW_t\ \Longrightarrow\ M_t = \int_0^t\ f(s)\ dW_s.$$

This clearly shows that M_t is a martingale w.r.t. $\mathcal{F}_t = \sigma(W_s,\ s \leq t)$.

$$dM_t dM_t = f(t)^2\ dt\ \Longrightarrow\ \langle M\rangle_t = \int_0^t\ f(s)^2\ ds.$$

This ends the proof of the exercise.

■

Solution to exercise 228:
Using an elementary Taylor expansion, we have

$$|V(y) - V(x) - \partial_x V(x)(y-x)| \leq \ c \ |x-y|^2 .$$

We also have

$$\begin{aligned} M_h(f)(x) - \overline{M}_h(f)(x) &= \int P_h(x,dy) \ (a-\overline{a})(x,y) \ f(y) \\ &\qquad + \left(\int P_h(x,dz) \ (\overline{a}-a)(x,z) \right) f(x) \\ &= \int P_h(x,dy) \ (a-\overline{a})(x,y) \ (f(y)-f(x)). \end{aligned}$$

This implies that

$$\left\| \left(M_h - \overline{M}_h \right)(f) \right\| \leq \mathrm{osc}(f) \ \sup_{x \in \mathbb{R}} \int P_h(x,dy) \ |a(x,y) - \overline{a}(x,y)| .$$

For any $u, v \in \mathbb{R}$, we have

$$|\min(1, e^u) - \min(1, e^v)| \leq 1 - e^{-|u-v|} \leq |u-v|.$$

We readily check this claim by considering all possible cases: When $u, v \geq 0$ the result is obvious. When $(u \wedge v) < 0 \leq (u \vee v)$ we have

$$\begin{aligned} |\min(1, e^u) - \min(1, e^v)| &= 1 - e^{(u \wedge v)} \\ &\leq 1 - e^{-(u \vee v)} \ e^{(u \wedge v)} \leq 1 - e^{-|u-v|}. \end{aligned}$$

Finally, when $(u \vee v) < 0$ we have

$$\begin{aligned} |\min(1, e^u) - \min(1, e^v)| &= e^{(u \vee v)} - e^{(u \wedge v)} \\ &= e^{(u \vee v)} \ \left(1 - e^{(u \wedge v) - (u \vee v)} \right) \leq 1 - e^{-|u-v|}. \end{aligned}$$

The above estimate implies that

$$|a(x,y) - \overline{a}(x,y)| \leq |V(y) - V(x) - \partial_x V(x)(y-x)| \leq c \ |x-y|^2$$

from which we check that

$$\left\| \left(M_h - \overline{M}_h \right)(f) \right\| \leq c \ h \ \mathrm{osc}(f).$$

In addition, for Lipschitz functions f s.t.

$$|f(x) - f(y)| \leq |x-y|$$

we have

$$\begin{aligned} \left| M_h(f)(x) - \overline{M}_h(f)(x) \right| &\leq \int P_h(x,dy) \ |(a-\overline{a})(x,y)| \ |x-y| \\ &\leq c \int P_h(x,dy) \ \ |x-y|^3 = c \ h^{1+1/2} \ \mathbb{E}\left(|W_1|^3 \right) . \end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 229:
We have

$$h^{-1}\left[M_h(f)(v,x)-f(v,x)\right]$$

$$= h^{-1}\left[f(v,x+vh)-f(v,x)\right]\ e^{-(U(x+hv)-U(x))_+}$$

$$+\left[f(-v,x)-f(v,x)\right]\ h^{-1}\left(1-e^{-(U(x+hv)-U(x))_+}\right).$$

Using the estimates provided in exercise 228 we check that

$$e^{-(U(x+hv)-U(x))_+}=1+\mathrm{O}(h)$$

and

$$h^{-1}\left(1-e^{-(U(x+hv)-U(x))_+}\right)=\left(v\ \partial_x U(x)\right)_+ +\mathrm{O}(h).$$

On the other hand we have

$$h^{-1}\left[f(v,x+vh)-f(v,x)\right]=v\ \partial_x f(v,x)+\mathrm{O}(h).$$

We conclude that

$$h^{-1}\left[M_h(f)(v,x)-f(v,x)\right]=v\ \partial_x f(v,x)+\left(v\partial_x U(x)\right)_+\ \left(f(-v,x)-f(v,x)\right)+\mathrm{O}(h).$$

This ends the proof of the exercise.

■

Solution to exercise 230:
For any $p\geq 1$, we have

$$\left|\int M_h(x,dy)\ |y-x|^p-\int \overline{M}_h(x,dy)\ |y-x|^p\ \right|$$

$$=\left|\int P_h(x,dy)\ \left(a(x,y)-\overline{a}(x,y)\right)\ |y-x|^p\right|=h^{1+p/2}\ \mathbb{E}\left(\ |W_1|^{p+2}\right)=\mathrm{O}(h^{1+\frac{p}{2}}).$$

We also notice that for any function g s.t. $g(0)=0$ we have

$$\begin{aligned}\mathbb{E}\left(g\left(X^h_{t_n+h}-X^h_{t_n}\right)\ \mid X_{t_n}=x\right) &= \int M_h(x,dy)\ g(y-x)\\ &= \int P_h(x,dy)\ a(x,y)\ g(y-x)\\ &= \mathbb{E}\left(a(x,x+\sqrt{h}\ W_1)\ g\left(\sqrt{h}\ W_1\right)\ \right),\end{aligned}$$

as well as

$$\left|\int\ \left[M_h-\overline{M}_h\right](x,dy)\ g(y-x)\ \right|=\left|\int P_h(x,dy)\ \left(a(x,y)-\overline{a}(x,y)\right)\ g(y-x)\right|.$$

This implies that

$$\begin{aligned}\mathbb{E}\left(\left(X^h_{t_n+h}-X^h_{t_n}\right) \mid X_{t_n}=x\right) &= \int M_h(x,dy)\ (y-x)\\ &= \int (y-x)\ P_h(x,dy)\ \overline{a}(x,y)+\mathrm{o}\left(h^{1+1/2}\right).\end{aligned}$$

To take the final step, we further assume that $\partial_x V(x)\geq 0$. In this situation, we observe that

$$\begin{aligned}&h^{-1/2}\int (y-x)\ P_h(x,dy)\ \overline{a}(x,y)\\ &=\mathbb{E}\left(W_1\min\left(1,e^{-\sqrt{h}\ \partial_x V(x)\ W_1}\right)\right)=\mathbb{E}\left(W_1\ 1_{W_1\leq 0}\right)+\mathbb{E}\left(W_1\ 1_{W_1>0}\ e^{-\sqrt{h}\ \partial_x V(x) W_1}\right)\\ &=\mathbb{E}\left(W_1\ 1_{W_1>0}\ \left[e^{-\sqrt{h}\ \partial_x V(x)\ W_1}-1\right]\right).\end{aligned}$$

In much the same way, when $\partial_x V(x)\leq 0$. In this situation, we observe that

$$\begin{aligned}&h^{-1/2}\int (y-x)\ P_h(x,dy)\ \overline{a}(x,y)\\ &=\mathbb{E}\left(W_1\min\left(1,e^{-\sqrt{h}\ \partial_x V(x)\ W_1}\right)\right)=\mathbb{E}\left(W_1\ 1_{W_1\geq 0}\right)+\mathbb{E}\left(W_1\ 1_{W_1<0}\ e^{-\sqrt{h}\ \partial_x V(x) W_1}\right)\\ &=\mathbb{E}\left(W_1\ 1_{W_1<0}\ \left[e^{-\sqrt{h}\ \partial_x V(x)\ W_1}-1\right]\right).\end{aligned}$$

There are many ways to estimate the above quantities.We further assume that $\partial_x V(x)\leq 0$. In this case, we can use the change of variable formula

$$\begin{aligned}\mathbb{E}\left(W_1\ 1_{W_1>0}\ e^{-\sqrt{h}\partial_x V(x) W_1}\right) &= e^{\frac{h}{2}\ (\partial_x V(x))^2}\ \mathbb{E}\left(U_h(x)\ 1_{U_h(x)>0}\right)\\ &= \mathbb{E}\left(U_h(x)\ 1_{U_h(x)>0}\right)+\mathrm{O}(h)\end{aligned}$$

with

$$U_h(x)=-\sqrt{h}\ \partial_x V(x)+W_1.$$

This implies that

$$\begin{aligned}&\sqrt{h}\ \mathbb{E}\left(W_1\ 1_{W_1>0}\ e^{-\sqrt{h}\partial_x V(x) W_1}\right)\\ &=-h\ \partial_x V(x)\ \underbrace{\mathbb{P}\left(W_1>\sqrt{h}\partial_x V(x)\right)}_{=\frac{1}{2}-\mathbb{P}\left(0\leq W_1\leq\sqrt{h}\partial_x V(x)\right)}+\sqrt{h}\ \underbrace{\mathbb{E}\left(W_1\ 1_{W_1>\sqrt{h}\partial_x V(x)}\right)}_{=\mathbb{E}\left(W_1\ 1_{W_1\geq 0}\right)-\mathbb{E}\left(W_1\ 1_{0\leq W_1\leq\sqrt{h}\partial_x V(x)}\right)}+\mathrm{O}(h^{1+1/2}),\end{aligned}$$

from which we prove the formula

$$\begin{aligned}&\sqrt{h}\ \mathbb{E}\left(W_1\ 1_{W_1>0}\ \left[e^{-\sqrt{h}\partial_x V(x) W_1}-1\right]\right)\\ &=-h\ \partial_x V(x)\ \left(\tfrac{1}{2}-\mathbb{P}\left(0\leq W_1\leq\sqrt{h}\partial_x V(x)\right)\right)-\sqrt{h}\ \mathbb{E}\left(W_1\ 1_{0\leq W_1\leq\sqrt{h}\partial_x V(x)}\right)+\mathrm{O}(h^{1+1/2}).\end{aligned}$$

Notice that

$$\mathbb{E}\left(W_1\ 1_{0\leq W_1\leq a}\right) = -\frac{1}{\sqrt{2\pi}}\int_0^a \partial_w e^{-w^2/2}\ dw=\frac{1}{\sqrt{2\pi}}\ \left[1-e^{-a^2/2}\right]=\mathrm{O}(a^2)$$

and

$$\mathbb{P}\left(0 \le W_1 \le a\right) \quad = \quad \frac{1}{\sqrt{2\pi}} \int_0^a \; e^{-w^2/2} \; dw \le \frac{a}{\sqrt{2\pi}}.$$

This yields the estimate

$$\sqrt{h}\;\mathbb{E}\left(W_1 \; 1_{W_1>0} \; \left[e^{-\sqrt{h}\partial_x V(x) W_1} - 1\right]\right) = -\frac{h}{2}\; \partial_x V(x) + \mathrm{O}(h^{1+1/2}).$$

In much the same way, when $\partial_x V(x) \le 0$ we have

$$\mathbb{E}\left(W_1 \; 1_{W_1<0} \; e^{-\sqrt{h}\partial_x V(x) W_1}\right) \quad = \quad \mathbb{E}\left(U_h(x) \; 1_{U_h(x)<0}\right) + \mathrm{O}(h),$$

as well as

$$\begin{aligned}
&\sqrt{h}\;\mathbb{E}\left(W_1 \; 1_{W_1<0} \; e^{-\sqrt{h}\partial_x V(x) W_1}\right)\\
&= -h\;\partial_x V(x)\; \underbrace{\mathbb{P}\left(W_1 < \sqrt{h}\;\partial_x V(x)\right)}_{=\frac{1}{2}-\mathbb{P}\left(\sqrt{h}\partial_x V(x) \le W_1 \le 0\right)} + \sqrt{h} \qquad \underbrace{\mathbb{E}\left(W_1 \; 1_{W_1 < \sqrt{h}\partial_x V(x)}\right)}_{=\mathbb{E}\left(W_1 \; 1_{W_1 \le 0}\right) - \mathbb{E}\left(W_1 \; 1_{\sqrt{h}\partial_x V(x) \le W_1 \le 0}\right)} \qquad + \mathrm{O}(h^{1+1/2}).
\end{aligned}$$

Hence we prove the formula

$$\begin{aligned}
&\sqrt{h}\;\mathbb{E}\left(W_1 \; 1_{W_1<0} \; \left[e^{-\sqrt{h}\partial_x V(x) W_1} - 1\right]\right)\\
&= -h\;\partial_x V(x)\; \left(\tfrac{1}{2} - \mathbb{P}\left(\sqrt{h}\partial_x V(x) \le W_1 \le 0\right)\right) - \sqrt{h}\;\mathbb{E}\left(W_1 \; 1_{\sqrt{h}\partial_x V(x) \le W_1 \le 0}\right) + \mathrm{O}(h^{1+1/2}).
\end{aligned}$$

This also yields the estimate

$$\sqrt{h}\;\mathbb{E}\left(W_1 \; 1_{W_1<0} \; \left[e^{-\sqrt{h}\partial_x V(x) W_1} - 1\right]\right) = -\frac{h}{2}\; \partial_x V(x) + \mathrm{O}(h^{1+1/2}).$$

We conclude that

$$h^{-1} \int \; (y-x)\; P_h(x, dy)\; a(x,y) = -\frac{1}{2}\; \partial_x V(x) + \mathrm{O}(h^{1/2}).$$

Arguing as above we have

$$\mathbb{E}\left(\left(X^h_{t_n+h} - X^h_{t_n}\right)^2 \;\mid X_{t_n} = x\right) \quad = \quad \int \; (y-x)^2\; P_h(x,dy)\; \overline{a}(x,y) + \mathrm{o}\left(h^2\right).$$

We further assume that $\partial_x V(x) \ge 0$. In this situation, we observe that

$$\begin{aligned}
&h^{-1} \int \; (y-x)^2 \; P_h(x,dy)\; \overline{a}(x,y)\\
&= \mathbb{E}\left(W_1^2 \min\left(1, e^{-\sqrt{h}\;\partial_x V(x)\;W_1}\right)\right) = \mathbb{E}\left(W_1^2 \; 1_{W_1 \le 0}\right) + \mathbb{E}\left(W_1^2 \; 1_{W_1>0} \; e^{-\sqrt{h}\;\partial_x V(x) W_1}\right)\\
&= 1 - \mathbb{E}\left(W_1^2 \; 1_{W_1>0} \; \left[1 - e^{-\sqrt{h}\;\partial_x V(x)\;W_1}\right]\right) = 1 + \mathrm{O}(h^{1/2}).
\end{aligned}$$

In the last display we have used the fact that $e^{-u} \geq 1-u$, for any $u \geq 0$. In much the same way, when $\partial_x V(x) \leq 0$ we have

$$\begin{aligned}
&h \int (y-x)^2 \; P_h(x,dy) \; \overline{a}(x,y) \\
&= \mathbb{E}\left(W_1^2 \min\left(1, e^{-\sqrt{h}\;\partial_x V(x)\;W_1}\right)\right) = \mathbb{E}\left(W_1^2 \; 1_{W_1 \geq 0}\right) + \mathbb{E}\left(W_1^2 \; 1_{W_1<0} \; e^{-\sqrt{h}\;\partial_x V(x) W_1}\right) \\
&= 1 + \mathbb{E}\left(W_1^2 \; 1_{W_1<0} \; \left[e^{-\sqrt{h}\;\partial_x V(x)\;W_1} - 1\right]\right) = 1 + \mathrm{O}(h^{1/2}).
\end{aligned}$$

We conclude that

$$\int (y-x)^2 \; P_h(x,dy) \; \alpha(x,y) = h + \mathrm{O}(h^{1+1/2})$$

with $\alpha = a$ or $\alpha = \overline{a}$. Similar computations imply that

$$\int |y-x|^3 \; P_h(x,dy) \; \alpha(x,y) = \mathrm{O}(h^{1+1/2}),$$

with $\alpha = a$ or $\alpha = \overline{a}$.

This ends the proof of the exercise.

■

Solution to exercise 231:

As $u \to 1$, we have the expansion

$$\begin{aligned}
\frac{1}{1+u} &= \frac{1}{2-(1-u)} = \frac{1}{2}\;\frac{1}{1-((1-u)/2)} \\
&= \frac{1}{2}\left(1+\frac{1-u}{2}\right) + \mathrm{O}((1-u)^2) = \frac{1}{2} + \frac{1-u}{4} + \mathrm{O}((1-u)^2)
\end{aligned}$$

Recalling that

$$|V(x)-V(y)| \leq c_1 \;\; |x-y| \Rightarrow \left|1-e^{V(y)-V(x)}\right| \leq c_2 \;\; |x-y|$$

we find that

$$V(y) \leq V(x) \Rightarrow b(x,y) = \frac{1}{2} + \frac{1-e^{-(V(x)-V(y))}}{4} + \mathrm{O}(|x-y|^2),$$

as well as

$$V(x) > V(y) \Rightarrow b(x,y) = e^{-(V(y)-V(x))} \;\; \left[\frac{1}{2} + \frac{1-e^{-(V(y)-V(x))}}{4}\right] + \mathrm{O}(|x-y|^2).$$

We conclude that

$$b(x,y) = \overline{b}(x,y) + \mathrm{O}(|x-y|^2),$$

with

$$\begin{aligned}
\overline{b}(x,y) &= \frac{1}{2} + \frac{1-e^{-(V(x)-V(y))}}{4} \; 1_{V(x)\geq V(y)} \\
&\qquad + e^{-(V(y)-V(x))} \;\; \left[\frac{1}{2} + \frac{1-e^{-(V(y)-V(x))}}{4}\right] \; 1_{V(y)>V(x)}.
\end{aligned}$$

Arguing as in exercise 228 we find that

$$\overline{b}(x,y) = \overline{\overline{b}}(x,y) + \mathrm{O}(|x-y|^2).$$

The end the proof of the exercise follows the same lines of arguments as the one of exercise 228, thus it is omitted.

This completes the proof of the exercise. ■

Solution to exercise 232:

The proof follows the same arguments as the solution of exercise 230, thus a sketch only is given below.

We further assume that $\partial_x V(x) \geq 0$. Using the same arguments as in exercise 230, we observe that

$$\begin{aligned}
& h^{-1/2} \int \; (y-x) \; P_h(x,dy) \; \overline{\overline{b}}(x,y) \\
&= -\mathbb{E}\left(W_1 \; 1_{W_1 \geq 0} \left[\frac{1}{2} + \frac{1-e^{-\sqrt{h}\partial_x V(x) W_1}}{4} \right] \right) + \\
& \qquad \mathbb{E}\left(W_1 \; 1_{W_1 \geq 0} \; e^{-\sqrt{h} \; \partial_x V(x) W_1} \; \left[\frac{1}{2} + \frac{1-e^{-\sqrt{h} \; \partial_x V(x) W_1}}{4} \right] \right).
\end{aligned}$$

This implies that

$$\begin{aligned}
& h^{-1/2} \int \; (y-x) \; P_h(x,dy) \; \overline{\overline{b}}(x,y) \\
&= \mathbb{E}\left(W_1 \; 1_{W_1 \geq 0} \; \left[e^{-\sqrt{h} \; \partial_x V(x) W_1} - 1 \right] \; \left[\frac{1}{2} + \frac{1-e^{-\sqrt{h} \; \partial_x V(x) W_1}}{4} \right] \right) \\
&= \frac{1}{2} \; \mathbb{E}\left(W_1 \; 1_{W_1 \geq 0} \; \left[e^{-\sqrt{h} \; \partial_x V(x) W_1} - 1 \right] \right) + \mathrm{O}\left(h\right).
\end{aligned}$$

Using the estimates derived in exercise 230 we conclude that

$$\int \; (y-x) \; P_h(x,dy) \; \overline{\overline{b}}(x,y) = -\frac{h}{4} \; \partial_x V(x) + \mathrm{O}(h^{1+1/2}),$$

as well as

$$\int \; (y-x)^2 \; P_h(x,dy) \; \overline{\overline{b}}(x,y) = \frac{h}{2} + \mathrm{O}(h^{1+1/2}).$$

This completes the proof of the exercise. ■

Solution to exercise 233:

The Brownian motion W_t is a martingale (w.r.t. its natural filtration $\mathcal{F}_t = \sigma(W_s, \; s \leq t)$) with angle bracket $\langle W \rangle = t$. The stopped martingale satisfies the property

$$W_{t \wedge T_D} \in [-a,a] \Longrightarrow |W_{t \wedge T_D}| \leq c = a^2.$$

Applying (12.23) to the martingale W_t and its angle bracket $\langle W \rangle_t = \; t$ we find that

$$\mathbb{E}(\langle W \rangle_{T_D}) = \mathbb{E}(T_D) \leq a^2.$$

The Brownian motion starting at some $x \in D$ is given by $W_t^x = x + W_t$. In this situation, we have

$$(W_t^x)^2 = x^2 + W_t^2 + 2xW_t$$

$$\overset{\mathbb{E}(W_{t\wedge T_D^x})=0}{\Longrightarrow} \quad \mathbb{E}\left(W^2_{t\wedge T_D^x}\right) = \mathbb{E}(t \wedge T_D^x) = \mathbb{E}\left(\left(W^x_{t\wedge T_D^x}\right)^2\right) - x^2 \leq a^2 - x^2.$$

Applying Fatou's lemma, this yields the estimate

$$\mathbb{E}(T_D^x) \leq (a^2 - x^2).$$

This ends the proof of the exercise. ■

Solution to exercise 234:

We have

$$\mathbb{E}\left(T_x^p\right) = \int_0^\infty \mathbb{P}\left(T_x^p \geq t\right)\ dt = \int_0^\infty \mathbb{P}\left(T_x \geq t^{1/p}\right)\ dt.$$

Using the change of variable

$$s = t^{1/p} \Leftrightarrow s^p = t \Rightarrow ds = \frac{1}{p}\ t^{\frac{1}{p}-1}\ dt = \frac{1}{p}\ s^{1-p}\ dt \Rightarrow dt = ps^{p-1}\ ds$$

we find that

$$\mathbb{E}\left(T_x^p\right) = \int_0^\infty \mathbb{P}\left(T_x \geq s\right)\ ps^{p-1}\ ds.$$

Notice that for any $x \in D$ we have

$$\|W_1^x - x\| > \mathrm{diam}(D)\ \Rightarrow W_1^x \notin D \Rightarrow T_x < 1.$$

Recalling that $W_1^x = x + W_1$ These inclusions imply that

$$\mathbb{P}\left(T_x < 1\right) \geq \mathbb{P}\left(\|W_1^x - x\| > \mathrm{diam}(D)\right) = \mathbb{P}\left(\|W_1\| > \mathrm{diam}(D)\right) := \epsilon > 0$$

$$\Longrightarrow \sup_{x\in D} \mathbb{P}\left(T_x \geq 1\right) \leq 1 - \epsilon < 1.$$

We check that

$$\sup_{x\in D} \mathbb{P}\left(T_x \geq n\right) \leq (1-\epsilon)^n$$

by induction w.r.t. the parameter $n \geq 1$. The result has already been checked at rank $n = 1$. Suppose it has been proved at rank n. In this situation, we have

$$\begin{aligned}
\mathbb{P}\left(T_x \geq n+1\right) &= \mathbb{E}\left(\mathbb{P}\left(T_x \geq n+1 \mid W_1^x\right)\ 1_{W_1^x \in D}\ \right) \\
&= \frac{1}{(2\pi)^{r/2}} \int_D \exp\left[-\frac{1}{2}\ \|x-y\|^2\right] \mathbb{P}\left(T_x \geq n+1 \mid W_1^x = y\right)\ dy \\
&= \frac{1}{(2\pi)^{r/2}} \int_D \exp\left[-\frac{1}{2}\ \|x-y\|^2\right] \mathbb{P}\left(T_y \geq n\right)\ dy \\
&= (1-\epsilon)^n\ \frac{1}{(2\pi)^{r/2}} \int_D \exp\left[-\frac{1}{2}\ \|x-y\|^2\right] dy \qquad \text{(by induction)} \\
&= (1-\epsilon)^n \times \mathbb{P}\left(x + W_1 \in D\right) \\
&\leq (1-\epsilon)^n \times \mathbb{P}\left(\|W_1\| \leq \mathrm{diam}(D)\right) = (1-\epsilon)^{n+1}.
\end{aligned}$$

This implies that

$$\begin{aligned}\mathbb{E}\left(T_x^p\right) = \int_0^\infty \mathbb{P}\left(T_x \geq s\right)\ ps^{p-1}\ ds &\leq \int_0^\infty \mathbb{P}\left(T_x \geq \lfloor s \rfloor\right)\ ps^{p-1}\ ds \\ &\leq \int_0^\infty (1-\epsilon)^{\lfloor s \rfloor}\ ps^{p-1}\ ds \\ &\leq (1-\epsilon)^{-1} \int_0^\infty (1-\epsilon)^s\ ps^{p-1}\ ds.\end{aligned}$$

We conclude that

$$\sup_{x \in D} \mathbb{E}\left(T_x^p\right) \leq p\ (1-\epsilon)^{-1} \int_0^\infty e^{-\epsilon s}\ s^{p-1}\ ds < \infty.$$

This ends the proof of the exercise. ■

Solution to exercise 235:
We have

$$X_t = \int_0^t a(u) dW_u \Rightarrow X_t - X_s = \int_s^t a(u) dW_u.$$

For piecewise constant functions a, the process X_t is clearly a Gaussian process with independent increments. This follows from the fact that linear combinations of joint Gaussian variables are themselves Gaussian. The general case follows by taking limits. On the other hand, we have

$$\mathbb{E}\left([X_t - X_s]^2\right) = \mathbb{E}\left(\left[\int_s^t a(u) dW_u\right]^2\right) = \int_s^t a^2(u) du = b(t) - b(s) = \mathbb{E}\left(\left[W_{b(t)} - W_{b(s)}\right]^2\right).$$

We conclude that the diffusion $dX_t = \sqrt{b'(t)} dW_t$ starting at $X_0 = 0$ has the same law as the time-changed Brownian motion $W_{b(t)}$.

In addition, X_t is a martingale w.r.t. $\mathcal{F}_t = \sigma(W_s,\ s \leq t)$ with angle bracket

$$\langle X \rangle_t = \int_s^t a^2(u) du = b(t).$$

Thus, the diffusion X_t starting at $X_0 = 0$ has the same law as the time-changed Brownian motion $W_{\langle X \rangle_t}$.

This ends the proof of the exercise. ■

Solution to exercise 236:
We have

$$\frac{\partial f}{\partial x} = (a + \frac{x}{3})^2 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} = \frac{2}{3}\ (a + \frac{x}{3}).$$

By applying the Doeblin-Itō formula, this leads us to

$$f(W_t) = (a + \frac{1}{3}\ W_t)^3 \Rightarrow df(W_t) = \left(a + \frac{1}{3}\ W_t\right)^2\ dW_t + \frac{1}{3}\ \left(a + \frac{W_t}{3}\right)\ dt.$$

On the other hand, we have

$$X_t := \left(a + \frac{1}{3}\ W_t\right)^3 \Rightarrow X_t^{1/3} = \left(a + \frac{1}{3}\ W_t\right) \quad \text{and} \quad X_t^{2/3} = \left(a + \frac{1}{3}\ W_t\right)^2.$$

This shows that $X_0 = a^3$ and

$$dX_t = \frac{1}{3}\, X_t^{1/3}\, dt + X_t^{2/3}\, dW_t.$$

On the other hand $X_t = 0$ is also a solution of the above equation. The problem arises due to the fact that the functions $x^{1/3}$ and $x^{2/3}$ are not smooth at the origin.

This ends the proof of the exercise.

■

Solution to exercise 237:

By applying the Doeblin-Itō formula to the function $f(t,x) = x/t$ we have

$$\begin{aligned} d\left(\frac{X_t}{t}\right) &= -\frac{X_t}{t^2}\, dt + \frac{1}{t}\, dX_t \\ &= -\frac{X_t}{t^2}\, dt + \frac{1}{t}\, \left(\frac{X_t}{t}\, dt + t\, dW_t\right) = dW_t. \end{aligned}$$

This implies that

$$\frac{X_t}{t} = x_1 + W_t - W_1 \Rightarrow X_t = t\left(x_1 + (W_t - W_1)\right).$$

This ends the proof of the exercise.

■

Solution to exercise 238:

By applying the Doeblin-Itō formula to the function $f(t,x) = a(t)\, x$ we have

$$\begin{aligned} d\left(a(t)X_t\right) &= a'(t)\; X_t\; dt + a(t)\; dX_t \\ &= a'(t)\; X_t\; dt + a(t)\; \left(-\frac{a'(t)}{a(t)}\; X_t\; dt + \frac{1}{a(t)}\; dW_t\right) = dW_t. \end{aligned}$$

This implies that

$$a(t)X_t = a(t_0)\; x_{t_0} + (W_t - W_{t_0}) \Rightarrow X_t = \left(a(t_0)\; x_{t_0} + (W_t - W_{t_0})\right)/a(t).$$

This ends the proof of the exercise.

■

Solution to exercise 239:

Notice that W_t^a is a continuous martingale with angle bracket

$$\langle W^a \rangle_t = t.$$

We check this claim by using the fact that

$$dW_t^i dW_t^j = 1_{i=j}\; dt \Longrightarrow dW_t^a dW_t^a = \sum_{1 \leq i \leq n} a_i^2\; dt$$

Levy's characterization of the Brownian motion ends the proof of the exercise.

■

Solution to exercise 240:

The exercise is a direct consequence of the invariance properties (14.21) of the Brownian motion and the Laplacian under orthogonal transformations stated in theorem 14.4.3. We check directly the exercise following the developments of section 14.4.3 applied to $r = 2$ and $O = R_\theta$. Notice that

$$O' = R'_\theta = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} = \begin{pmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{pmatrix} = R_{-\theta} = R_\theta^{-1}.$$

The end of the proof of the exercise is now easily completed.

■

Solution to exercise 241:

As exercise 240, this exercise is also a direct consequence of the invariance properties (14.21) of the Brownian motion and the Laplacian under orthogonal transformations stated in theorem 14.4.3. We check directly the exercise following the developments of section 14.4.3 applied to $r = 2$ and $O = \overline{R}_\theta$. Notice that

$$O' = \overline{R}'_\theta = \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix} = \overline{R}_\alpha^{-1} = \overline{R}_\alpha.$$

The end of the proof of the exercise is now easily completed.

■

Solution to exercise 242:

By applying the Doeblin-Itō formula to the function $f(x) = \log \frac{x}{1-x}$, we find that

$$\begin{aligned} \partial_x f(x) &= \frac{1-x}{x}\, \partial_x \left(\frac{x}{1-x}\right) = \frac{1-x}{x}\, \frac{(1-x)+x}{(1-x)^2} = \frac{1}{x(1-x)} \\ \partial_x^2 f(x) &= -\frac{1}{x^2(1-x)^2}\ (1-2x), \end{aligned}$$

and

$$\begin{aligned} df(X_t) &= \frac{1}{X_t(1-X_t)}\, dX_t - \frac{1}{2X_t^2(1-X_t)^2}\ (1-2X_t)\ dX_t dX_t \\ &= \left(\frac{1}{2} - X_t\right) dt + dW_t - \frac{1}{2X_t^2(1-X_t)^2}\ (1-2X_t)\ X_t^2(1-X_t)^2\ dt \\ &= \left(\frac{1}{2} - X_t\right) dt + dW_t - \frac{1}{2}\ (1-2X_t)\ dt = dW_t. \end{aligned}$$

The end of the proof of the exercise is now clearly completed.

■

Solution to exercise 243:

We have

$$\frac{\partial f}{\partial x} = -a\sin(x) = -\frac{a}{b}\ g(x) \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} = -a\cos(x) = -f(x).$$

In much the same way, we have

$$\frac{\partial g}{\partial x} = b\cos(x) = \frac{b}{a}\ f(x) \quad \text{and} \quad \frac{\partial^2 g}{\partial x^2} = -b\sin(x) = -g(x).$$

We set

$$X_t := f(W_t) \quad \text{and} \quad Y_t = g(W_t).$$

Applying the Doeblin-Itō formula, we find that

$$dX_t = df(W_t) = -\frac{a}{b}\, g(W_t)\, dW_t - \frac{1}{2}\, f(W_t)\, dt = -\frac{a}{b}\, Y_t\, dW_t - \frac{1}{2}\, X_t\, dt$$

and

$$dY_t = \frac{b}{a}\, f(W_t)\, dW_t - \frac{1}{2}\, g(W_t)\, dt = \frac{b}{a}\, X_t\, dW_t - \frac{1}{2}\, Y_t\, dt.$$

Finally, we observe that

$$\begin{array}{rcl} Y_t & = & b\, \sin(W_t) = b\sqrt{1-\cos^2(W_t)} = b\sqrt{1-a^{-2}(a\cos(W_t))^2} \\ & = & \frac{b}{a}\sqrt{a^2-X_t^2} \Rightarrow dX_t = -\frac{1}{2}\, X_t\, dt - \frac{b}{a}\, \sqrt{(a-X_t)(a+X_t)}\, dW_t. \end{array}$$

This ends the proof of the exercise.

■

Solution to exercise 244:

We have

$$\frac{\partial f}{\partial x} = a\alpha\, \sinh(\alpha x) = \alpha\, \frac{a}{b}\, g(x) \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} = a\alpha^2\, \cosh(\alpha x) = \alpha^2\, f(x).$$

In much the same way, we have

$$\frac{\partial g}{\partial x} = b\alpha\, \cosh(\alpha x) = \alpha\, \frac{b}{a}\, f(x) \quad \text{and} \quad \frac{\partial^2 g}{\partial x^2} = \alpha^2\, b\, \sinh(\alpha x) = \alpha^2\, g(x).$$

This implies that

$$\begin{array}{rcl} df(W_t) & = & \frac{\partial f}{\partial x}(W_t)\, dW_t + \frac{1}{2}\, \frac{\partial^2 f}{\partial x^2}(W_t)\, dt = \alpha\, \frac{a}{b}\, g(W_t)\, dW_t + \frac{\alpha^2}{2}\, f(W_t)\, dt \\ dg(W_t) & = & \frac{\partial g}{\partial x}(W_t)\, dW_t + \frac{1}{2}\, \frac{\partial^2 g}{\partial x^2}(W_t)\, dt = \alpha\, \frac{b}{a}\, f(W_t)\, dW_t + \frac{\alpha^2}{2}\, g(W_t)\, dt. \end{array}$$

Replacing $f(W_t)$ and $g(W_t)$ by X_t and Y_t this yields the stochastic differential equation

$$\left\{ \begin{array}{rcl} dX_t & = & \frac{\alpha^2}{2}\, X_t\, dt + \alpha\, \frac{a}{b}\, Y_t\, dW_t \\ dY_t & = & \frac{\alpha^2}{2}\, Y_t\, dt + \alpha\, \frac{b}{a}\, X_t\, dW_t. \end{array} \right.$$

This ends the proof of the exercise.

■

Solution to exercise 245:

The process satisfies the continuity and the Gaussian properties discussed in definition 14.1.2. Note that for any $s \leq t$ the increments

$$W_t^\alpha - W_s^\alpha = \frac{1}{\alpha}\, \left(W_{t\,\alpha^2} - W_{s\,\alpha^2}\right)$$

are centered Gaussian random variables, with variance $\frac{1}{\alpha^2}\, \left(t\,\alpha^2 - s\,\alpha^2\right) = (t-s)$.

In much the same way, for any $s \leq t$ we have

$$\begin{array}{rcl}\mathbb{E}\left(W_t^- W_s^-\right) & = & ts\mathbb{E}(W(1/t)W(1/s)) \\ & = & ts\mathbb{E}(W(1/t)[W(1/s) - W(1/t)] + W(1/t)^2) = st\mathbb{E}(W(1/t)^2) = \dfrac{st}{t} = s.\end{array}$$

This implies that $W_t^- - W_s^-$ centered Gaussian random variables, with variance

$$\mathbb{E}\left(\left[W_t^- - W_s^-\right]^2\right) = t^2\mathbb{E}(W(1/t)^2) + s^2\mathbb{E}(W(1/s)^2) - 2s = t + s - 2s = t - s.$$

This ends the proof of the exercise.

■

Solution to exercise 246:

We have

$$\mathbb{P}(T \leq a) = \mathbb{P}(T \leq t \mid W_t < a)\ \mathbb{P}(W_t < a) + \underbrace{\mathbb{P}(T \leq t \mid W_t \geq a)}_{=1}\ \mathbb{P}(W_t \geq a)$$

and

$$\mathbb{P}(W_t < a \,|\, T \leq t) = \frac{1}{2} \Rightarrow \begin{array}{rcl}\mathbb{P}(T \leq t \mid W_t < a)\ \mathbb{P}(W_t < a) & = & \mathbb{P}(W_t < a \mid T \leq t)\ \mathbb{P}(T < a) \\ & = & \frac{1}{2}\mathbb{P}(T < a).\end{array}$$

This ends the proof of the exercise.

■

Solution to exercise 247:

For any $\tau_t(x, y)$, and for any function $F(x, y) = f(x)$ that depends on the first coordinate, we clearly have have

$$\partial_{x_i, y_j} F(x, y) = 0 \Rightarrow \mathcal{L}_t(F)(x, y) = L_t(F(\,.\,, y))(x) = L_t(f)(x).$$

By symmetry arguments we also have $\mathcal{L}_t(F)(x, y) = L_t(g)(y)$ for any function $F(x, y) = g(y)$ that depends on the second coordinate.

Assume that

$$\tau_t(x, y) = 2^{-1}\left[\sigma_t(x)\sigma_t'(y) + \sigma_t(y)\sigma_t'(x)\right].$$

Using the Doeblin-Itō formula we check that the generator of the diffusion

$$\left\{\begin{array}{l} d\mathcal{X}_t = b_t(\mathcal{X}_t)\ dt + \sigma_t(\mathcal{X}_t)\ dW_t \\ d\mathcal{Y}_t = b_t(\mathcal{Y}_t)\ dt + \sigma_t(\mathcal{Y}_t)\ dW_t \end{array}\right.$$

is given by

$$\mathcal{L}_t(F)(x, y) = L_t(F(\,.\,, y))(x) + L_t(F(x, .))(y) + \sum_{1 \leq i,j \leq r} \tau_t(x, y)^{i,j}\ \partial_{x_i, y_j} F(x, y).$$

The fact that $\mathcal{X}_t$ and $\mathcal{Y}_t$ have the same law as X_t is immediate from the above representation. This ends the proof of the exercise.

■

Solution to exercise 248:

Observe that

$$\left[I-2\,U\left(\mathcal{X}_t-\mathcal{Y}_t\right)U\left(\mathcal{X}_t-\mathcal{Y}_t\right)'\right]=\left[I-2\,U\left(\mathcal{X}_t-\mathcal{Y}_t\right)U\left(\mathcal{X}_t-\mathcal{Y}_t\right)'\right]'$$

and

$$\left[I-2\,U\left(\mathcal{X}_t-\mathcal{Y}_t\right)U\left(\mathcal{X}_t-\mathcal{Y}_t\right)'\right]^2=I.$$

This shows that $\left[I-2\,U\left(\mathcal{X}_t-\mathcal{Y}_t\right)U\left(\mathcal{X}_t-\mathcal{Y}_t\right)'\right]$ is an orthogonal transformation so that

$$dV_t:=\left[I-2\,U\left(\mathcal{X}_t-\mathcal{Y}_t\right)U\left(\mathcal{X}_t-\mathcal{Y}_t\right)'\right]\;dW_t$$

is the increment of a (standard) r-dimensional Brownian motion. Recall that $dW_t(dW_t)'=I\;dt$ We can also observe that

$$\begin{aligned}dV_t(dV_t)' &= \left[I-2\,U\left(\mathcal{X}_t-\mathcal{Y}_t\right)U\left(\mathcal{X}_t-\mathcal{Y}_t\right)'\right]\;dW_t(dW_t)'\left[I-2\,U\left(\mathcal{X}_t-\mathcal{Y}_t\right)U\left(\mathcal{X}_t-\mathcal{Y}_t\right)'\right]\\ &= \left[I-2\,U\left(\mathcal{X}_t-\mathcal{Y}_t\right)U\left(\mathcal{X}_t-\mathcal{Y}_t\right)'\right]\left[I-2\,U\left(\mathcal{X}_t-\mathcal{Y}_t\right)U\left(\mathcal{X}_t-\mathcal{Y}_t\right)'\right]=Id.\end{aligned}$$

This clearly implies that $\mathcal{X}_t$ and $\mathcal{Y}_t$ have the same law as X_t.

Arguing as above, we have

$$d\mathcal{X}_t\;d\mathcal{Y}_t'=\underbrace{\sigma_t(\mathcal{X}_t)\;\left[I-2\,U\left(\mathcal{X}_t-\mathcal{Y}_t\right)U\left(\mathcal{X}_t-\mathcal{Y}_t\right)'\right]\sigma_t(\mathcal{Y}_t)'}_{:=\overline{\sigma}_t(\mathcal{X}_t,\mathcal{Y}_t)}\;dt.$$

This implies that

$$\mathcal{L}_t(F)(x,y)=L_t(F(\,.\,,y))(x)+L_t(F(x,\,.\,))(y)+\sum_{1\leq i,j\leq r}\tau_t(x,y)^{i,j}\;\partial_{x_i,y_j}F(x,y)$$

with $\tau(x,y)=(\overline{\sigma}_t(x,y)+\overline{\sigma}_t(y,x))/2$.

This ends the proof of the exercise.

■

Solution to exercise 249:

Consider a 1-dimensional Brownian motion W_t (starting at the origin) and set $W_t^\star:=\sup_{0\leq s\leq t}W_t$. Let T_x the first time it reaches the value x so that $\{T_x\leq t\}=\{W_t^\star\geq x\}$.

For any $\epsilon>0$ and any $y>x+\epsilon$, we have

$$\begin{aligned}\mathbb{P}\left(W_t^\star\geq y\;,\;W_t\in[x,x+\epsilon]\right) &= \mathbb{P}\left(W_t\in[x,x+\epsilon]\,,\;T_y\leq t\right)\\ &= \mathbb{P}\left(W_t\in[x,x+\epsilon]\,,\;T_y\leq t\right).\end{aligned}$$

On the other hand, we have the key observation

$$\text{Law}\left[\left(W_t-W_{T_y}\right)\;\mid T_y\leq t\right]=\text{Law}\left[-\left(W_t-W_{T_y}\right)\;\mid T_y\leq t\right].$$

this implies that

$$\begin{aligned}\mathbb{P}\left(W_t\in[x,x+\epsilon]\,,\;T_y\leq t\right) &= \mathbb{P}\left(W_t-y\in[x-y,x-y+\epsilon]\,,\;T_y\leq t\right)\\ &= \mathbb{P}\left(W_t-y\in[-(x-y+\epsilon),-(x-y)]\,,\;T_y\leq t\right)\\ &= \mathbb{P}\left(W_t\in[-(x-y+\epsilon)+y,-(x-y)+y]\,,\;T_y\leq t\right)\\ &= \mathbb{P}\left(W_t\in[2y-x-\epsilon,2y-x]\right)\\ &\qquad\Uparrow\quad(2y-x>y+\epsilon>y\quad\text{and}\quad(2y-x)-\epsilon>y)\\ &\qquad\Uparrow\quad\;(y>x+\epsilon)\,.\end{aligned}$$

Hence

$$\mathbb{P}\left(W_t^\star \geq y\ ,\ W_t \in [x, x+\epsilon]\right) = \mathbb{P}\left(W_t \in [2y - x - \epsilon, 2y - x]\right)$$

from which we readily prove that

$$\mathbb{P}\left(W_t^\star \geq y \mid W_t \in [x, x+\epsilon]\right) = \frac{\mathbb{P}\left(W_t \in [2y - x - \epsilon, 2y - x]\right)}{\mathbb{P}\left(W_t \in [x, x+\epsilon]\right)}.$$

Letting $\epsilon \downarrow 0$ we conclude that for any $y \geq x$

$$\begin{aligned}
\mathbb{P}\left(W_t^\star \geq y \mid W_t = x\right) &= \frac{\exp\left[-\frac{1}{2t}(2y-x)^2\right]}{\exp\left[-\frac{1}{2t}\ x^2\right]} \\
&= \exp\left[\frac{1}{2t}[x^2 - (2y-x)^2]\right] = \exp\left[-2y(y-x)/t\right].
\end{aligned}$$

The end of the proof is now completed.

■

Solution to exercise 250:
Using exercise 223 we have

$$\mathbb{P}\left(X_t^\star \geq y \mid X_t = x\right) = \mathbb{P}\left(\sigma\ W_t^\star \geq y \mid \sigma\ W_t = x\right).$$

On the other hand, using exercise 249 we have

$$\mathbb{P}\left(W_t^\star \geq y/\sigma \mid W_t = x/\sigma\right) \quad = \quad \exp\left[-2y(y-x)/(\sigma^2 t)\right]$$

for any $y \geq x$. Otherwise

$$y < x \Rightarrow \mathbb{P}\left(X_t^\star \geq y \mid X_t = x\right) = 1.$$

Finally for any $0 \leq y$ we have

$$\begin{aligned}
\mathbb{P}\left(X_t^\star \geq y \mid X_0 = 0\right) &= \int_{-\infty}^{y} \exp\left[-2y(y-x)/(\sigma^2 t)\right]\ \frac{1}{\sqrt{2\pi\sigma^2 t}}\ \exp\left[-\frac{1}{2\sigma^2 t}\ (x - bt)^2\right] dx \\
&\quad \underbrace{+\int_{y}^{+\infty} \exp\left[-\frac{1}{2\sigma^2 t}\ (x-bt)^2\right] dx}_{=\mathbb{P}(X_t \geq y\ \mid\ X_0 = 0)}.
\end{aligned}$$

To take the final step, observe that

$$\begin{aligned}
4y(y-x) + (x-bt)^2 &= 4y^2 + x^2 - 2x(bt + 2y) + (bt)^2 \\
&= (x - [bt + 2y])^2 + 4y^2 + (bt)^2 - [bt + 2y]^2 \\
&= (x - [bt + 2y])^2 - 4ybt.
\end{aligned}$$

This yields

$$\begin{aligned}
&\exp\left[-\tfrac{1}{2\sigma^2 t}\ \left[4y(y-x) + (x-bt)^2\right]\right] \\
&= \exp\left[-\tfrac{1}{2\sigma^2 t}\ (x - [bt - 2y])^2\right] \times \exp\left[2yb/\sigma^2\right],
\end{aligned}$$

from which we conclude that

$$\begin{aligned}
&\mathbb{P}\left(X_t^\star \geq y \mid X_0 = 0\right)\\
&= \exp\left[2yb/\sigma^2\right]\, \mathbb{P}\left(X_t + 2y \leq y \mid X_0 = 0\right) + \mathbb{P}(X_t \geq y \mid X_0 = 0)\\
&= \exp\left[2yb/\sigma^2\right]\, \mathbb{P}\left(X_t + y \leq 0 \mid X_0 = 0\right) + \mathbb{P}(X_t \geq y \mid X_0 = 0).
\end{aligned}$$

Finally, given $X_0 = 0$ we have

$$\begin{array}{rcl}
X_t + y \leq 0 & \Longleftrightarrow & y + bt + \sigma\sqrt{t}\; W_t/\sqrt{t} \leq 0\\
& \Longleftrightarrow & \underbrace{-W_t/\sqrt{t}}_{\stackrel{law}{=} W_1} \geq (y+bt)\,/(\sigma\sqrt{t}).
\end{array}$$

This implies that

$$\mathbb{P}(X_t + y \leq 0 \mid X_0 = 0) = \mathbb{P}\left(W_1 \geq (y+bt)\,/(\sigma\sqrt{t})\right).$$

In much the same way, we have

$$\begin{array}{rcl}
\mathbb{P}(X_t \geq y \mid X_0 = 0) & = & \mathbb{P}\left(W_t/\sqrt{t} \geq [y - bt]\,/(\sigma\sqrt{t})\right)\\
& = & \mathbb{P}\left(W_1 \geq [y-bt]\,/(\sigma\sqrt{t})\right).
\end{array}$$

The end of the proof of the exercise is now easily completed.

■

Solution to exercise 251:
We set $X'_t := -X_t$ and $b' = -b$. In this notation given $X_0 = 0 = X'_0$ we have

$$X'_t := -X_t = -bt - \sigma\; W_t \stackrel{law}{=} b't + \sigma\; W_t.$$

On the other hand, recalling that

$$\inf_{0\leq s\leq t} X_t = -\sup_{0\leq s\leq t} (-X_t)$$

for any $y \leq 0 = X_0$ we have

$$\inf_{0\leq s\leq t} X_t \leq y \Longleftrightarrow \sup_{0\leq s\leq t} X'_t \geq y' = -y\ \ (\geq X'_0 = 0)\,.$$

This implies that

$$\mathbb{P}\left(\inf_{0\leq s\leq t} X_t \leq y \mid X_0 = 0\right) = \mathbb{P}\left(\sup_{0\leq s\leq t} X'_t \geq y' \mid X'_0 = 0\right).$$

Using exercise 250 and symmetry arguments we prove that

$$\begin{array}{rcl}
\mathbb{P}\left(\inf\limits_{0\leq s\leq t} X_t \leq y \mid X_0 = 0\right) & = & \mathbb{P}\left(\sup\limits_{0\leq s\leq t} X'_t \geq y' \mid X'_0 = 0\right)\\
& = & \exp\left[2y'b'/\sigma^2\right]\, \mathbb{P}\left(W_1 \geq [y' + b't]\,/(\sigma\sqrt{t})\right)\\
& & \qquad + \mathbb{P}\left(W_1 \geq [y' - b't]\,/(\sigma\sqrt{t})\right)\\
& = & \exp\left[2yb\sigma^2\right]\, \mathbb{P}\left(W_1 \leq [y + bt]\,/(\sigma\sqrt{t})\right)\\
& & \qquad + \mathbb{P}\left(W_1 \leq [y - bt]\,/(\sigma\sqrt{t})\right).
\end{array}$$

This ends the proof of the exercise. ∎

Solution to exercise 252:
Notice that $g\ :\ x \mapsto \frac{2}{\pi}\arctan x$ maps $\mathbb{R}$ into $[-1,1]$ and its derivative is given by

$$g'(x) = \frac{2}{\pi}\ \frac{1}{1+x^2}.$$

By applying the Doeblin-Itō formula to the function

$$\begin{aligned}&f(x) = \tfrac{2}{\pi}\arctan\left(a(x+b)\right)\\ &\Rightarrow f'(x) = \tfrac{2a}{\pi}\ \tfrac{1}{1+(a(x+b))^2} \quad \text{and} \quad f''(x) = -\tfrac{2a}{\pi}\ \tfrac{1}{(1+(a(x+b))^2)^2}\ 2a(a(x+b))\end{aligned}$$

we find

$$dU_t = \frac{2a}{\pi}\ \frac{1}{1+(a(W_t+b))^2}\ dW_t - \frac{a}{\pi}\ \frac{1}{(1+(a(W_t+b))^2)^2}\ 2a(a(W_t+b))\ dt.$$

Recalling that

$$U_t := \frac{2}{\pi}\arctan\left(a(W_t+b)\right) \Leftrightarrow \tan\left(\frac{\pi}{2}U_t\right) = a(W_t+b) = \frac{\sin\left(\frac{\pi}{2}U_t\right)}{\cos\left(\frac{\pi}{2}U_t\right)}$$

we conclude that

$$\begin{aligned} dU_t &= \frac{2a}{\pi}\ \cos^2\left(\frac{\pi}{2}U_t\right)\ dW_t - \frac{2a^2}{\pi}\ \cos^4\left(\frac{\pi}{2}U_t\right)\ \frac{\sin\left(\frac{\pi}{2}U_t\right)}{\cos\left(\frac{\pi}{2}U_t\right)}\ dt \\ &= \frac{2a}{\pi}\ \cos^2\left(\frac{\pi}{2}U_t\right)\ dW_t - \frac{2a^2}{\pi}\ \cos^3\left(\frac{\pi}{2}U_t\right)\ \sin\left(\frac{\pi}{2}U_t\right)\ dt. \end{aligned}$$

The end of the proof is now easily completed. ∎

Solution to exercise 253:
Notice that the increments

$$\overline{W}_t - \overline{W}_s := \int_s^t\ U_r\ dW_r^{(2)} + \int_s^t\ \sqrt{1-U_r^2}\ dW_r^{(3)}$$

and independent of $\overline{W}_s$ and

$$\mathbb{E}(\overline{W}_t - \overline{W}_s \mid \mathcal{F}_s) = 0$$

with the filtration $\mathcal{F}_t$ generated by the stochastic processes $W_t^{(i)}$, with $1 \leq i \leq 3$. In addition, we have that

$$\begin{aligned} \mathbb{E}\left((\overline{W}_t - \overline{W}_s)^2\right) &= \mathbb{E}\left[\left(\int_s^t\ U_r\ dW_r^{(2)} + \int_s^t\ \sqrt{1-U_r^2}\ dW_r^{(3)}\right)^2\right] \\ &= \mathbb{E}\left[\left(\int_s^t\ U_r\ dW_r^{(2)}\right)^2\right] + \mathbb{E}\left[\left(\int_s^t\ \sqrt{1-U_r^2}\ dW_r^{(3)}\right)^2\right] \\ &= \mathbb{E}\left[\int_s^t\ U_r^2\ dr + \int_s^t\ (1-U_r^2)\ dr\right] = (t-s). \end{aligned}$$

Finally, we have

$$\mathbb{E}\left(\overline{W}_t\ W_t^{(2)}\right) = \mathbb{E}\left(\left[\int_0^t\ U_r\ dW_r^{(2)}\right]\ \int_0^t dW_s^{(2)}\right) = \mathbb{E}\left(\int_0^t\ U_s\ ds\right)$$

and

$$\mathbb{E}\left(\overline{W}_t\ W_t^{(3)}\right) = \mathbb{E}\left(\left[\int_0^t\ \sqrt{1-U_r^2}\ dW_r^{(3)}\right]\ \int_0^t dW_s^{(3)}\right) = \mathbb{E}\left(\int_0^t\ \sqrt{1-U_r^2}\ ds\right).$$

This ends the proof of the exercise is now easily completed.

■

Solution to exercise 254:
Notice that

$$X_t = f(W_t^1, \ldots, W_t^n) \quad \text{with} \quad f(w^1, \ldots, w^n) = \sum_{1\leq i\leq n} (w^i)^2.$$

For any $\lambda > 0$,

$$\mathbb{E}\left(e^{-\lambda X_t} \mid \mathcal{F}_s\right) = \prod_{1\leq i\leq n} \mathbb{E}(e^{-\lambda (W_t^i)^2} \mid W_s^i).$$

On the other hand, we have

$$\mathbb{E}(e^{-\lambda (W_t^i)^2}) = \mathbb{E}(e^{-\lambda (W_s^i+[W_t^i-W_s^i])^2}) = \frac{1}{\sqrt{2\pi(t-s)}}\ \int\ e^{-\lambda(W_s^i+x)^2-\frac{x^2}{2(t-s)}}\ dx.$$

We also observe that

$$\begin{aligned}
&\lambda(W_s^i+x)^2 + \tfrac{x^2}{2(t-s)}\\
&= \lambda(W_s^i)^2 + x^2\left(\lambda + \tfrac{1}{2(t-s)}\right) + 2\lambda x W_s^i\\
&= \lambda\ (W_s^i)^2 + \tfrac{1+2\lambda(t-s)}{2(t-s)}\left(x^2 + 2x\ \tfrac{2\lambda(t-s)W_s^i}{1+2\lambda(t-s)}\right)\\
&= (W_s^i)^2\left[\lambda\ -\ \tfrac{(2(t-s))\lambda^2}{(1+2\lambda(t-s))}\right] + \tfrac{1+2\lambda(t-s)}{2(t-s)}\left(x + \tfrac{2\lambda(t-s)W_s^i}{1+2\lambda(t-s)}\right)^2.
\end{aligned}$$

This implies that

$$\lambda(W_s^i+x)^2 + \frac{x^2}{2(t-s)} = \frac{\lambda}{1+2\lambda(t-s)}\ (W_s^i)^2\ +\ \frac{1+2\lambda(t-s)}{2(t-s)}\left(x + \frac{2\lambda(t-s)W_s^i}{1+2\lambda(t-s)}\right)^2.$$

We conclude that

$$\mathbb{E}(e^{-\lambda(W_t^i)^2}) = (1+2\lambda(t-s))^{-1/2}\ \exp\left(-\frac{\lambda}{1+2\lambda(t-s)}\ (W_s^i)^2\right)$$

and therefore

$$\mathbb{E}\left(e^{-\lambda X_t} \mid \mathcal{F}_s\right) = (1+2\lambda(t-s))^{-n/2}\ \exp\left(-\frac{\lambda X_s}{1+2\lambda(t-s)}\right).$$

This yields

$$\forall s \leq t \qquad \mathbb{P}(X_t = 0 \mid \mathcal{F}_s) = \lim_{\lambda \to \infty} \mathbb{E}\left(e^{-\lambda X_t} \mid \mathcal{F}_s\right) = 0.$$

By applying Doeblin-Itō formula we have

$$dX_t = 2 \sum_{1 \leq i \leq n} W_t^i dW_t^i + n\ dt.$$

This implies that

$$d\overline{W}_t := \sum_{1 \leq i \leq n} 1_{X_t \neq 0} \frac{W_t^i}{\sqrt{X_t}}\ dW_t^i \Longrightarrow d\overline{W}_t d\overline{W}_t = \sum_{1 \leq i \leq n} 1_{X_t \neq 0} \frac{(W_t^i)^2}{X_t} = dt$$

and we now end up with

$$\begin{aligned} dX_t &= 2 \sum_{1 \leq i \leq n} W_t^i dW_t^i + n\ dt \\ &= 2\ \sqrt{X_t} \sum_{1 \leq i \leq n} 1_{X_t \neq 0} \frac{W_t^i}{\sqrt{X_t}}\ dW_t^i + n\ dt = 2\ \sqrt{X_t}\ d\overline{W}_t + n\ dt. \end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 255:
The first assertion is a direct consequence of theorem 14.3.1.
By applying the Doeblin-Itō lemma to the function $f(t,x) = e^{at}\ x$ we find that

$$\begin{aligned} d\left(e^{at}\ X_t\right) &= \frac{\partial}{\partial t}\left(e^{at}\ x\right)_{|x=X_t}\ dt + \frac{\partial}{\partial x}\left(e^{at}\ x\right)_{|x=X_t}\ dX_t \\ &\qquad + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(e^{at}\ x\right)_{|x=X_t}\ dX_t dX_t \\ &= a\ e^{at}\ X_t\ dt + e^{at}\ \left(a\ (b - X_t)\ dt + \sigma\ dW_t\right) \\ &= e^{at}\ \left(ab\ dt + \sigma dW_t\right). \end{aligned}$$

Integrating from 0 to t we find that

$$\begin{aligned} e^{at}\ X_t &= X_0 + b \int_0^t a e^{as}\ ds + \int_0^t e^{as}\ \sigma dW_s \\ &= X_0 + b\ \left(e^{at} - 1\right) + \int_0^t e^{as}\ \sigma dW_s. \end{aligned}$$

Hence we conclude that

$$X_t = e^{-at}\ X_0 + b\ \left(1 - e^{-at}\right) + \sigma \int_0^t e^{-a(t-s)}\ dW_s.$$

From this formula, we easily prove that

$$\mathbb{E}(X_t \mid X_0) = e^{-at}\ X_0 + b\ \left(1 - e^{-at}\right).$$

In addition, by using the fact that

$$X_t - \mathbb{E}(X_t \mid X_0) = \sigma \int_0^t e^{-a(t-s)} \, dW_s$$

we prove that

$$\begin{aligned} &\mathbb{E}\left([X_t - \mathbb{E}(X_t \mid X_0)]^2 \mid X_0\right) \\ &= \sigma^2 \int_0^t e^{-2a(t-s)} \, ds \\ &= \tfrac{\sigma^2}{2a} \, e^{-2at} \int_0^t 2a \, e^{2a\,s} \, ds = \tfrac{\sigma^2}{2a} \, \left(1 - e^{-2at}\right). \end{aligned}$$

On the other hand, given X_0 the random variable X_t is Gaussian. This ends the proof of the exercise.

■

Solution to exercise 256: Using exercise 235, we check that $Y_t := \sigma^2 \int_0^t e^{-a(t-s)} \, dW_s$ has the same law as the time-changed Brownian motion $W_{\langle Y \rangle_t}$ with

$$\langle Y \rangle_t = \frac{\sigma^2}{2a} \, \left(e^{2at} - 1\right) = \sigma^2 \int_0^t e^{2as} \, ds.$$

Recalling that $\mathbb{E}(W_1^2) = 1$ and using the fact that $d(t) \leq \frac{\sigma^2}{2a}$, the moments estimates (14.22) are direct consequences of (5.10) Using (14.22) we have

$$\begin{aligned} \mathbb{E}\left(X_t^{2n}\right)^{1/(2n)} &\leq \left(\frac{\sigma^2}{2a}\right)^{1/2} \left(\frac{(2n)!}{n!2^n}\right)^{1/(2n)} + e^{-at} \, \mathbb{E}\left((X_0)^{2n}\right)^{1/(2n)} + b \, \left(1 - e^{-at}\right) \\ &= \left(\frac{\sigma^2}{2a}\right)^{1/2} \left(\frac{(2n)!}{n!2^n}\right)^{1/(2n)} + \mathbb{E}\left((X_0)^{2n}\right)^{1/(2n)} + b. \end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 257:
We have

$$dX_t = AX_t \, dt + B \, dW_t \implies dX_t dX_t' = B \, dW_t dW_t' B' = BB' \, dt$$

where $(.)'$ stands for the transpose of some vector or some matrix. We have

$$X_t = e^{At} X_0 + \int_0^t e^{(t-s)A} \, B \, dW_s.$$

Notice that

$$\begin{aligned} \mathbb{E}\left(\left[\int_0^t e^{(t-s)A} \, B \, dW_s\right]\left[\int_0^t e^{(t-s)A} \, B \, dW_s\right]'\right) &= = \int_0^t e^{(t-s)A} \, BB' \, e^{(t-s)A'} \, ds \\ &= \int_0^t e^{sA} \, BB' \, e^{sA'} \, ds. \end{aligned}$$

This shows that X_t is a gaussian random variable with mean

$$\mathbb{E}(X_t) = e^{At}\ \mathbb{E}(X_0).$$

Using the decomposition

$$X_t - \mathbb{E}(X_t) = e^{At}\ (X_0 - \mathbb{E}(X_0)) + \int_0^t e^{(t-s)A}\ B\ dW_s$$

we prove that

$$P_t = \mathbb{E}\left([X_t - \mathbb{E}(X_t)]\,[X_t - \mathbb{E}(X_t)]'\right) = e^{At}\ P_0\ e^{A't} + \int_0^t e^{sA}\ BB'\ e^{sA'}\ ds.$$

This shows that

$$\begin{aligned}\dot{P}_t &= Ae^{At}\ P_0\ e^{A't} + e^{At}\ P_0\ e^{A't}A' + e^{tA}\ BB'\ e^{tA'}\\ &= AP_t + P_tA' - \int_0^t \partial_s\left(e^{sA}\ BB'\ e^{sA'}\right)\ ds + e^{tA}\ BB'\ e^{tA'}\\ &= AP_t + P_tA' + BB'.\end{aligned}$$

The last assertion is immediate. Observe that

$$\begin{aligned}AP_\infty + P_\infty A' + BB' &= \lim_{t\to\infty}\int_0^t \partial_s\left(e^{sA}\ BB'\ e^{sA'}\right)\ ds + BB'\\ &= \lim_{t\to\infty}\left(e^{tA}\ BB'\ e^{tA'}\right) = 0.\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 258:

Using exercise 255 the first assertion is immediate. The infinitesimal generator of the process is given by

$$L(f)(x) = -x\ f'(x) + f''(x).$$

The formula $\pi\left(f_1\ P_t(f_2)\right) = \pi\left(P_t(f_1)\ f_2\right)$ is a direct consequence of the fact that

$$X_t \stackrel{law}{=} \sqrt{\epsilon_t}\ X_0 + \sqrt{1-\epsilon_t}\ W_1.$$

Indeed, using the above formula we check that the transition $P_t(x,dy) = \mathbb{P}(X_t \in dy \mid X_0 = x)$ is reversible with respect to the Gaussian distribution π, in the sense that

$$\pi(dx)\ P_t(x,dy) = \pi(dy)\ P_t(y,dx).$$

We end the proof by a simple integration of the function $f_1(x)f_2(y)$.

On the other hand, we have

$$\frac{d}{dt}P_t(f)(x) = P_t(L(f))(x) \stackrel{t=0}{=} L(f)(x).$$

This implies that

$$\frac{d}{dt}\pi\left(f_1\ P_t(f_2)\right) = \pi\left(f_1\ \frac{d}{dt}P_t(f_2)\right) = \pi\left(\frac{d}{dt}P_t(f_1)\ f_2\right).$$

Choosing $t = 0$ we conclude that

$$\pi\left(f_1\ L(f_2)\right) = \pi\left(L(f_1)\ f_2\right)$$

with

$$L(f)(x) = -xf'(x) + f''(x).$$

In addition using the Gaussian integration by parts formula stated in exercise (54) we find that

$$\begin{aligned}\pi\left(f_1\ L(f_2)\right) &= -\mathbb{E}\left(W_1\left(f_1 f_2'\right)(W_1)\right) + \mathbb{E}\left((f_1 f_2'')(W_1)\right)\\ &= -\mathbb{E}\left(\left(f_1 f_2'\right)'(W_1)\right) + \mathbb{E}\left((f_1 f_2'')(W_1)\right) = -\mathbb{E}\left(\left(f_1' f_2'\right)(W_1)\right).\end{aligned}$$

In terms of Dirichlet forms we have proved that

$$\mathcal{E}\left(f_1, f_2\right) = \mathcal{E}\left(f_2, f_1\right) = \pi\left(f_1' f_2'\right).$$

Notice that

$$\begin{aligned}\frac{\partial}{\partial x} P_t(f_1)(x) &= \frac{\partial}{\partial x}\mathbb{E}\left(f_1\left(e^{-t}\ x + \sqrt{1-e^{-2t}}\ W_1\right)\right)\\ &= e^{-t}\ \mathbb{E}\left(f_1'\left(e^{-t}\ x + \sqrt{1-e^{-2t}}\ W_1\right)\right) = e^{-t}\ P_t\left(f'\right)(x).\end{aligned}$$

This implies that

$$\mathcal{E}\left(P_t(f), P_t(f)\right) = \pi\left(P_t(f)' P_t(f)'\right) = e^{-2t}\int \pi(dx)\ \left(P_t\left(f'\right)(x)\right)^2.$$

Recalling that $\pi = \pi P_t$ we conclude that

$$\mathcal{E}\left(P_t(f), P_t(f)\right) \leq e^{-2t}\ \pi\left[\left(f'\right)^2\right] = e^{-2t}\ \mathcal{E}\left(f, f\right).$$

To end the proof, we recall the formula

$$\frac{d}{dt}\mathrm{Var}_\pi(P_t(f)) = -2\mathcal{E}\left(P_t(f), P_t(f)\right)$$

which we proved in exercise 187 in the context of finite state spaces. The same proof is valid for general state spaces.

$$\begin{aligned}\mathrm{Var}_\pi(P_t(f)) \to_{t\to\infty} 0 \Longrightarrow \mathrm{Var}_\pi(f) &= 2\int_0^\infty \mathcal{E}\left(P_t(f), P_t(f)\right)\ dt\\ &\leq 2\ \mathcal{E}\left(f, f\right)\ \int_0^\infty e^{-2t} dt = \mathcal{E}\left(f, f\right).\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 259:

- For any function f with compact support on S we clearly have that

$$\mathrm{Var}_\pi(f) := \pi[(f - \pi(f))^2] = \inf_{c\in\mathbb{R}} \pi[(f-c)^2] \leq \pi[(f - f(0))^2].$$

- For any function f with compact support on S s.t. $f(0)=0$, by a simple integration by parts we check that

$$\begin{aligned}
\pi(f^2) &= \int_0^\infty f(x)^2\ e^{-x}\ dx \\
&= -\int_0^\infty \left(\int_0^x \frac{\partial f^2}{\partial y}(y)\ dy\right)\ \frac{\partial}{\partial x}(e^{-x})\ dx \\
&= \int_0^\infty\ 2f(x)f'(x)\ e^{-x}\ dx = 2\ \pi(ff') \leq 2\pi(f^2)^{1/2}\ \pi((f')^2)^{1/2}.
\end{aligned}$$

  The last assertion is a consequence of the Cauchy-Schwartz inequality.

- We readily deduce the Poincaré inequality from the fact that

$$\left(\pi(f^2)\right)^2 \leq 4\ \pi(f^2)\ \pi((f')^2)\ \Leftrightarrow\ \mathrm{Var}_\pi(f) \leq 4\ \|f'\|_{\mathbb{L}_2(\pi)}.$$

This ends the proof of the exercise.

■

Solution to exercise 260:
The generator of X_t is defined by

$$L(f)(x) = -\frac{\lambda}{2}\ \mathrm{sign}(x)\ \partial_x f(x) + \frac{1}{2}\ \partial_x^2 f(x).$$

Observe that

$$\begin{aligned}
&\int e^{-\lambda|x|}\ L(f)(x)\ dx \\
&= \int_0^\infty e^{-\lambda x}\ \left(-\frac{\lambda}{2}\ \partial_x f(x) + \frac{1}{2}\ \partial_x^2 f(x)\right)\ dx + \int_{-\infty}^x\ e^{\lambda x}\ \left(\frac{\lambda}{2}\ \partial_x f(x) + \frac{1}{2}\ \partial_x^2 f(x)\right)\ dx.
\end{aligned}$$

A integration by parts yields

$$\begin{aligned}
&\int_0^\infty\ e^{-\lambda x}\ \left(-\frac{\lambda}{2}\ \partial_x f(x) + \frac{1}{2}\ \partial_x^2 f(x)\right)\ dx \\
&= -\frac{\lambda}{2}\ \left(\left[e^{-\lambda x} f(x)\right]_{x=0}^{x=\infty} + \int_0^\infty\ \lambda\ e^{-\lambda x}\ f(x)\ dx\right) \\
&\qquad + \frac{1}{2}\ \left(\left[e^{-\lambda x}\partial_x f(x)\right]_{x=0}^{x=\infty} + \lambda\left[e^{-\lambda x} f(x)\right]_{x=0}^{x=\infty} + \lambda^2\int_0^\infty e^{-\lambda x} f(x)\right) \\
&= \frac{\lambda}{2}\ \left(f(0) - \int_0^\infty\ \lambda\ e^{-\lambda x}\ f(x)\ dx\right) - \frac{1}{2}\ \left(\partial_x f(0) + \lambda f(0) - \lambda^2\int_0^\infty e^{-\lambda x} f(x)\right).
\end{aligned}$$

This implies that

$$\int_0^\infty\ e^{-\lambda x}\ \left(-\frac{\lambda}{2}\ \partial_x f(x) + \frac{1}{2}\ \partial_x^2 f(x)\right)\ dx = -\frac{1}{2}\partial_x f(0).$$

By symmetry arguments, we also have

$$\int_{-\infty}^{0} e^{-\lambda x} \left(-\frac{\lambda}{2}\,\partial_x f(x)+\frac{1}{2}\,\partial_x^2 f(x)\right)\,dx$$

$$= -\frac{\lambda}{2}\,\left(\left[e^{-\lambda x} f(x)\right]_{x=-\infty}^{x=0}+\int_{-\infty}^{0}\lambda\, e^{-\lambda x}\, f(x)\,dx\right)$$

$$+\frac{1}{2}\,\left(\left[e^{-\lambda x}\partial_x f(x)\right]_{x=-\infty}^{x=0}+\lambda\left[e^{-\lambda x} f(x)\right]_{x=-\infty}^{x=0}+\lambda^2\int_{-\infty}^{0} e^{-\lambda x} f(x)\right)$$

$$= -\frac{\lambda}{2}\,\left(f(0)+\int_{0}^{\infty}\lambda\, e^{-\lambda x}\, f(x)\,dx\right)+\frac{1}{2}\,\left(\partial_x f(0)+\lambda f(0)+\lambda^2\int_{0}^{\infty} e^{-\lambda x} f(x)dx\right).$$

$$\int_{\infty}^{0} e^{-\lambda x}\,\left(-\frac{\lambda}{2}\,\partial_x f(x)+\frac{1}{2}\,\partial_x^2 f(x)\right)\,dx=\frac{1}{2}\partial_x f(0).$$

We conclude that $\pi L(f)=0$.

This ends the proof of the exercise.

■

Solution to exercise 261:

We have

$$\begin{aligned}
dX_t &= d\left(U_t^2+V_t^2\right)\\
&= 2U_t\,dUt+2V_t\,dV_t+2\,dt\\
&= 2\left(1-(U_t^2+V_t^2)\right)\,dt+2\sqrt{U_t^2+V_t^2}\,\left(\frac{U_t}{\sqrt{U_t^2+V_t^2}}\,dW_t+\frac{V_t}{\sqrt{U_t^2+V_t^2}}\,dW_t'\right)\\
&= 2\left(1-X_t\right)\,dt+2\sqrt{X_t}\,d\overline{W}_t
\end{aligned}$$

with $d\overline{W}_t=\frac{U_t}{\sqrt{U_t^2+V_t^2}}\,dW_t+\frac{V_t}{\sqrt{U_t^2+V_t^2}}\,dW_t'$. By Levy's characterization, it is readily checked that $\overline{W}_t$ has the same distribution as a standard Brownian motion. We can also check this property by recalling that the Gaussian distributions are rotation-invariant. This ends the proof of the exercise.

■

Solution to exercise 262:

- The generator of X_t is given by

$$L(f)=2(1-x)f'(x)+2x\,f''(x).$$

For any functions f_1,f_2 with compact support on S by a simple integration by parts we check that

$$\begin{aligned}
\pi(f_1L(f_2)) &= \int_0^{\infty} f_1(x)\,\left(2(1-x)f_2'(x)+2x\,f_2''(x)\right)\,e^{-x}\,dx\\
&= 2\int_0^{\infty} f_2(x)\,\left[-\left(f_1(x)\,(1-x)e^{-x}\right)'+\left(x\,f_1(x)\,e^{-x}\right)''\right]\,dx\\
&= \pi(L(f_1)f_2).
\end{aligned}$$

The last assertion is a consequence of

$$-\left(f_1(x)\ (1-x)e^{-x}\right)' + \left(x\ f_1(x)\ e^{-x}\right)''$$

$$= e^{-x}(1-x)\ f_1(x) + e^{-x} f_1(x) - (1-x)e^{-x} f_1'(x)$$

$$+ \left(f_1(x)\ e^{-x} + x\ f_1'(x) e^{-x} - x\ f_1(x)\ e^{-x}\right)'$$

$$= e^{-x}(1-x)\ f_1(x) + e^{-x} f_1(x) - (1-x)e^{-x} f_1'(x)$$

$$+ f_1'(x)e^{-x} - f_1(x)e^{-x}$$

$$+ f_1'(x)e^{-x} + x f''(x) e^{-x} - x\ f_1'(x)e^{-x}$$

$$- f_1(x)\ e^{-x} - x f_1'(x)\ e^{-x} + x\ f_1(x)\ e^{-x}$$

$$= \tfrac{1}{2}\ L(f_1)(x)\ e^{-x}.$$

- Arguing as above, we have

$$\begin{aligned}\pi(f_1 L(f_2)) &= 2\int_0^\infty f_2'(x)\ \left((1-x)f_1(x)e^{-x}\right)\ dx - 2\int_0^\infty f_2'(x)\ \left(x f_1(x) e^{-x}\right)'\ dx \\ &= 2\int_0^\infty x\ f_2'(x)\ f_1'(x)\ e^{-x}\ dx.\end{aligned}$$

Here we used the fact that

$$\left(x f_1(x) e^{-x}\right)' = \left((1-x)f_1(x)e^{-x}\right) + x\ f_1'(x)\ e^{-x}.$$

This ends the proof of the exercise.

■

Solution to exercise 263:

Using the Doeblin-Itō formula to the function $f(x) = 1/x$ ($\Rightarrow f'(x) = -1/x^2$ and $f''(x) = 2/x^3$) we prove that

$$\begin{aligned} d(X_t^{-1}) &= f'(X_t)\ dX_t\ + \frac{1}{2}\ f''(X_t) dX_t dX_t \\ &= -\frac{1}{X_t^2}\ (b_t\ X_t\ dt + \ \sigma_t\ X_t\ dW_t) + \frac{1}{X_t^3}\ \sigma_t^2\ X_t^2\ dt \\ &= \left(\sigma_t^2 - b_t\right)\ X_t^{-1}\ dt\ -\ \sigma_t\ X_t^{-1}\ dW_t. \end{aligned}$$

This clearly yields

$$X_t^{-1} \ = \ e^{\int_0^t \left(\left[\sigma_s^2 - b_s\right] - \frac{\sigma_s^2}{2}\right) ds - \int_0^t \ \sigma_s\ dW_s} = e^{-\int_0^t \left(b_s - \frac{\sigma_s^2}{2}\right) ds - \int_0^t \ \sigma_s\ dW_s}.$$

By the integration by parts formula (14.20) we have

$$\begin{aligned} dZ_t &= d\left(Y_t X_t^{-1}\right) \\ &= Y_t\ d(X_t^{-1}) + X_t^{-1}\ dY_t + dY_t d(X_t^{-1}) \\ &= \left(\sigma_t^2 - b_t\right)\ Z_t\ dt\ -\ \sigma_t\ Z_t\ dW_t + \left(a_t X_t^{-1}\ + b_t\ Z_t\right)\ dt \\ &\qquad + \ \left(\tau_t X_t^{-1}\ + \sigma_t\ Z_t\right)\ dW_t - \ \left(\tau_t X_t^{-1} + \sigma_t\ Z_t\right) \sigma_t\ dt \\ &= \left(a_t - \sigma_t \tau_t\right) X_t^{-1}\ dt + \tau_t X_t^{-1}\ dW_t. \end{aligned}$$

Recalling that $Z_0 = Y_0/X_0 = Y_0$ we obtain

$$Z_t = Y_0 + \int_0^t \left[(a_s - \sigma_t \tau_t)\, X_s^{-1}\; ds + \tau_s X_s^{-1}\; dW_s \right].$$

Therefore

$$\begin{aligned} Y_t = \quad & X_t \left[Y_0 + \int_0^t \left(X_s^{-1}\; (a_s - \sigma_t \tau_t)\; ds + X_s^{-1}\; \tau_s\; dW_s \right) \right] \\ = \quad & X_t Y_0 + \int_0^t\; X_t X_s^{-1}\; (a_s - \sigma_s \tau_s)\; ds + \int_0^t\; X_t X_s^{-1}\; \tau_s\; dW_s \end{aligned}$$

with

$$X_t X_s^{-1} = \exp\left(\int_s^t \left(b_r - \frac{\sigma_r^2}{2} \right) dr + \int_s^t \sigma_r dW_r \right).$$

This ends the proof of the exercise. ■

Solution to exercise 264:
We have

$$\mathbb{E}\left(\left(W_s - \frac{s}{t}\; W_t \right)\; W_t \right) = \mathbb{E}(W_s W_t) - \frac{s}{t}\; \mathbb{E}(W_t^2).$$

Recalling that $W_0 = 0$, and the increments are centered and independent we prove

$$\mathbb{E}(W_s W_t) = \mathbb{E}(W_s[(W_t - W_s) + W_s]) = \underbrace{\mathbb{E}((W_s - W_0)(W_t - W_s))}_{=0} + \mathbb{E}(W_s^2) = s.$$

This implies that

$$\mathbb{E}\left(\left(W_s - \frac{s}{t}\; W_t \right)\; W_t \right) = s - \frac{s}{t}\; t = 0 = \mathbb{E}(W_s - \frac{s}{t}\; W_t) \times \mathbb{E}(W_t).$$

This shows that the Gaussian random variables $\left(\left(W_s - \frac{s}{t}\; W_t \right),\; W_t \right)$ are uncorrelated, thus independent.

Since $\left(W_s - \frac{s}{t}\; W_t \right)$ and W_t are independent, we have

$$0 = \mathbb{E}(W_s - \frac{s}{t}\; W_t) = \mathbb{E}(W_s - \frac{s}{t}\; W_t \mid\; W_t) = \mathbb{E}(W_s | W_t) - \frac{s}{t}\; W_t.$$

We conclude that

$$\mathbb{E}(W_s | W_t) = \frac{s}{t}\; W_t.$$

In much the same way, the variance is given by

$$\begin{aligned} \mathrm{Var}(W_s \mid W_t) \quad &= \quad \mathbb{E}\left((W_s - \mathbb{E}(W_s \mid W_t))^2 \mid \right) \\ &= \quad \mathbb{E}\left((W_s - \frac{s}{t}\; W_t)^2 \mid W_t \right) = \mathbb{E}\left(\left(W_s - \frac{s}{t}\; W_t \right)^2 \right) \quad \text{(by independence)} \\ &= \quad \mathbb{E}\left(W_s^2 \right) + \left(\frac{s}{t} \right)^2\; \mathbb{E}(W_t^2) - 2\; \frac{s}{t}\; \mathbb{E}(W_s W_t) \\ &= \quad s + \left(\frac{s}{t} \right)^2\; t - 2s\; \left(\frac{s}{t} \right) = s\left(1 - \frac{s}{t} \right). \end{aligned}$$

We conclude that $W_s - \frac{s}{t}\; W_t$ is a Gaussian random variable with

$$\mathrm{Law}\left(W_s \mid W_t \right) = \mathcal{N}\left(\frac{s}{t}\; W_t, s\left(1 - \frac{s}{t} \right) \right).$$

This implies that

$$\text{Law}\left(W_s - \frac{s}{t}\, W_t \mid W_t\right) = \text{Law}\left(W_s - \frac{s}{t}\, W_t\ \right) = \mathcal{N}\left(0, s\left(1-\frac{s}{t}\right)\right)$$

and

$$\text{Law}\left(W_s \mid W_t = 0\right) = \mathcal{N}\left(0, s\left(1-\frac{s}{t}\right)\right) = \text{Law}\left(W_s - \frac{s}{t}\, W_t\ \right).$$

The last assertion follows from the fact that $s \leq t \Rightarrow W_s - \frac{s}{t}\, W_t = W_s - sW_1$. This ends the proof of the exercise.

■

Solution to exercise 265:

For any fixed time horizon t, and $0 \leq s \leq t$ by applying the Doeblin-Itō formula to the function $f(s, X_s) = \dfrac{X_s - b}{t-s}$ we have

$$\begin{aligned} d\left(\frac{X_s - b}{t-s}\right) &= \frac{X_s - b}{(t-s)^2}\, ds + \frac{dX_s}{t-s} \\ &= \frac{X_s - b}{(t-s)^2}\, ds + \frac{1}{t-s}\left(\frac{b - X_s}{t-s}\, ds + dW_s\right) = \frac{dW_s}{t-s}. \end{aligned}$$

This implies that

$$\forall 0 \leq r \leq s \leq t \qquad \frac{X_s - b}{t-s} - \frac{X_r - b}{t-r} = \int_r^s \frac{dW_u}{t-u}.$$

We conclude that

$$\begin{aligned} X_s &= b + \frac{t-s}{t-r}\,(X_r - b) + \int_r^s \frac{t-s}{t-u}\, dW_u \\ &= b\left(1 - \frac{t-s}{t-r}\right) + \frac{(t-s)}{t-r}\, X_r + \int_r^s \frac{(t-s)}{t-u}\, dW_u \\ &= \frac{s-r}{t-r}\, b + \frac{t-s}{t-r}\, X_r + \int_r^s \frac{t-s}{t-u}\, dW_u. \end{aligned}$$

This ends the proof of the first assertion. We deduce that $X_t = b$, as well as

$$\mathbb{E}\left(X_s \mid X_r\right) = \frac{s-r}{t-r}\, b + \frac{t-s}{t-r}\, X_r$$

and

$$\begin{aligned} \text{Var}\left(X_s \mid X_r\right) &= \mathbb{E}\left(\left[\int_r^s \frac{t-s}{t-u}\, dW_u\right]^2\right) \\ &= \int_r^s \left(\frac{t-s}{t-u}\right)^2\, du = (t-s)^2 \int_r^s \frac{1}{(t-u)^2}\, du \\ &= (t-s)^2 \int_r^s \frac{\partial}{\partial u}\left(\frac{1}{t-u}\right)\, du \\ &= (t-s)^2\left[\frac{1}{t-s} - \frac{1}{t-r}\right] = (t-s)\, \frac{s-r}{t-r}. \end{aligned}$$

We conclude that the conditional distribution of X_s given X_r is given by the Gaussian distribution

$$\mathrm{Law}\,(X_s \mid X_r) = \mathcal{N}\left(\frac{(t-s)X_r + (s-r)b}{t-r},\ (s-r)\ \frac{t-s}{t-r}\right).$$

This ends the proof of the exercise.

■

Solution to exercise 266:

Applying the Doeblin-Itō formula to the functions $b(X_t)$ and $\sigma(X_t)$ we have

$$\begin{aligned} db(X_t) &= L(b)(X_t)dt + b'(X_t)\sigma(X_t)dW_t \\ d\sigma(X_t) &= L(\sigma)(X_t)dt + \sigma'(X_t)\sigma(X_t)dW_t \end{aligned}$$

with the infinitesimal generator L given for any smooth function f by

$$L(f) = bf' + \frac{1}{2}\sigma^2 f''.$$

This shows that for any $s \leq t$ we have

$$\begin{aligned} b(X_s) &= b(X_t) + \int_t^s L(b)(X_r)dr + \int_t^s b'(X_r)\sigma(X_r)dW_r \\ \sigma(X_s) &= \sigma(X_t) + \int_t^s L(\sigma)(X_r)dr + \int_t^s \sigma'(X_r)\sigma(X_r)dW_r \end{aligned}$$

from which we prove that

$$\begin{aligned} X_{t+h} - X_t &= \int_t^{t+h} b(X_s)ds + \int_t^{t+h} \sigma(X_s)dW_s \\ &= \int_t^{t+h} \left[b(X_t) + \int_t^s L(b)(X_r)dr + \int_t^s b'(X_r)\sigma(X_r)dW_r\right] ds \\ &\quad + \int_t^{t+h} \left[\sigma(X_t) + \int_t^s L(\sigma)(X_r)dr + \int_t^s \sigma'(X_r)\sigma(X_r)dW_r\right] dW_s. \end{aligned}$$

This yields the second order approximation

$$\begin{aligned} X_{t+h} - X_t &= b(X_t)\ h + \sigma(X_t)\ (W_{t+h} - W_t) \\ &\quad + \int_t^{t+h} \left[\int_t^s L(b)(X_r)dr + \int_t^s b'(X_r)\sigma(X_r)dW_r\right] ds \\ &\quad + \int_t^{t+h} \left[\int_t^s L(\sigma)(X_r)dr + \int_t^s \sigma'(X_r)\sigma(X_r)dW_r\right] dW_s. \end{aligned}$$

Using the fact that

$$\begin{aligned} \mathbb{E}\left(\left[\frac{1}{h}\int_t^{t+h}\left[\int_t^s f(X_r)\sigma(X_r)dW_r\right] ds\right]^2\right) &\leq \frac{1}{h}\int_t^{t+h} \mathbb{E}\left(\left[\int_t^s f(X_r)\sigma(X_r)dW_r\right]^2\right) \\ &= \frac{1}{h}\int_t^{t+h} \mathbb{E}\left(\int_t^s f(X_r)^2 dr\right) \leq \|f\|^2\ h \end{aligned}$$

for any smooth function f with bounded derivatives, we check that

$$X_{t+h} - X_t = b(X_t)\ h + \sigma(X_t)\ (W_{t+h} - W_t) + \int_t^{t+h} \left[\int_t^s\ \overline{\sigma}(X_r)dW_r\right] dW_s + R_{t,t+h}$$

with $\overline{\sigma} := \sigma'\sigma$ and some second order remainder random function $R_{t,t+h}$ such that

$$\mathbb{E}\left(|R_{t,t+h}|^2\right)^{1/2} \leq (\|L(b)\| + \|L(\sigma)\| + \|b'\sigma\|)\ \ h^{3/2}.$$

To take the final step, we apply the Doeblin-Itō formula to the functions $\overline{\sigma}$

$$d\overline{\sigma}(X_t) = L(\overline{\sigma})(X_t)dt + \overline{\sigma}'(X_t)\sigma(X_t)dW_t.$$

This yields for any $t \leq r$

$$\overline{\sigma}(X_r) = \overline{\sigma}(X_t) + \int_t^r L(\overline{\sigma})(X_u)du + \int_t^r \overline{\sigma}'(X_u)\sigma(X_u)dW_u$$

$$\Rightarrow \int_t^s\ \overline{\sigma}(X_r)dW_r = \overline{\sigma}(X_t)\ (W_s - W_t) + \int_t^s \left[\int_t^r L(\overline{\sigma})(X_u)du + \int_t^r \overline{\sigma}'(X_u)\sigma(X_u)dW_u\right] dW_r$$

$$\begin{aligned}
&\mathbb{E}\left(\left[\int_t^{t+h} \left[\int_t^s\ \left(\int_t^r \overline{\sigma}'(X_u)\sigma(X_u)dW_u\right) dW_r\right] dW_s\right]^2\right)\\
&= \int_t^{t+h} \mathbb{E}\left(\left[\int_t^s\ \left(\int_t^r \overline{\sigma}'(X_u)\sigma(X_u)dW_u\right) dW_r\right]^2\right)\ ds\\
&= \int_t^{t+h} \left[\int_t^s\ \mathbb{E}\left(\left(\int_t^r \overline{\sigma}'(X_u)\sigma(X_u)dW_u\right)^2\right)\ dr\right]\ ds\\
&= \int_t^{t+h} \left[\int_t^s\ \left(\int_t^r \mathbb{E}\left((\overline{\sigma}'(X_u)\sigma(X_u))^2\right)\ du\right]\ dr\right]\ ds \leq \|\overline{\sigma}'\sigma\|^2\ \ \frac{h^3}{3!}.
\end{aligned}$$

In much the same way, we find that

$$\begin{aligned}
&\mathbb{E}\left(\left[\int_t^{t+h} \left[\int_t^s\ \left(\int_t^r L(\overline{\sigma})(X_u)du\right) dW_r\right] dW_s\right]^2\right)\\
&= \int_t^{t+h} \left[\int_t^s\ \mathbb{E}\left(\left(\int_t^r L(\overline{\sigma})(X_u)du\right)^2\right)\ dr\right]\ ds \leq \|L(\overline{\sigma})\|^2\ \ \frac{h^4}{12}.
\end{aligned}$$

We conclude that

$$X_{t+h} - X_t = b(X_t)\ h + \sigma(X_t)\ (W_{t+h} - W_t) + \overline{\sigma}(X_t)\ \int_t^{t+h} (W_s - W_t)\ dW_s + \overline{R}_{t,t+h}$$

with some second order remainder random function $\overline{R}_{t,t+h}$ such that

$$\mathbb{E}\left(|R_{t,t+h}|^2\right)^{1/2} \leq (\|L(b)\| + \|L(\sigma)\| + \|b'\sigma\| + \|\overline{\sigma}'\sigma\| + \|L(\overline{\sigma})\|)\ \ h^{3/2}.$$

Finally, for any fixed t and $t \leq s \leq t+h$ we have

$$d(W_s - W_t)^2 = 2(W_s - W_t)dW_s + dW_s dW_s = 2(W_s - W_t)dW_s + ds$$

$$\Rightarrow \int_t^{t+h} (W_s - W_t)\ dW_s = \frac{1}{2}\ \left[(W_{t+h} - W_t)^2 - h\right].$$

This ends the proof of the exercise.

■

Chapter 15

Solution to exercise 267:

Recalling that the jumps times of a Poisson process are uniform on a given interval, we sample first a Poisson random variable N_t with parameter (λt). Given $N_t = n$ we sample n uniform random variables $T_1, \ldots, T_n$ in the interval $[0, t]$, and n independent random variables $Y_1, \ldots, Y_n$ with common distribution μ. The process V_t is now easily simulated. To sample the Brownian motion we simply sample a sequence of independent and centered Gaussian random variables $(W_{t_{k+1}} - W_{t_k})$ with variance $(t_{k+1} - t_k)$. The process X_t is now easily simulated on the time mesh.

Since (N_t, W_t, Y) are independent, we have

$$\begin{aligned}\phi_t(u) &:= \mathbb{E}(e^{iuX_t}) = e^{iuat}\ \mathbb{E}(e^{iubV_t})\ \mathbb{E}(e^{iucW_t})\\ &= e^{iuat - \frac{(uc)^2 t}{2}}\ \mathbb{E}(\mathbb{E}(e^{iubV_t} \mid N_t)).\end{aligned}$$

On the other hand, we also have that

$$\mathbb{E}(\mathbb{E}(e^{iubV_t} \mid N_t)) = \mathbb{E}(\mathbb{E}(e^{iubY_1})^{N_t}) = e^{-\lambda t}\ e^{\lambda t\ \mathbb{E}(e^{iubY_1})}.$$

This implies that

$$\phi_t(u) = \exp\left(t \left\{ iua - \frac{(uc)^2}{2} + \lambda\left(\phi_Y(u) - 1\right)\right\}\right).$$

This ends the proof of the exercise.

■

Solution to exercise 268:

We have

$$Z_t = Z_0\ e^{X_t} = Z_0\ e^{at+bV_t+cW_t} = Z_0\ e^{at+cW_t} \prod_{1\leq n\leq N_t} e^{bY_n}.$$

We let T_n be the random jump times of the Poisson process. In this notation, we have

$$Z_{T_n} - Z_{T_n-} = Z_{T_n-}\ \left(e^{bY_n} - 1\right) = Z_{T_n-}\ dU_{T_n}$$

with

$$U_t = \sum_{1\leq n\leq N_t} \left(e^{bY_n} - 1\right).$$

Between the jump times $t \in [T_n, T_{n+1}[$, applying the Doeblin-Itō formula to the function $f(t,x) = e^{at+cx}$ we find that

$$Z_t = Z_0\ e^{at+cW_t} \prod_{1\leq k\leq n} e^{bY_k} \Rightarrow dZ_t = Z_0 \left(\prod_{1\leq k\leq n} e^{bY_k}\right) \left(\left(a + \frac{c^2}{2}\right)\ dt +\ c\ dW_t\right) f(t, W_t).$$

This implies that

$$\forall t \in [T_n, T_{n+1}[\qquad dZ_t = Z_t \, \left(\left(a + \frac{c^2}{2} \right) \, dt + \, c \, dW_t \right).$$

We conclude that

$$dZ_t = Z_t \, \left(a + \frac{c^2}{2} \right) \, dt + \, c \, Z_t \, dW_t + Z_t \, dU_t.$$

This ends the proof of the exercise.

■

Solution to exercise 269:
The infinitesimal generator of the Levy process X_t is given by

$$L_t^X(f)(x) = a \, f' + \frac{c^2}{2} \, f''(x) + \lambda \int \, (f(x+by) - f(x)) \, \mu(dy).$$

The infinitesimal generator of the exponential Levy process Z_t is given by

$$L_t^Z(f)(x) = \left(a + \frac{c^2}{2} \right) \, x \, f'(x) + \frac{(cx)^2}{2} \, f''(x) + \lambda \int \, (f(x(e^{by}-1)) - f(x)) \, \mu(dy).$$

This ends the proof of the exercise.

■

Solution to exercise 270:
We set

$$dX_t^c = a_t(X_t)dt + b_t(X_t)dW_t.$$

In this notation, the Doeblin-Itō formula (15.11) takes the form

$$df(t,X_t) = \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t^c + \frac{1}{2} \, \frac{\partial^2 f}{\partial x^2}(t,X_t)dX_t^c dX_t^c + \Delta f(t,X_t)$$

with the jump increment

$$\begin{array}{rcl} \Delta f(t,X_t) & := & f(t, X_t + (X_{t+dt} - X_t)) - f(t,X_t) \\ & = & f(t, X_t + \Delta X_t) - f(t,X_t) \\ & = & [f(t, X_t + c_t(X_t)) - f(t,X_t)] \ \ dN_t. \end{array}$$

We notice that

$$\mathbb{E}\left(\Delta f(t,X_t) \mid \mathcal{F}_t\right) = \lambda_t(X_t) \ \ [f(t, X_t + c_t(X_t)) - f(X_t)] \ \ dt.$$

This implies that

$$\begin{array}{rcl} df(t,X_t) & = & \left(\frac{\partial}{\partial t} + L_t^c \right) f(t,X_t) \ dt + \frac{\partial f}{\partial x}(t,X_t) b_t(X_t) dW_t \\ & & + [f(t, X_t + c_t(X_t)) - f(t,X_t)] \ \ dN_t \end{array}$$

with

$$L_t^c(f)(t,x) = a_t(x) \ \frac{\partial f}{\partial x}(t,x) + \frac{1}{2} \ b_t(x)^2 \ \frac{\partial^2 f}{\partial x^2}(t,x).$$

In terms of the martingales

$$\begin{array}{rcl} dM_t^c(f) & := & \frac{\partial f}{\partial x}(t,X_t)b_t(X_t)dW_t \\ dM_t^d(f) & := & [f(t,X_t+c_t(X_t))-f(t,X_t)]\ \ [dN_t-\lambda_t(X_t)dt] \end{array}$$

the above formula takes the following form:

$$df(t,X_t)=\left(\frac{\partial}{\partial t}+L_t\right)f(t,X_t)\ dt+dM_t(f)$$

with the martingale $dM_t(f)=dM_t^c(f)+dM_t^d(f)$, and the infinitesimal generator

$$L_t(f)=L_t^c(f)+L_t^d(f)\quad \text{with}\quad L_t^d(f)(t,x)=\lambda_t(x)\ (f(t,x+c_t(x))-f(t,x)).$$

This ends the proof of the exercise.

■

Solution to exercise 271:
For each $1\leq i\leq r$ we have

$$dX_t^i:=b_t^i(X_t)\ dt+\sum_{1\leq j\leq d}\sigma_{j,t}^i(X_t)\ dW_t^j+\sum_{1\leq j\leq d}c_{t,j}^i(x)\ dN_t^j$$

with a collection of Poisson processes N_t^i with intensity $\lambda_t^{(i)}(X_t)$. At rate $\lambda_t^{(i)}(X_t)$ we have $dN_t^i=1$ and the jump of the process is defined by

$$x=\left(\begin{array}{c}x_1\\ \vdots\\ x_r\end{array}\right)\rightsquigarrow x=\left(\begin{array}{c}x_1+c_{t,i}^1(x)\\ \vdots\\ x_r+c_{t,i}^r(x)\end{array}\right)=x+c_{t,i}(x)$$

with the column vector $c_{t,i}(x)=\left(c_{t,i}^j(x)\right)_{1\leq j\leq r}^T$. If we set $M_t^{(i)}(x,dy)=\delta_{x+c_{t,i}(x)}(dy)$ then the generator is given by

$$L_t=L_t^c+L_t^d$$

with the generator L_t^c of the pure diffusion process

$$dX_t\ =\ a_t(X_t)\ dt+b_t(X_t)\ dW_t$$

and the jump generator L_t^d defined in (15.17). This ends the proof of the exercise.

■

Solution to exercise 272:
For any sufficiently regular functions $g(t,x)$ we have

$$dg(s,X_s)=[\partial_s+L_s]\,g(s,X_s)ds+dM_s(g)$$

for some martingale $M_s(f)$ with angle bracket

$$\langle M(g)\rangle_s=\int_0^s\ \Gamma_{L_u}(P_{u,s}(g),P_{u,s}(g))(X_u)\ du.$$

We fix a given time horizon $t \geq 0$. By applying this formula to the function $g(s,x) = P_{s,t}(f)(x)$ for $s \in [0,t]$ and using the backward evolution equations (15.16) we have

$$\partial_s g(s,x) + L_s g_s(x) = 0 \Rightarrow P_{s,t}(f)(X_s) = M_s(g).$$

This shows that

$$f(X_t) - P_{0,t}(f)(X_0) = M_t(g) - M_0(g)$$

$$\Rightarrow \ \mathbb{E}\left[(f(X_t) - P_{0,t}(f)(X_0))^2\right] = \mathbb{E}\left(\langle M(g)\rangle_t\right) = \int_0^t \ \eta_s\left(\Gamma_{L_s}(P_{s,t}(g), P_{s,t}(g))\right) \ ds.$$

On the other hand, we have

$$\underbrace{\mathbb{E}\left[(f(X_t) - \mathbb{E}(f(X_t)))^2\right]}_{=\eta_t[(f-\eta_t(f))^2]} = \mathbb{E}\left[(f(X_t) - P_{0,t}(f)(X_0))^2\right] + \underbrace{\mathbb{E}\left[(P_{0,t}(f)(X_0) - \mathbb{E}(f(X_t)))^2\right]}_{=\eta_0\left[(P_{0,t}(f)-\eta_0[P_{0,t}(f)])^2\right]}$$

and this ends the proof of the exercise.

■

Solution to exercise 273:

The first assertion is immediate since $N^{-1}\ \mathcal{X}_t = N^{-1}\ \sum_{1\leq i\leq N} \delta_{\xi_t^i}$ is the empirical measure associated with N independent copies of X_t.

For functions of the form $f(\xi_t) = F(\mathcal{X}_t(\varphi))$, the Doeblin-Itō formula (15.22) takes the form

$$df(\xi_t) = \mathcal{L}_t(f)(\xi_t)dt + d\mathcal{M}_t(f)$$

with a martingale with angle bracket defined by the formulae

$$\langle \mathcal{M}(f), \mathcal{M}(f)\rangle_t = \int_0^t \ \Gamma_{L_s}(f(s,.), f(s,.))(\xi_s) \ ds.$$

If we choose the empirical type functions

$$f_1(\xi_t) = \mathcal{X}_t(\varphi) \quad \text{and} \quad f_2(\xi_t) = (\mathcal{X}_t(\varphi))^2 = f_1(\xi_t)^2$$

then we have

$$f_1(x_1,\ldots,x_i,\ldots,x_N) = f(x_1) + \ldots + f(x_i) + \ldots + f(x_N) \qquad \text{and} \quad L_t(1) = 0$$

$$\Rightarrow L_t^{(i)} f_1(x_1,\ldots,x_i,\ldots,x_N) = L(\varphi)(x_i)$$

$$\Rightarrow \mathcal{L}_t(f_1)(x) = \sum_{1\leq i\leq N} L_t^{(i)} f(x_1,\ldots,x_i,\ldots,x_N) = \sum_{1\leq i\leq N} L(f)(x_i) = Nm(x)(f).$$

In much the same way, we check that

$$[f_1(x_1,\ldots x_{i-1},y_i,x_{i+1},\ldots,x_N) - f_1(x)]^2 = [\varphi(y_i) - \varphi(x_i)]^2$$

$$\Rightarrow \mathcal{L}_t[(f_1 - f_1(x))^2](x) = \sum_{1\leq i\leq N} L_t\left[\varphi - \varphi(x_i)\right]^2(x_i)$$

$$\Rightarrow \Gamma_{\mathcal{L}_t}(f_1,f_1)(x) = \mathcal{L}_t[(f_1 - f_1(x))^2](x) = Nm(x)\left(\Gamma_{L_t}(\varphi,\varphi)\right).$$

This shows that

$$d\mathcal{X}_t(\varphi) = \mathcal{X}_t(L_t(\varphi))\ dt + d\mathcal{M}_t(f_1)$$

with a martingale $\mathcal{M}_t(f_1)$ with angle bracket

$$\langle \mathcal{M}(f_1), \mathcal{M}(f_1)\rangle_t = \int_0^t \mathcal{X}_s\left(\Gamma_{L_s}(\varphi,\varphi)\right)\ ds.$$

This yields the evolution equations

$$\mu_t(f) := \mathbb{E}\left(\mathcal{X}_t(\varphi)\right) \Rightarrow \partial_t \mathbb{E}\left(\mathcal{X}_t(\varphi)\right) = \partial_t \mu_t(\varphi) = \mu_t(L_t(\varphi)) = \mathbb{E}\left(\mathcal{X}_t(L_t(\varphi))\right)\ dt.$$

We conclude that

$$\mu_0(\varphi) = N\ \eta_0(\varphi) \Rightarrow \mu_t(\varphi) = N\ \eta_t(\varphi).$$

We fix the time horizon t. Applying the Doeblin-Itō formula to the function $f(s,x) = \sum_{1\leq i\leq N} P_{s,t}(\varphi)(x_i)$ w.r.t. $s \in [0,t]$ we have

$$d\mathcal{X}_s(P_{s,t}(\varphi)) \quad = \quad \mathcal{X}_s(\underbrace{\partial_s P_{s,t}(\varphi) + L_s(P_{s,t}(\varphi))}_{=0})\ ds + dM_s^{[t]}(\varphi) = dM_s^{[t]}(\varphi)$$

with a martingale $M_s^{[t]}(\varphi)$ with angle bracket

$$\forall s \in [0,t] \qquad \left\langle M^{[t]}(\varphi), M^{[t]}(\varphi)\right\rangle_s = \int_0^s \mathcal{X}_\tau\left(\Gamma_{L_\tau}(P_{\tau,t}(\varphi), P_{\tau,t}(\varphi))\right)\ d\tau.$$

We conclude that

$$\mathcal{X}_t(P_{t,t}(\varphi)) - \mathcal{X}_s(P_{s,t}(\varphi)) = \mathcal{X}_t(\varphi) - \mathcal{X}_s(P_{s,t}(\varphi)) = M_t^{[t]}(\varphi) - M_s^{[t]}(\varphi)$$

$$\Longrightarrow\ \mathbb{E}\left(\mathcal{X}_t(\varphi) \mid \mathcal{X}_s\right) = \mathcal{X}_s(P_{s,t}(\varphi),$$

as well as

$$\begin{aligned}
\mathbb{E}\left(\left[\mathcal{X}_t(\varphi) - \mathcal{X}_0(P_{0,t}(\varphi))\right]^2\right) &= \mathbb{E}\left(\int_0^t \mathcal{X}_\tau\left(\Gamma_{L_\tau}(P_{\tau,t}(\varphi), P_{\tau,t}(\varphi)\right)\ d\tau\right) \\
&= N \int_0^t \eta_\tau\left(\Gamma_{L_\tau}(P_{\tau,t}(\varphi), P_{\tau,t}(\varphi)\right)\ d\tau.
\end{aligned}$$

On the other hand, applying the Doeblin-Itō formula to the function $f(s,x) = P_{s,t}(\varphi)(x)$ we find that

$$dP_{s,t}(\varphi)(X_s) = \underbrace{(\partial_s P_{s,t}(\varphi) + L_s(P_{s,t}(\varphi))(X_s)\ )}_{=0}\ ds + dM_s(P_{\bullet,t}(\varphi))$$

with a martingale $M_s(P_{\bullet,t}(\varphi))$ with angle bracket

$$\forall s \in [0,t] \qquad \langle M(P_{\bullet,t}(\varphi)), M(P_{\bullet,t}(\varphi))\rangle_s = \int_0^s \left(\Gamma_{L_\tau}(P_{\tau,t}(\varphi), P_{\tau,t}(\varphi)\right)(X_\tau)\ d\tau.$$

This implies that

$$\begin{aligned}
\mathbb{E}\left(\left[\mathcal{X}_t(\varphi) - \mathcal{X}_0(P_{0,t}(\varphi))\right]^2\right) &= N \int_0^s \mathbb{E}\left(\left(\Gamma_{L_\tau}(P_{\tau,t}(\varphi), P_{\tau,t}(\varphi)\right)(X_\tau)\right)\ d\tau \\
&= N\ \mathbb{E}\left(\left[P_{t,t}(\varphi)(X_t) - P_{0,t}(\varphi)(X_0)\right]^2\right) \\
&= N\ \mathbb{E}\left(\left[\varphi(X_t) - P_{0,t}(\varphi)(X_0)\right]^2\right).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathbb{E}\left([\mathcal{X}_0(P_{0,t}(\varphi)) - N\ \eta_t(\varphi)]^2\right) &= \mathbb{E}\left([\mathcal{X}_0([P_{0,t}(\varphi) - \eta_t(\varphi)]]^2\right)\\ &= N\ \eta_0\left([P_{0,t}(\varphi) - \eta_t(\varphi)]^2\right)\end{aligned}$$

from which we conclude that

$$\begin{aligned}\mathbb{E}\left([\mathcal{X}_t(\varphi) - N\ \eta_t(\varphi)]^2\right) &= \mathbb{E}\left([[\mathcal{X}_t(\varphi) - \mathcal{X}_0(P_{0,t}(\varphi))] + [\mathcal{X}_0(P_{0,t}(\varphi)) - N\ \eta_t(\varphi)]]^2\right)\\ &= \mathbb{E}\left([\mathcal{X}_t(\varphi) - \mathcal{X}_0(P_{0,t}(\varphi))]^2\right) + \mathbb{E}\left([\mathcal{X}_0(P_{0,t}(\varphi)) - N\ \eta_t(\varphi)]^2\right)\\ &= N\ \left(\mathbb{E}\left([\varphi(X_t) - P_{0,t}(\varphi)(X_0)]^2\right) + \eta_0\left([P_{0,t}(\varphi) - \eta_t(\varphi)]^2\right)\right)\\ &= N\ \eta_t\left([\varphi - \eta_t(\varphi)]^2\right).\end{aligned}$$

Using the integration by parts formula (15.24) (or equivalently, the definition of the carré du champ operator Γ_{L_t} associated with some generator L_t) we have

$$\mathcal{L}_t(f_2) = \mathcal{L}_t(f_1^2) = 2f_1\ \mathcal{L}_t(f_1) + \Gamma_{\mathcal{L}_t}(f_1, f_1).$$

This shows that

$$d\left(\mathcal{X}_t(\varphi)\right)^2 = [2\ \mathcal{X}_t(\varphi)\ \mathcal{X}_t(L_t(\varphi)) + \mathcal{X}_t\left(\Gamma_{L_t}(\varphi,\varphi)\right)]\ dt + d\mathcal{M}_t(f_2)$$

with a martingale $\mathcal{M}_t(f_2)$. We fix the time horizon t. An application of the Doeblin-Itō formula to the function $f(s,x) = \left(\sum_{1\le i\le N} P_{s,t}(\varphi)(x_i)\right)^2$ w.r.t. $s \in [0,t]$ leads us to

$$\begin{aligned}d\left(\mathcal{X}_s P_{s,t}(\varphi)\right)^2 &= [2\ \mathcal{X}_s(P_{s,t}(\varphi))\ \mathcal{X}_s(\partial_s P_{s,t}(\varphi)) + 2\ \mathcal{X}_s(P_{s,t}(\varphi))\ \mathcal{X}_s(L_s(P_{s,t}(\varphi)))\\ &\qquad + \mathcal{X}_s\left(\Gamma_{L_s}(\varphi,\varphi)\right)]\ ds + d\mathcal{M}_s^{[t]}(f)\\ &= \mathcal{X}_s\left(\Gamma_{L_s}(P_{s,t}(\varphi), P_{s,t}(\varphi)\right)\ ds + d\mathcal{M}_s^{[t]}(f)\end{aligned}$$

with some martingale $\mathcal{M}_s^{[t]}(f)$, $s \in [0,t]$. We conclude that

$$\begin{aligned}\mathbb{E}\left((\mathcal{X}_t(\varphi))^2\right) - \mathbb{E}\left((\mathcal{X}_0 P_{0,t}(\varphi))^2\right) &= \mathbb{E}\left((\mathcal{X}_t P_{t,t}(\varphi))^2\right) - \mathbb{E}\left((\mathcal{X}_0 P_{0,t}(\varphi))^2\right)\\ &= \int_0^t \mathbb{E}\left(\mathcal{X}_s\left(\Gamma_{L_s}(P_{s,t}(\varphi), P_{s,t}(\varphi))\right)\right)\ ds\\ &= N\int_0^t \eta_s\left(\Gamma_{L_s}(P_{s,t}(\varphi), P_{s,t}(\varphi)\right)\ ds\\ &= N\ \mathbb{E}\left([\varphi(X_t) - P_{0,t}(\varphi)(X_0)]^2\right).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathbb{E}\left((\mathcal{X}_0 P_{0,t}(\varphi))^2\right) &= N\ \eta_0\left([P_{0,t}(\varphi)]^2\right) + N(N-1)\ \left(\eta_t\left(\varphi\right)\right)^2\\ &= N\ \eta_0\left([P_{0,t}(\varphi) - \eta_t(\varphi)^2]\right) + N^2\ \left(\eta_t\left(\varphi\right)\right)^2\end{aligned}$$

and

$$\begin{aligned}\eta_t\left([\varphi - \eta_t(\varphi)]^2\right) &= \mathbb{E}\left([\varphi(X_t) - \eta_t(\varphi)]^2\right)\\ &= \mathbb{E}\left([[\varphi(X_t) - P_{0,t}(\varphi)(X_0)] + [P_{0,t}(\varphi)(X_0) - \eta_t(\varphi)]]^2\right)\\ &= \mathbb{E}\left([\varphi(X_t) - P_{0,t}(\varphi)(X_0)]^2\right) + \eta_0\left([P_{0,t}(\varphi) - \eta_t(\varphi)]^2\right),\end{aligned}$$

from which we prove that

$$\mathbb{E}\left((\mathcal{X}_t(\varphi))^2\right) = N^2\ \left(\eta_t\left(\varphi\right)\right)^2 + N\ \eta_t\left([\varphi-\eta_t(\varphi)]^2\right).$$

This ends the proof of the exercise.

■

Solution to exercise 274:

We follow the arguments provided in exercise 270. We have $X_t := (X_t^1, X_t^2) \in S = \mathbb{R}^{r=r_1+r_2} = (S_1 \times S_2) = (\mathbb{R}^{r_1} \times \mathbb{R}^{r_2})$.

$$\left\{\begin{array}{rcl} dX_t^1 &=& a_t^1(X_t^1)\ dt + b_t^1(X_t^1)\ dW_t^1 + c_t^1(X_t^1)\ dN_t^1 \\ dX_t^2 &=& a_t^2(X_t)\ dt + b_t^2(X_t)\ dW_t^2 + c_t^2(X_t)\ dN_t^2. \end{array}\right.$$

The jumps

$$(x^1,x^2) \mapsto (x^1 + c_t^1(x^1), x^2) \quad \text{and} \quad (x^1,x^2) \mapsto (x^1, x^2 + c_t^2(x^1,x^2))$$

occur at rate $\lambda_t^1(x^1)$, respectively at rate $\lambda_t^2(x^1,x^2)$. Between these jumps the system evolves according to the diffusion process

$$\left\{\begin{array}{rcl} dX_t^1 &=& a_t^1(X_t^1)\ dt + b_t^1(X_t^1)\ dW_t^1 \\ dX_t^2 &=& a_t^2(X_t)\ dt + b_t^2(X_t)\ dW_t^2. \end{array}\right.$$

We conclude that the generator of X_t is defined by

$$L_t = L_t^c + L_t^d$$

with

$$\begin{array}{rcl} L_t^c(f)(x) &=& \displaystyle\sum_{1\leq i_1\leq r_1} a_t^{1,i_1}(x^1)\ \partial_{x^1_{i_1}} f(x) + \sum_{1\leq i_2\leq r_2} a_t^{2,i_2}(x^1)\ \partial_{x^2_{i_2}} f(x) \\ && \displaystyle +\frac{1}{2}\sum_{1\leq i_1,j_1\leq r_1} \left(b_t^1\ (b_t^1)^T\right)(x^1)\ \partial_{x^1_{i_1},x^1_{j_1}} f(x) \\ && \displaystyle +\frac{1}{2}\sum_{1\leq i_2,j_2\leq r_1} \left(b_t^2\ (b_t^2)^T\right)(x)\ \partial_{x^2_{i_2},x^2_{j_2}} f(x) \end{array}$$

and the jump generator

$$\begin{array}{rcl} L_t^d(f)(x) &=& \lambda_t^1(x^1)\ \left(f\left(x^1+c_t^1(x^1),x^2\right) - f\left(x^1,x^2\right)\right) \\ && +\lambda_t^2(x)\ \left(f\left(x^1,x^2+c_t^2(x)\right) - f\left(x^1,x^2\right)\right). \end{array}$$

The generator $L_t^1 = L_t^{1,c} + L_t^{1,d}$ of the process $X_t^1 \in \mathbb{R}^{r_1}$ reduces to the sum of the generator of the diffusion

$$dX_t^1 = a_t^1(X_t^1)\ dt + b_t^1(X_t^1)\ dW_t^1$$

and the jump generator

$$L_t^{1,d}(f)(x_1) = \lambda_t^1(x^1)\ \left(f\left(x^1+c_t^1(x^1)\right) - f\left(x^1\right)\right).$$

This ends the proof of the exercise.

■

Solution to exercise 275:

We further assume that $N_t^1 = (N_t^{1,j})_{1\leq i\leq r_1}$, respectively $N_t^2 = (N_t^{2,j})_{1\leq i\leq r_2}$, is a column vector of independent Poisson entries with intensities $\lambda_t^{1,j}(X_t^1)$, respectively $\lambda_t^{2,j}(X_t)$.

At rate $\lambda_t^{1,i}(X_t^1)$ we have $dN_t^{1,i} = 1$ and the jump of the process X_t^1 is defined by

$$x^1 = \begin{pmatrix} x_1^1 \\ \vdots \\ x_{r_1}^1 \end{pmatrix} \rightsquigarrow x = \begin{pmatrix} x_1^1 + c_{t,i}^{1,1}(x^1) \\ \vdots \\ x_{r_1}^1 + c_{t,i}^{1,r_1}(x^1) \end{pmatrix} = x^1 + c_{t,i}^1(x^1),$$

with the column vector $c_{t,i}^1(x^1) = \left(c_{t,i}^{1,j}(x^1)\right)_{1\leq j\leq r_1}^T$. In much the same way, at a rate $\lambda_t^{2,i}(X_t)$ we have $dN_t^{2,i} = 1$ and the jump of the process X_t^2 is defined by

$$x^2 = \begin{pmatrix} x_1^2 \\ \vdots \\ x_{r_2}^2 \end{pmatrix} \rightsquigarrow x = \begin{pmatrix} x_1^2 + c_{t,i}^{2,1}(x) \\ \vdots \\ x_{r_2}^2 + c_{t,i}^{2,r_2}(x) \end{pmatrix} = x^2 + c_{t,i}^2(x),$$

with the column vector $c_{t,i}^2(x) = \left(c_{t,i}^{2,j}(x)\right)_{1\leq j\leq r_2}^T$. We conclude that the generator of X_t is defined by

$$L_t = L_t^c + L_t^d \quad \text{with} \quad L_t^d = L_t^{1,d} + L_t^{2,d}.$$

The diffusion generator L_t^c is the same as the one presented in exercise 274. The jump generators $L_t^{1,d}$ and $L_t^{2,d}$ are defined by

$$\begin{aligned} L_t^{1,d}(f)(x) &= \sum_{1\leq i_1\leq r_1} \lambda_t^{1,i_1}(x^1)\ \left(f\left(x^1 + c_{t,i_1}^1(x^1), x^2\right)\right) - f\left(x^1, x^2\right)) \\ L_t^{2,d}(f)(x) &= \sum_{1\leq i_2\leq r_2} \lambda_t^{2,i_2}(x)\ \left(f\left(x^1, x^2 + c_{t,i_2}^2(x)\right)\right) - f\left(x^1, x^2\right)). \end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 276:

At some rate, say $\lambda(X_t)$ the jump of the process $X_t \in \{0,1\}$ is defined by

$$x \rightsquigarrow 1_{x=0}\ 1 + 1_{x=1}\ 0 = 1_{x=0}\ 1.$$

Between the jumps the process evolves as

$$\begin{cases} dX_t &= 0 \\ dY_t &= b_t(Y_t)\ dt + X_t\ \sigma_t(Y_t)\ dW_t. \end{cases}$$

We conclude that the generator of $Z_t = (X_t, Y_t)$ is defined by

$$L_t = L_t^c + L^d$$

with

$$L_t^c(f)(x,y) = \sum_{1\leq i\leq r} b_t^i(x,y)\ \partial_{y_i} f(x,y) + \frac{1}{2} \sum_{1\leq i,j\leq r} x^2\ \left(\sigma_t\,(\sigma_t)^T\right)(x,y)\ \partial_{y_i,y_j} f(x,y)$$

and

$$\begin{aligned} L^d(f)(x,y) &= \lambda(x)\ \left(f(1_{x=0},y)-f(x,y)\right) \\ &= \lambda(x) \int \left(f(x',y)-f(x,y)\right)\ \left(1_{x=0}\ \delta_1(dx')+1_{x=1}\ \delta_0(dx')\right). \end{aligned}$$

In other words

$$L^d(f)(0)=\lambda(0)\ \left(f(1)-f(0)\right) \quad \text{and} \quad L^d(f)(1)=\lambda(1)\ \left(f(0)-f(1)\right).$$

This ends the proof of the exercise.

■

Solution to exercise 277: Notice that

$$\begin{aligned} \sum_{x\in\{0,1\}} \int_{\mathbb{R}^r} p_t(x,y)\ L^d(f)(x,y)\ dy &= \int_{\mathbb{R}^r} \lambda(0) \int \left(f(1,y)-f(0,y)\right)\ p_t(0,y)\ dy \\ &\quad + \int_{\mathbb{R}^r} \lambda(1) \int \left(f(0,y)-f(1,y)\right)\ p_t(1,y)\ dy. \end{aligned}$$

Choosing $f(x,y)=1_{x=0}\ g(y)$ we find that

$$\sum_{x\in\{0,1\}} \int_{\mathbb{R}^r} p_t(x,y)\ L^d(f)(x,y)\ dy = \int g(y)\ \left(\lambda(1)\ p_t(1,y)-\lambda(0)p_t(0,y)\right)\ dy.$$

In this situation, for any smooth function g with compact support we have

$$\begin{aligned} \sum_{x\in\{0,1\}} \int_{\mathbb{R}^r} p_t(x,y)\ L_t^c(f)(x,y)\ dy &= \sum_{1\leq i\leq r} \int_{\mathbb{R}^r} p_t(0,y)\ b_t^i(x,y)\ \partial_{y_i} g(y)\ dy \\ &= -\sum_{1\leq i\leq r} \int_{\mathbb{R}^r} \partial_{y_i}\left(p_t(0,y)\ b_t^i(0,y)\right)\ g(y)\ dy. \end{aligned}$$

This yields

$$\begin{aligned} \partial_t \mathbb{E}(f(X_t,Y_t)) &= \partial_t \mathbb{E}(1_{X_t=0}\ g(Y_t)) = \int_{\mathbb{R}^r} g(y)\ \partial_t p_t(0,y)\ dy \\ &= \int g(y) \left[\left(\lambda(1)\ p_t(1,y)-\lambda(0)p_t(0,y)\right) - \sum_{1\leq i\leq r} \partial_{y_i}\left(p_t(0,y)\ b_t^i(0,y)\right)\right]\ dy. \end{aligned}$$

This implies that

$$\partial_t p_t(0,y) = \left[\lambda(1)\ p_t(1,y)-\lambda(0)\ p_t(0,y)\right] - \sum_{1\leq i\leq r} \partial_{y_i}\left(p_t(0,y)\ b_t^i(0,y)\right).$$

In much the same way, by choosing $f(x,y)=1_{x=1}\ g(y)$ we find that

$$\sum_{x\in\{0,1\}} \int_{\mathbb{R}^r} p_t(x,y)\ L^d(f)(x,y)\ dy = \int_{\mathbb{R}^r} g(y)\ \left[\lambda(0)\ p_t(0,y)-\lambda(1)\ p_t(1,y)\right]\ dy.$$

In this situation, for any smooth function g with compact support, we have

$$\begin{aligned}\sum_{x\in\{0,1\}}\int_{\mathbb{R}^r} p_t(x,y)\ L_t^c(f)(x,y)\ dy \quad &= \quad -\sum_{1\leq i\leq r}\int_{\mathbb{R}^r} g(y)\ \partial_{y_i}\left(p_t(0,y)\ b_t^i(1,y)\right)\ dy \\ &\quad +\frac{1}{2}\sum_{1\leq i,j\leq r}\int g(y)\ \left[\left(\sigma_t\,(\sigma_t)^T\right)(x,y)\ \partial_{y_i,y_j}p_t(1,y)\right]\ dy.\end{aligned}$$

This yields

$$\begin{aligned}\partial_t\mathbb{E}(f(X_t,Y_t)) &= \partial_t\mathbb{E}(1_{X_t=1}\ g(Y_t)) = \int_{\mathbb{R}^r} g(y)\ \partial_t p_t(1,y)\ dy \\ &= \int g(y)\ \Bigg[[\lambda(0)\ p_t(0,y)-\lambda(1)\ p_t(1,y)] - \sum_{1\leq i\leq r}\partial_{y_i}\left(p_t(1,y)\ b_t^i(1,y)\right) \\ &\qquad +\frac{1}{2}\sum_{1\leq i,j\leq r}\partial_{y_i,y_j}\left[\left(\sigma_t\,(\sigma_t)^T\right)(1,y)\ p_t(1,y)\right]\Bigg]\ dy.\end{aligned}$$

This implies that

$$\begin{aligned}&\partial_t p_t(1,y)\\ &= [\lambda(0)\ p_t(0,y)-\lambda(1)\ p_t(1,y)] - \textstyle\sum_{1\leq i\leq r}\partial_{y_i}\left(p_t(1,y)\ b_t^i(1,y)\right)\\ &\qquad +\tfrac{1}{2}\textstyle\sum_{1\leq i,j\leq r}\partial_{y_i,y_j}\left[\left(\sigma_t\,(\sigma_t)^T\right)(1,y)\ p_t(1,y)\right].\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 278:
At the jump times T_n of the Poisson process N_t we have

$$\begin{aligned}dZ_{T_n} &= \sum_{1\leq p\leq N_{T_n}}(Y_p-1) - \sum_{1\leq p\leq N_{T_n-}}(Y_p-1)\\ &= \sum_{1\leq p\leq n}(Y_p-1) - \sum_{1\leq p\leq n-1}(Y_p-1) = Y_n-1\end{aligned}$$

so that

$$X_{T_n}-X_{T_n-} = X_{T_n-}\ dZ_{T_n} = X_{T_n-}\ (Y_n-1)\ \Rightarrow X_{T_n} = Y_n\ X_{T_n-}.$$

Between the jumps $T_n\leq t<T_{n+1}$, the process satisfies the stochastic differential equation

$$dX_t \;=\; a\ X_t\ dt \;+\; b\ X_t\ dW_t.$$

Applying the Doeblin-Itō formula to the function $f(X_t)=\log X_t$ we have

$$\begin{aligned}d\log X_t &= \frac{1}{X_t}\ dX_t - \frac{1}{2}\ \frac{1}{X_t^2}\ dX_t dX_t\\ &= \left[a-\frac{b^2}{2}\right]\ dt +\ b\ dW_t.\end{aligned}$$

This means that

$$\forall t \in [T_n, T_{n+1}[\qquad \log X_t = \log X_{T_n} + \left[a - \frac{b^2}{2}\right] (t - T_n) + b\ (W_t - W_{T_n})$$

so that

$$\begin{aligned} X_t &= X_{T_n} \exp\left(\left[a - \frac{b^2}{2}\right] (t - T_n) + b\ (W_t - W_{T_n})\right) \\ &= X_{T_n-} \left(Y_n \exp\left(\left[a - \frac{b^2}{2}\right] (t - T_n) + b\ (W_t - W_{T_n})\right)\right). \end{aligned}$$

This shows that

$$\forall t \in [0, T_1[\qquad X_t = X_0 \exp\left(\left[a - \frac{b^2}{2}\right] t + b\ W_t\right)$$

and

$$X_{T_1} = Y_1\ X_{T_1-} = X_0\ Y_1 \exp\left(\left[a - \frac{b^2}{2}\right] T_1 + b\ W_{T_1}\right).$$

For $t \in [T_1, T_2[$ we also have

$$\begin{aligned} X_t &= X_{T_1} \exp\left(\left[a - \frac{b^2}{2}\right] (t - T_1) + b\ (W_t - W_{T_1})\right) \\ &= X_0\ Y_1 \exp\left(\left[a - \frac{b^2}{2}\right] t + b\ W_t\right) \end{aligned}$$

and

$$X_{T_2} = Y_2\ X_{T_2-} = X_0\ Y_1 Y_2 \exp\left(\left[a - \frac{b^2}{2}\right] T_2 + b\ W_{T_2}\right).$$

For $t \in [T_2, T_3[$ we also have

$$\begin{aligned} X_t &= X_{T_2} \exp\left(\left[a - \frac{b^2}{2}\right] (t - T_2) + b\ (W_t - W_{T_2})\right) \\ &= X_0\ Y_1\ Y_2 \exp\left(\left[a - \frac{b^2}{2}\right] t + b\ W_t\right) \end{aligned}$$

and

$$X_{T_3} = Y_3\ X_{T_3-} = X_0\ Y_1 Y_2 Y_3 \exp\left(\left[a - \frac{b^2}{2}\right] T_3 + b\ W_{T_3}\right).$$

Iterating this procedure, we find that

$$X_t = X_0 \left\{ \prod_{1 \leq n \leq N_t} Y_n \right\} \exp\left(\left[a - \frac{b^2}{2}\right] t + b\ W_t\right).$$

This ends the proof of the exercise.

■

Solution to exercise 279:
Using the fact that $dN_t \times dN_t = dN_t$ and $dt \times dN_t = 0 = dN_t dW_t$, we readily check that

$$dX_t^{(1)} dX_t^{(2)} = b_t^{(1)}(X_t) b_t^{(2)}(X_t)\ dN_t + c_t^{(1)}(X_t) c_t^{(2)}(X_t)\ dt.$$

Observe that the increments of the process $(X^{(1)}X^{(2)})_t = X_t^{(1)}X_t^{(2)}$ are given by

$$\begin{aligned} d(X^{(1)}X^{(2)})_t &:= (X^{(1)}X^{(2)})_{t+dt} - (X^{(1)}X^{(2)})_t = X_{t+dt}^{(1)}X_{t+dt}^{(2)} - X_t^{(1)}X_t^{(2)} \\ &= X_t^{(1)}dX_t^{(2)} + X_t^{(2)}dX_t^{(1)} + \underbrace{dX_t^{(1)}dX_t^{(2)}}_{=b_t^{(1)}(X_t)b_t^{(2)}(X_t)\ dN_t + c_t^{(1)}(X_t)c_t^{(2)}(X_t)\ dt}. \end{aligned}$$

This ends the proof of the exercise is now easy completed.

∎

Solution to exercise 280:
The continuous and the pure jump parts of (X_t, Y_t) are given by

$$\left\{ \begin{array}{lcl} dX_t^c &=& -X_t\ dt + \sqrt{2}\ Y_t\ dW_t \\ dY_t^c &=& -Y_t\ dt - \sqrt{2}\ X_t\ dW_t \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{lcl} \Delta X_t &=& a(X_t, Y_t)\ X_t\ dN_t \\ \Delta Y_t &=& a(X_t, Y_t)\ Y_t\ dN_t. \end{array} \right.$$

By applying the Doeblin-Itō formula (15.11) to the function $f(X_t, Y_t) = X_t^2 + Y_t^2$ we have

$$\begin{aligned} df(X_t, Y_t) &= 2X_t\ dX_t^c + dX_t^c\ dX_t^c + 2Y_t\ dY_t^c + dY_t^c\ dY_t^c \\ &\quad + \left(X_t^2\, (1 + a(X_t, Y_t))^2 - X_t^2 + Y_t^2\, (1 + a(X_t, Y_t))^2 - Y_t^2 \right) dN_t. \end{aligned}$$

We observe that

$$2X_t dX_t^c + dX_t^c dX_t^c + 2Y_t dY_t^c + dY_t^c dY_t^c$$

$$= 2X_t \left(-X_t\ dt + \sqrt{2}\ Y_t\ dW_t \right) + 2\ Y_t^2\ dt + 2Y_t \left(-Y_t\ dt - \sqrt{2}\ X_t\ dW_t \right) + 2\ X_t^2\ dt = 0.$$

This yields

$$\begin{aligned} df(X_t, Y_t) &= \left(X_t^2 \left[(1 + a(X_t, Y_t))^2 - 1 \right] + Y_t^2 \left[(1 + a(X_t, Y_t))^2 - 1 \right] \right) dN_t \\ &= \left(X_t^2 + Y_t^2 \right) \left[(1 + a(X_t, Y_t))^2 - 1 \right] dN_t \end{aligned}$$

Choosing

$$a(X_t, Y_t) = -1 + \epsilon\ \sqrt{1 + \frac{b(X_t, Y_t)}{X_t^2 + Y_t^2}}$$

we find that

$$(1 + a(X_t, Y_t))^2 - 1 = \frac{b(X_t, Y_t)}{X_t^2 + Y_t^2}$$

from which we conclude that

$$df(X_t, Y_t) = b(X_t, Y_t)\ dN_t.$$

Between the jump times T_{n-1} and T_n of the Poisson process N_t, the process $Z_t = (X_t, Y_t)$, with $t \in [T_{n-1}, T_n[$ is given by the 2-dimensional diffusion

$$\left\{ \begin{array}{lcl} dX_t &=& -X_t\ dt + \sqrt{2}\ Y_t\ dW_t \\ dY_t &=& -Y_t\ dt - \sqrt{2}\ X_t\ dW_t \end{array} \right.$$

starting at $Z_{T_{n-1}}$. At the jump time T_n the process jumps from Z_{T_n-} to

$$Z_{T_n} = Z_{T_n-} + b\left(Z_{T_n-} \right).$$

For any $t \in [T_0, T_1[$ we have

$$d\|Z_t\|^2 = b(Z_t)\ dN_t = 0 \Rightarrow \|Z_t\|^2 = \|Z_0\|^2.$$

At the jump time T_1 the process jumps from Z_{T_1-} to

$$Z_{T_1} = Z_{T_1-} + b\left(Z_{T_1-}\right).$$

For any $t \in [T_1, T_2[$ we have

$$d\|Z_t\|^2 = b(Z_t)\ dN_t = 0 \Rightarrow \|Z_t\|^2 = \|Z_{T_1}\|^2.$$

At the jump time T_2 the process jumps from Z_{T_2-} to

$$Z_{T_2} = Z_{T_2-} + b\left(Z_{T_2-}\right).$$

This ends the proof of the exercise.

■

Solution to exercise 281: We have

$$\begin{aligned}
u_s(x) &= \mathbb{E}\left[f_t(X_t) \mid X_s = x\right] = \mathbb{E}\left[f_t\left(x + \sigma\ (W_t - W_s)\right)\right] \\
&= \mathbb{E}\left[f_t\left(x + \sigma\sqrt{t-s}\ W_1\right)\right] = \frac{1}{\sigma\sqrt{2\pi}}\ \frac{1}{\sqrt{t-s}} \int f_t\left(x+w\right)\ \exp\left[-\frac{w^2}{2\sigma^2(t-s)}\right] dw.
\end{aligned}$$

Using an elementary change of variable ($y = x + w$ ($\Rightarrow dy = dw$)) we find that

$$u_s(x) = \frac{1}{\sigma\sqrt{2\pi}}\ \frac{1}{\sqrt{t-s}} \int f_t\left(y\right)\ \exp\left[-\frac{(y-x)^2}{2\sigma^2(t-s)}\right] dy.$$

It is now readily checked that

$$\begin{aligned}
\partial_s u_s(x) &= \frac{1}{\sigma\sqrt{2\pi}}\ \partial_s\left(\frac{1}{\sqrt{t-s}}\right) \int f_t\left(x+w\right)\ \exp\left[-\frac{w^2}{2\sigma^2(t-s)}\right] dw \\
&\quad + \frac{1}{\sigma\sqrt{2\pi}}\ \frac{1}{\sqrt{t-s}} \int f_t\left(y\right)\ \partial_s\left(\exp\left[-\frac{(y-x)^2}{2\sigma^2(t-s)}\right]\right) dy.
\end{aligned}$$

On the other hand, we have

$$\partial_s\left(\frac{1}{\sqrt{t-s}}\right) = \frac{1}{2(t-s)}\ \frac{1}{\sqrt{t-s}}$$

and

$$\begin{aligned}
\exp\left[\frac{(y-x)^2}{2\sigma^2(t-s)}\right] \partial_s\left(\exp\left[-\frac{(y-x)^2}{2\sigma^2(t-s)}\right]\right) &= -\frac{(y-x)^2}{2\sigma^2}\ \partial_s\left(\frac{1}{(t-s)}\right) \\
&= -\frac{(y-x)^2}{2\sigma^2}\ \frac{1}{(t-s)^2}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\partial_s u_s(x) &= \frac{1}{2(t-s)}\ u_s(x) \\
&\quad - \frac{1}{2(\sigma(t-s))^2}\ \frac{1}{\sigma\sqrt{2\pi}}\ \frac{1}{\sqrt{t-s}} \int f_t\left(y\right)\ (y-x)^2\ \exp\left[-\frac{(y-x)^2}{2\sigma^2(t-s)}\right] dy.
\end{aligned}$$

To take the final step, observe that

$$\frac{\sigma^2}{2}\partial_x^2 u_s(x) \quad = \quad \frac{1}{\sigma\sqrt{2\pi}}\ \frac{1}{\sqrt{t-s}}\ \int\ f_t(y)\ \frac{\sigma^2}{2}\partial_x^2 \exp\left[-\frac{(y-x)^2}{2\sigma^2(t-s)}\right]\ dy$$

with

$$\begin{aligned}\frac{\sigma^2}{2}\partial_x^2\left(\exp\left[-\frac{(y-x)^2}{2\sigma^2(t-s)}\right]\right) &= \frac{1}{2(t-s)}\ \partial_x\left((y-x)\ \exp\left[-\frac{(y-x)^2}{2\sigma^2(t-s)}\right]\right)\\ &= -\frac{1}{2(t-s)}\ \exp\left[-\frac{(y-x)^2}{2\sigma^2(t-s)}\right]\\ &\qquad +\frac{1}{2(t-s)}\ (y-x)\ \partial_x\left(\exp\left[-\frac{(y-x)^2}{2\sigma^2(t-s)}\right]\right)\\ &= -\frac{1}{2(t-s)}\ \exp\left[-\frac{(y-x)^2}{2\sigma^2(t-s)}\right]\\ &\qquad +\frac{1}{2(\sigma(t-s))^2}\ (y-x)^2\ \exp\left[-\frac{(y-x)^2}{2\sigma^2(t-s)}\right].\end{aligned}$$

This implies that

$$\begin{aligned}\frac{\sigma^2}{2}\partial_x^2 u_s(x) &= -\frac{1}{2(t-s)}u_s(x)\\ &\quad +\frac{1}{2(\sigma(t-s))^2}\ \frac{1}{\sigma\sqrt{2\pi}}\ \frac{1}{\sqrt{t-s}}\ \int\ f_t(y)\ (y-x)^2\exp\left[-\frac{(y-x)^2}{2\sigma^2(t-s)}\right]\ dy.\end{aligned}$$

The end of the proof of the exercise is now easily completed.

■

Solution to exercise 282:

The first assertion is proved in section 15.5. If we choose $g(t,x) = e^{-\lambda t}$ for some $\lambda \in \mathbb{R}$ then we have

$$g(0,x) = 1 \qquad [\partial_t + L_t]\, g(t,.) = -\lambda\ e^{-\lambda t} \quad \text{and} \quad \Gamma_{L_t}(f(t,.), g(t,.)) = 0.$$

In this situation, the first assertion implies that

$$M_t = f(t,X_t)e^{-\lambda t} - f(0,X_0) + \int_0^t\ e^{-\lambda s}\ \left[\lambda\ f(s,.) - [\partial_s + L_s]\, f(s,.)\right](X_s)\ ds.$$

In this situation, we have

$$M_t = \int_0^t\ e^{-\lambda s}\ d\mathcal{M}_s(f)$$

with the martingale $d\mathcal{M}_s(f) = df(s,X_s) - [\partial_s + L_s]\, f(s,X_s)\ ds$. This ends the proof of the exercise is now easy completed.

■

Solution to exercise 283:

Arguing as in (15.32) we have

$$Z_{s,t}(f) = \int_s^t e^{-\int_s^r V_u(X_u)du}\ dM_r(f)$$

with the martingale

$$M_t(f) = f(t, X_t) - f(0, X_0) - \int_0^t (\partial_s + L_s) f(s, X_s) ds.$$

We conclude that

$$\partial_t P^V_{s,t}(f(t,.)) = P^V_{s,t}((\partial_t + L^V_t) f(t,.)).$$

This ends the proof of the exercise.

■

Solution to exercise 284:
Using (15.25) we have

$$\begin{aligned}
P_{r,t} &= P^{(1)}_{r,t} + \int_r^t P_{r,s_1} L^{(2)}_{s_1} P^{(1)}_{s_1,t}\, ds_1 \\
&= P^{(1)}_{r,t} + \int_r^t \left[P^{(1)}_{r,s_1} + \int_r^{s_1} P_{r,s_2} L^{(2)}_{s_2} P^{(1)}_{s_2,s_1}\, ds_2 \right] L^{(2)}_{s_1} P^{(1)}_{s_1,t}\, ds_1 \\
&= P^{(1)}_{r,t} + \int_r^t P^{(1)}_{r,s_1} L^{(2)}_{s_1} P^{(1)}_{s_1,t}\, ds_1 + \int_r^t \int_r^{s_1} P_{r,s_2} L^{(2)}_{s_2} P^{(1)}_{s_2,s_1} L^{(2)}_{s_1} P^{(1)}_{s_1,t}\, ds_1\, ds_2.
\end{aligned}$$

Iterating and letting $t = s_0$ we find the formula

$$P_{r,s_0} = \sum_{n \geq 0} \int_r^{s_0} \ldots \int_r^{s_{n-1}} P^{(1)}_{r,s_n} L^{(2)}_{s_n} P^{(1)}_{s_n,s_{n-1}} \ldots L^{(2)}_{s_1} P^{(1)}_{s_1,s_0} \quad ds_n \ \ldots \ ds_1.$$

This ends the proof of the exercise.

■

Solution to exercise 285:
Using (15.27) we have

$$\begin{aligned}
P_{s,t} &= Q_{s,t} + \int_s^t Q_{s,s_1} \overline{K}_{s_1} P_{s_1,t}\, ds_1 \\
&= Q_{s,t} + \int_s^t Q_{s,s_1} \overline{K}_{s_1} \left[Q_{s_1,t} + \int_{s_1}^t Q_{s_1,s_2} \overline{K}_{s_2} P_{s_2,t}\, ds_2 \right] ds_1 \\
&= Q_{s,t} + \int_s^t Q_{s,s_1} \overline{K}_{s_1} Q_{s_1,t}\, ds_1 + \int_s^t \int_{s_1}^t Q_{s,s_1} \overline{K}_{s_1} Q_{s_1,s_2} \overline{K}_{s_2} P_{s_2,t}\, ds_2\, ds_1.
\end{aligned}$$

Iterating and letting $s = s_0$ we find the formula

$$P_{s_0,t} = \sum_{n \geq 0} \int_{s_0}^t \ldots \int_{s_{n-1}}^t Q_{s_0,s_1} \overline{K}_{s_1} Q_{s_1,s_2} \ldots \overline{K}_{s_n} Q_{s_n,t}\, ds_n \ \ldots \ ds_1.$$

This ends the proof of the exercise.

■

Solution to exercise 286:
Given a spatially homogeneous jump rate function $\lambda_t(x) = \lambda$ we have

$$Q_{s,t}(f)(x) \quad := \quad \exp\left[-\lambda(t-s)\right] \mathbb{E}\left(f(X^{(1)}_t) \mid X^{(1)}_s = x \right) = \exp\left[-\lambda(t-s)\right] P^{(1)}_{s,t}(f)(x).$$

In this situation we have

$$\begin{aligned}
P_{s_0,t} &= \sum_{n\geq 0} \lambda^n \int_{s_0}^t \ldots \int_{s_{n-1}}^t Q_{s_0,s_1} K_{s_1} Q_{s_1,s_2} \ldots K_{s_n} Q_{s_n,t}\, ds_n\ \ldots\ ds_1\\
&= \sum_{n\geq 0} \lambda^n \int_{s_0}^t \ldots \int_{s_{n-1}}^t \exp\left[-\lambda \left\{(t-s_n)+\ldots+(s_1-s_0)\right\}\right]\\
&\qquad\qquad \times P^{(1)}_{s_0,s_1} K_{s_1} P^{(1)}_{s_1,s_2} \ldots K_{s_n} P^{(1)}_{s_n,t}\, ds_n\ \ldots\ ds_1\\
&= \sum_{n\geq 0} \lambda^n\ e^{-\lambda(t-s_0)} \int_{s_0}^t \ldots \int_{s_{n-1}}^t P^{(1)}_{s_0,s_1} K_{s_1} P^{(1)}_{s_1,s_2} \ldots K_{s_n} P^{(1)}_{s_n,t}\, ds_n\ \ldots\ ds_1.
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 287:
Observe that

$$\exp\left(\int_s^t W_u(X_u)du\right) = 1 + \int_s^t W_u(X_u)\ \exp\left(\int_s^u W_v(X_v)dv\right) du.$$

This yields the integral decomposition

$$\begin{aligned}
Q_{s,t}(f)(x) &= \mathbb{E}\left(f(X_t)\ \exp\left(\int_s^t W_u(X_u)du\right) \mid X_s = x\right)\\
&= \mathbb{E}\left(f(X_t)\ \left[1 + \int_s^t W_u(X_u)\ \exp\left(\int_s^u W_v(X_v)dv\right) du\right] \mid X_s = x\right)\\
&= P_{s,t}(f) + \int_s^t Q_{s,u}\left(W_u P_{u,t}(f)\right)\ du.
\end{aligned}$$

This implies that

$$Q_{s,t} = P_{s,t} + \int_s^t Q_{s,s_1} \overline{W}_{s_1} P_{s_1,t}\ ds_1$$

thus finishing the proof of the first assertion. Observe that

$$\begin{aligned}
Q_{s,t} &= P_{s,t} + \int_s^t Q_{s,s_1} \overline{W}_{s_1} P_{s_1,t}\ ds_1\\
&= P_{s,t} + \int_s^t \left[P_{s,s_1} + \int_s^{s_1} Q_{s,s_2} \overline{W}_{s_2} P_{s_2,s_1}\ ds_2\right] \overline{W}_{s_1} P_{s_1,t}\ ds_1\\
&= P_{s,t} + \int_s^t P_{s,s_1} \overline{W}_{s_1} P_{s_1,t}\ ds_1 + \int_s^t \int_s^{s_1} Q_{s,s_2} \overline{W}_{s_2} P_{s_2,s_1} \overline{W}_{s_1} P_{s_1,t}\ ds_2\ ds_1.
\end{aligned}$$

Iterating the argument (as in exercise 284) and letting $s = r$ and $t = s_0$ we find that

$$Q_{r,s_0} = \sum_{n\geq 0} \int_r^{s_0} \ldots \int_r^{s_{n-1}} P_{r,s_n} \overline{W}_{s_n} P_{s_n,s_{n-1}} \ldots \overline{W}_{s_1} P_{s_1,s_0}\ ds_n \ldots ds_1.$$

This ends the proof of the exercise.

■

Solution to exercise 288:
By exercise 283, the process

$$Z_t(f) \quad := \quad e^{\int_0^t V(X_s)ds}\ f(X_t) - f(X_0) - \int_0^t e^{\int_0^s V(X_r)dr}\ L^V f(X_s)\ ds$$

is a martingale w.r.t. $\mathcal{F}_t = \sigma(X_s,\ s \leq t)$, with

$$L^V f(i,y) := L(f)(i,y) + V(i,y)f(i,y) = L(f)(i,y) + v(i)\ f(i,y).$$

We have

$$\int_0^t V(X_s)ds = \langle v, \overline{I}_t\rangle := \sum_{i\in S_1} v(i)\ \overline{I}_t^i(i) \quad \text{with} \quad \overline{I}_t^i(i) := \int_0^t 1_{I_s=i}\ ds.$$

This implies that

$$\begin{aligned}
Z_t(f) \quad &:= \quad e^{\langle v,\overline{I}_t\rangle}\ f(X_t) - f(X_0) - \int_0^t e^{\langle v,\overline{I}_s\rangle}\ (v(I_s)\ f(X_s) + L(f)(X_s))\ ds \\
&= \quad e^{\langle v,\overline{I}_t^i\rangle}\ f(X_t) - f(X_0) - \sum_{i\in S_1} \int_0^t e^{\langle v,\overline{I}_s\rangle} \\
&\qquad \times\ (v(i)\ f(i,Y_s) + \mathcal{L}_i(f(i,\,.))(Y_s) + w(i)\ (f(i+1,Y_s) - f(i,Y_s)))\ 1_{I_s=i}\ ds \\
&= \quad e^{\langle v,\overline{I}_t\rangle}\ f(X_t) - f(X_0) - \sum_{i\in S_1} \int_0^t e^{\langle v,\overline{I}_s\rangle} \\
&\qquad \times\ [w(i)\ f(i+1,Y_s) + (v(i) - w(i))\ f(i,Y_s) + \mathcal{L}_i(f(i,\,.))(Y_s)]\ 1_{I_s=i}\ ds.
\end{aligned}$$

We conclude that

$$\begin{aligned}
Z_t(f) \quad &= \quad e^{\langle v,\overline{I}_t\rangle}\ f(X_t) - f(X_0) - \sum_{i\in S_1} \int_0^t e^{\langle v,\overline{I}_s\rangle} \\
&\qquad\qquad \times\ \left[w(i)\ f(i+1,Y_s) + L^{v-w} f(i,Y_s)\right]\ 1_{I_s=i}\ ds
\end{aligned}$$

with

$$\begin{aligned}
L^{v-w}(f)(i,y) \quad &:= \quad L(f)(i,y) + (v(i) - w(i))\ f(i,y) \\
&= \quad \mathcal{L}_i(f(i,\,.))(y) + (v(i) - w(i))\ f(i,y) := \mathcal{L}_i^{v-w}\left(f(i,\,.)\right).
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 289:
By construction we have

$$Z_t(f) \quad = \quad e^{\langle v,\overline{I}_t\rangle}\ f(I_t,Y_t) - f(I_0,Y_0) + \int_0^t e^{\langle v,\overline{I}_s\rangle}\ h(Y_s)\ 1_{I_s=m-1}\ ds.$$

This ends the proof of the exercise.

■

Solution to exercise 290:

We have

$$
\begin{aligned}
-L_t(w_{c,y})(x) &= w_{c,y}\left[\sum_{1\leq i,j\leq d}\left(\sigma_t\sigma_t^T\right)_{i,j}(x)\; y_i\; y_j - \sum_{1\leq i\leq d} y_i b_t^i(x)\right]\\
&\geq c\; e^{-\|y\|_1\delta}\;\left[\rho\;\|y\|_1^2 - \|y\|_1\;\|b\|_D\right]
\end{aligned}
$$

with

$$
\delta = \max_{1\leq i\leq d}\sup_{x\in D} x_i \quad \text{and} \quad \|b\|_D = \max_{1\leq i\leq d}\sup_{x\in D}|b_t^i(x)|.
$$

Choosing y s.t. $\|y\|_1 = \rho^{-1}\left[1+\|b\|_D\right]$ we have

$$
\rho\;\|y\|_1^2 - \|y\|_1\;\|b\|_D = \|y\|_1\;\left[\rho\;\|y\|_1 - \;\|b\|_D\right] = \|y\|_1
$$

and

$$
c\; e^{-\|y\|_1\delta}\;\left[\rho\;\|y\|_1^2 - \|y\|_1\;\|b\|_D\right] = c\; e^{-\|y\|_1\delta}\;\|y\|_1.
$$

This implies that

$$
c \geq e^{\rho^{-1}[1+\|b\|_D]\delta}\;\rho\left[1+\|b\|_D\right]^{-1} \Rightarrow c\; e^{-\|y\|_1\delta}\;\|y\|_1 \geq 1.
$$

This ends the proof of the exercise.

■

Solution to exercise 291:

We have the almost sure convergence of the stopped martingale

$$
N_t := M_{t\wedge T} = \;\rightarrow_{t\rightarrow\infty}\; N_\infty = M_T.
$$

In addition, using Doob's stopping theorem (theorem 15.6.1), N_t is a martingale null at the origin, so that we have $\mathbb{E}(N_t) = 0$, for any $t\geq 0$. In addition, we have

$$
\mathbb{E}\left((N_t - N_s)^2\right) = \mathbb{E}\left(\langle M,M\rangle_{t\wedge T} - \langle M,M\rangle_{s\wedge T}\right) \leq c\;\mathbb{E}\left(|(t\wedge T)-(s\wedge T)|\right).
$$

Using the dominated convergence theorem, we also have

$$
\mathbb{E}(T)<\infty \implies \lim_{s,t\rightarrow\infty}\mathbb{E}\left(|(t\wedge T)-(s\wedge T)|\right) = 0 \implies \lim_{s,t\rightarrow\infty}\mathbb{E}\left((N_t-N_s)^2\right) = 0.
$$

We conclude that $N_t = M_{t\wedge T}$ is a Cauchy sequence in $\mathbb{L}_2(\mathbb{P})$. By completeness $M_{t\wedge T}$ converges in $\mathbb{L}_2(\mathbb{P})$ to M_T so that $\mathbb{E}(M_T) = 0$.

When M_t is defined by

$$
M_t = \int_0^t X_s\; dW_s
$$

for any $s\leq t$, we have

$$
\langle M,M\rangle_{t\wedge T} - \langle M,M\rangle_{t\wedge T} = \int_{s\wedge T}^{t\wedge T} X_s^2\; ds \leq c\;\left((t\wedge T)-(s\wedge T)\right).
$$

From the first part of the exercise, $\int_0^{t\wedge T} X_s\; dW_s$ converges to $\int_0^T X_s\; dW_s$ in $\mathbb{L}_2(\mathbb{P})$ as $t\uparrow\infty$, and we have $\mathbb{E}\left(\int_0^T X_s\; dW_s\right) = 0$.

This ends the proof of the exercise.

■

Solution to exercise 292:

The Cauchy problem is a particular case of the one discussed in (15.36). Using the backward equation (15.16) we have

$$\begin{aligned}
\partial_s v_s &= \partial_s P_{s,t}(f_t) - P_{s,s}(g_s) + \int_s^t \partial_s P_{s,u}(g_u)\, du \\
&= -g_s - L_s P_{s,t}(f_t) - \int_s^t L_s P_{s,u}(g_u)\, du = -g_s - L_s\left(P_{s,t}(f_t) + \int_s^t P_{s,u}(g_u)\, du\right) \\
&= -g_s - L_s(v_s).
\end{aligned}$$

We also have

$$v_t(x) = P_{t,t}(f_t) + \int_t^t P_{s,u}(g_u)\, du = f_t.$$

This ends the proof of the exercise.

■

Solution to exercise 293:

We have

$$\begin{aligned}
w_t(x) &= \mathbb{E}\left(f(X_t) + \int_0^t g_s(X_{t-s})\, ds \mid X_0 = x\right) \\
&= P_t(f)(x) + \int_0^t P_{t-s}(g_s)(x)\, du
\end{aligned}$$

with the Markov semigroup

$$P_t(f)(x) = \mathbb{E}\left(f(X_t) \mid X_0 = x\right).$$

This implies that

$$\begin{aligned}
\partial_t w_t &= \partial_t P_t(f_0) + \partial_t \int_0^t P_{t-s}(g_s)\, du + \int_0^t \partial_t P_{t-s}(g_s)\, du \\
&= LP_t(f) + g_t + \int_0^t LP_{t-s}(g_s)\, du = L\left[P_{0,t}(f) \int_0^t P_{t-s}(g_s)\, du\right] + g_t \\
&= Lw_t + g_t
\end{aligned}$$

with the initial condition $w_0 = f$.

This ends the proof of the exercise.

■

Solution to exercise 294:

By a simple conditioning argument, we have

$$\begin{aligned}
\gamma_t(f) &= \mathbb{E}\left(\mathbb{E}\left(f(X_t)\ \exp\left(-\int_s^t V_u(X_u)du\right)|X_s\right)\ \exp\left(-\int_0^s V_u(X_u)du\right)\right) \\
&= \mathbb{E}\left(Q_{s,t}(f)(X_s)\ \exp\left(-\int_0^s V_u(X_u)du\right)\right) = \gamma_s(Q_{s,t}(f)).
\end{aligned}$$

This implies that

$$\gamma_t(f) = \gamma_s(Q_{s,t}(f)) = (\gamma_s Q_{s,t})(f)$$

from which we conclude that $\gamma_t = \gamma_s Q_{s,t}$. On the other hand we have

$$\begin{array}{rcl} (15.31) \Rightarrow \ \partial_t \gamma_t(f) & = & \partial_t \eta_0(Q_{0,t}(f)) \\ & = & \eta_0(\partial_t Q_{0,t}(f)) = \eta_0(Q_{0,t}(L_t^V(f))) \\ & = & \gamma_t(L_t^V(f)) = \gamma_t(L_t(f)) - \gamma_t(fV_t). \end{array}$$

We also notice that

$$\begin{array}{rcl} \partial_t \log \gamma_t(1) & = & \dfrac{1}{\gamma_t(1)} \ \partial_t \gamma_t(1) \\ & = & \dfrac{1}{\gamma_t(1)} \ \partial_t \mathbb{E}\left(\exp\left(-\displaystyle\int_0^t V_u(X_u)du\right)\right) \\ & = & -\dfrac{1}{\gamma_t(1)} \ \mathbb{E}\left(V(X_t) \ \exp\left(-\displaystyle\int_0^t V_u(X_u)du\right)\right) = -\dfrac{\gamma_t(V_t)}{\gamma_t(1)} = -\eta_t(V_t). \end{array}$$

This implies that

$$\eta_t(f) = \mathbb{E}\left(f(X_t) \ \exp\left(-\int_0^t (V_u(X_u) - \eta_u(V_u)) \ du\right)\right).$$

This shows that η_t is defined as γ_t by replacing V_t by $(V_t - \eta_t(V_t))$. We conclude that

$$\begin{array}{rcl} \partial_t \eta_t(f) & = & \eta_t(L_t(f)) - \eta_t\left(f(V_t - \eta_t(V_t)\right) \\ & = & \partial_t \eta_t(f) = \eta_t(L_t(f)) + \eta_t(f)\eta_t(V_t) - \eta_t\left(fV_t\right). \end{array}$$

This ends the proof of the exercise.

■

Solution to exercise 295:

We let T^s be the first jump time of X_t arriving at rate $\lambda_t(X_t)$ after time s. By construction, for any function f on $([0,\infty[\times S)$ we have

$$\begin{array}{rcl} \mathcal{K}(f)(s,x) & = & \mathbb{E}\left(K_{T^s}(f(T^s,.))(X^{(1)}_{T^s-}) \ 1_{T^s<\infty} \mid X_s = x\right) \\ & = & \displaystyle\int_s^\infty \mathbb{E}\left(K_t(f(t,.))(X^{(1)}_t) \ \lambda_t(X^{(1)}_t) \ e^{-\int_s^t \lambda_r(X^{(1)}_r) \ dr} \mid X^{(1)}_s = x\right) \ dt. \end{array}$$

For time homogeneous models, we have

$$\begin{array}{rcl} \mathcal{K}(f)(s,x) & = & \mathbb{E}\left(K(f(T^0,.))(X^{(1)}_{T^0-}) \ 1_{T^0<\infty} \mid X_0 = x\right) \\ & = & \displaystyle\int_0^\infty \mathbb{E}\left(K(f(t,.))(X^{(1)}_t) \ \lambda(X^{(1)}_t) \ e^{-\int_0^t \lambda(X^{(1)}_r) \ dr} \mid X^{(1)}_0 = x\right) \ dt. \end{array}$$

This shows that $\mathcal{K}(f)(s,x)$ doesn't depend on the parameter s. The Markov transition $\mathcal{M}$ is given for any function f on S by the formula

$$\mathcal{M}(f)(x) \ = \ \int_0^\infty \mathbb{E}\left(K(f)(X^{(1)}_t) \ \lambda(X^{(1)}_t) \ e^{-\int_0^t \lambda(X^{(1)}_r) \ dr} \mid X^{(1)}_0 = x\right) \ dt.$$

For a constant rate function $\lambda(x) = \lambda > 0$ we have

$$\mathcal{M}(f)(x) = \int_0^\infty \lambda\, e^{-\lambda t}\, \mathbb{E}\left(K(f)(X_t^{(1)}) \mid X_0^{(1)} = x\right)\, dt = \mathbb{E}\left(K(f)(X_T^{(1)}) \mid X_0^{(1)} = x\right)$$

where T stands for an exponential random variable (independent of $X_t^{(1)}$) with parameter λ. This ends the proof of the exercise.

■

Solution to exercise 296:

Under the minorization conditions, for any non negative function f on S we have construction, we have

$$\begin{aligned}\mathcal{M}(f)(x) &= \int_0^\infty \mathbb{E}\left(K(f)(X_t^{(1)})\ \lambda(X_t^{(1)})\ e^{-\int_0^t \lambda(X_r^{(1)})\, dr} \mid X_0^{(1)} = x\right)\, dt \\ &\geq (\lambda_\star/\lambda^\star) \int_0^\infty \lambda^\star\, e^{-\lambda^\star t}\, \nu(f)\, dt = \overline{\epsilon}\, \nu(f) \quad \text{with} \quad \overline{\epsilon} = (\lambda_\star/\lambda^\star).\end{aligned}$$

The last assertion is a direct consequence of (8.15) and the contraction theorem 8.2.13. This ends the proof of the exercise.

■

Solution to exercise 297:

We have

$$\mathcal{M}(f) = \int_0^\infty Q_t\left(\lambda\ K(f)\right)\, dt.$$

This implies that

$$\mathcal{M}(f) = \mathcal{Q}(\lambda K(f)).$$

Observe that

$$(15.31) \Rightarrow \partial_t Q_t(f) = Q_t(L^{(1)}(f) - \lambda f) \Rightarrow \int_0^\infty Q_t(L^{(1)}(f) - \lambda f)(x)\, dt = [Q_t(f)(x)]_0^\infty = -f(x).$$

This implies that

$$\begin{aligned}&L^{(2)}(f) = \lambda(K(f) - f) \\ &\Rightarrow \mathcal{Q}(L(f)) = \textstyle\int_0^\infty\ Q_t\left(L^{(1)}(f) + L^{(2)}(f)\right)\, dt = -f + \int_0^\infty\ Q_t(\lambda K(f))\, dt\end{aligned}$$

from which we find that

$$\mathcal{M}(f) = \mathcal{Q}(\lambda K(f)) \Rightarrow \mathcal{Q}(L(f)) = \mathcal{Q}(\lambda K(f)) - f = \mathcal{M}(f) - f.$$

This clearly implies that

$$\mu L(f) \propto \pi \mathcal{Q}(L(f)) = -\pi(f) + \pi(\mathcal{M}(f)) = 0.$$

This ends the proof of the exercise.

■

Solution to exercise 298:

Given $t \in [T_{n-1}, T_n[$ we have the Doeblin-Itō formula

$$f(X_t) - f(X_{T_{n-1}}) = \int_{T_{n-1}}^t L_s(f)(X_s)\, ds + \int_{T_{n-1}}^t dM_s(f)$$

for some (conditional) martingale $(M_s(f))_{s\in[T_{n-1},T_n[}$ with (conditional) angle bracket

$$d\langle M(f)\rangle_s = \Gamma_{L_s}(f,f)(X_s)\, ds.$$

At every time T_n we also have

$$f(X_{T_n}) - f(X_{T_n-}) = \underbrace{\mathbb{E}\left(f(X_{T_n}) - f(X_{T_n-}) \mid X_{T_n-}\right)}_{L^D_{T_n-}(f)(X_{T_n-})} + M^D_{T_n}(f) - M^D_{T_n-}(f)$$

with the martingale increment

$$\begin{array}{rcl} M^D_{T_n}(f) - M^D_{T_n-}(f) & = & f(X_{T_n}) - f(X_{T_n-}) - \mathbb{E}\left(f(X_{T_n}) - f(X_{T_n-}) \mid X_{T_n-}\right) \\ & = & f(X_{T_n}) - \mathbb{E}\left(f(X_{T_n}) \mid X_{T_n-}\right) = f(X_{T_n}) - K_{T_n-}(f)(X_{T_n-}) \end{array}$$

and the operator L^D_s defined by

$$L^D_s(f)(x) = 1_D(x) \int (f(y) - f(x))\, K_s(x, dy).$$

We set

$$\forall t \in [T_{n-1}, T_n[\qquad M^D_t(f) = M^D_{T_n-}(f).$$

In this notation we have

$$M^D_{T_n}(f) - M^D_{T_n-}(f) = \int_{T_{n-1}}^{T_n} dM^D_s(f).$$

This yields the decomposition

$$\begin{array}{l} f(X_t) - f(X_0) \\ = \displaystyle\sum_{n\geq 1} \left(\int_{T_{n-1}\wedge t}^{T_n\wedge t} (L_s(f)(X_s)\, ds + dM_s(f)] + L^D_{t\wedge T_n-}(f)(X_{t\wedge T_n-}) + \int_{T_{n-1}\wedge t}^{T_n\wedge t} dM^D_s(f) \right) \\ = \displaystyle\int_0^t \left[L_s(f)(X_s)\, ds + L^D_s(f)(X_s)\, \mu^D(ds)\right] + \overline{M}_t(f) \end{array}$$

with the martingale

$$\overline{M}_t(f) = \int_0^t \left(dM_s(f) + dM^D_s(f)\right) \quad \text{and the random empirical measure} \quad \mu^D = \sum_{n\geq 1} \delta_{T_n-}\,.$$

The last assertion is immediate. This ends the proof of the exercise.

■

Solution to exercise 299:

$$L(f)(x) = x^2\, f''(x) + x\, f'(x) \quad \text{and} \quad V(x) = -(x^2 - n^2) = (n-x)(n+x).$$

We look for a solution of the form

$$v(x) = x^n \sum_{i \geq 0} a_i\ x^i.$$

We use the convention $a_{-2} = a_{-1} = 0$ so that

$$x^2 v(x) = x^n \sum_{i \geq 0} a_i\ x^{i+2} = x^n \sum_{i \geq 2} a_{i-2}\ x^i.$$

We have

$$\begin{array}{rcl} v'(x) & = & n\ x^{n-1} \displaystyle\sum_{i \geq 0} a_i\ x^i + x^n \sum_{i \geq 0} i\ a_i\ x^{i-1} \\ v''(x) & = & n(n-1)\ x^{n-2} \displaystyle\sum_{i \geq 0} a_i\ x^i + 2\ n\ x^{n-1} \sum_{i \geq 0} i\ a_i\ x^{i-1} + x^n \sum_{i \geq 0} i\ (i-1)\ a_i\ x^{i-2}. \end{array}$$

This yields

$$\begin{array}{rcl} x\ v'(x) & = & n\ x^n \displaystyle\sum_{i \geq 0} a_i\ x^i + x^n \sum_{i \geq 0} i\ a_i\ x^i \\ x^2\ v''(x) & = & n(n-1)\ x^n \displaystyle\sum_{i \geq 0} a_i\ x^i + 2\ n\ x^n \sum_{i \geq 0} i\ a_i\ x^i + x^n \sum_{i \geq 0} i\ (i-1)\ a_i\ x^i \end{array}$$

from which we prove that

$$x^2\ v''(x) + x\ v'(x) + (x^2 - n^2)\ v(x) = 0$$

$$\Longleftrightarrow\ x^n \sum_{i \geq 0} \left[n\ a_i + i\ a_i + n(n-1)\ a_i + 2n\ i\ a_i\ + i\ (i-1)\ a_i + a_{i-2} - n^2\ a_i\right]\ x^i = 0.$$

We conclude that

$$i(i+2n)\ a_i + a_{i-2} = 0.$$

On the other hand, we have

$$a_{-1} = 0\ \Rightarrow\ a_1 = 0\ \Rightarrow\ a_3 = 0\ \Rightarrow\ a_{2k+1} = 0.$$

For the even indices we find the recursion

$$2i(2i+2n)\ a_{2i} + a_{2i-2} = 0\ \Rightarrow\ a_{2i} = \frac{(-1)}{2^2\ i\ (i+n)}\ a_{2(i-1)} = \ldots = \frac{(-1)^i}{2^{2i}\ i!\ (n+i)!}\ n!\ a_0.$$

By choosing $a_0 = 2^{-n}\ n!$ we conclude that

$$a_{2i} = \frac{(-1)^i}{2^{2i+n}\ i!\ (n+i)!}\ \Rightarrow\ v(x) = \left(\frac{x}{2}\right)^n \sum_{i \geq 0} \frac{(-1)^i}{i!(n+i)!}\ \left(\frac{x}{2}\right)^{2i} = B_n(x).$$

This ends the proof of the exercise.

■

Solution to exercise 300:

The Dirichlet-Poisson problem has the form (15.38) with $(L, V, h) = \left(\frac{1}{2}\partial_x^2, 0, 0\right)$. The operator $L = \frac{1}{2}\partial_x^2$ is the generator of $X_t = X_0 + W_t$. Using (15.42) we have

$$v(x) = \mathbb{E}\left(\int_0^{T_D} g(X_s)\ ds \mid X_0 = x\right).$$

The second assertion is immediate after choosing the unit function $g = 1$.

This ends the proof of the exercise.

■

Solution to exercise 301:

We have

$$\begin{array}{rcl} d\left(u(X_t) - tL(u)(X_t)\right) & = & L(u)(X_t)dt + dM_t(u) - L(u)(X_t)dt - t\ L^2(u)(X_t)dt - tdM_t(L(u)) \\ & = & -t\ L^2(u)(X_t)dt - tdM_t(L(u)) + dM_t(u) \end{array}$$

from which we prove that

$$N_t(u) = u(X_t) - tL(u)(X_t) - u(X_0) + \int_0^t s\ L^2(u)(X_s)ds = \int_0^t \left(sdM_s(L(u)) + dM_s(u)\right)$$

and

$$\mathbb{E}\left(N_t(u) \mid X_0\right) = \mathbb{E}\left(u(X_t) - tL(u)(X_t) + \int_0^t s\ L^2(u)(X_s)ds \mid X_0\right) - u(X_0) = 0.$$

On the other hand, using the fact that

$$\left\{\begin{array}{rcll} L^2u(x) & = & 0 & \text{if} \quad x \in D \\ (u(x), Lu(x)) & = & (f(x), g(x)) & \text{if} \quad x \in \partial D \end{array}\right.$$

we prove that

$$\begin{array}{rcl} u(x) & = & \mathbb{E}\left(u(X_{T_D}) - T_D L(u)(X_{T_D}) + \int_0^t s\ L^2(u)(X_s)ds \mid X_0 = x\right) \\ & = & \mathbb{E}\left(f(X_{T_D}) - T_D\ g(X_{T_D}) \mid X_0 = x\right). \end{array}$$

This ends the proof of the exercise.

■

Solution to exercise 302:

We fix the time horizon t and we let u_s, $s \in [0, t]$ be the solution of

$$\left\{\begin{array}{rcll} \partial_s u_s(x) + \dfrac{\sigma^2}{2}\partial_x^2 u_s(x) & = & 0 & \text{for any} \quad (s, x) \in ([0, t] \times] - a, a[) \\ u_s(x) & = & 0 & \text{for any} \quad (s, x) \in ([0, t] \times \{-a, +a\}) \\ u_t(x) & = & 1 & \text{for any} \quad x \in] - a, a[. \end{array}\right.$$

By (15.50) the solution is given by

$$D =] - a, a[\Rightarrow u_s(x) = \mathbb{E}\left(1_{T_D^{(s)} > t} \mid X_s = x\right) = \mathbb{E}\left(1_{T_D^{(0)} > (t-s)} \mid X_0 = x\right) = v_{t-s}(x)$$

with the function

$$v_t(x) := \mathbb{E}\left(1_{T_D^{(0)}>t} \mid X_0 = x\right).$$

By construction, we have

$$\forall s \leq t \quad \partial_s u_s(x) = -\partial_r v_r(x)_{|r=t-s} = -\frac{\sigma^2}{2}\, \partial_x^2 v_r(x)_{|r=t-s} \Rightarrow \forall r \geq 0 \quad \partial_r v_r = \frac{\sigma^2}{2}\, \partial_x^2 v_r$$

as well as the boundary conditions

$$\forall x \in]-a,a[\quad v_0(x) = u_t(x) = 1 \quad \text{ and } \quad \forall (t,x) \in ([0,\infty[\times\{-a,+a\}) \quad v_0(x) = u_t(x) = 0.$$

This ends the proof of the first assertion. Finally we have

$$\begin{aligned}&\sum_{n\geq 0} b_n\, \cos\left((2n+1)\,\frac{\pi}{2}\,\frac{x}{a}\right)\partial_t \exp\left(-\frac{\sigma^2}{2}\left((2n+1)\,\frac{\pi}{2}\,\frac{1}{a}\right)^2\, t\right)\\ &= -\frac{\sigma^2}{2}\sum_{n\geq 0} b_n\, \partial_x^2\left(\cos\left((2n+1)\,\frac{\pi}{2}\,\frac{x}{a}\right)\right)\, \exp\left(-\frac{\pi\sigma^2}{8a^2}\,(2n+1)^2\, t\right).\end{aligned}$$

For $x \in \{-a,a\}$ we have

$$\forall n \geq 0 \quad \cos\left((2n+1)\,\frac{\pi}{2}\right) = 0 = \cos\left(-(2n+1)\,\frac{\pi}{2}\right)$$

and for $t=0$ and $x \in]-a,a[$ we have the boundary condition

$$v_0(x) = \sum_{n\geq 0} b_n\, \cos\left((2n+1)\,\frac{\pi}{2}\,\frac{x}{a}\right) = 1,$$

with the Fourier series coefficients

$$b_n = \frac{1}{a}\int_{-a}^{a} 1\, \cos\left((2n+1)\,\frac{\pi}{2}\,\frac{x}{a}\right)\, dx = \frac{4}{(2n+1)\pi}\,(-1)^n.$$

Further details on these Fourier expansions can be found in section 5.8 in the textbook by Russel L. Herman [148].

This ends the proof of the exercise.

■

Solution to exercise 303:

We have

$$v_n(x) = \sin\left((n\pi/a)x\right) \Rightarrow v_n'(x) = (n\pi/a)\, \cos\left((n\pi/a)x\right) \Rightarrow v_n''(x) = -(n\pi/a)^2\, \sin\left((n\pi/a)x\right).$$

This shows that

$$L = \partial_x^2 \Rightarrow L(v_n) = \lambda_n\, v_n \quad \text{with} \quad \lambda_n = -(n\pi/a)^2.$$

On the other hand, we clearly have

$$v_n(0) = \sin(0) = 0 = \sin(n\pi) = v_n(a).$$

This ends the proof of the first assertion. The proof of the last assertion follows the same arguments. This ends the proof of the exercise. ■

Solution to exercise 304:

$$D = ([0,a_1]\times[0,a_2])$$

$$\Rightarrow \partial D = \underbrace{(\{a_1\}\times[0,a_2])}_{\partial_1 D}\cup\underbrace{([0,a_2]\times\{a_2\})}_{\partial_2 D}\cup\underbrace{(\{0\}\times[0,a_2])}_{\partial_3 D}\cup\underbrace{([0,a_2]\times\{0\})}_{\partial_4 D}.$$

We also have

$$N^\perp = 1_{\partial_1 D}\begin{pmatrix}1\\0\end{pmatrix}+1_{\partial_2 D}\begin{pmatrix}0\\1\end{pmatrix}+1_{\partial_3 D}\begin{pmatrix}-1\\0\end{pmatrix}+1_{\partial_4 D}\begin{pmatrix}0\\-1\end{pmatrix}.$$

Arguing as in exercise 303

$$\begin{aligned}
& v_n(x) := v_{n_1,n_2}(x_1,x_2) = \sin\left((n_1\pi/a_1)x_1\right)\sin\left((n_2\pi/a_2)x_2\right)\\
& \Rightarrow \forall i\in\{1,2\}\quad \partial^2_{x_i}v_n = \lambda_{n_i}\ v_n \quad\text{with}\quad \lambda_{n_i} := -(n_i\pi/a_i)^2\\
& \Rightarrow \sum_{1\leq i\leq 2}\partial^2_{x_i}v_n = \lambda_n\ v_n \quad\text{with}\quad \lambda_n = \sum_{1\leq i\leq 2}\lambda_{n_i}
\end{aligned}$$

and clearly $v_n(x) = 0$ for any $x\in\partial D$. This ends the proof of the first assertion. The proof of the last assertion follows the same arguments. To check that Neuman condition, we observe that

$$\begin{aligned}
& v_n(x) = \cos\left(n_1\pi x_1/a_1\right)\ \cos\left(n_2\pi x_2/a_2\right)\\
& \Rightarrow \nabla v_n(x) = \begin{pmatrix}\partial_{x_1}v_n\\ \partial_{x_2}v_n\end{pmatrix} = -\begin{pmatrix}(n_1\pi/a_1)\ \sin\left(n_1\pi x_1/a_1\right)\ \cos\left(n_2\pi x_2/a_2\right)\\ (n_2\pi/a_2)\ \cos\left(n_1\pi x_1/a_1\right)\ \sin\left(n_2\pi x_2/a_2\right)\end{pmatrix}.
\end{aligned}$$

This yields

$$\begin{aligned}
-\nabla v_n(x) \quad = \quad & 1_{\partial_1 D}(x)\ (n_2\pi/a_2)\ (-1)^{n_1}\begin{pmatrix}0\\ \sin\left(n_2\pi x_2/a_2\right)\end{pmatrix}\\
& +1_{\partial_2 D}(x)\ (n_1\pi/a_1)\ (-1)^{n_2}\begin{pmatrix}\sin\left(n_1\pi x_1/a_1\right)\\ 0\end{pmatrix}\\
& +1_{\partial_3 D}(x)\ (n_2\pi/a_2)\begin{pmatrix}0\\ \sin\left(n_2\pi x_2/a_2\right)\end{pmatrix}\\
& +1_{\partial_4 D}(x)\ (n_1\pi/a_1)\begin{pmatrix}\sin\left(n_1\pi x_1/a_1\right)\\ 0\end{pmatrix}.
\end{aligned}$$

This clearly implies that $\langle\nabla v(x), N^\perp(x)\rangle = 0$ for any $x\in\partial D$.

This ends the proof of the exercise. ■

Solution to exercise 305:

We fix the time horizon t and we let u_s, $s\in[0,t]$ be the solution of

$$\left\{\begin{array}{rcll}
\partial_s u_s(x)+\dfrac{\sigma^2}{2}\partial^2_x u_s(x) & = & 0 & \text{for any}\quad (s,x)\in([0,t]\times]0,a[)\\
u_s(x) & = & 0 & \text{for any}\quad (s,x)\in([0,t]\times\{0,a\})\\
u_t(x) & = & f & \text{for any}\quad x\in]0,a[.
\end{array}\right.$$

By (15.50) the solution is given by

$$D =]0,a[\Rightarrow u_s(x) = \mathbb{E}\left(1_{T_D^{(s)}>t}\ f(X_t) \mid X_s = x\right) = \mathbb{E}\left(1_{T_D^{(0)}>(t-s)}\ f(X_{t-s}) \mid X_0 = x\right) = v_{t-s}(x)$$

with the function

$$v_t(x) := \mathbb{E}\left(1_{T_D^{(0)}>t}\ f(X_t) \mid X_0 = x\right).$$

By construction, we have

$$\forall s \leq t \quad \partial_s u_s(x) = -\partial_r v_r(x)_{|r=t-s} = -\frac{\sigma^2}{2}\ \partial_x^2 v_r(x)_{|r=t-s} \ \Rightarrow\ \forall r \geq 0 \quad \partial_r v_r = \frac{\sigma^2}{2}\ \partial_x^2 v_r$$

as well as the boundary conditions

$$\forall x \in]0,a[\quad v_0(x) = u_t(x) = f \quad \text{ and } \quad \forall (t,x) \in ([0,\infty[\times\{0,+a\}) \quad v_0(x) = u_t(x) = 0.$$

This ends the proof of the first assertion. Finally we have

$$\lambda_n = -\left(\frac{n\pi}{a}\right)^2 \Rightarrow \partial_t\left(e^{\sigma^2\lambda_n t/2}\right)\ \sin(n\pi x/a) = \left(e^{\lambda_n t}\right)\ \frac{\sigma^2}{2}\ \partial_x^2 \sin(n\pi x/a)$$

as well as

$$\forall n \geq 0 \quad \sin(n\pi) = 0.$$

This shows that

$$v_t(x) = \sum_{n\geq 1} b_n(f)\ e^{\frac{\sigma^2}{2}\ \lambda_n t}\ \sin(n\pi x/a)$$

satisfies the desired boundary conditions as soon as for $t = 0$ and $x \in]0,a[$

$$v_0(x) = \sum_{n\geq 1} b_n(f)\ \sin(n\pi x/a) = f(x).$$

The coefficients $b_n(f)$ are determined by the Fourier Sine series on $[0,a]$

$$b_n(f) = \frac{2}{a}\int_0^a f(x)\ \sin(n\pi x/a)\ dx.$$

Further details on these Fourier expansions can be found in section 5.8, p.301, and section 5.10 in the textbook by Russel L. Herman [148].

When $f(x) = \sin(x)$ and $a = \pi$ we have directly

$$v_0(x) = \sin(x) = b_1(f)\ \sin(1\ \pi x/\pi) \Rightarrow b_1(f) = 1.$$

This yields the solution

$$\lambda_1 = -1 \implies v_t(x) = e^{-\frac{\sigma^2 t}{2}}\ \sin(x).$$

When $f(x) = x(1-x)$ and $a = 1$, using twice an integration by parts we find that

$$\begin{aligned} b_n(f) &= 2 \int_0^1 x(1-x) \ \sin(n\pi x) \ dx \\ &= \left[2x(1-x)\frac{\cos(n\pi x)}{n\pi}\right]_1^0 + \frac{2}{n\pi} \int_0^1 (1-2x) \ \cos(n\pi x) \ dx \\ &= \frac{2}{n\pi} \int_0^1 (1-2x) \ \cos(n\pi x) \ dx \\ &= \frac{2}{n^2\pi^2} \ [(1-2x)\sin(n\pi x)]_1^0 + \left(\frac{2}{n\pi}\right)^2 \int_0^1 \sin(n\pi x) \ dx \\ &= \left(\frac{2}{n\pi}\right)^2 \int_0^1 \sin(n\pi x) \ dx \\ &= \frac{1}{n\pi} \left(\frac{2}{n\pi}\right)^2 [\cos(n\pi x)]_0^1 = \frac{1}{n\pi} \left(\frac{2}{n\pi}\right)^2 (\cos(n\pi) - 1). \end{aligned}$$

This implies that

$$b_{2n}(f) = 0 \quad \text{and} \quad b_{2n+1}(f) = -\left(\frac{2}{(2n+1)\pi}\right)^3.$$

We conclude that

$$v_t(x) = -\sum_{n\geq 0} \left(\frac{2}{(2n+1)\pi}\right)^3 \ e^{-\frac{\sigma^2}{2} \ (2n+1)^2\pi^2 t} \ \sin((2n+1)\pi x).$$

This ends the proof of the exercise. ∎

Solution to exercise 306:
We follow the same arguments as the ones we used in solving exercise 305.

$$Q_t(f)(x) = \mathbb{E}\left(f(X_t) \ 1_{T_D > t} \mid X_0 = x_0\right) = \int_0^1 \ q_t(x,y) \ f(y) \ dy.$$

We have the forward and backward formulae

$$\partial_t Q_t(f)(x) = \frac{1}{2}\partial_x Q_t(f)(x) = Q_t\left(\frac{1}{2}\partial_y f\right).$$

Using an integration by parts, for any function f s.t. $f(0) = 0 = f(1) = 0$ we have

$$Q_t\left(\frac{1}{2}\partial_y^2 f\right)(x) = \int_0^1 \ q_t(x,y) \ \frac{1}{2}\partial_y^2 f(y) \ dy = \int_0^1 \ \frac{1}{2}\partial_y^2 q_t(x,y) \ f(y) \ dy.$$

This implies that

$$\partial_t q_t(x,y) = \frac{1}{2}\partial_y^2 q_t(x,y).$$

We try to find a solution of the form

$$q_t(x,y) = \sum_{n\geq 1} \ \alpha_n(t) \ u_n(x) \ u_n(y)$$

with the basis functions $u_n(y) = \sin(n\pi y)$ (recall that $u_n/\sqrt{2}$ form an orthonormal basis). Observe that

$$\begin{aligned}\sum_{n\geq 1} \partial_t \alpha_n(t)\ u_n(x)\ u_n(y) &= \partial_t q_t(x,y) \\ &= 2^{-1}\sigma^2\ \partial_y^2 q_t(x,y) = -2^{-1}\sigma^2 \sum_{n\geq 1} (n\pi)^2\ \alpha_n(t)\ u_n(x)\ u_n(y).\end{aligned}$$

This implies that

$$\partial_t \alpha_n(t) = -2^{-1}\sigma^2\ (n\pi)^2\ \alpha_n(t)$$

from which we conclude that

$$\alpha_n(t) = \alpha_n(0)\ \exp\left[-2^{-1}\sigma^2\ (n\pi)^2\ t\right].$$

This shows that

$$q_t(x,y) = \sum_{n\geq 1} \alpha_n(0)\ \exp\left[-2^{-1}\sigma^2\ (n\pi)^2\ t\right]\ u_n(x)\ u_n(y).$$

For $t = 0$ we have

$$\begin{aligned} f(x) &= 2 \sum_{n\geq 1} u_n(x) \int f(y)\ u_n(y)\ dy \\ &= Q_0(f)(x) = \sum_{n\geq 1} \alpha_n(0)\ u_n(x) \int f(y)\ u_n(y)\ dy \ \Rightarrow\ \alpha_n(0) = 2.\end{aligned}$$

We conclude that

$$q_t(x,y) = \sum_{n\geq 1} \exp\left[-2^{-1}\sigma^2\ (n\pi)^2\ t\right]\ \sin(n\pi x)\ \sin(n\pi y).$$

This ends the proof of the exercise.

■

Chapter 16

Solution to exercise 307:

By (16.8), we have the evolution equations

$$\partial_t \gamma_t(f) = \gamma_t\left(L^V(f)\right) \quad \text{and} \quad \partial_t \eta_t(f) = \eta_t\left(L_{\eta_t}(f)\right)$$

with the Schrödinger operator

$$L^V = L - V$$

and the collection of jump type generators

$$\begin{array}{rcl} L_\eta(f)(0) & = & L(f)(0) + V(0)\ (f(1) - f(0))\ \eta(1) \\ L_\eta(f)(1) & = & L(f)(1) + V(1)\ (f(0) - f(1))\ \eta(0). \end{array}$$

The nonlinear jump process X_t in $S = \{0,1\}$ with generator L_{η_t} changes its state 0 with a rate $V(0)\eta_t(1)$. In other words, it jumps from 0 to 1 at rate $V(0)\eta_t(1)$), and from 1 to 0 at rate $V(1)\eta_t(0)$. Between these jumps it evolves as a Markov process on S with generator L.

The mean field particle model is defined by a Markov process $\xi_t = (\xi_t^i)_{1 \leq i \leq N} \in \{0,1\}$. Each particle ξ_t^i jumps from 0 to 1 at rate $V(0)\eta_t^N(1)$, and from 1 to 0 at rate $V(1)\eta_t^N(0)$, with $\eta_t^N = \frac{1}{N}\sum_{1 \leq i \leq N} \delta_{\xi_t^i}$. Notice that

$$V(0)\eta_t^N(1) = V(0)\ \frac{1}{N} \sum_{1 \leq i \leq N} 1_{\xi_t^i = 1} \quad \text{and} \quad V(1)\eta_t^N(0) = V(1)\ \frac{1}{N} \sum_{1 \leq i \leq N} 1_{\xi_t^i = 0}.$$

Between these interacting jumps, each particle evolves independently as as a Markov process on S with generator L.

The N-mean field particle model can be interpreted as an epidemic propagation process. The state 1 represents the infected individuals, while 0 represents the susceptible ones. The transitions $0 \rightsquigarrow 1 \rightsquigarrow 0$ represent transitions of a **S**usceptibe-**I**nfected-**S**usceptible epidemic model. These models are called SIS models in biology and statistical inference. In these settings, the parameters of the Feynman-Kac model can be interpreted as follows:

$$\begin{array}{rcl} \lambda(0) & = & \text{rate of infection from an external source} \\ \lambda(1) & = & \text{rate of recovering of infected individuals} \\ V(0)\eta_t^N(1) & = & \text{infection rate of susceptible individuals by the infected ones} \\ V(1)\eta_t^N(0) & = & \text{recovering rate of infected individuals interacting with susceptible ones} \\ & = & \text{(often null).} \end{array}$$

This ends the proof of the exercise.

∎

Solution to exercise 308: Notice that

$$L(1_0)(0) = -\lambda(0) = L(1_1)(0) \quad \text{and} \quad L(1_0)(1) = \lambda(1) = -L(1_1)(1).$$

This yields

$$\begin{array}{rcl} L_\eta(1_0)(0) &=& -\lambda(0) - V(0)\eta(1) = -L_\eta(1_1)(0) \\ L_\eta(1_0)(1) &=& \lambda(1) + V(1)\eta(0) = -L_\eta(1_1)(1) \end{array}$$

from which we conclude that

$$\begin{array}{rcl} \eta L_\eta(1_0) &=& \eta(0)\ L_\eta(1_0)(0) + \eta(1)\ L_\eta(1_0)(1) \\ &=& -\eta(0)\ \left[\lambda(0) + V(0)\,(1-\eta(0))\right] + (1-\eta(0))\ \left[\lambda(1) + V(1)\eta(0)\right] \\ &=& (V(0) - V(1))\ \eta(0)^2 - \eta(0)\ \left((V(0)-V(1)) + (\lambda(0)+\lambda(1))\right) + \lambda(1). \end{array}$$

This yields the evolution equation

$$\partial_t \eta_t(0) \quad = \quad \partial_t \eta_t(1_0) = \eta_t L_{\eta_t}(1_0) = a\ \eta_t(0)^2 - b\ \eta_t(0) + c$$

with

$$a := [V(0) - V(1)]\,, \quad b := a + c + \lambda(0), \quad \text{and} \quad c := \lambda(1).$$

When $V(0) = V(1)$ (i.e. $a = 0$) the equation takes the form

$$\partial_t \eta_t(0) = -b\ \eta_t(0) + c \quad \text{with} \quad b := c + \lambda(0).$$

In this situation, the solution is given by

$$\begin{array}{rcl} \mathbb{P}(X_t = 0) &=& \eta_t(0) = e^{-bt}\left[\ \eta_0(0) + \dfrac{c}{b}\displaystyle\int_0^t\ be^{bs}\ ds\right] \\ &=& e^{-bt}\ \eta_0(0) + \dfrac{c}{b}\ \left(1 - e^{-bt}\right). \end{array}$$

Another simple case occurs when $c = 0$, and $b \geq a > 0$. In this case, the evolution equation takes the form

$$\partial_t \eta_t(0) = \partial_t \eta_t(1_0) = \eta_t L_{\eta_t}(1_0) = a\ \eta_t(0)^2 - b\ \eta_t(0).$$

The solution is given by the formula

$$\eta_t(0) \quad = \quad \frac{b}{a}\ \frac{\eta_0(0)}{\eta_0(0) + e^{bt}\ \left(\frac{b}{a} - \eta_0(0)\right)} = e^{-bt}\ \frac{\eta_0(0)}{1 - \frac{a}{b}\eta_0(0)\,(1 - e^{-bt})}.$$

To check this claim we observe that

$$\begin{array}{rcl} \partial_t \eta_t(0) &=& \dfrac{b}{a}\ \dfrac{\eta_0(0)}{\left[\eta_0(0) + e^{bt}\ \left(\frac{b}{a} - \eta_0(0)\right)\right]^2}\ \left(-b\left[e^{bt}\ \left(\dfrac{b}{a} - \eta_0(0)\right) + \eta_0(0)\right] + b\ \eta_0(0)\right) \\ &=& -b\ \eta_t(0) + a\eta_t(0)^2. \end{array}$$

More generally, we need to solve the Riccati equation

$$x' = a\ x^2 - b\ x + c \quad \text{with} \quad a \wedge c > 0 \quad \text{and} \quad b \geq a + c > 0.$$

The solution of this equation has the form

$$x(t) = -\frac{1}{a}\ \frac{y'(t)}{y(t)} \quad \text{with} \quad y'' + by' + acy = 0.$$

We check this claim by using the fact that

$$x = -\frac{1}{a}\,\frac{y'}{y} \quad \Rightarrow \quad x' = -\frac{1}{a}\,\frac{y''}{y} + \frac{1}{a}\,\left(\frac{y'}{y}\right)^2 = -\frac{1}{a}\,\frac{y''}{y} + a\;x^2$$
$$\Rightarrow \quad -\frac{1}{a}\,\frac{y''}{y} = -b\;x + c = \frac{b}{a}\,\frac{y'}{y} + c \Rightarrow y'' + by' + acy = 0.$$

The characteristic polynomial $p(z)$ of the second order differential equation

$$y'' + by' + acy = 0$$

is given by

$$p(z) = z^2 + bz + ac = \left(z + \frac{b}{2}\right)^2 - \left(\left(\frac{b}{2}\right)^2 - ac\right).$$

Under our assumptions, using the fact that $a^2 + c^2 \geq 2ac$ we have

$$b \geq (a+c) \Rightarrow b^2 \geq a^2 + c^2 + 2ac \geq 4ac \Rightarrow \left(\frac{b}{2}\right)^2 - ac \geq 0.$$

Observe that

$$b^2 = 4ac \Rightarrow b = 2\sqrt{a}\;\sqrt{c} \geq a + c = \left(\sqrt{a} - \sqrt{c}\right)^2 + 2\sqrt{a}\;\sqrt{c} \Rightarrow a = c.$$

In other words, when $b \geq (a+c)$ we have

$$b^2 = 4ac \iff a = c = b/2.$$

We examine the two cases

$$b^2 = 4ac \qquad \text{and} \qquad b^2 > 4ac.$$

- The case $b^2 = 4ac$ is associated with the parameters $a = c = b/2$. In this case, we have

$$x' = a\;x^2 - b\;x + c = a\;\left(x^2 - 2\;x + 1\right) = a(1-x)^2.$$

  If we set $\overline{x} = (1-x)$, then we have

$$\overline{x}' = -a\overline{x}^2 \Leftrightarrow \overline{x}(t) = \frac{\overline{x}(0)}{1 + a\overline{x}(0)t}.$$

  Notice that

$$\begin{aligned} a = c = b/2 \quad &\iff \quad [V(0) - V(1)] = \lambda(1) = \frac{1}{2}\left\{[V(0) - V(1)] + \lambda(1) + \lambda(0)\right\} \\ &\iff \quad [V(0) - V(1)] = \lambda(1) \quad \text{and} \quad \lambda(0) = 0. \end{aligned}$$

  We conclude that

$$\eta_t(1) = 1 - \eta_t(0) = \frac{\eta_0(1)}{1 + \lambda(1)\;\eta_0(1)\;t} \quad \longrightarrow_{t\uparrow\infty} \quad 0$$

  as soon as $\eta_0(1) > 0$.

- In the case $b^2 > 4ac$, the solution of $y'' + by' + acy = 0$ is given by

$$y(t) = c_1\ e^{z_1 t} + c_2\ e^{z_2 t}$$

with the two different roots

$$z_1 = -\frac{1}{2}\left(b + \sqrt{b^2 - 4ac}\right) \leq z_2 = -\frac{1}{2}\left(b - \sqrt{b^2 - 4ac}\right) < 0$$

of the characteristic polynomial

$$p(z) = \left(z + \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - ac}\right)\left(z + \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - ac}\right) = (z - z_2)(z - z_1),$$

and a couple of constants c_1, c_2 determined by the initial conditions

$$y(0) = c_1 + c_2 \quad \text{and} \quad y'(0) = c_1 z_1 + c_2 z_2.$$

A simple calculation shows that

$$\left.\begin{array}{rcl} z_1 y(0) & = & c_1 z_1 + c_2 z_1 \\ y'(0) & = & c_1 z_1 + c_2 z_2 \\ z_2 y(0) & = & c_1 z_2 + c_2 z_2 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} c_1(z_2 - z_1) & = & z_2 y(0) - y'(0) \\ c_2(z_2 - z_1) & = & y'(0) - z_1 y(0). \end{array}\right.$$

To check that $y(t) = c_1\ e^{z_1 t} + c_2\ e^{z_2 t}$ satisfies $y'' + by' + acy = 0$ for any choice of c_1, c_2 we notice that

$$\left\{\begin{array}{rcl} acy(t) & = & c_1\ ac\ e^{z_1 t} + c_2\ ac\ e^{z_2 t} \\ by' & = & c_1\ bz_1\ e^{z_1 t} + c_2\ bz_2\ e^{z_2 t} \\ y' & = & c_1\ z_1^2\ e^{z_1 t} + c_2\ z_2^2\ e^{z_2 t}. \end{array}\right.$$

Recalling that

$$p(z_1) = z_1^2 + bz_1 + ac = 0 = z_2^2 + bz_2 + ac = p(z_2)$$

we end the proof of the desired result.

On the other hand, using the fact that $y'(0) = -ax(0)y(0)$, we find that

$$\begin{array}{rcl} c_1(z_2 - z_1) & = & z_2 y(0) - y'(0) = y(0)\ (ax(0) + z_2) \\ c_2(z_2 - z_1) & = & y'(0) - z_1 y(0) = -y(0)\ (ax(0) + z_1). \end{array}$$

This shows that

$$\begin{array}{rcl} -a\ x(t) & = & \dfrac{c_1\ z_1\ e^{z_1 t} + c_2\ z_2\ e^{z_2 t}}{c_1\ e^{z_1 t} + c_2\ e^{z_2 t}} \\ & = & \dfrac{(ax(0) + z_2)\ z_1\ e^{z_1 t} - (ax(0) + z_1)\ z_2\ e^{z_2 t}}{(ax(0) + z_2)\ e^{z_1 t} - (ax(0) + z_1)\ e^{z_2 t}} \\ & = & \dfrac{(ax(0) + z_1)\ z_2 - (ax(0) + z_2)\ (z_1 - z_2 + z_2)\ e^{-(z_2 - z_1)t}}{(ax(0) + z_1) - (ax(0) + z_2)\ e^{-(z_2 - z_1)t}} \\ & = & z_2 + \dfrac{(ax(0) + z_2)\ (z_2 - z_1)\ e^{-(z_2 - z_1)t}}{(ax(0) + z_1) - (ax(0) + z_2)\ e^{-(z_2 - z_1)t}}. \end{array}$$

Observe that the function

$$\theta(t) = (ax(0) + z_1) - (ax(0) + z_2)\ e^{-(z_2 - z_1)t}$$

is monotone, starts at $\theta(0) = -(z_2 - z_1) < 0$ and converges to $(ax(0) + z_1)$ as $t \uparrow \infty$. Let us show that $(ax(0) + z_1) < 0$ so that $\theta(t)$ cannot change its sign and never crosses the null axis. We have

$$ax(0) + z_1 \leq a - \frac{1}{2}\,\left(b + \sqrt{b^2 - 4ac}\right) = \left(a - \frac{b}{2}\right) - \sqrt{\left(\frac{b}{2}\right)^2 - ac}.$$

Notice that

$$a - \frac{b}{2} \leq 0 \Rightarrow \left(a - \frac{b}{2}\right) - \sqrt{\left(\frac{b}{2}\right)^2 - ac} < 0 \implies ax(0) + z_1 < 0$$

since $b^2 > 4ac$.

On the other hand, if $a - \frac{b}{2} > 0$ then we have

$$\begin{aligned}\left(a - \frac{b}{2}\right) < \sqrt{\left(\frac{b}{2}\right)^2 - ac} \quad &\Leftrightarrow \quad a^2 + \left(\frac{b}{2}\right) - ab < \left(\frac{b}{2}\right)^2 - ac \\ &\Leftrightarrow \quad a^2 - ab < -ac \Leftrightarrow b > (a+c) \Rightarrow ax(0) + z_1 < 0.\end{aligned}$$

Last but not least, we need to examine the case $b = (a + c)$. Then we have

$$b^2 = (a^2 + c^2 + 2ac) > 4ac \iff (a-c)^2 > 0 \iff a \neq c$$

and

$$\left(a - \frac{b}{2}\right) - \sqrt{\left(\frac{b}{2}\right)^2 - ac} = \frac{1}{2}\,\left((a-c) - \sqrt{a^2 + c^2}\right).$$

It is readily checked that

$$a < c \Rightarrow \frac{1}{2}\,\left((a-c) - \sqrt{a^2 + c^2}\right) < 0 \implies ax(0) + z_1 < 0$$

and when $a > c$ we have

$$\begin{aligned}(a-c) < \sqrt{a^2 + c^2} \quad &\iff \quad a^2 + c^2 - 2ac \leq a^2 + c^2 \\ &\iff \quad ac > 0 \implies ax(0) + z_1 < 0.\end{aligned}$$

We conclude that $x(t)$ is well defined for any t and an initial condition, and it is given by the formula

$$x(t) = -\frac{z_2}{a} + \left(x(0) + \frac{z_2}{a}\right)\ \underbrace{\frac{(z_2 - z_1)\ e^{-(z_2 - z_1)t}}{(ax(0) + z_2)\ e^{-(z_2 - z_1)t} - (ax(0) + z_1)}}_{>0}.$$

Finally we observe that

$$a > 0 \ \Rightarrow \ -\frac{z_2}{a} = \frac{b}{2a} - \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \geq 0.$$

Let us check that

$$\frac{b}{2a} - \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \leq 1.$$

When $\frac{b}{2a} < 1$ the result is trivial. When $\frac{b}{2a} \geq 1$ we have

$$0 \leq \frac{b}{2a} - 1 \leq \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \quad \Leftrightarrow \quad \left(\frac{b}{2a}\right)^2 + 1 - \frac{b}{a} \leq \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$
$$\Leftrightarrow \quad 1 + \frac{c}{a} \leq \frac{b}{a} \Leftrightarrow b \geq a + c.$$

This ends the proof of the exercise. ∎

Solution to exercise 309:
We have

$$L_{\eta_t}(1_0)(0) = -\lambda(\eta_t, 0) = -a(u_t) \quad \text{and} \quad L_{\eta_t}(1_0)(1) = \lambda(\eta_t, 1) = b(u_t).$$

This implies that

$$\partial_t \eta_t(0) \quad = \quad \eta_t(0)\ L_{\eta_t}(1_0)(0) + \eta_t(1)\ L_{\eta_t}(1_0)(1) = -u_t\ a(u_t) + v_t\ b(u_t)$$

and

$$\begin{aligned}
\partial_t \eta_t(1) &= -\partial_t \eta_t(0) \\
&= -\eta_t(0)\ L_{\eta_t}(1_0)(0) - \eta_t(1)\ L_{\eta_t}(1_0)(1) = u_t\ a(u_t) - v_t\ b(u_t).
\end{aligned}$$

The nonlinear jump process X_t in $S = \{0,1\}$ with generator L_{η_t} changes its state x with a rate $\lambda(\eta_t, x)$. In other words it jumps from 0 to 1 at rate $\lambda(\eta_t, 0)$ and from 1 to 0 at rate $\lambda(\eta_t, 1)$.

The mean field particle model is defined by a Markov process $\xi_t = (\xi_t^i)_{1\leq i\leq N} \in \{0,1\}$. Each particle ξ_t^i jumps from 0 to 1 at rate $\lambda(\eta_t^N, 0)$ and from 1 to 0 at rate $\lambda(\eta_t^N, 1)$, with $\eta_t^N = \frac{1}{N}\sum_{1\leq i\leq N} \delta_{\xi_t^i}$.

When $\lambda(\eta_t, 0) = \eta_t(1)$ and $\lambda(\eta_t, 1) = \eta_t(0)$. The limiting system takes the form

$$\partial_t \eta_t(0) = -\eta_t(0)\ \eta_t(0) + \eta_t(1)\ \eta_t(1) = 1 - 2\eta_t(0).$$

In this situation, we have

$$\begin{aligned}
\eta_t(0) &= e^{-2t}\left[\eta_0(0) + \int_0^t e^{2s} ds\right] = e^{-2t}\eta_0(0) + \frac{1}{2}\ (1 - e^{-2t}) \\
&= \frac{1}{2} + e^{-2t}\left(\eta_0(0) - \frac{1}{2}\right).
\end{aligned}$$

When $\lambda(\eta_t, 0) = \eta_t(1)$ and $\lambda(\eta_t, 1) = \eta_t(0)$. The limiting system takes the form

$$\partial_t \eta_t(0) = -\eta_t(1)\ \eta_t(0) + \eta_t(0)\ \eta_t(1) = 0 = \partial_t \eta_t(1).$$

Hence in this case we have $\eta_t(x) = \eta_0(x)$, for any $x \in S$.

This ends the proof of the exercise. ∎

Solution to exercise 310:
By integration by parts, we have

$$\begin{aligned}
\partial_t \eta_t(f) &= \int f(x)\ \partial_t p_t(x)\ dx \\
&= \int f(x)\ \partial_x^2 p_t(x)\ dx - \int f(x)\ \partial_x \left(p_t\ (p_t \star a)\right)(x)\ dx \\
&= \int \partial_x^2 f(x)\ p_t(x)\ dx + \int \partial_x f(x)\ b(x, \eta_t)\ p_t(x)\ dx = \int L_{\eta_t}(f)(x)\ p_t(x)\ dx.
\end{aligned}$$

The corresponding nonlinear Markov process is defined by

$$dX_t = b(X_t, \eta_t)dt + \sqrt{2}\, dW_t.$$

The mean field model $\xi_t = (\xi_t^i)_{1\leq i\leq N}$ associated with this nonlinear diffusion process is given by

$$\left\{\begin{array}{rcl} d\xi_t^i &=& b(\xi_t^i, \eta_t^N)dt + \sqrt{2}\, dW_t^i \\ i &=& 1,\ldots,N \end{array}\right. \quad \text{with} \quad \eta_t^N := \frac{1}{N}\sum_{1\leq i\leq N} \delta_{\xi_t^i}$$

with N independent copies W_t^i of W_t. When $a(u) = \alpha u + \beta$ we have

$$b(x, \eta_t) = \alpha(x - \int \int y\ \eta_t(dy)) = \alpha(x - \mathbb{E}(X_t)).$$

In this situation, the nonlinear model takes the form

$$dX_t = \alpha\,(X_t - \mathbb{E}(X_t))\ \ dt + \sqrt{2}\ dW_t$$

and the N-mean field particle model is defined by

$$d\xi_t^i \;=\; \alpha\left(\xi_t^i - \frac{1}{N}\sum_{1\leq i\leq N}\xi_t^j\right)\ dt + \sqrt{2}\ dW_t^i = \frac{\alpha}{N}\sum_{1\leq i\leq N}\left(\xi_t^i - \xi_t^j\right)\ dt + \sqrt{2}\ dW_t^i.$$

Notice that

$$dX_t = \alpha\,(X_t - \mathbb{E}(X_t))\ \ dt + \sqrt{2}\ dW_t \Rightarrow d\mathbb{E}(X_t) = 0 \Rightarrow \mathbb{E}(X_t) = \mathbb{E}(X_0).$$

This ends the proof of the exercise.

■

Solution to exercise 311:
Reversing the integration order, we have

$$\begin{aligned}
\partial_t\eta_t(f) &= -\eta_t(fH(\,.\,,p_t)) + \int f(x)\ \left[\int_{-\infty}^{x} q(x-y)\ \ p_t(y)\ dy\right]\ dx \\
&= -\eta_t(fH(\,.\,,p_t)) + \int H(x,p_t)\ \left[\int f(y)\ 1_{[x,\infty[}\ q(y-x)\ dy\ \right]\ p_t(x)\ dx \\
&= -\eta_t(f\lambda(\,.\,,\eta_t)) + \int \lambda(x,\eta_t)\ \left[\int f(y)\ 1_{[x,\infty[}\ q(y-x)\ dy\ \right]\ \eta_t(dx) \\
&= \eta_t(L_{\eta_t}(f)).
\end{aligned}$$

The nonlinear jump process X_t with generator L_{t,η_t} evolves as a time inhomogeneous pure jump model. At jumps times T_n arriving at rate $\lambda(x,\eta_t)$ it jumps $X_{T_n-} \rightsquigarrow X_{T_n} = X_{T_n-} + U_n$ where U_n stands for a sequence of independent random variables with distribution $q(u)du$. When $h = 1$ the jump intensity resumes to

$$\lambda(x,\eta_t) = \int_x^{+\infty} \eta_t(dy) = \mathbb{P}(X_t > x)\ \ \uparrow_{x\uparrow\infty}\ \ 1.$$

Using (16.5), the generator of the mean field model $\xi_t = (\xi_t^i)_{1\leq i\leq N}$ associated with this nonlinear jump process is given for any sufficiently regular function F by the formula

$$\begin{aligned}
&L^{(i)}_{t,m(x)}(F)(x^1,\ldots,x^N)\\
&:= \sum_{1\leq i\leq N} \int_{x^i}^{+\infty} h\left(y-\int_{-\infty}^{+\infty} z\ m(x)(dz)\right)\ m(x)(dy)\\
&\qquad\qquad \times \int \left[F(x^1,\ldots,x^{i-1},y,x^{i+1},\ldots,x^N)-F(x^1,\ldots,x^i,\ldots,x^N)\right]\ M(x^i,dy),
\end{aligned}$$

with $m(x) := \frac{1}{N}\sum_{j=1}^N \delta_{x^j}$. For each $1\leq i\leq N$, at rate

$$\begin{aligned}
&\int_{\xi_t^i}^{+\infty} h\left(y-\int_{-\infty}^{+\infty} z\ m(\xi_t)(dz)\right)\ m(\xi_t)(dy)\\
&= \frac{1}{N}\sum_{1\leq j\leq N} 1_{[\xi_t^i,\infty[}(\xi_t^j)\ h\left(\xi_t^j-\frac{1}{N}\sum_{1\leq k\leq N}\xi_t^k\right)
\end{aligned}$$

each particle ξ_t^i performs a jump to the right with an amplitude U. When $h=1$, the jump rate of each particle ξ_t^i coincides with the proportion of particles ξ_t^j in the r.h.s. of ξ_t^i; more formally, we have

$$\int_{\xi_t^i}^{+\infty} h\left(y-\int_{-\infty}^{+\infty} z\ m(\xi_t)(dz)\right)\ m(\xi_t)(dy) \stackrel{h=1}{=} \frac{1}{N}\sum_{1\leq j\leq N} 1_{[\xi_t^i,\infty[}(\xi_t^j).$$

This ends the proof of the exercise. ■

Solution to exercise 312:
We have

$$\begin{aligned}
L_\eta(f)(x) &= \int (f((1-\epsilon)x+\epsilon y)-f(x))\ \kappa(x,y)\ \eta(dy)\\
&= \lambda(x,\eta_t) \int (f((1-\epsilon)x+\epsilon y)-f(x))\ K_{\eta_t}(x,dy),
\end{aligned}$$

with the intensity function

$$\lambda(x,\eta_t) := \int \kappa(x,y)\ \eta_t(dy)$$

and with the Markov jump transitions

$$K_{\eta_t}(x,dy) = \frac{\kappa(x,y)\ \eta_t(dy)}{\int\ \kappa(x,z)\ \eta_t(dz)}.$$

This shows that the process X_t is a pure jump process, with a jump rate $\lambda(X_t,\eta_t)$. When a jump occurs at some time t the process jumps from X_{t-} to $X_t=((1-\epsilon)X_{t-}+\epsilon Y)$, where Y is chosen according to the Markov transition $K_{\eta_t}(X_{t-},dy)$.

The N-mean field particle model is defined in terms of a system of N particles $\xi_t = \left(\xi_t^i\right)_{1\leq i\leq N}\in S^N$. We set $\eta_t^N=\frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi_t^i}$. At rate

$$\lambda\left(\xi_t^i,\eta_t^N\right) = \int \kappa(\xi_t^i,y)\ \eta_t^N(dy) = \frac{1}{N}\sum_{1\leq i\leq N} \kappa(\xi_t^i,\xi_t^j)$$

the particle ξ^i_{t-} jumps to a new opinion $\xi^i_t = ((1-\epsilon)\xi^i_{t-} + \epsilon Y^i_t)$ where Y^i_t is chosen according to the Markov transition

$$K_{\eta^N_t}(\xi^i_{t-}, dy) := \sum_{1\leq j\leq N} \frac{\kappa(\xi^i_{t-}, \xi^j_{t-})}{\sum_{1\leq l\leq N} \kappa(\xi^i_{t-}, \xi^l_{t-})} \; \delta_{\xi^i_{t-}}(dy).$$

Solution to exercise 313:

For symmetric densities $k(x,y) = k(y,x)$, choosing the multidimensional function $f_1(x) = x$ we have

$$\partial_t \eta_t(f_1) \;=\; \epsilon \int (y-x)\; \kappa(x,y)\; \eta_t(dx)\eta_t(dy) = 0.$$

This shows that

$$\int x\; \eta_t(dx) = \int x\; \eta_0(dx) = 0.$$

In addition, choosing $f_2(x) = \|x\|^2 = \langle x, x\rangle$ we have

$$\partial_t \eta_t(f_2) \;=\; \epsilon^2 \int \|y-x\|^2\; \kappa(x,y)\; \eta_t(dx)\eta_t(dy) - 2\epsilon \int \langle x-y, x\rangle\; \kappa(x,y)\; \eta_t(dx)\eta_t(dy).$$

By symmetry arguments, observe that

$$\begin{aligned}\int \langle x-y, x\rangle\; \kappa(x,y)\; \eta_t(dy)\eta_t(dy) &= \int \langle y-x, y\rangle\; \kappa(x,y)\; \eta_t(dx)\eta_t(dy)\\ &= -\int \langle x-y, y\rangle\; \kappa(x,y)\; \eta_t(dx)\eta_t(dy).\end{aligned}$$

This yields

$$\partial_t \eta_t(f_2) \;=\; -\epsilon(1-\epsilon) \int \|y-x\|^2\; \kappa(x,y)\; \eta_t(dx)\eta_t(dy) \leq 0 \Rightarrow \eta_t(f_2) \leq \eta_0(f_2).$$

This ensures the existence of the limiting second moment $\lim_{t\uparrow\infty} \eta_t(f_2) < \infty$ as soon as $\eta_0(f_2) < \infty$.

When $\kappa = 1$ we have

$$\partial_t \eta_t(f_2) = -2\epsilon(1-\epsilon)\; \eta_t(f_2) \;\Rightarrow\; \eta_t(f_2) = e^{-2\epsilon(1-\epsilon)t}\; \eta_0(f_2).$$

In addition, for any Lipschitz function $\|f(x) - f(y)\| \leq \|x-y\|$ using Cauchy-Schwartz inequality we prove that

$$\begin{aligned}\frac{1}{2}\, \partial_t \|\eta_t(f)\|^2 &= \langle \partial_t \eta_t(f), \eta_t(f)\rangle\\ &= \int \langle (f((1-\epsilon)x + \epsilon y) - f(x)), \eta_t(f)\rangle\; \eta_t(dx)\eta_t(dy)\\ &\leq \epsilon\; \|\eta_t(f)\| \int \|y-x\|\; \eta_t(dx)\eta_t(dy) \leq \sqrt{2}\; \epsilon\; \|\eta_t(f)\| e^{-\epsilon(1-\epsilon)t}\; \left(\eta_0(f_2)\right)^{1/2}.\end{aligned}$$

In the last assertion we have used the fact that

$$\left(\int \|y-x\|\; \eta_t(dx)\eta_t(dy)\right)^2 \leq \int \|y-x\|^2\; \eta_t(dx)\eta_t(dy) \leq 2\; \eta_t(f_2).$$

This implies that

$$\partial_t\|\eta_t(f)\| \left(= \partial_t\sqrt{\|\eta_t(f)\|^2}\right) = \frac{1}{2}\ \frac{1}{\|\eta_t(f)\|}\ \partial_t\|\eta_t(f)\|^2 \le \sqrt{2}\ \epsilon\ e^{-\epsilon(1-\epsilon)t}\ \ (\eta_0(f_2))^{1/2}$$

from which we conclude that

$$\begin{aligned}\|\eta_t(f)\| &\le \|\eta_0(f)\| + (2\eta_0(f_2))^{1/2}\ \epsilon\ \int_0^t e^{-\epsilon(1-\epsilon)s}\ ds \\ &= \|\eta_0(f)\| + (2\eta_0(f_2))^{1/2}\ (1-\epsilon)\ \left(1 - e^{-\epsilon(1-\epsilon)t}\right).\end{aligned}$$

This ends the proof of the exercise. ∎

Solution to exercise 314: A simple integration by parts shows that

$$\begin{aligned}\partial_t\eta_t(f) &= \int f(x)\ \partial_t p_t(x)\ dx \\ &= -\int f(x)\ \partial_x\left(B_{\eta_t}\ p_t\right)\ dx + \frac{1}{2}\int f(x)\ \partial_x^2\left(D^2_{\eta_t}\ p_t\right)\ dx \\ &= \int B_{\eta_t}(x)\ \partial_x f(x)\ p_t(x)\ dx + \frac{1}{2}\int D^2_{\eta_t}(x)\ \partial_x^2 f(x)\ p_t(x)\ dx = \eta_t(L_{\eta_t}(f)).\end{aligned}$$

The nonlinear process X_t with distribution η_t is given by the stochastic differential equation

$$dX_t = B_{\eta_t}(X_t)\ dt + D_{\eta_t}(X_t)\ dW_t.$$

Here W_t denotes the 1-dimensional Brownian motion. The N-mean field particle approximation of the nonlinear diffusion is defined by

$$\left\{\begin{array}{rcl} d\xi^i_t &=& B_{\eta^N_t}(\xi^i_t)\ dt + D_{\eta^N_t}(\xi^i_t)\ dW^i_t \\ i &=& 1,\ldots,N \quad \text{with} \quad \eta^N_t = \frac{1}{N}\sum_{1\le j\le N}\delta_{\xi^j_t}, \end{array}\right.$$

where $W^i_t = (W^{i,j})_{1\le j\le r}$ stands for N independent copies of the r-dimensional Brownian motion W_t. Notice that

$$B_{\eta^N_t}(x) = \alpha\left(\int_{-\infty}^x \eta^N_t(dy)\right) = \alpha\left(\frac{1}{N}\sum_{1\le j\le N} 1_{]-\infty,x]}(\xi^j_t)\right)$$

and

$$D_{\eta^N_t}(x) = \beta\left(\frac{1}{N}\sum_{1\le j\le N} 1_{]-\infty,x]}(\xi^j_t)\right).$$

This shows that

$$\left\{\begin{array}{rcl} d\xi^i_t &=& \alpha\left(\frac{1}{N}\sum_{1\le j\le N}\ 1_{]-\infty,\xi^i_t]}(\xi^j_t)\right)\ dt + \beta\left(\frac{1}{N}\sum_{1\le j\le N}\ 1_{]-\infty,\xi^i_t]}(\xi^j_t)\right)\ dW^i_t \\ i &=& 1,\ldots,N. \end{array}\right.$$

This ends the proof of the exercise. ∎

Solution to exercise 315:

We have

$$\begin{cases} dX_t^i &= \left[\partial_{x_i} V_1(X_t) + \int \partial_{x_i} V_2(X_t - x)\ \eta_t(dx)\right]\ dt + \sum_{1\leq j\leq r} \sigma_j^i(X_t)\ dW_t^j \\ i &= 1,\ldots,r \end{cases}$$

with $\eta_t(dx) = \mathbb{P}(X_t \in dx)$. The generator of X_t is given by

$$\begin{aligned} L_{t,\eta_t}(f)(x) &= \sum_{1\leq i\leq r} \left[\partial_{x_i} V_1(x) + \int \partial_{x_i} V_2(x-y)\ \eta_t(dy)\right]\ \partial_{x_i} f(x) \\ &\quad + \frac{1}{2} \sum_{1\leq i,j\leq r} (\sigma\sigma^T)_{i,j}(x)\ \partial_{x_i,x_j} f(x). \end{aligned}$$

The N-mean field particle approximation of the nonlinear diffusion is defined by

$$\begin{cases} d\xi_t^i &= \partial V_1(\xi_t^i)\ dt + \frac{1}{N} \sum_{1\leq j\leq N} \partial V_2(\xi_t^i - \xi_t^j)\ dt + \sigma(\xi_t^i)\ dW_t^i \\ i &= 1,\ldots,N \end{cases}$$

where $W_t^i = (W^{i,j})_{1\leq j\leq r}$ stands for N independent copies of the r-dimensional Brownian motion W_t. As discussed in the beginning of section 16.1, the nonlinear diffusion can be interpreted as a time inhomogeneous diffusion. Therefore, by (15.21) the density $p_t(x)$ satisfies the nonlinear Fokker-Planck equation

$$\partial_t p_t(x) = -\sum_{i=1}^d \partial_i \left(b_{t,p_t}^i\ p_t\right) + \frac{1}{2} \sum_{i,j=1}^d \partial_{i,j} \left(\left(\sigma_t(\sigma_t)^T\right)_{i,j}\ p_t\right),$$

with the drift functions

$$b_{t,p_t}^i(x) = \partial_{x_i} V_1(x) + \int \partial_{x_i} V_2(X_t - x)\ p_t(x)\ dx.$$

This ends the proof of the exercise.

■

Solution to exercise 316:
Notice that

$$x + \rho_{1/N}^i(x) = \left(x_1 + \frac{1}{N}, \ldots, x_{i-1} + \frac{1}{N}, \underbrace{0}_{i-th}, x_{i+1} + \frac{1}{N}, \ldots, x_N + \frac{1}{N}\right)$$

so that

$$\rho_{1/N}^i(x) = \rho_0^i(x) + \epsilon^i \quad \text{with} \quad \epsilon^i = (\epsilon_j^i)_{1\leq j\leq N} \quad \text{and} \quad \epsilon_j^i = \frac{1}{N}\ 1_{j\neq i} = \frac{1}{N} - \frac{1}{N}\ 1_{i=j}.$$

We recall the first order Taylor expansions

$$\begin{aligned} f(y) - f(x) &= \sum_{1\leq j\leq N} \int_0^1 \partial_{x_j} f(x + t(y-x))\ (y_j - x_j)\ dt \\ &= \sum_{1\leq j\leq N} \partial_{x_j} f(x)\ (y_j - x_j) \\ &\quad + \sum_{1\leq j,k\leq N} \left[\int_0^1 (1-t)\ \partial_{x_j,x_k} f(x + t(y-x))\ dt\right]\ (y_j - x_j)\ (y_k - x_k). \end{aligned}$$

This implies that

$$
\begin{aligned}
& f(x+\rho^i_{1/N}(x)) - f(x+\rho^i_0(x)) \\
&= \sum_{1\leq j\leq N} \partial_{x_j} f(x+\rho^i_0(x))\ \epsilon^i_j + \sum_{1\leq j,k\leq N} \left[\int_0^1 (1-t)\ \partial_{x_j,x_k} f(x+\rho^i_0(x)+t\epsilon^i)\ dt\right]\ \epsilon^i_j\ \epsilon^i_k \\
&= \frac{1}{N}\sum_{1\leq j\leq N} \partial_{x_j} f(x) - \frac{1}{N}\ \partial_{x_i} f(x) - \sum_{1\leq j\leq N} \left[\int_0^1 \partial_{x_i,x_j} f(x+t\rho^i_0(x))\ dt\right] x_i\ \epsilon^i_j \\
&\qquad + \sum_{1\leq j,k\leq N} \left[\int_0^1 (1-t)\ \partial_{x_j,x_k} f(x+\rho^i_0(x)+t\epsilon^i)\ dt\right]\ \epsilon^i_j\ \epsilon^i_k
\end{aligned}
$$

from which we find that

$$
\begin{aligned}
& \sum_{1\leq i\leq N} \lambda(x_i)\ \left(f(x+\rho^i_{1/N}(x)) - f(x+\rho^i_0(x))\right) \\
&= \sum_{1\leq j\leq N} \left(\frac{1}{N}\sum_{1\leq i\leq N} \lambda(x_i)\right)\ \partial_{x_j} f(x) - \frac{1}{N}\sum_{1\leq i\leq N} \lambda(x_i)\ \partial_{x_i} f(x) \\
&\qquad - \sum_{1\leq i,j\leq N} \lambda(x_i)\ \left[\int_0^1 \partial_{x_i,x_j} f(x+t\rho^i_0(x))\ dt\right] x_i\ \epsilon^i_j \\
&\qquad + \sum_{1\leq i,j,k\leq N} \lambda(x_i)\ \left[\int_0^1 (1-t)\ \partial_{x_j,x_k} f(x+\rho^i_0(x)+t\epsilon^i)\ dt\right]\ \epsilon^i_j\ \epsilon^i_k.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
\mathcal{L}(f)(x_1,\ldots,x_N) &= \mathcal{G}(f)(x_1,\ldots,x_N) - \frac{1}{N}\sum_{1\leq i\leq N} \lambda(x_i)\ \partial_{x_i} f(x) \\
&\quad - \sum_{1\leq i,j\leq N} \lambda(x_i)\ \left[\int_0^1 \partial_{x_i,x_j} f(x+t\rho^i_0(x))\ dt\right] x_i\ \epsilon^i_j \\
&\quad + \sum_{1\leq i,j,k\leq N} \lambda(x_i)\ \left[\int_0^1 (1-t)\ \partial_{x_j,x_k} f(x+\rho^i_0(x)+t\epsilon^i)\ dt\right]\ \epsilon^i_j\ \epsilon^i_k.
\end{aligned}
$$

For empirical functions $f(x) = m(x)(\varphi) = \frac{1}{N}\sum_{1\leq i\leq N} \varphi(x_i)$ we have

$$
\partial_{x_j} f(x) = \frac{1}{N}\ \varphi'(x_j) \quad \text{and} \quad \partial_{x_j,x_k} f(x) = 1_{j=k}\ \frac{1}{N}\ \varphi''(x_j).
$$

A simple calculation shows that

$$
\begin{aligned}
N\left[\mathcal{L}-\mathcal{G}\right](f)(x_1,\ldots,x_N) &= \frac{1}{N}\sum_{1\leq i\leq N} \lambda(x_i)\ \varphi'(x_i) \\
&\quad + \frac{1}{N^2}\sum_{1\leq i\neq j\leq N} \lambda(x_i)\ \left[\int_0^1 (1-t)\ \varphi''(x_j+t/N)\ dt\right].
\end{aligned}
$$

Observe that

$$f(x+\rho_0^i(x))-f(x)=m(x+\rho_0^i(x))(\varphi)-m(x)(f)=\frac{1}{N}\;(\varphi(0)-\varphi(x^i)).$$

This implies that

$$\begin{aligned}\mathcal{G}(f)(x) &= \frac{1}{N}\sum_{1\leq i\leq N}\lambda(x_i)\;(\varphi(0)-\varphi(x^i))\\ &\quad+\frac{1}{N}\sum_{1\leq i\leq N}\left[a\;\left(x_i-\frac{1}{N}\sum_{1\leq j\leq N}x_j\right)+\frac{1}{N}\sum_{1\leq i\leq N}\lambda(x_i)\right]\;\varphi'(x_i)\\ &= m(x)\left(L_{m(x)}(\varphi)\right),\end{aligned}$$

with

$$L_\eta(\varphi)(u)=\lambda(u)\;(\varphi(0)-\varphi(u))+\left[a\;\left(u-\int\;v\;\eta(dv)\right)+\int\;\lambda(v)\;\eta(dv)\right]\;\varphi'(u).$$

At a rate $\lambda(X_t)$, the process jumps from X_t to 0. The resulting jump increment is given by $\Delta X_t=X_t-X_{t-}=0-X_{t-}$. Between the jumps, the process evolves according to the deterministic evolution equation

$$dX_t=[a\;\;(X_t-\mathbb{E}(X_t))+\mathbb{E}\left(\lambda(X_t)\right)]\;\;dt.$$

We conclude that

$$dX_t=[a\;\;(X_t-\mathbb{E}(X_t))+\mathbb{E}\left(\lambda(X_t)\right)]\;\;dt\;\;-\;\;X_{t-}\;dN_t$$

where N_t stands for a Poisson process with intensity $\lambda(X_t)$.

This ends the proof of the exercise.

■

Solution to exercise 317:

We have

$$\eta_t(f)\eta_t(V_t)-\eta_t\left(fV_t\right)=\int\eta_t(dx)\;V_t(x)\;\left[\int\;\;(f(y)-f(x))\;\eta_t(dy)\right].$$

This implies that

$$\partial_t\eta_t(f)=\eta_t(L_{t,\eta_t}(f))$$

with

$$L_{t,\eta_t}(f)=L_t(f)(x)+V_t(x)\;\int\;\;(f(y)-f(x))\;\eta_t(dy).$$

The nonlinear jump-diffusion process X_t with generator L_{t,η_t} evolves as a jump-diffusion model with generator L_t between jumps times T_n arriving at rate $V_t(X_t)$. At these jump times, it jumps $X_{T_n-}\rightsquigarrow X_{T_n}$ to a new state $X_{T_n}=x$ distributed with the probability measure $\eta_{T_n}(dx)$.

Using (16.5), the generator of the mean field $\xi_t=(\xi_t^i)_{1\leq i\leq N}$ associated with this model is given for any sufficiently regular function F by the formula

$$\begin{aligned}&L^{(i)}_{t,m(x)}(F)(x^1,\ldots,x^N)\\ &:=\sum_{1\leq i\leq N}L^{(i)}_t(F)(x^1,\ldots,x^i,\ldots,x^N)+\sum_{1\leq i\leq N}V_t(x^i)\\ &\qquad\times\int\left[F(x^1,\ldots,x^{i-1},y,x^{i+1},\ldots,x^N)-F(x^1,\ldots,x^i,\ldots,x^N)\right]\;m(x)(dy),\end{aligned}$$

with $m(x) := \frac{1}{N}\sum_{j=1}^N \delta_{x^j}$. By construction, between jumps the particles evolve independently as a jump-diffusion process with generator L_t. At rate V_t, the particles jump to a new, randomly selected location in the current population.

This ends the proof of the exercise.

■

Solution to exercise 318:

Replacing V_t by $(-V_t)$, the evolution of the normalized Feyman-Kac measures η_t is given by

$$\partial_t\eta_t(f) = \eta_t(L_t(f)) + \eta_t\left(fV_t\right) - \eta_t(f)\eta_t(V_t).$$

In this situation, we use the formula

$$\eta_t\left(fV_t\right) - \eta_t(f)\eta_t(V_t) = \int \eta_t(dx)\ \left[\int\ (f(y)-f(x))\ V_t(y)\ \eta_t(dy)\right].$$

This implies that

$$\partial_t\eta_t(f) = \eta_t(L_{t,\eta_t}(f))$$

with

$$L_{t,\eta_t}(f) := L_t(f)(x) + \int\ (f(y)-f(x))\ V_t(y)\ \eta_t(dy).$$

The nonlinear jump-diffusion process X_t with generator L_{t,η_t} evolves as a jump-diffusion model with generator L_t between jumps times T_n arriving at rate $V_t(X_t)$. At these jump times, it jumps $X_{T_n-} \rightsquigarrow X_{T_n}$ to a new state $X_{T_n} = x$ distributed with the probability measure $\eta_{T_n}(dx)$.

Using (16.5), the generator of the mean field model $\xi_t = (\xi_t^i)_{1\leq i\leq N}$ associated with this model is given for any sufficiently regular function F by the formula

$$\begin{aligned}
&L^{(i)}_{t,m(x)}(F)(x^1,\ldots,x^N)\\
&:= \sum_{1\leq i\leq N} L_t^{(i)}(F)(x^1,\ldots,x^i,\ldots,x^N) + \sum_{1\leq i\leq N} m(x)(V_t)\\
&\qquad\times\int\left[F(x^1,\ldots,x^{i-1},y,x^{i+1},\ldots,x^N) - F(x^1,\ldots,x^i,\ldots,x^N)\right]\ \frac{V_t(y)\ m(x)(dy)}{m(x)(V_t)},
\end{aligned}$$

with $m(x) := \frac{1}{N}\sum_{j=1}^N \delta_{x^j}$. By construction, between jumps the particles $\xi_t = (\xi_t^i)_{1\leq i\leq N}$ evolve independently as a jump-diffusion process with generator L_t. At rate $m(\xi_t)(V_t)$, the particles jump to a new location in the current population with distribution

$$\Psi_{V_t}(m(\xi_t))(dy) = \sum_{1\leq i\leq N} \frac{V_t(\xi^i)}{\sum_{1\leq j\leq N} V_t(\xi^j)}\ \delta_{\xi_t^i}(dy).$$

This ends the proof of the exercise.

■

Solution to exercise 319:

Take empirical test functions of the form

$$F(x) = m(x)(f) = \frac{1}{N}\sum_{j=1}^N f(x^j) \overset{\text{for each fixed } i}{=} \frac{1}{N}\ f(x^i) + \frac{1}{N}\sum_{1\leq j\neq i\leq N} f(x^j).$$

Recalling that $L(1) = 0$ for any infinitesimal generator, we find that

$$\begin{aligned} L^{(i)}_{t,m(x)}(F)(x^1,\ldots,x^i,\ldots,x^N) &= \frac{1}{N}\, L_{t,m(x)}(f)(x^i) + \frac{1}{N}\sum_{1\leq j\not= i\leq N} f(x^j)\ \underbrace{L^{(i)}_{t,m(x)}(1)}_{=0} \\ &= \frac{1}{N}\, L_{t,m(x)}(f)(x^i). \end{aligned}$$

This implies that

$$\mathcal{L}_t(F)(x^1,\ldots,x^N) := \frac{1}{N}\sum_{1\leq i\leq N} L_{t,m(x)}(f)(x^i) = m(x)\left(L_{t,m(x)}(f)\right).$$

In much the same way, we have

$$\begin{aligned} \Gamma_{\mathcal{L}_t}\left(F,F\right)(x) &:= \mathcal{L}_t\left[(F-F(x))^2\right](x) \\ &= \sum_{1\leq i\leq N} L^{(i)}_{t,m(x)}\left(F-F(x^1,\ldots,x^N)\right)^2(x^1,\ldots,x^N). \end{aligned}$$

Notice that

$$F(x) = \frac{1}{N}\sum_{j=1}^N f(x^j) \Rightarrow F(y)-F(x) = \frac{1}{N}\sum_{j=1}^N (f(y^j)-f(x^j))$$

and for any fixed x we have

$$G_x(y) := (F(y)-F(x))^2 = \left(\frac{1}{N}\sum_{j=1}^N (f(y^j)-f(x^j))\right)^2.$$

On the other hand, for each fixed i we have

$$\begin{aligned} G_x(x^1,\ldots,x^{i-1},y^i,x^{i+1},\ldots,x^N) &= \left(F(x^1,\ldots,x^{i-1},y^i,x^{i+1},\ldots,x^N)-F(x^1,\ldots,x^N)\right)^2 \\ &= \left(\frac{1}{N}(f(y^i)-f(x^i))\right)^2. \end{aligned}$$

This implies that

$$\begin{aligned} \Gamma_{\mathcal{L}_t}\left(F,F\right)(x) &= \sum_{1\leq i\leq N} L^{(i)}_{t,m(x)}(G_x)(x^1,\ldots,x^N) \\ &= \frac{1}{N}\sum_{1\leq i\leq N} L_{t,m(x)}((f-f(x^i))^2)(x^i) = \frac{1}{N}\ m(x)\left(\Gamma_{L_{t,m(x)}}(f,f)\right). \end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 320:
Using (16.10) the Bolztmann-Gibbs measures

$$t \mapsto \pi_{\beta_t}(dx) = \frac{1}{\mathcal{Z}_{\beta_t}}\ e^{-\beta_t V(x)}\ \lambda(dx)$$

on $\mathbb{R}^d$, coincide with the Feynman-Kac model

$$pi_{\beta_t}(f) = \eta_t(f) := \gamma_t(f)/\gamma_t(1) \quad \text{with} \quad \gamma_t(f) := \mathbb{E}\left(f(X_t)\ \exp\left(-\int_0^t \beta'_s\ V(X_s)ds\right)\right).$$

In addition, we have the exponential formula

$$\exp\left(-\int_0^t \beta'_s\ \eta_s(V)ds\right) = \mathcal{Z}_{\beta_t}/\mathcal{Z}_{\beta_0}.$$

On the other hand, using the Feynman-Kac jump interpretations developed in section 16.1.3, we have

$$\frac{d}{dt}\eta_t(f) = \eta_t\left(L_{t,\eta_t}(f)\right)$$

with the generator

$$L_{t,\eta_t}(f)(x) = L^c_t(f)(x) + \beta'_t\ V(x)\int\ (f(y) - f(x))\ \eta_t(dy).$$

In the above display, $L^c_t = -\beta_t\ \nabla V \cdot \nabla + \triangle$ stands for the infinitesimal generator of the Langevin diffusion

$$dX_t = -\beta_t\ \nabla V(X_t) + \sqrt{2}\ dB_t.$$

Using (16.12), the infinitesimal generator of the N-mean field particle model $\xi_t := (\xi^i_t)_{1\leq i\leq N}$, is given

$$\begin{aligned}
&L^{(i)}_{t,m(x)}(F)(x^1,\ldots,x^N)\\
&:= \sum_{1\leq i\leq N} L^{c,(i)}_t(F)(x^1,\ldots,x^i,\ldots,x^N) + \sum_{1\leq i\leq N} \beta'_t V_t(x^i)\\
&\qquad\qquad \times \int \left[F(x^1,\ldots,x^{i-1},y,x^{i+1},\ldots,x^N) - F(x^1,\ldots,x^i,\ldots,x^N)\right]\ m(x)(dy).
\end{aligned}$$

Between the jumps, the particles ξ^i_t follow independently the same evolution as the diffusion X_t with generator L^c_t. At jump times T^i_n, occurring with the stochastic rate $\beta'_t V_t(\xi^i_t)$, the i-th particle $\xi^i_{T^i_n-} \rightsquigarrow \xi^i_{T^i_n}$ jumps to a new location, say $\xi^i_{T^i_n} = y$, randomly chosen with the distribution $m\left(\xi_{T^i_n-}\right)(dy) = \frac{1}{N}\sum_{1\leq j\leq N} \xi^j_{T^i_n-}(dy)$.

This ends the proof of the exercise.

■

Chapter 17

Solution to exercise 321:
We apply (17.6) to the functions

$$b(x) = -(\alpha + \beta\ x) \quad \text{and} \quad \sigma^2(x) = \tau^2.$$

In this situation, we have

$$2\int_0^x \frac{b(y)}{\sigma^2(y)}\ dy \quad = \quad -\frac{\beta}{\tau^2}\quad \int_0^x \left(2\ \frac{\alpha}{\beta}\ y + y^2\right)'\ dy = -\frac{\beta}{\tau^2}\ \left[\left(x+\frac{\alpha}{\beta}\right)^2 - \left(\frac{\alpha}{\beta}\right)^2\right].$$

We conclude that the Gaussian distribution

$$\pi(dx) = \frac{1}{\sqrt{\pi\ \tau^2/\beta}}\ \exp\left[-\frac{\beta}{\tau^2}\ \left(x+\frac{\alpha}{\beta}\right)^2\right]\ dx$$

is the invariant measure of the diffusion (17.25).
This ends the proof of the exercise.

■

Solution to exercise 322: Notice that $X_t = 0$ is a solution of the Landau-Stuart diffusion process (17.26). Therefore, starting with $X_0 = x_0 > 0$, the solution will remain positive, so that $X_t \in S = [0,\infty[$ for any $X_0 \in S$. We apply (17.6) to the functions

$$b(x) = \alpha\ x\ (1-x^2) \quad \text{and} \quad \sigma(x) = \sqrt{2}\ \tau\ x \quad \Rightarrow\ \sigma^2(x)/2 = \tau^2 x^2.$$

Observe that

$$\begin{aligned}\int_c^x \frac{2b(y)}{\sigma^2(y)}\ dy \quad &\overset{c=1}{=} \quad \frac{\alpha}{\tau^2}\ \int_1^x \left[\frac{1}{y} - y\right]\ dy \\ &= \quad \frac{\alpha}{\tau^2}\ \left(\log(x) - \frac{1}{2}(x^2-1)\right).\end{aligned}$$

This implies that

$$\pi(dx) \propto 1_S(x)\ x^{\frac{\alpha}{\tau^2}-2}\ e^{-\frac{\alpha}{2\tau^2}\ x^2}\ dx.$$

This ends the proof of the exercise.

■

Solution to exercise 323: We apply (17.6) to the functions

$$b(x) = -(\alpha + \beta\ x) \quad \text{and} \quad \sigma^2(x) = \tau^2 + \rho\ x$$

for some parameters $(\alpha, \beta, \tau, \rho)$, with $\alpha < 0$, and $\beta, \rho > 0$. In this situation, we have

$$\tau^2 + \rho\ x \geq 0 \iff x \in S := [m, +\infty[\quad \text{with } m := -\frac{\tau^2}{\rho}.$$

The corresponding diffusion is clearly given by the square root process (17.27) In addition, for any $y > m$ we have

$$\frac{b(y)}{\sigma^2(y)} = -\frac{\beta}{\rho}\ \frac{y+\alpha/\beta}{y-m} = -\frac{\beta}{\rho}\ \left[1+\frac{m+\alpha/\beta}{y-m}\right].$$

This implies that

$$\begin{aligned} &2\int_{m+\epsilon}^{x} \frac{b(y)}{\sigma^2(y)}dy \\ &= -\frac{2\beta}{\rho}\ \left[(x-m)+(m+\alpha/\beta)\ \log{(x-m)}\right] + \frac{2\beta}{\rho}\ \left(\epsilon+(m+\alpha/\beta)\ \log{(\epsilon)}\right) \end{aligned}$$

for any $\epsilon > 0$. Using the fact that $\frac{2}{\sigma^2(x)} \propto (x-m)^{-1}$, we conclude that the invariant measure is given by the shifted Gamma distribution

$$\pi(dx) \propto 1_{[m,\infty[}(x)\ e^{-\nu_1(x-m)}\ \ (x-m)^{\nu_0-1}\ dx$$

with

$$\nu_0 := -\frac{2\beta}{\rho}\ \left(\frac{\alpha}{\beta}+m\right) = \frac{2\beta}{\rho}\ \left(\frac{|\alpha|}{\beta}+\frac{\tau^2}{\rho}\right) \quad \text{and} \quad \nu_1 := \frac{2\beta}{\rho}(>0).$$

The refined analysis of general square root diffusion processes of the form (17.27) is rather technical, thus it will not be discussed in this book. For $|\alpha| \geq \beta m + \frac{\rho}{2}$ it can be shown that the state m is unattainable. Otherwise it acts as a reflecting boundary [160].

The particular case $(\alpha,\beta,\tau,\rho) = (-2,2,0,4)$ has been worked out in exercise 262

■

Solution to exercise 324:

Notice that the centered process $Y_t = (X_t - m)$ satisfies the diffusion equation

$$dY_t = (\gamma - \beta Y_t)\ dt + \sqrt{\rho Y_t}\ dW_t \quad \text{with} \quad \gamma := |\alpha| - \beta m. \tag{30.34}$$

Let us suppose that $\gamma = n\rho/4$, for some integer $n \geq 1$. We consider a sequence $U_t = \left(U_t^{(1)},\ldots,U_t^{(n)}\right)$ of n independent Ornstein-Uhlenbeck processes of the following form

$$\forall 1 \leq i \leq n \qquad dU_t^{(i)} = -\frac{\beta}{2}\ U_t^{(i)}\ dt + \frac{\sqrt{\rho}}{2}\ dW_t^{(i)}$$

with n independent Brownian motion $W_t^{(i)}$. If we set $Z_t := \|U_t\|^2 = \sum_{1\leq i\leq n}(U_t^{(i)})^2$, then we have

$$d\left(U_t^{(i)}\right)^2 = \left(\frac{\rho}{4}\ -\ \beta\ (U_t^{(i)})^2\ \right)\ dt + \sqrt{\rho}\ U_t^{(i)}\ dW_t^{(i)}$$

from which we conclude that

$$dZ_t = \left(\frac{n\rho}{4}\ -\ \beta\ Z_t\ \right)\ dt + \sqrt{\rho\ Z_t}\ d\overline{W}_t \quad \text{with} \quad d\overline{W}_t := \sum_{1\leq i\leq n} \frac{U_t^{(i)}}{\sqrt{Z_t}}\ dW_t^{(i)}.$$

Arguing as in exercise 254, we check that $\overline{W}_t$ is a Brownian motion. This shows that

$$Y \overset{law}{=} Z \quad \text{and} \quad X \overset{law}{=} m+Z.$$

This ends the proof of the exercise.

■

Solution to exercise 325: We apply (17.6) to the functions

$$b(x) = -(\alpha + \beta\ x) = -\beta \left(x + \frac{\alpha}{\beta}\right) \quad \text{and} \quad \sigma^2(x) = \tau^2\ \ (x - \gamma_1) \quad (\gamma_2 - x)$$

for $x \in S := [\gamma_1, \gamma_2]$ and some parameters $(\alpha, \beta, \tau, \gamma_1, \gamma_2)$, with $\gamma_1 < \gamma_2$ and $\alpha + \beta\gamma_1 < 0 < \alpha + \beta\gamma_2$. The corresponding diffusion is clearly given by the Jacobi process (17.28).

In this situation, we have

$$\begin{aligned}
\frac{2b(y)}{\sigma^2(y)} &= -\frac{2\beta}{\tau^2}\ \frac{y + \alpha/\beta}{(y - \gamma_1)\quad (\gamma_2 - y)} \\
&= -\frac{2\beta}{\tau^2}\ \left[\frac{\gamma_1 + \alpha/\beta}{\gamma_2 - \gamma_1}\ \frac{1}{y - \gamma_1} + \frac{\gamma_2 + \alpha/\beta}{\gamma_2 - \gamma_1}\ \frac{1}{\gamma_2 - y}\right].
\end{aligned}$$

This implies that

$$\int_{\frac{\gamma_1+\gamma_2}{2}}^{x} \frac{2b(y)}{\sigma^2(y)} dy \quad = \quad \alpha_0\ \log \frac{y - \gamma_1}{(\gamma_2 - \gamma_1)/2} + \beta_0\ \log \frac{\gamma_2 - y}{(\gamma_2 - \gamma_1)/2}$$

with

$$\alpha_0 := -\frac{2}{\tau^2}\ \frac{\alpha + \beta\gamma_1}{\gamma_2 - \gamma_1} \quad \text{and} \quad \beta_0 := \frac{2}{\tau^2}\ \frac{\beta\gamma_2 + \alpha}{\gamma_2 - \gamma_1},$$

from which we conclude that π is given by the Beta distribution

$$\pi(dx) \propto 1_S(x)\ \sigma^{-2}(x)\ \exp\left[\int_{\frac{\gamma_1+\gamma_2}{2}}^{x} \frac{2b(y)}{\sigma^2(y)} dy\right] dx\ \propto\ \left(\frac{x - \gamma_1}{\gamma_2 - \gamma_1}\right)^{\alpha_0 - 1}\ \left(\frac{\gamma_2 - x}{\gamma_2 - \gamma_1}\right)^{\beta_0 - 1}\ dx$$

for any $x \in S$. For instance, for

$$(\gamma_1, \gamma_2) = (0, 1) \qquad \beta > 0 \qquad -\frac{\alpha}{\beta} = m \in]0, 1[\quad \text{and} \quad \tau^2 = 2\beta\nu \quad \text{with} \quad \nu > 0$$

we have $\alpha + \beta\gamma_1 = \alpha < 0$ and $\alpha + \beta\gamma_2 = \alpha + \beta = \beta\ (1 - m) > 0$

$$\alpha_0 := \frac{m}{\nu} \quad \text{and} \quad \beta_0 := \frac{1 - m}{\nu}.$$

When $(\alpha, \beta) = (0, 1)$, $\tau^2 = 2$ and $(\gamma_1, \gamma_2) = (-1, 1)$, we have

$$S := [-1, 1] \quad \text{and} \quad \alpha + \beta\gamma_1 = -1 < 0 < \alpha + \beta\gamma_2 = 1.$$

In this situation, the diffusion process X_t on $S = [-1, 1]$ is defined by

$$dX_t = -X_t\ dt + \sqrt{2(1 - X_t^2)}\ dW_t$$

and its generator is defined by

$$L(f)(x) = -x\ f'(x) + \ (1 - x^2)\ f''(x) = \sqrt{1 - x^2}\ \partial_x \left(\sqrt{1 - x^2}\ \partial_x f\right). \tag{30.35}$$

In addition, we have $\pi(dx) \propto 1_S(x)\ (1 - x^2)^{-1/2}$. The r.h.s. formula in the above display can be checked directly or can be proved using the general Sturm-Liouville formula (17.7).

When $(\alpha,\beta)=(0,2)$, $\tau^2=2$ and $(\gamma_1,\gamma_2)=(-1,1)$, we have

$$S:=[-1,1] \quad \text{and} \quad \alpha+\beta\gamma_1=-2<0<\alpha+\beta\gamma_2=2.$$

In this situation, the diffusion process X_t on $S=[-1,1]$ is defined by

$$dX_t=-2X_t\,dt+\sqrt{2(1-X_t^2)}\,dW_t$$

and its generator is defined by

$$L(f)(x)=-2x\ f'(x)+\ (1-x^2)\ f''(x)=\ \partial_x\left((1-x^2)\ \partial_x f\right). \tag{30.36}$$

In addition, we have $\pi(dx)=\frac{1}{2}\ 1_{[-1,1]}(x)$. Here again, the r.h.s. formula in the above display can be checked directly or can be proved using the general Sturm-Liouville formula (17.7).

This ends the proof of the exercise.

■

Solution to exercise 326: We apply (17.6) to the functions

$$b(x)=-(\alpha+\beta\ x)=-\beta\left(x+\frac{\alpha}{\beta}\right) \quad \text{and} \quad \sigma(x)=\tau\ x$$

for some parameters (α,β,τ) with $\tau>0$ and $\alpha<0$. The corresponding diffusion process is clearly given by the equation (17.29). In this situation, we have

$$\begin{aligned}\int_1^x \frac{2b(y)}{\sigma^2(y)}\ dy \quad &= \quad -\frac{2}{\tau^2}\ \left(\alpha\ \int_1^x \frac{1}{y^2}\ dy+\beta\ \int_1^x \frac{1}{y}\ dy\right)\\ &= \quad -\frac{2}{\tau^2}\ \left(\alpha\ \left(1-\frac{1}{x}\right)+\beta\ \log x\ \right).\end{aligned}$$

This implies that the invariant measure is the inverse Gamma distribution

$$\pi(dx)\propto 1_S(x)\ \sigma^{-2}(x)\ \exp\left[\int_1^x \frac{2b(y)}{\sigma^2(y)}dy\right]\ dx\ \propto\ 1_S(x)\ \left(\frac{1}{x}\right)^{\frac{2\beta}{\tau^2}+2}\ \exp\left(\frac{2\alpha}{\tau^2}\ \frac{1}{x}\right)\ dx.$$

This ends the proof of the exercise.

■

Solution to exercise 327:
We apply (17.6) to the functions

$$b(x)=-(\alpha+\beta\ x)=-\beta\left(x+\frac{\alpha}{\beta}\right) \quad \text{and} \quad \sigma^2(x)=\tau^2\ \ (x+\gamma_1)\quad (x+\gamma_2)$$

for some parameters $(\alpha,\beta,\tau,\gamma_1,\gamma_2)$, s.t. $\alpha/\beta<\gamma_1<\gamma_2$ and $2\beta+\tau^2>0$. The corresponding diffusion is clearly given by the diffusion process (17.30) on $S:=]-\gamma_1,\infty[$. In this situation, we have

$$\begin{aligned}\frac{2b(y)}{\sigma^2(y)} \quad &= \quad -\frac{2\beta}{\tau^2}\ \frac{y+\alpha/\beta}{(y+\gamma_1)\quad(\gamma_2+y)}\\ &= \quad -\frac{2\beta}{\tau^2}\ \left[\frac{\gamma_2-\frac{\alpha}{\beta}}{\gamma_2-\gamma_1}\ \frac{1}{y+\gamma_2}+\frac{\frac{\alpha}{\beta}-\gamma_1}{\gamma_2-\gamma_1}\ \frac{1}{\gamma_1+y}\right].\end{aligned}$$

This implies that

$$\begin{aligned}\pi(dx) &\propto 1_S(x)\ \sigma^{-2}(x)\ \exp\left[\int_1^x \frac{2b(y)}{\sigma^2(y)}dy\right]\ dx\\ &\propto 1_S(x)\ \frac{1}{x+\gamma_1}\ (x+\gamma_2)^{-\frac{2\beta}{\tau^2}\left(1-\frac{\frac{\alpha}{\beta}-\gamma_1}{\gamma_2-\gamma_1}\right)-1}\quad (x+\gamma_1)^{-\frac{2\beta}{\tau^2}\ \frac{\frac{\alpha}{\beta}-\gamma_1}{\gamma_2-\gamma_1}}\ dx.\end{aligned}$$

This formulae can be rewritten in the following form

$$\pi(dx) \propto 1_S(x)\ \frac{1}{x+\gamma_1}\ \left(\frac{x+\gamma_1}{x+\gamma_1+\delta}\right)^{d_1/2}\ \left(\frac{1}{x+\gamma_1+\delta}\right)^{d_2/2}\ dx,$$

with

$$\delta := \gamma_2-\gamma_1 \qquad d_1/2 = \frac{2\beta}{\tau^2}\ \frac{\gamma_1-\frac{\alpha}{\beta}}{\gamma_2-\gamma_1} > 0 \quad \text{and} \quad d_2/2 = 1+\frac{2\beta}{\tau^2} > 0.$$

If X denotes a random variable with distribution π, then $Y = \frac{d_2}{d_1}\ \left(\frac{X+\gamma_1}{\delta}\right)$ is distributed according to the Fisher distribution

$$\mathbb{P}\left(Y\in dy\right) \propto 1_{]0,\infty[}(y)\ \frac{1}{y}\ \left(\frac{d_1 y}{d_1 y+d_2}\right)^{d_1/2}\ \left(\frac{d_2}{d_1 y+d_2}\right)^{d_2/2}\ dy.$$

For instance when

$$(\gamma_1,\gamma_2) = (0,1) \qquad \beta>0 \qquad -\frac{\alpha}{\beta} = m > 0 \quad \text{and} \quad \tau^2 = 2\beta\nu \quad \text{with} \quad \nu>0$$

we have $S :=]0,\infty[$, as well as $\alpha/\beta = -m < \gamma_1 = 0$ and $2\beta+\tau^2>0$. In this case, we also have

$$d_1/2 = \frac{m}{\nu} > 0 \quad \text{and} \quad d_2/2 = 1+\frac{1}{\nu} > 0.$$

This ends the proof of the exercise.

■

Solution to exercise 328: We apply (17.6) to the functions

$$b(x) = -(\alpha+\beta\ x) \quad \text{and} \quad \sigma^2(x) = \tau^2\ \left((\alpha+\beta\ x)^2+\gamma^2\right)$$

for some parameters $(\alpha,\beta,\tau,\gamma))$, with $\beta>0$. The corresponding diffusion is clearly given by the equation (17.31). In this situation, we have

$$\frac{2b(y)}{\sigma^2(y)} = -\frac{1}{\tau^2\beta}\ \frac{2\beta(\alpha+\beta\ y)}{((\alpha+\beta\ x)^2+\gamma^2)} = -\frac{1}{\tau^2\beta}\ \left(\log\left[(\alpha+\beta\ y)^2+\gamma^2\right]\right)'.$$

This implies that

$$\begin{aligned}\pi(dx) &\propto 1_S(x)\ \sigma^{-2}(x)\ \exp\left[\int_0^x \frac{2b(y)}{\sigma^2(y)}dy\right]\ dx \propto \left[(\alpha+\beta\ x)^2+\gamma^2\right]^{-\left(\frac{1}{\tau^2\beta}+1\right)}\ dx\\ &\propto \left[1+\left(\frac{x+\frac{\alpha}{\beta}}{\gamma}\right)^2\right]^{-\frac{\left(\frac{2}{\tau^2\beta}+1\right)+1}{2}} = \left[1+\left(\frac{x+\frac{\alpha}{\beta}}{\gamma}\right)^2\right]^{-\frac{\kappa+1}{2}} \quad \text{with} \quad \kappa := 1+\frac{2}{\tau^2\beta}.\end{aligned}$$

This shows that π is a Student's t-distribution $T\left(\kappa,\gamma,-\frac{\alpha}{\beta}\right)$ with a scaling parameter γ, a tail index κ, and a location parameter $-\frac{\alpha}{\beta}$. This ends the proof of the exercise.

■

Solution to exercise 329:
We have

$$\overline{\mathbb{H}}_n(x) := \mathbb{H}_n(h(x)) \Rightarrow \overline{\mathbb{H}}'_n(x) = \mathbb{H}'_n(h(x))\ h'(x) \Rightarrow \mathbb{H}'_n(h(x)) = \overline{\mathbb{H}}'_n(x)/h'(x).$$

Therefore

$$2\ (h')^2(x)\ h(x)\ \mathbb{H}'_n(h(x)) = 2h(x)h'(x)\ \overline{\mathbb{H}}'_n(x) \ = \ (h^2)'(x)\ \overline{\mathbb{H}}'_n(x).$$

On the other hand, we also have

$$\begin{aligned}
\overline{\mathbb{H}}''_n(x) &= \mathbb{H}''_n(h(x))\ (h'(x))^2 + \mathbb{H}'_n(h(x))\ h''(x) \\
&= \mathbb{H}''_n(h(x))\ (h'(x))^2 + \overline{\mathbb{H}}'_n(x)\ \ h''(x)/h'(x) \\
&= \mathbb{H}''_n(h(x))\ (h'(x))^2 + \overline{\mathbb{H}}'_n(x)\ \ (\log h')'(x).
\end{aligned}$$

This implies that

$$\mathbb{H}''_n(h(x)) = (h'(x))^{-2}\ \left[\overline{\mathbb{H}}''_n(x) - \overline{\mathbb{H}}'_n(x)\ \ (\log h')'(x)\right].$$

By (17.20) we have

$$\mathbb{H}''_n(h(x)) - 2h(x)\mathbb{H}'_n(h(x)) = -2n\ \mathbb{H}_n(h(x)).$$

In terms of $\overline{\mathbb{H}}_n$, this can be rewritten as

$$(h')^{-2}\ \left[\overline{\mathbb{H}}''_n - \left(h^2 + \log h'\right)'\ \overline{\mathbb{H}}'_n\right] = -2n\ \overline{\mathbb{H}}_n.$$

We conclude that

$$\overline{L}_h(\overline{\mathbb{H}}_n) = -2n\ \overline{\mathbb{H}}_n$$

with the generator

$$\overline{L}_h(f) := (h')^{-2}\ L_h(f) \quad \text{and} \quad L_h := f'' - \left(h^2 + \log h'\right)'\ f'.$$

By (17.5), the reversible probability measure π_h of L_h is given by

$$\pi_h(dx)\ \propto\ e^{-(h^2+\log h')}\ dx = \frac{1}{h'(x)}\ e^{-h^2(x)}\ dx.$$

Using (17.3), the reversible probability measure $\overline{\pi}_h$ of $\overline{L}_h$ is given by

$$\overline{\pi}_h(dx) \propto (h'(x))^2\ \pi_h(dx)\ \propto\ h'(x)\ e^{-h^2(x)}\ dx.$$

The claim that $2^{n/2}\sqrt{n!}\overline{\mathbb{H}}_n$ forms an orthonormal basis of $\mathbb{L}_2(\overline{\pi}_h)$ is a consequence of the fact that

$$\begin{aligned}
\int \overline{\mathbb{H}}_n(x)\ \overline{\mathbb{H}}_m(x)\ \overline{\pi}_h(dx) &\propto \int \mathbb{H}_n(h(x))\ \mathbb{H}_m(h(x))\ h'(x)\ e^{-h^2(x)}\ dx \\
&= \int \mathbb{H}_n(y)\ \mathbb{H}_m(y)\ e^{-y^2}\ dy = 1_{m=n}\ 2^n n!.
\end{aligned}$$

Applying this result to

$$h(x) = \sqrt{\frac{a}{\sigma^2}} \left(x + \frac{b}{a}\right) \Rightarrow h'(x) = \sqrt{\frac{a}{\sigma^2}} \quad \text{and} \quad L_h := \frac{2}{\sigma^2}\left(\frac{1}{2}\,\sigma^2\, f'' - (ax+b)\, f'\right)$$

we have

$$\overline{L}_h(f) = \frac{2}{a}\left(\frac{1}{2}\,\sigma^2\, f'' - (ax+b)\, f'\right) = \left(\frac{\sigma^2}{a}\, f'' - 2\,\left(x + \frac{b}{a}\right)\, f'\right).$$

We conclude that the reversible probability measure of any of the generators $\overline{L}_h, L_h$ or

$$L(f) = -\,(ax+b)\;f' + \frac{1}{2}\,\sigma^2\, f''$$

is given by the Gaussian distribution

$$\pi_h(dx) = \overline{\pi}_h(dx) = \pi(dx) := \sqrt{\frac{a}{\pi\sigma^2}}\; e^{-\frac{a}{\sigma^2}\,\left(x+\frac{b}{a}\right)^2}\; dx.$$

In addition, we have

$$L(\overline{\mathbb{H}}_n) = -2n\,\frac{a}{2}\,\overline{\mathbb{H}}_n = -na\,\overline{\mathbb{H}}_n.$$

This implies that

$$L(\widehat{\mathbb{H}}_n) = -na\,\widehat{\mathbb{H}}_n$$

with the orthonormal sequence of polynomials in $\mathbb{L}_2(\pi)$ defined by

$$\widehat{\mathbb{H}}_n\,(x) := 2^{n/2}\sqrt{n!}\;\mathbb{H}_n\left(\sqrt{\frac{a}{\sigma^2}}\;\left(x + \frac{b}{a}\right)\right).$$

This ends the proof of the exercise.

■

Solution to exercise 330:
Observe that

$$t = t_m := \frac{m}{m+1} \Rightarrow 1 - t = 1 - \frac{m}{m+1} = \frac{1}{m+1} \Rightarrow \frac{t}{1-t} = m.$$

This yields

$$e^{-mx} = (1-t_m)^{(\alpha+1)}\;\mathbb{S}_{t_m}(x) = (1-t_m)^{(\alpha+1)}\sum_{n\geq 0}\;\mathbb{I}_n(x)\;\frac{t_m^n}{n!}.$$

The Sturm-Liouville formula

$$L(f) = x^{-\alpha}\;e^x\;\partial_x\left(x^{\alpha+1}\;e^{-x}\;\partial_x(f)\right)$$

is a direct consequence of (17.7) applied to $\sigma^2(x) = 2x$ and $b(x) = ((\alpha+1) - x)$.
Also notice that $\mathbb{I}_0(x) = 1$. We have

$$-\frac{xt}{1-t} = x\,\left(1 - \frac{1}{1-t}\right) \;\Rightarrow\; \partial_t\left(-\frac{xt}{1-t}\right) = -\frac{x}{(1-t)^2}.$$

Therefore

$$\begin{aligned}\partial_t \mathbb{S}_t(x) &= e^{-\frac{xt}{1-t}} \left[(\alpha+1)\ (1-t)^{-(\alpha+2)} - (1-t)^{-(\alpha+1)}\ \frac{x}{(1-t)^2}\right] \\ &= e^{-\frac{xt}{1-t}}\ (1-t)^{-(\alpha+2)}\ \left(\alpha+1-\frac{x}{1-t}\right) = \frac{1}{1-t}\ \left(\alpha+1-\frac{x}{1-t}\right)\ \mathbb{S}_t(x).\end{aligned}$$

This implies that

$$(1-t)^2\ \partial_t \mathbb{S}_t(x) + (x-(\alpha+1)(1-t))\ \mathbb{S}_t(x) = 0. \tag{30.37}$$

On the other hand, we have

$$\mathbb{S}_t'(x) = \partial_x \mathbb{S}_t(x) = -\frac{t}{1-t}\ (1-t)^{-(\alpha+1)}\ e^{-\frac{xt}{1-t}} = -\frac{t}{1-t}\ \mathbb{S}_t(x).$$

Therefore

$$(1-t)\ \mathbb{S}_t'(x) + t\ \mathbb{S}_t(x) = 0. \tag{30.38}$$

Using (30.38) we have

$$\sum_{n\geq 0} \mathbb{I}_n'(x)\ (1-t)\frac{t^n}{n!} + \sum_{n\geq 0} \mathbb{I}_n(x)\ \frac{t^{n+1}}{n!} = 0.$$

In other words, we have that

$$\underbrace{\mathbb{I}_0'(x)}_{=0} + \sum_{n\geq 0} \mathbb{I}_{n+1}'(x)\ \frac{t^{n+1}}{(n+1)!} = \sum_{n\geq 0} \mathbb{I}_n'(x)\ \frac{t^n}{n!} = \sum_{n\geq 0}\ (n+1)\ (\mathbb{I}_n'(x) - \mathbb{I}_n(x))\ \frac{t^{n+1}}{(n+1)!}$$

from which we conclude that

$$\mathbb{I}_{n+1}' = (n+1)\ (\mathbb{I}_n' - \mathbb{I}_n) \iff ((n+1)\ \mathbb{I}_n - \mathbb{I}_{n+1})' = (n+1)\mathbb{I}_n. \tag{30.39}$$

On the other hand, using (30.37) we have

$$\begin{aligned}&(1-2t+t^2)\ \partial_t \mathbb{S}_t(x) + (x-(\alpha+1))\ \mathbb{S}_t(x) + (\alpha+1)\ t\ \mathbb{S}_t(x) = 0 \\ &= (1-2t+t^2)\ \textstyle\sum_{n\geq 0}\ \mathbb{I}_{n+1}(x)\ \frac{t^n}{n!} \\ &+ (x-(\alpha+1))\ \textstyle\sum_{n\geq 0}\ \mathbb{I}_n(x)\ \frac{t^n}{n!} + (\alpha+1)\ \sum_{n\geq 0}\ (n+1)\mathbb{I}_n(x)\ \frac{t^{n+1}}{(n+1)!} \\ &= (1-2t+t^2)\ \textstyle\sum_{n\geq 0}\ \mathbb{I}_{n+1}(x)\ \frac{t^n}{n!} \\ &+ (x-(\alpha+1))\ \textstyle\sum_{n\geq 0}\ \mathbb{I}_n(x)\ \frac{t^n}{n!} + (\alpha+1)\ \sum_{n\geq 1}\ n\mathbb{I}_{n-1}(x)\ \frac{t^n}{n!} \\ &= (1-2t+t^2)\ \textstyle\sum_{n\geq 0}\ \mathbb{I}_{n+1}(x)\ \frac{t^n}{n!} \\ &+ (x-(\alpha+1))\underbrace{\mathbb{I}_0(x)}_{=1} + \textstyle\sum_{n\geq 1}\ \left((x-(\alpha+1))\,\mathbb{I}_n(x) + n(\alpha+1)\mathbb{I}_{n-1}(x)\right)\ \frac{t^n}{n!}.\end{aligned}$$

Using the fact that

$$\begin{aligned}(1-2t+t^2)\ \sum_{n\geq 0}\ \mathbb{I}_{n+1}(x)\ \frac{t^n}{n!} &= \sum_{n\geq 0} \mathbb{I}_{n+1}(x)\ \frac{t^n}{n!} - \sum_{n\geq 1}\ 2n\mathbb{I}_n(x)\ \frac{t^n}{n!} \\ &+ \sum_{n\geq 2}\ n(n-1)\mathbb{I}_{n-1}(x)\ \frac{t^n}{n!}\end{aligned}$$

we check that

$$\mathbb{I}_{n+1}(x) - 2n\mathbb{I}_n(x) + n(n-1)\mathbb{I}_{n-1}(x) + \left((x - (\alpha+1))\,\mathbb{I}_n(x) + n(\alpha+1)\mathbb{I}_{n-1}(x)\right)$$

$$= \mathbb{I}_{n+1}(x) + \underbrace{(x - (\alpha+1) - 2n)}_{=x-(n+1)-(\alpha+n)}\mathbb{I}_n(x) + \ n\ \ (n+\alpha)\,\mathbb{I}_{n-1}(x) = 0$$

for any $n \geq 2$. From the above decompositions, we also have

$$\mathbb{I}_1(x) = ((\alpha+1) - x)$$

and

$$\begin{array}{rcl} & & \mathbb{I}_2(x) - 2\mathbb{I}_1(x) + \left((x - (\alpha+1))\,\mathbb{I}_1(x) + (\alpha+1)\mathbb{I}_0(x)\right) = 0 \\ \Rightarrow\ \mathbb{I}_2(x) & = & (x - (\alpha+1))^2 - 2\,(x - (\alpha+1)) - (\alpha+1) \\ & = & (x - (\alpha+2))^2 - (\alpha+2). \end{array}$$

This yields

$$x\mathbb{I}_n(x) + (n+\alpha)\ \ (n\ \mathbb{I}_{n-1}(x) - \mathbb{I}_n(x)) = (n+1)\ \mathbb{I}_n(x) - \mathbb{I}_{n+1}(x).$$

Taking the derivative w.r.t. x, using (30.39) we deduce that

$$\begin{array}{l} \mathbb{I}_n(x) + x\mathbb{I}_n'(x) + (n+\alpha)\ \ n\mathbb{I}_{n-1} = (n+1)\mathbb{I}_n(x) \\ \Longleftrightarrow (n+\alpha)\ \ n\mathbb{I}_{n-1} = n\mathbb{I}_n(x) - x\mathbb{I}_n'(x) \\ \Longleftrightarrow \quad x\mathbb{I}_n'(x) + (n+\alpha)\ \ (n\mathbb{I}_{n-1} - \mathbb{I}_n(x)) + \alpha\mathbb{I}_n(x) = 0. \end{array} \tag{30.40}$$

Taking once more the derivative w.r.t. x we have

$$\mathbb{I}_n'(x) + x\mathbb{I}_n''(x) + (n+\alpha)\ \ n\mathbb{I}_{n-1} + \alpha\mathbb{I}_n'(x) = 0.$$

Finally, using (30.40) we conclude that

$$\mathbb{I}_n'(x) + x\mathbb{I}_n''(x) + n\mathbb{I}_n(x) - x\mathbb{I}_n'(x) + \alpha\mathbb{I}_n'(x)$$

$$= x\mathbb{I}_n''(x) + ((\alpha+1) - x)\,\mathbb{I}_n'(x) + n\mathbb{I}_n(x) = 0.$$

We consider the Gamma distribution

$$\pi(dx) = \frac{1}{\Gamma(\alpha+1)}\ 1_{[0,\infty[}(x)\ x^\alpha\ e^{-x}\ dx.$$

We have

$$\begin{array}{rcl} \pi\left(\mathbb{S}_s\mathbb{S}_t\right) & = & \displaystyle\sum_{m,n\geq 0} \pi\left(\mathbb{I}_n\mathbb{I}_m\right)\ \frac{s^m}{m!}\ \frac{t^n}{n!} \\ & = & \displaystyle\frac{((1-t)(1-s))^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^\infty e^{-x\frac{t}{1-t}}\ e^{-x\frac{s}{1-s}}\ x^\alpha\ e^{-x}\ dx \\ & = & \displaystyle\frac{((1-t)(1-s))^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^\infty e^{-x\left[1+\frac{t}{1-t}+\frac{s}{1-s}\right]}\ x^\alpha\ dx \\ & = & \displaystyle\frac{((1-t)(1-s))^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^\infty e^{-x\ \frac{1-st}{(1-t)(1-s)}}\ x^\alpha\ dx. \end{array}$$

The last assertion follows from the fact that

$$1+\frac{t}{1-t}+\frac{s}{1-s}=\frac{1}{1-t}+\frac{s}{1-s}=\frac{(1-s)+s(1-t)}{(1-t)(1-s)}=\frac{1-st}{(1-t)(1-s)}.$$

Changing the integration variable

$$y=x\ \frac{1-st}{(1-t)(1-s)}\Rightarrow dx=\frac{(1-t)(1-s)}{1-st}\ dy\quad \text{and}\quad x=\frac{(1-t)(1-s)}{1-st}\ y$$

we check that

$$\begin{aligned}\pi\left(\mathbb{S}_s\mathbb{S}_t\right) &= \frac{((1-t)(1-s))^{-(\alpha+1)}}{\Gamma(\alpha+1)}\ \left(\frac{(1-t)(1-s)}{1-st}\right)^{1+\alpha}\ \int_0^\infty\ e^{-y}\ y^\alpha\ dy\\ &= \frac{1}{(1-st)^{1+\alpha}}=\sum_{n\geq 0}\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)\Gamma(n+1)}\ (st)^n.\end{aligned}$$

The last assertion follows from the fact that for any $u\neq 1$ we have

$$\left(\frac{1}{1-u}\right)^{\alpha+1}=\left(\sum_{n\geq 0}\ u^n\right)^{\alpha+1}=\sum_{n\geq 0}\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)\Gamma(n+1)}\ u^n. \tag{30.41}$$

We check this claim by using the fact that

$$\begin{aligned}\partial_u\left(\frac{1}{1-u}\right)^{\alpha+1} &= (\alpha+1)\ \left(\frac{1}{1-u}\right)^{\alpha}\times\partial_u\left(\frac{1}{1-u}\right)=(\alpha+1)\ \left(\frac{1}{1-u}\right)^{\alpha+2}\\ \partial_u^2\left(\frac{1}{1-u}\right)^{\alpha+1} &= (\alpha+1)\ \partial_u\left(\frac{1}{1-u}\right)^{\alpha+2}=(\alpha+1)(\alpha+2)\ \left(\frac{1}{1-u}\right)^{\alpha+3},\end{aligned}$$

and by a simple induction

$$\begin{aligned}\partial_u^n\left(\frac{1}{1-u}\right)^{\alpha+1} &= (\alpha+1)(\alpha+2)\dots(\alpha+n)\ \left(\frac{1}{1-u}\right)^{\alpha+n+1}\\ &= \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)}\ \left(\frac{1}{1-u}\right)^{\alpha+n+1}\overset{u=0}{=}\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)}.\end{aligned}$$

This yields

$$\sum_{m,n\geq 0}\pi\left(\mathbb{I}_n\mathbb{I}_m\right)\ \frac{s^m}{m!}\ \frac{t^n}{n!}=\sum_{n\geq 0}\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)\Gamma(n+1)}\ (st)^n$$

from which we conclude that

$$\pi\left(\mathbb{I}_n\mathbb{I}_m\right)=1_{m=n}\ \frac{\Gamma(\alpha+n+1)\Gamma(n+1)}{\Gamma(\alpha+1)}.$$

This shows that

$$\sqrt{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)\Gamma(n+1)}}\ \mathbb{I}_n$$

is an orthonormal subset of $\mathbb{L}_2(\pi)$. A simple induction shows that

$$\mathbb{I}_n(x)-(-1)^n\ x^n=\text{a polynomial of order }(n-1).$$

Therefore to check that the normalized Laguerre polynomials are a dense subset of $\mathbb{L}_2(\pi)$ we use the same arguments as the ones we used on page 491 to prove that the Hermite polynomials form a Schauder basis.

Using (30.41) we have

$$\begin{aligned}\mathbb{S}_t(x) &= \sum_{n,m\geq 0} \frac{(-xt)^n}{n!} \overbrace{\frac{\Gamma(\alpha+m+n+1)}{\Gamma(\alpha+n+1)\Gamma(m+1)}}^{=\frac{1}{m!}(\alpha+n+m)(\alpha+n+(m-1))\ldots(\alpha+n+1)} t^m \\ &= \sum_{k,m\geq 0}\left[\prod_{1\leq l\leq m}(\alpha+k+l)\right]\frac{(-xt)^k}{k!}\,\frac{t^m}{m!}.\end{aligned}$$

On the other hand, using the Leibniz formula

$$\partial_x^n(fg) = \sum_{0\leq m\leq n}\frac{n!}{(n-m)!m!}\,\partial_x^{n-m}f\ \ \partial_x^m g \tag{30.42}$$

we have

$$\partial_x^n\left(e^{-x}\ x^{n+\alpha}\right) = \sum_{0\leq m\leq n}\frac{n!}{(n-m)!m!}\,\partial_x^{n-m}(e^{-x})\ \partial_x^m\left(x^{n+\alpha}\right).$$

Recalling that

$$\partial_x^{n-m}(e^{-x}) = (-1)^{n-m} \quad \text{and} \quad \partial_x^m\left(x^{n+\alpha}\right) = \underbrace{\left[\prod_{0\leq p<m}(\alpha+n-p)\right]}_{\prod_{1\leq l\leq m}(\alpha+(n-m)+l)} x^{\alpha+(n-m)}$$

we check that

$$e^x\ x^{-\alpha}\partial_x^n\left(e^{-x}\ x^{n+\alpha}\right) = \sum_{0\leq m\leq n}\frac{n!}{m!}\left[\prod_{1\leq l\leq m}(\alpha+(n-m)+l)\right]\frac{(-x)^{(n-m)}}{(n-m)!}$$

and therefore

$$\sum_{n\geq 0} e^x\ x^{-\alpha}\partial_x^n\left(e^{-x}\ x^{n+\alpha}\right)\ \frac{t^n}{n!} = \sum_{m\geq 0}\ \sum_{k(:=m-n)\geq 0}\left[\prod_{1\leq l\leq m}(\alpha+k+l)\right]\frac{t^m}{m!}\ \frac{(-xt)^k}{k!}.$$

This implies that

$$\mathbb{I}_n(x) = e^x\ x^{-\alpha}\partial_x^n\left(e^{-x}\ x^{n+\alpha}\right).$$

This ends the proof of the exercise.

■

Solution to exercise 331:

Using the fact that

$$\frac{1}{1-te^{i\theta}} = \sum_{n\geq 0}\left(te^{i\theta}\right)^n = \sum_{n\geq 0}\left(\cos(n\theta)+i\sin(n\theta)\right)t^n$$

and

$$\frac{1}{1-te^{i\theta}} = \frac{1}{(1-t\cos(\theta)) - it\sin(\theta)} = \frac{(1-t\cos(\theta)) + it\sin(\theta)}{(1-t\cos(\theta))^2 + t^2\sin^2(\theta)},$$

by taking the real part, we check that

$$\sum_{n\geq 0} \cos(n\theta)\ t^n = \frac{(1-t\cos(\theta))}{(1-t\cos(\theta))^2 + t^2(1-\cos^2(\theta))} = \frac{(1-t\cos(\theta))}{1-2t\cos(\theta)+t^2}.$$

This shows that

$$\sum_{n\geq 0} \cos(n\theta)\ t^n = \sum_{n\geq 0} \mathbb{T}_t(\cos\theta)\ t^n \implies \mathbb{T}_n(\cos\theta) = \cos(n\theta).$$

Using the change of variable

$$x = \cos(\theta) \ \Rightarrow \ dx = \ -\sin(\theta)\ d\theta \Rightarrow \frac{1}{\sqrt{1-x^2}}\ dx = -d\theta$$

it is readily checked that π is the probability distribution of $\cos\Theta$ where Θ is an uniform random variable on $[0,\pi]$.

This implies that

$$\int \pi(dx)\ \mathbb{T}_n(x)\ \mathbb{T}_m(x) = \frac{1}{\pi}\int_0^\pi \mathbb{T}_n(\cos\theta)\ \mathbb{T}_m(\cos\theta)\ d\theta = \frac{1}{\pi}\int_0^\pi \cos(n\theta)\ \cos(m\theta)\ d\theta.$$

Recalling that

$$\cos(\theta_1+\theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$$

we readily check that

$$\cos(n\theta)\ \cos(m\theta) = \frac{\cos((n+m)\theta) - \cos(n-m)\theta)}{2}.$$

On the other hand, we have

$$\frac{1}{2\pi}\int_0^\pi \cos((n+m)\theta)\ d\theta = \frac{1}{2\pi(n+m)}\ [\sin((n+m)\theta)]_0^\pi = 0$$

and for any $m \neq m$

$$\frac{1}{2\pi}\int_0^\pi \cos((n-m)\theta)\ d\theta = \frac{1}{2\pi(n-m)}\ [\sin((n-m)\theta)]_0^\pi = 0.$$

For $n = m$ we also have

$$\frac{1}{2\pi}\int_0^\pi \cos((n-m)\theta)\ d\theta = \frac{1}{2}.$$

This implies that

$$\int \pi(dx)\ \mathbb{T}_n(x)\ \mathbb{T}_m(x) = 1_{m=n}\ \frac{1}{2}.$$

We conclude that $\sqrt{2}\mathbb{T}_n$ forms an orthonormal subset of $\mathbb{L}_2(\pi)$.

For instance, we have

$$\mathbb{T}_0(x) = 1 \quad \mathbb{T}_1(x) = x \quad \text{and} \quad \mathbb{T}_2(x) = 2x^2 - 1.$$

The r.h.s. formula follows from

$$\cos(2\theta) = 2\ \cos^2(\theta) - 1.$$

In the same vein, it is also readily checked that

$$e^{(n+1)i\theta} + e^{(n-1)i\theta} = e^{ni\theta}\ \left(e^{i\theta} + e^{-i\theta}\right) \Rightarrow 2\cos(\theta)\cos(n\theta) = \cos((n+1)\theta) + \cos((n-1)\theta).$$

This implies that

$$x\mathbb{T}_n(x) = \frac{1}{2}\left(\mathbb{T}_{n+1}(x) + \mathbb{T}_{n-1}(x)\right) \quad \left(\text{and} \quad x\mathbb{T}_0(x) = \mathbb{T}_1(x)\right).$$

$$\begin{aligned}
x^2 &= x\mathbb{T}_1(x) = \frac{1}{2}\left(\mathbb{T}_2(x) + \mathbb{T}_0(x)\right)\\
x^3 &= x(x\mathbb{T}_1)(x) = \frac{1}{2}\left(x\mathbb{T}_2(x) + x\mathbb{T}_0(x)\right)\\
&= \frac{1}{2}\left(\frac{1}{2}\left(\mathbb{T}_3(x) + \mathbb{T}_1(x)\right) + \mathbb{T}_1(x)\right)\\
x^4 &= x(x(x\mathbb{T}_1)(x)) = \frac{1}{2}\left(\frac{1}{2}\left(x\mathbb{T}_3(x) + x\mathbb{T}_1(x)\right) + x\mathbb{T}_1(x)\right)\\
&= \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}\left(\mathbb{T}_4(x) + \mathbb{T}_2(x)\right) + \frac{1}{2}\left(\mathbb{T}_2(x) + \mathbb{T}_0(x)\right)\right) + \frac{1}{2}\left(\mathbb{T}_2(x) + \mathbb{T}_0(x)\right)\right).
\end{aligned}$$

A simple induction shows that

$$x^n = \sum_{0\leq k\leq n} a_{k,n}\ \mathbb{T}_k(x).$$

Therefore to check that the normalized Tchebyshev polynomials are a dense subset of $\mathbb{L}_2(\pi)$ we use the same arguments as the ones we used on page 491 to prove that the Hermite polynomials form a Schauder basis.

Finally we have

$$\partial_\theta\left(\mathbb{T}_n(\cos(\theta))\right) = -\left(\partial_x\mathbb{T}_n\right)(\cos(\theta))\ (\sin(\theta))$$

and

$$\partial_\theta^2\left(\mathbb{T}_n(\cos(\theta))\right) = \left(\partial_x^2\mathbb{T}_n\right)(\cos(\theta))\ (\sin(\theta))^2 - \left(\partial_x\mathbb{T}_n\right)(\cos(\theta))\ (\cos(\theta)).$$

Recalling that

$$\mathbb{T}_n(\cos(\theta)) = \cos(n\theta) \Rightarrow \partial_\theta^2\left(\mathbb{T}_n(\cos(\theta))\right) = -n\partial_\theta\left(\sin(n\theta)\right) = -n^2\cos(n\theta) = -n^2\ \mathbb{T}_n(\cos(\theta))$$

we conclude that

$$(1-x^2)\ \partial_x^2\mathbb{T}_n(x) - x\ \partial_x\mathbb{T}_n(x) = -n^2\mathbb{T}_n(x).$$

Finally, using (30.35), we also have the Sturm-Liouville formula

$$-\sqrt{1-x^2}\ \partial_x\left(\sqrt{1-x^2}\ \partial_x\mathbb{T}_n(x)\right) = n^2\mathbb{T}_n(x).$$

This ends the proof of the exercise.

■

Solution to exercise 332:
Notice that

$$\partial_x^k \mathbb{J}_n = \frac{(-1)^n}{2^n n!} \partial_x^{n+k} J_n \propto \partial_x^{n+k} J_n. \tag{30.43}$$

On the other hand, we have

$$J_n'(x) = -2nx\,(1-x^2)^{n-1} \Rightarrow (1-x^2)\,J_n'(x) + 2nxJ_n(x) = 0.$$

Thus, using the Leibniz formula (30.42) we check that

$$\begin{aligned}
&\partial_x^{n+1}\left((1-x^2)\,J_n'(x)\right)\\
&= \sum_{0\le k\le 2} \binom{n+1}{k} \partial_x^k(1-x^2)\,\partial_x^{((n+1)-k)+1} J_n(x)\\
&= (1-x^2)\,\partial_x^{(n+2)} J_n(x) - 2x(n+1)\,\partial_x^{(n+1)} J_n(x) - n(n+1)\,\partial_x^n J_n(x).
\end{aligned}$$

In much the same way, we have

$$\begin{aligned}
\partial_x^{n+1}(2nxJ_n(x)) &= 2n \sum_{0\le k\le 1} \binom{n+1}{k} \partial_x^k x\,\partial_x^{((n+1)-k)} J_n(x)\\
&= 2nx\,\partial_x^{(n+1)} J_n(x) + 2n(n+1)\,\partial_x^n J_n(x).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\partial_x^n\left((1-x^2)\,J_n'(x) + 2nxJ_n(x)\right)\\
&= (1-x^2)\,\partial_x^{(n+2)} J_n(x) - 2x(n+1)\,\partial_x^{(n+1)} J_n(x) - n(n+1)\,\partial_x^n J_n(x)\\
&\qquad +2nx\,\partial_x^{(n+1)} J_n(x) + 2n(n+1)\,\partial_x^n J_n(x)\\
&= (1-x^2)\,\partial_x^{(n+2)} J_n(x) - 2x\,\partial_x^{(n+1)} J_n(x) + n(n+1)\,\partial_x^n J_n(x) = 0
\end{aligned}$$

and therefore

$$(30.43) \Longrightarrow (1-x^2)\,\mathbb{J}_n''(x) - 2x\,\mathbb{J}_n'(x) = -n(n+1)\,\mathbb{J}_n(x).$$

Using the Leibniz formula (30.42), for any $k < n$ we have

$$\begin{aligned}
\partial_x^k J_n(x) &= \partial_x^k((x-1)^n(x+1)^n) = \sum_{0\le l\le k} \binom{k}{l} \partial_x^l(x-1)^n\,\partial_x^{k-l}(x+1)^n\\
&= \sum_{0\le l\le k} \frac{k!}{l!(k-l)!}\,\frac{n!^2}{(n-l)!(n-(k-l))!}\,(x-1)^{n-l}\,(x+1)^{n-(k-l)}.
\end{aligned}$$

The last assertion follows from the fact that for any $k < n$ we have

$$\partial_x^k(x-1)^n = \frac{n!}{(n-k)!}\,(x-1)^{n-k} \quad \text{and} \quad \partial_x^k(x+1)^n = \frac{n!}{(n-k)!}\,(x+1)^{n-k}.$$

This clearly implies that

$$\forall k < n \qquad \partial_x^k J_n(-1) = \partial_x^k J_n(+1) = 0.$$

Thus, for any $n > m$ using a simple integration by parts we find that

$$\begin{aligned}
&\int_{-1}^{1} \partial_x^n J_n(x)\ \partial_x^m J_m(x)\ dx \\
&= \left[\partial_x^{n-1} J_n(x)\ \partial_x^m J_m(x)\right]_{-1}^{1} - \int_{-1}^{1} \partial_x^{n-1} J_n(x)\ \partial_x^{m+1} J_m(x)\ dx \\
&= -\int_{-1}^{1} \partial_x^{n-1} J_n(x)\ \partial_x^{m+1} J_m(x)\ dx \\
&= (-1)^2 \int_{-1}^{1} \partial_x^{n-2} J_n(x)\ \partial_x^{m+2} J_m(x)\ dx = \cdots = (-1)^n \int_{-1}^{1} J_n(x)\ \partial_x^{m+n} J_m(x)\ dx.
\end{aligned}$$

Recalling that $J_m(x)$ is a polynomial of degree $2m$ we have

$$\forall n > m\ (\Rightarrow m+n > 2m) \quad \partial_x^{m+n} J_m = 0 \quad \text{and} \quad \partial_x^{2m} J_m = (-1)^m\ (2m)!.$$

This shows the orthogonality property

$$\forall m \neq n \qquad \pi(\mathbb{J}_m \mathbb{J}_n) = 0.$$

For $m = n$ we have

$$\pi(\mathbb{J}_n^2) = \frac{1}{2^{2n+1} n!^2} \int_{-1}^{1} \left(\partial_x^n J_n\right)^2 (x)\ dx = \frac{(2n)!}{2^{2n+1} n!^2} \int_{-1}^{1} J_n(x)\ dx.$$

With the change of variables

$$y = \frac{x+1}{2} \ \Rightarrow\ dx = 2dy \quad (x+1) = 2y \quad \text{and} \quad (x-1) = 2(y-1)$$

we can easily see that

$$\begin{aligned}
&\int_{-1}^{1} (1-x)^n\ (1+x)^n\ dx = 2^{2n+1} \int_{0}^{1} y^n\ (1-y)^n\ dy \\
&= 2^{2n+1}\ \tfrac{n!^2}{(2n+1)!}\ \left(\Leftarrow \text{Beta distribution with parameters } (n+1, n+1)\right).
\end{aligned}$$

Hence we conclude that

$$\pi(\mathbb{J}_n^2) = \frac{1}{2n+1}.$$

Since any monomial x^n can be described in terms of Legendre polynomials, the completeness of the Legendre polynomials can be proved using Weierstrass approximation theorem that states that the set of polynomials on $[-1, 1]$ is dense in $\mathbb{L}_2(\pi)$.

By construction we have

$$J_n(x) = (1-x^2)^n = \sum_{0 \leq m \leq n} \begin{pmatrix} n \\ m \end{pmatrix} (-1)^{n-m}\ x^{2(n-m)}.$$

Recalling that

$$\partial_x^k x^m = 1_{m \geq k}\ (m)_k\ x^{(m-k)}$$

with the Pochhammer symbol representing the falling factorial

$$(m)_k := m\ (m-1) \ldots (m-(k-1)) = m!/(m-k)!,$$

we check that

$$\begin{aligned}
\partial_x^n J_n(x) &= \sum_{0\leq m\leq n} \left(\begin{array}{c} n \\ m \end{array}\right) (-1)^{n-m}\ 1_{2(n-m)\geq n}\ (2(n-m))_n\ x^{2(n-m)-n} \\
&= \sum_{0\leq m\leq \lfloor n/2\rfloor} (-1)^{n+m}\ \frac{n!}{(n-m)!m!}\ \frac{(2(n-m))!}{(n-2m)!}\ x^{n-2m}.
\end{aligned}$$

This implies that

$$\sum_{n\geq 0} \mathbb{J}_n(x)\ t^n = \sum_{n\geq 0} \left[\sum_{0\leq m\leq \lfloor n/2\rfloor} j_{m,n-m}(x)\right]\ t^n$$

with the polynomials

$$\forall 0\leq m\leq q \qquad j_{m,q}(x) := \frac{(2q)!}{(q-m)!\ q!\ m!}\ \frac{(-1)^m}{2^{2q}}\ (2x)^{q-m}.$$

Changing the summation order with the summation indices $p = n-m \geq 0$ and $m \geq n-m = p$ we have

$$\sum_{0\leq n}\sum_{0\leq m\leq \lfloor n/2\rfloor} j_{m,n-m}(x)\ t^{(n-m)+m} = \sum_{m\geq 0}\ \sum_{0\leq p\leq m} j_{p,m}(x)\ t^{p+m}.$$

This yields

$$\begin{aligned}
\sum_{n\geq 0} \mathbb{J}_n(x)\ t^n &= \sum_{m\geq 0}\ \sum_{0\leq p\leq m} \frac{(2m)!}{(m-p)!\ m!\ p!}\ \frac{(-1)^p}{2^{2m}}\ (2x)^{m-p}\ t^p\ t^m \\
&= \sum_{m\geq 0} \frac{(-t)^m}{2^{2m}}\ \frac{(2m)!}{m!^2}\ \sum_{0\leq p\leq m} \frac{m!}{(m-p)!p!}\ (-2x)^{m-p}\ t^p \\
&= \sum_{m\geq 0} \frac{(-1)^m}{2^{2m}}\ \frac{(2m)!}{m!^2}\ u_t(x)^m \quad \text{with} \quad u_t(x) = t^2 - 2xt.
\end{aligned}$$

We recall the binomial formula

$$(1+u)^\alpha = \sum_{m\geq 0} \left(\begin{array}{c} \alpha \\ m \end{array}\right)\ u^m \quad \text{with} \quad \left(\begin{array}{c} \alpha \\ m \end{array}\right) = \frac{(\alpha)_m}{m!}$$

which is valid for any $\alpha \in \mathbb{R}$. In the above display $(\alpha)_n$ stands for the extended Pochhammer symbol

$$(\alpha)_m := \alpha\ (\alpha-1)\ldots(\alpha-(m-1)).$$

Notice that

$$\begin{aligned}
\alpha = -1/2\ \Rightarrow\quad (\alpha)_m &= (-1)^m\ \frac{1}{2}\ (\frac{1}{2}+1)(\frac{1}{2}+2)\ldots(\frac{1}{2}+(m-1)) \\
&= (-1)^m\ \frac{1}{2^m}\ (1\times 3\times 5\times (1+2(m-1))) \\
&= (-1)^m\ \frac{1}{2^m}\ \frac{(2m)!}{2\times 4\times\ldots\times 2m} = (-1)^m\ \frac{1}{2^{2m}}\ \frac{(2m)!}{m!}.
\end{aligned}$$

Using this formula, we have

$$(1+u_t(x))^{-\frac{1}{2}} = \sum_{m\geq 0} \frac{(-1)^m}{2^{2m}}\ \frac{(2m)!}{m!}\ u_t(x)^m = \sum_{n\geq 0} \mathbb{J}_n(x)\ t^n.$$

This ends the proof of the exercise.

■

Chapter 18

Solution to exercise 333:

We consider the 1-dimensional diffusion given by

$$(dX_t = b\ dt + \ dW_t \quad \text{and} \quad X_0 = W_0 = 0) \Leftrightarrow X_t = bt + W_t$$

for some given parameter b and a Wiener process.

We fix a time horizon t and we set $X = (X_s)_{s\in[0,t]}$ and $W = (W_s)_{s\in[0,t]}$. Using the Cameron-Martin density formula (18.9) we have

$$\mathbb{P}(X \in d\omega) = \exp\left[b\ \omega_t - \frac{b^2}{2}\ t\right] \mathbb{P}(W \in d\omega).$$

Equivalently, the Girsanov theorem yields the integration formulae

$$\mathbb{E}\left[F\left((W_s)_{s\in[0,t]}\right)\ Z_t^{(b)}\right] = \mathbb{E}\left[F\left((X_s)_{s\in[0,t]}\right)\right] = \mathbb{E}\left[F\left((W_s + bs)_{s\in[0,t]}\right)\right]$$

for any function F on the path space $C([0,t],\mathbb{R})$, with the change of probability measure

$$Z_t^{(b)} = \exp\left[b\ W_t - \frac{b^2}{2}\ t\right] \quad \Rightarrow \quad dZ_t^{(b)} = Z_t^{(b)}\ b\ dW_t.$$

This clearly implies that $Z_t^{(b)}$ is a martingale w.r.t. $\mathcal{F}_t = \sigma(W_s,\ s \leq t)$. We can also check the martingale property using the fact that

$$\forall s \leq t \quad Z_t^{(b)} = Z_s^{(b)} \times \exp\left[b\ (W_t - W_s) - \frac{b^2}{2}\ (t-s)\right].$$

Recalling that $(W_t - W_s)$ is independent of W_s and $(W_t - W_s)$ is a centered Gaussian random variable with variance $(t-s)$ we have

$$\begin{aligned}
\mathbb{E}\left(Z_t^{(b)} \mid \mathcal{F}_s\right) &= Z_s^{(b)} \times \mathbb{E}\left(\exp\left[b\ (W_t - W_s) - \frac{b^2}{2}\ (t-s)\right] \mid \mathcal{F}_s\right) \\
&= Z_s^{(b)} \times \underbrace{\mathbb{E}\left(\exp\left[b\ (W_t - W_s) - \frac{b^2}{2}\ (t-s)\right]\right)}_{=1}.
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 334:

Using exercise exercise 223 we have

$$\mathbb{E}\left[F\left((X_s)_{s\in[0,t]}\right)\ \mid X_t = x\right] = \mathbb{E}\left[F\left((\sigma W_s)_{s\in[0,t]}\right)\ \mid \sigma W_t = x\right].$$

This implies that

$$\begin{aligned}
&\mathbb{E}\left[F\left((X_s)_{s\in[0,t]}\right)\right]\\
&= \int \mathbb{E}\left[F\left((X_s)_{s\in[0,t]}\right) \mid X_t = x\right] \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2\sigma^2 t}(x-bt)^2} dx\\
&= \int \mathbb{E}\left[F\left((\sigma W_s)_{s\in[0,t]}\right) \mid \sigma W_t = x\right] \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2\sigma^2 t}x^2} e^{\frac{1}{2\sigma^2}(2xb-b^2t)} dx\\
&= e^{-\frac{t}{2}\left(\frac{b}{\sigma}\right)^2} \mathbb{E}\left[F\left((\sigma W_s)_{s\in[0,t]}\right) e^{\frac{bW_t}{\sigma}}\right].
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 335: Observe that

$$\mathbb{E}\left[F\left((X_s)_{s\in[0,t]}\right)\right] = e^{-\frac{t}{2}\left(\frac{b}{\sigma}\right)^2} \mathbb{E}\left[F\left((\sigma\ W_s)_{s\in[0,t]}\right) e^{\left(\frac{b}{\sigma^2}\right)\sigma W_t}\right].$$

Replacing $F\left((X_s)_{s\in[0,t]}\right)$ by

$$F\left((X_s)_{s\in[0,t]}\right) \times \exp\left(-\left(\frac{b}{\sigma^2}\right)(X_t - bt) - \frac{t}{2}\left(\frac{b}{\sigma}\right)^2\right)$$

and recalling that $\sigma W_t = (X_t - bt)$ we find that

$$\begin{aligned}
&\mathbb{E}\left[F\left((X_s)_{s\in[0,t]}\right)\exp\left[-\left(\frac{b}{\sigma}\right) W_t - \frac{t}{2}\left(\frac{b}{\sigma}\right)^2\right]\right]\\
&= e^{-\frac{t}{2}\left(\frac{b}{\sigma}\right)^2} \mathbb{E}\left[F\left((\sigma\ W_s)_{s\in[0,t]}\right) \exp\left(-\left(\frac{b}{\sigma^2}\right)(\sigma W_t - bt) - \frac{t}{2}\left(\frac{b}{\sigma}\right)^2 + \left(\frac{b}{\sigma^2}\right)\sigma W_t\right)\right]\\
&= \mathbb{E}\left[F\left((\sigma\ W_s)_{s\in[0,t]}\right)\right].
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 336:

Following the remark at the end of section 18.2.3, the exercise is a direct consequence of theorem 18.2.2.

This ends the proof of the exercise.

■

Solution to exercise 337:

We set

$$X' := \mathbb{E}(ZX|\mathcal{G})/\mathbb{E}(Z|\mathcal{G}).$$

In this notation, for any $A \subset \mathcal{G}$ we have

$$\begin{aligned}
\mathbb{E}'(1_A\ X') &= \mathbb{E}(1_A\ Z\ X')\\
&= \mathbb{E}(\mathbb{E}(1_A\ Z \mid \mathcal{G})\ X') = \mathbb{E}(1_A\ \mathbb{E}(Z \mid \mathcal{G})\ X')\\
&= \mathbb{E}(1_A\ \mathbb{E}(Z|\mathcal{G})\ \mathbb{E}(ZX|\mathcal{G})/\mathbb{E}(Z|\mathcal{G}))\\
&= \mathbb{E}(1_A\ \mathbb{E}(ZX|\mathcal{G})) = \mathbb{E}(\mathbb{E}(ZX1_A|\mathcal{G})) = \mathbb{E}(ZX1_A) = \mathbb{E}'(X1_A).
\end{aligned}$$

This implies that

$$\mathbb{E}'(X|\mathcal{G}) = \mathbb{E}(ZX|\mathcal{G})/\mathbb{E}(Z|\mathcal{G}).$$

This ends the proof of the exercise.

■

Solution to exercise 338: The process Y_t is a particular case of the diffusion process (18.8). In our situation we have

$$b(y) = x \Rightarrow dY_t = x\ dt + dV_t \iff Y_t = xt + V_t.$$

Using the time inhomogeneous version of (18.11) (discussed in the end of section 18.2.3) we have

$$\mathbb{E}\left(F\left((Y_s)_{s\leq t}\right)\ \exp\left(-\int_0^t h_s\ dY_s + \frac{1}{2}\int_0^t h_s^2\ ds\right)\right) = \mathbb{E}\left(F\left((V_s)_{s\leq t}\right)\right).$$

We observe that

$$\begin{aligned} & dY_s := h_s ds + dV_s \\ & \Rightarrow \exp\left(-\textstyle\int_0^t h_s\ dV_s - \frac{1}{2}\ \int_0^t h_s^2\ ds\right) = \exp\left(-\textstyle\int_0^t h_s\ dY_s + \frac{1}{2}\ \int_0^t h_s^2\ ds\right). \end{aligned}$$

Using (18.10) we also have

$$\mathbb{E}\left(F\left((Y_s)_{s\leq t}\right)\right) = \mathbb{E}\left(F\left((V_s)_{s\leq t}\right)\ \exp\left(\int_0^t h_s\ dV_s - \frac{1}{2}\int_0^t h_s^2\ ds\right)\right).$$

This ends the proof of the exercise.

■

Solution to exercise 339:
Observe that

$$L_t(1_i)(k) = \sum_{1\leq j\leq n} (1_i(j) - 1_i(k))\ Q_t(k,j) = Q_t(k,i) - 1_i(k)\ \lambda_t(k)$$

with

$$\lambda_t(k) := \sum_{1\leq j\leq n} Q_t(k,j).$$

This yields

$$\begin{aligned} \eta_t(L_t(1_i)) &= \sum_{1\leq k\leq n} \eta_t(k)\left(Q_t(k,i) - 1_i(k)\ \lambda_t(k)\right) \\ &= (\eta_t Q_t)(i) - \eta_t(i)\lambda_t(i) = \left(\eta_t\left[Q_t - \mathrm{diag}(\lambda_t)\right]\right)(i) \end{aligned}$$

with the diagonal matrix

$$\mathrm{diag}(\lambda_t) := \left[\begin{array}{ccccc} \lambda_t(1) & 0 & \cdots & 0 & 0 \\ \vdots & & \cdots & & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_t(n) \end{array}\right].$$

On the other hand, we also have

$$\eta_t(1_i(h_t - \eta_t(h_t)) = \eta_t(i)\left(h_t(i) - \eta_t(h_t)\right) = \left(\eta_t \mathrm{diag}(h_t - \eta_t(h_t))\right)(i)$$

with the diagonal matrix

$$\mathrm{diag}(h_t-\eta_t(h_t)) := \left[\begin{array}{ccccc} h_t(1)-\eta_t(h_t) & 0 & \cdots & 0 & 0 \\ \vdots & & \ldots & & \vdots \\ 0 & 0 & \cdots & 0 & h_t(n)-\eta_t(h_t) \end{array}\right].$$

Using (18.25) we find that

$$d\eta_t(i) = (\eta_t\,[Q_t-\mathrm{diag}(\lambda_t)])\,(i)\;dt + (\eta_t\mathrm{diag}(h_t-\eta_t(h_t)))\,(i)\;\sigma_t^{-2}\;(dY_t-\eta_t(h_t)dt),$$

with the vector and matrix notation

$$\eta_t(h_t) = [\eta_t(1),\cdots,\eta_t(n)]\left[\begin{array}{c} h_t(1) \\ \vdots \\ h_t(n) \end{array}\right] = \sum_{1\le i\le n}\eta_t(i)\;h_t(i)$$

and

$$\eta_t Q_t = [\eta_t(1),\cdots,\eta_t(n)]\left[\begin{array}{ccc} Q_t(1,1) & \cdots & Q_t(1,n) \\ \vdots & \ldots & \vdots \\ Q_t(n,1) & \cdots & Q_t(n,n) \end{array}\right] = [(\eta_t Q_t)(1),\cdots,(\eta_t Q_t)(n)]\,.$$

This implies that

$$d\eta_t = \eta_t\,[Q_t-\mathrm{diag}(\lambda_t)]\;\;dt + \eta_t\mathrm{diag}(h_t-\eta_t(h_t))\;\sigma_t^{-2}\;(dY_t-\eta_t(h_t)\;dt).$$

This ends the proof of the exercise.

∎

Solution to exercise 340:
We apply the filtering equation derived in exercise 339 to the situation

$$Q_t = \lambda\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \Rightarrow \mathrm{diag}(\lambda_t) := \lambda\times I_{2\times 2}$$

and

$$h_t(x) = x \Rightarrow \eta_t(h_t) = \eta_t(1) \Rightarrow \mathrm{diag}(h_t-\eta_t(h_t)) = \left[\begin{array}{cc} 0-\eta_t(1) & 0 \\ 0 & 1-\eta_t(1) \end{array}\right].$$

In this case, we have

$$\eta_t\,[Q_t-\mathrm{diag}(\lambda_t)] = \lambda\times[\eta_t(0),\eta_t(1)]\left[\begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array}\right] = \lambda\times[\eta_t(1)-\eta_t(0),\eta_t(0)-\eta_t(1)]$$

and

$$\eta_t\mathrm{diag}(h_t-\eta_t(h_t)) = [\eta_t(0),\eta_t(1)]\left[\begin{array}{cc} 0-\eta_t(1) & 0 \\ 0 & 1-\eta_t(1) \end{array}\right] = [-\eta_t(0)\eta_t(1),\eta_t(1)(1-\eta_t(1))]\,.$$

This implies that

$$d\eta_t(1) = \lambda\;\;(1-2\eta_t(1))\;\;dt + \eta_t(1)(1-\eta_t(1))\;(dY_t-\eta_t(1)\;dt)$$

and

$$d\eta_t(0) = \lambda\ (1 - 2\eta_t(0))\ dt - \eta_t(0)(1 - \eta_t(0))\ (dY_t - (1 - \eta_t(0))\ dt).$$

This ends the proof of the exercise.

■

Solution to exercise 341:
Using (18.27) we have

$$\left\{\begin{array}{rcl} d\widehat{X}_t & = & \left(A\ \widehat{X}_t + a\right)\ dt + P_t\ (C/\sigma_2^2)\ \left(dY_t - \left(C\widehat{X}_t + c\right) dt\right) \\ \partial_t P_t & = & -P_t^2 (C/\sigma_2)^2 + 2AP_t + \sigma_1^2, \end{array}\right.$$

with the initial conditions $(\widehat{X}_0, P_0)$ given by the mean and covariance matrix $(\mathbb{E}(X_0), P_0)$ of the initial condition X_0.

- When $C = 0$ the signal and the observations are independent. In this situation, the Kalman filter resumes to

$$\left(\widehat{X}_t, P_t\right) = (\mathbb{E}(X_t), \mathrm{Var}(X_t)) \Longrightarrow \left\{\begin{array}{rcl} d\widehat{X}_t & = & \left(A\ \widehat{X}_t + a\right)\ dt \\ \partial_t P_t & = & 2AP_t + \sigma_1^2. \end{array}\right.$$

The solution is given by

$$\begin{array}{rcl} \widehat{X}_t & = & e^{At}\left[\widehat{X}_0 + a\displaystyle\int_0^t\ e^{-As}\ ds\right] \\ & = & e^{At}\left[\widehat{X}_0 + \dfrac{a}{A}\ \left(1 - e^{-At}\right)\right] = e^{At}\ \widehat{X}_0 + \dfrac{a}{A}\ \left(e^{At} - 1\right) \end{array}$$

and

$$\begin{array}{rcl} P_t & = & e^{2At}\left[P_0 + \sigma_1^2\displaystyle\int_0^t\ e^{-2As}\ ds\right] \\ & = & e^{2At}\left[P_0 + \dfrac{\sigma_1^2}{2A}\ \left(1 - e^{-2At}\right)\right] = e^{2At}\ P_0 + \dfrac{\sigma_1^2}{2A}\ \left(e^{2At} - 1\right). \end{array}$$

We clearly have

$$\lim_{t\to\infty}\left(\widehat{X}_t, P_t\right) = \left\{\begin{array}{cl} (0,0) & \text{if}\quad A < 0 \\ (\mathbb{E}(X_0), P_0) & \text{if}\quad A = 0 \\ (+\infty, +\infty) & \text{if}\quad A > 0. \end{array}\right.$$

- When $\sigma_1 = 0$, the signal is purely deterministic. In this situation we have

$$X_t = \left\{\begin{array}{cl} e^{At}\ X_0 + \frac{a}{A}\ \left(e^{At} - 1\right) & \text{if}\quad A \neq 0 \\ X_0 + at & \text{if}\quad A = 0. \end{array}\right.$$

In this situation, the randomness in the signal only comes from the initial condition X_0. In addition, if $(CP_0) \wedge (A\sigma_2) \neq 0$ we have

$$P_t = \frac{2A\sigma_2^2 P_0}{C^2 P_0\left(1 - e^{-2At}\right) + 2A\sigma_2^2 e^{-2At}}.$$

Notice that

$$|A| \wedge \sigma_2^2 \wedge P_0 = 0 \Longrightarrow P_t = 0.$$

To check that P_t satisfies the Riccati equation we use the fact that

$$\begin{aligned}\partial_t P_t &= \frac{2A\sigma_2^2 P_0\left(2A\left[\left(2A\sigma_2^2 - C^2P_0\right)e^{-2At} + C^2P_0\right] - 2AC^2P_0\right)}{\left[C^2P_0\left(1-e^{-2At}\right) + 2A\sigma_2^2 e^{-2At}\right]^2}\\ &= 2AP_t - \left(C/\sigma_2\right)^2 P_t^2.\end{aligned}$$

Notice that

$$A > 0 \Longrightarrow C^2P_0\left(1-e^{-2At}\right) + 2A\sigma_2^2 e^{-2At} > 0$$

and

$$(A > 0 \quad \text{and} \quad P_0 \neq 0) \Longrightarrow \lim_{t\to\infty} P_t = 2A(\sigma_2/C)^2.$$

In the reverse angle we have

$$\begin{aligned}A < 0 \quad &\Longrightarrow \quad C^2P_0\left(e^{2|A|t} - 1\right) + 2|A|\sigma_2^2 e^{2|A|t} > 0\\ &\Longrightarrow \quad P_t = \frac{2|A|\sigma_2^2 P_0}{C^2P_0\left(e^{2|A|t}-1\right) + 2|A|\sigma_2^2 e^{2|A|t}} > 0\end{aligned}$$

as soon as $P_0 \neq 0$. In this situation, we have

$$(A < 0 \quad \text{and} \quad P_0 \neq 0) \Longrightarrow \lim_{t\to\infty} P_t = 0.$$

- Assume that $C \neq 0$ and $\sigma_2 > 0$ and set $Q_t = \left(C/\sigma_2\right)^2 P_t$. We have

$$\partial_t Q_t = 2A\left(C/\sigma_2\right)^2 P_t - \left(C/\sigma_2\right)^4 P_t^2 + \left(C/\sigma_2\right)^2 \sigma_1^2 = -Q_t^2 + 2AQ_t + B^2$$

with $B := |C(\sigma_1/\sigma_2)| \geq 0$. In this notation the Kalman-Bucy filter takes the following form

$$\left\{\begin{array}{rcl} d\widehat{X}_t &=& \left(A\,\widehat{X}_t + a\right)\,dt + Q_tC^{-1}\,\left(dY_t - \left(C\widehat{X}_t + c\right)dt\right)\\ \partial_t Q_t &=& -Q_t^2 + 2AQ_t + B^2.\end{array}\right.$$

When $A = 0 = \sigma_1$ we have $A = 0 = B$ and the Riccati equation resumes to

$$\partial_t Q_t = -Q^2 \Rightarrow Q_t = \frac{Q_0}{1 + Q_0 t} \Rightarrow P_t = \frac{P_0}{1 + \left(C/\sigma_2\right)^2 P_0 t} \longrightarrow_{t\to\infty} 0.$$

When $|A| \vee \sigma_1^2 > 0$ $(\Rightarrow |A| \vee |B| > 0)$, the characteristic polynomial of the equation is given by

$$\begin{aligned}q(z) &= z^2 - 2Az - B^2\\ &= (z-A)^2 - \left(A^2 + B^2\right) = (z - z_1)(z - z_2),\end{aligned}$$

with the two different roots

$$z_1 = A - \sqrt{A^2 + B^2} < 0 < z_2 = A + \sqrt{A^2 + B^2}.$$

In this situation, the solution of the Riccati equation is given by

$$Q_t = \partial_t \log R_t = \frac{\partial_t R_t}{R_t} \quad \text{with} \quad \partial_t^2 R_t = 2A\;\partial_t R_t + B^2 R_t.$$

To check this claim, we recall that

$$\begin{aligned}\partial_t Q_t &= -Q_t^2 + 2AQ_t + B^2 = -Q_t^2 + 2A\frac{\partial_t R_t}{R_t} + B^2\\ &= \partial_t\left(\frac{\partial_t R_t}{R_t}\right) = -\left(\frac{\partial_t R_t}{R_t}\right)^2 + \frac{\partial_t^2 R_t}{R_t} = -Q_t^2 + \frac{\partial_t^2 R_t}{R_t}.\end{aligned}$$

This implies that

$$2A\frac{\partial_t R_t}{R_t} + B^2 = \frac{\partial_t^2 R_t}{R_t} \iff 2A\partial_t R_t + B^2 R_t = \partial_t^2 R_t.$$

The solution of the second order differential equation is given by

$$\begin{aligned} R_t = c_1\, e^{z_1 t} + c_2\, e^{z_2 t} \quad &\Rightarrow \left\{\begin{array}{lcl} R_0 &=& c_1 + c_2\\ \dot{R}_0 &=& c_1 z_1 + c_2 z_2 = R_0 Q_0\end{array}\right.\\ &\Rightarrow \left\{\begin{array}{lcl} c_1(z_2 - z_1) &=& R_0(z_2 - Q_0)\\ c_2(z_2 - z_1) &=& R_0(Q_0 - z_1).\end{array}\right.\end{aligned}$$

This yields the formulae

$$\begin{aligned} Q_t &= \frac{c_1 z_1\, e^{-(z_2 - z_1)t} + c_2 z_2}{c_1\, e^{-(z_2-z_1)t} + c_2}\\ &= z_2 - \frac{\left[c_1(z_2 - z_1)\, e^{-(z_2-z_1)t}\right] \times (z_2 - z_1)}{\left[c_1\, e^{-(z_2-z_1)t} + c_2\right] \times (z_2 - z_1)}\\ &= z_2 - (z_2 - z_1)\, \frac{(z_2 - Q_0)\, e^{-(z_2-z_1)t}}{(z_2 - Q_0)\, e^{-(z_2-z_1)t} + (Q_0 - z_1)}.\end{aligned}$$

Notice that

$$(z_2 - Q_0)\, e^{-(z_2-z_1)t} + (Q_0 - z_1) \;=\; z_2 e^{-(z_2-z_1)t} \underbrace{-z_1}_{>0} + Q_0\, \left(1 - e^{-(z_2-z_1)t}\right) > 0.$$

We conclude that

$$\lim_{t\to\infty} Q_t := Q_\infty = z_2 = A + \sqrt{A^2 + B^2} = A + \sqrt{A^2 + \left(\frac{\sigma_1}{\sigma_2}\right)^2\, C^2}$$

and

$$\lim_{t\to\infty} P_t := P_\infty \;=\; (\sigma_2/C)^2\, Q_\infty.$$

The final assertion is a consequence of the fact that

$$Q_t - z_2 = Q_t - Q_\infty = \partial_t \log\left((z_2 - Q_0)\, e^{-(z_2-z_1)t} + (Q_0 - z_1)\right).$$

This yields

$$\begin{aligned}\exp\left(\int_0^t (Q_\infty - Q_s)\, ds\right) &= \exp\left(-\int_0^t \partial_s \log\left((z_2 - Q_0)\, e^{-(z_2-z_1)s} + (Q_0 - z_1)\right)\, ds\right)\\ &= \left(\frac{z_2 - Q_0}{z_2 - z_1}\, e^{-(z_2-z_1)t} + \frac{Q_0 - z_1}{z_2 - z_1}\right)^{-1}.\end{aligned}$$

Recalling that

$$A - Q_\infty = -\sqrt{A^2 + B^2} = -(z_2 - z_1)/2$$

we conclude that

$$\begin{aligned}
\exp\left(\int_0^t (A - Q_s)\, ds\right) &= \exp\left(\int_0^t (A - Q_\infty)\, ds\right) \times \exp\left(\int_0^t (Q_\infty - Q_s)\, ds\right) \\
&= e^{-(z_2 - z_1)t/2} \frac{(z_2 - z_1)}{(z_2 - Q_0)\; e^{-(z_2 - z_1)t} + (Q_0 - z_1)} \\
&\leq e^{-(z_2 - z_1)t/2} \; (1 + |z_2/z_1|)\,.
\end{aligned}$$

The last assertion follows from the fact that

$$(z_2 - Q_0)\; e^{-(z_2 - z_1)t} + (Q_0 - z_1) = z_2\; e^{-(z_2 - z_1)t} - z_1 + Q_0\; \left(1 - e^{-(z_2 - z_1)t}\right) \geq -z_1 > 0.$$

This ends the proof of the exercise.

■

Solution to exercise 342:
Using exercise 337, we have

$$\begin{aligned}
\mathbb{E}'(M_t \mid \mathcal{F}_s) = M_s \quad &\Leftrightarrow \quad \mathbb{E}(Z_t M_t | \mathcal{F}_s)/\mathbb{E}(Z_t | \mathcal{F}_s) = M_s \\
&\Leftrightarrow \quad \mathbb{E}(Z_t M_t | \mathcal{F}_s) = M_s\; \mathbb{E}(Z_t | \mathcal{F}_s) = M_s Z_s.
\end{aligned}$$

The r.h.s. formula in the above display follows from the fact that Z_t is a martingale. The last assertion follows from the fact that

$$Z_s^{-1}\mathbb{E}(M_t Z_t | \mathcal{F}_s) = M_s = \mathbb{E}'(M_t \mid \mathcal{F}_s).$$

This ends the proof of the exercise.

■

Solution to exercise 343:
Using exercise 342, we need to check that $(M'_s Z_s)_{s\in[0,t]}$ is a martingale on $(\Omega_t, \mathcal{F}_t, \mathbb{P}_t)$. We have

$$M'_s Z_s = \underbrace{Z_s M_s - [Z, M]_s}_{\text{martingale on } (\Omega_t, \mathcal{F}_t, \mathbb{P}_t)} + [Z, M]_s - Z_s \int_0^s Z_{r+}^{-1}\; d\,[Z, M]_r$$

and

$$\begin{aligned}
U_s = Z_s \int_0^s Z_{r+}\; d\,[Z, M]_r \Rightarrow \Delta U_s &= Z_s\; Z_s^{-1}\; d\,[Z, M]_s + \left(\int_0^s Z_{r+}^{-1}\; d\,[Z, M]_r\right)\; \Delta Z_s \\
&= d\,[Z, M]_s + \underbrace{\left(\int_0^s Z_{r+}^{-1}\; d\,[Z, M]_r\right)\; \Delta Z_s}_{\text{martingale increment}}.
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 344:
We apply exercise 343 to the model

$$M_s = W_s \Rightarrow M'_s := W_s - \int_0^t \; Z_s^{-1} \; d\left\langle Z, W\right\rangle_s.$$

Recalling that

$$dZ_s = Z_s \; b(W_s) \; dW_s \Rightarrow d\left\langle Z, W_s\right\rangle_s = dZ_s dW_s = Z_s \; b(W_s) ds$$

we check that

$$M'_s := W_s - \int_0^s \; Z_r^{-1} \; Z_r \; b(W_r) \; dW_r dW_r = W_s - \int_0^s \; b(W_r) \; dr.$$

This ends the proof of the exercise.

■

Solution to exercise 345:
We apply exercise 343 to the model

$$M_s = N_s - s \Rightarrow M'_s := (N_s - s) - \int_0^s \; Z_{r+}^{-1} \; d\left[Z, M\right]_r.$$

At jump times of the exponential martingale Z_s defined in (18.4) we have

$$Z_{r+} = Z_r \; \lambda'_r(N_r).$$

Arguing as in section 18.1.3 we have

$$\begin{aligned}
& dZ_r = Z_r \; \left[(\lambda'_r(N_r) - 1) \; dN_r - (\lambda'_r(N_r) - 1) \; dr\right] \\
\Rightarrow \;\; & (Z_r \; \lambda'_r(N_r))^{-1} \; dZ_r dM_r \;\; = \;\; (Z_r \; \lambda'_r(N_r))^{-1} \, Z_r \; (\lambda'_r(N_r) - 1) \; dN_r = \frac{1}{\lambda'_r(N_r)} \; dN_r - dN_r.
\end{aligned}$$

This implies that

$$M'_s := \int_0^s \frac{1}{\lambda'_r(N_r)} \; dN_r - s$$

is a martingale on $(\Omega_t, \mathcal{F}_t, \mathbb{P}'_t)$. The last assertion follows from the formula

$$\int_0^s \lambda'_r(N_r) \; dM'_r = N_s - \int_0^s \lambda'_r(N_r) dr.$$

This ends the proof of the exercise.

■

Solution to exercise 346:
We have

$$M_t^{(1)} = M_t := \exp\left(-\lambda \int_0^t (e^{f(s)} - 1) ds\right) \prod_{0 \leq s \leq t} \left(1 + \left(e^{f(s)} - 1\right) \Delta N_s\right).$$

In terms of the martingale

$$\overline{M}_t = \left(e^{f(t)} - 1\right)(dN_t - \lambda t) = \underbrace{\left(e^{f(t)} - 1\right) dN_t}_{=\Delta \overline{M}_t} - \lambda \left(e^{f(t)} - 1\right) dt$$

we also have the decomposition

$$M_t = e^{\overline{M}_t} \prod_{0 \le s \le t} \left((1 + \Delta \overline{M}_s)\ e^{-\Delta \overline{M}_s} \right).$$

This implies that

$$\begin{aligned}
dM_t &= M_{t+dt} - M_t \\
&= M_t \left(e^{(\overline{M}_{t+dt} - \overline{M}_t) - \Delta \overline{M}_t} (1 + \Delta \overline{M}_t) - 1 \right) \\
&= M_t \left(e^{-\lambda \left(e^{f(t)} - 1\right) dt} (1 + \Delta \overline{M}_t) - 1 \right) \\
&= M_t \left(\left(1 - \lambda \left(e^{f(t)} - 1 \right) dt \right) (1 + \Delta \overline{M}_t) - 1 \right).
\end{aligned}$$

Using the fact that $dN_t \times dt = 0$ we conclude that

$$dM_t = M_t \left(-\lambda \left(e^{f(t)} - 1 \right) dt + \Delta \overline{M}_t \right) = M_t\ d\overline{M}_t.$$

Notice that

$$f = \log(1 + g) \implies M_t^{(1)} = M_t^{(2)}.$$

This ends the proof of the exercise.

■

Solution to exercise 347:

Two different proofs can be used. The first one is based on the fact that $X_{s,t} = \int_s^t f(r) dW_r,\ s \le t$ is independent of $\mathcal{F}_s$. In addition, $X_{s,t}$ is a centered Gaussian random variable with variance

$$\mathbb{E}(X_{s,t}^2 \mid \mathcal{F}_s) = \mathbb{E}(X_{s,t}^2) = \mathbb{E}\left(\left[\int_s^t f(r) dW_r \right]^2 \right) = \frac{1}{2} \int_s^t f(r)^2 dr := \sigma_{s,t}^2.$$

In this case, we have

$$\mathbb{E}\left(e^{X_{s,t}^2} \mid \mathcal{F}_s \right) = e^{\sigma_{s,t}^2/2} \Longleftrightarrow \mathbb{E}(M_t \mid \mathcal{F}_s) = M_s.$$

The second proof is based on Doeblin-Itō lemma applied to the function

$$g(t, X_t) = \exp\left(X_t - \frac{1}{2} \int_0^t f(s)^2 ds \right) \quad \text{with} \quad X_t = \int_0^t f(s) dW_s.$$

We have

$$\begin{aligned}
dg(t, X_t) &= \frac{\partial g}{\partial t}(t, X_t) dt + \frac{\partial g}{\partial x}(t, X_t)\ dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X_t)\ dX_t dX_t \\
&= g(t, X_t) \left[-\frac{1}{2}\ f(t)^2\ dt\ +\ f(t)\ dW_t + \frac{1}{2}\ f(t)^2\ dt \right] \\
&= g(t, X_t)\ f(t)\ dW_t.
\end{aligned}$$

On the other hand, for any $0 \leq s \leq t$ we have

$$\mathbb{E}(\int_s^t g(r,X_r)\ f(r)\ dW_r)\left(= \mathbb{E}\left(\int_s^t g(r,X_r)\ f(r)\ \mathbb{E}(\ dW_r \mid \mathcal{F}_r) \mid \mathcal{F}_s\right)\right) = 0.$$

This shows that

$$M_t = M_s + \int_s^t g(r,X_r)\ f(r)\ dW_r \quad \text{is a } \mathcal{F}_t\text{-martingale starting at } M_0 = 1.$$

This ends the proof of the exercise.

■

Solution to exercise 348: Using (18.15) and (18.16), we have

$$\begin{aligned}&\mathbb{E}\left(F((X_s)_{s\leq t}) \mid X_0 = x_0\right)\\ &= \varphi_0(x_0)\ \mathbb{E}\left(F((X_s^\varphi)_{s\leq t})\varphi_t^{-1}(X_t^\varphi)\ \exp\left(\int_0^t V_s^\varphi(X_s^\varphi)ds\right) \mid X_0^\varphi = x_0\right)\end{aligned}$$

with $V_t^\varphi = \varphi^{-1}L_t(\varphi)$, and the process X_t^φ has infinitesimal generator

$$L_t^{[\varphi]}(f) := L_t(f) + \varphi^{-1}\Gamma_{L_t}(\varphi, f).$$

After integrating w.r.t. $\eta_0(dx_0)$, the proof of the exercise is easily completed.

■

Solution to exercise 349:

The first part of the exercise is a direct consequence of exercise 348 applied to the function

$$F((X_s)_{s\leq t}) = f(X_t)\ \exp\left(-\int_0^t V(X_s)ds\right).$$

Notice that

$$V^\varphi = V - \varphi^{-1}L(\varphi) = \varphi^{-1}\mathcal{H}(\varphi)$$

with the Hamiltonian operator $\mathcal{H}$ defined for any sufficiently regular function g by $\mathcal{H}(g) := -L(g) + Vg$.

This ends the proof of the exercise.

■

Solution to exercise 350:

For any couple of sufficiently regular functions f_1 and f_2 we notice that

$$\begin{aligned}\mu\left(\varphi^2\ f_1\ L^{[\varphi]}(f_2)\right) &= \mu\left(\varphi^2\ f_1\ L(f_2)\right) + \mu\left(\varphi^2\ f_1\ \left[\varphi^{-1}\Gamma_L(\varphi, f_2)\right]\right)\\ &= \mu\left(\varphi^2\ f_1\ L(f_2)\right) + \mu\left(\varphi^2\ f_1\ \left[\varphi^{-1}L(\varphi f_2) - L(f_2) - \varphi^{-1}f_2L(\varphi)\right]\right)\\ &= \underbrace{\mu\left(\varphi\ f_1\ L(\varphi f_2)\right)}_{=\mu(\varphi\ f_2\ L(\varphi f_1))} - \mu\left(\varphi\ f_1\ f_2L(\varphi)\right) = \mu\left(\varphi^2\ f_2\ L^{[\varphi]}(f_1)\right).\end{aligned}$$

This shows that $L^{[\varphi]}$ is reversible w.r.t. $\Psi_{\varphi^2}(\mu)$.

This ends the proof of the exercise.

■

Chapter 19

Solution to exercise 352:

In the Euclidian space $\mathbb{R}^2 = \mathcal{V} = \mathrm{Vect}(e_1, e_2)$ we have $\pi = Id$ so that $\nabla = \partial$ and $\nabla^2 = \delta^2$. This implies that

$$\nabla F = \begin{pmatrix} \partial_{x_1} F \\ \partial_{x_2} F \end{pmatrix} \quad \text{and} \quad \nabla^2 F = \begin{pmatrix} \partial_{x_1,x_1} F & \partial_{x_1,x_2} F \\ \partial_{x_2,x_1} F & \partial_{x_2,x_2} F \end{pmatrix}.$$

We also have that

$$\Delta F = \mathrm{tr}\left(\nabla^2 F\right) = \partial_{x_1,x_1} F + \partial_{x_2,x_2} F := \partial^2_{x_1} F + \partial^2_{x_2} F.$$

This ends the proof of the exercise.

■

Solution to exercise 352:

We have

$$\|\nabla F\|^2 = (\partial_{x_1} F)^2 + (\partial_{x_2} F)^2$$

and

$$\frac{1}{2}\,\Delta\left(\|\nabla F\|^2\right) = \frac{1}{2}\,\partial^2_{x_1}\left((\partial_{x_1} F)^2 + (\partial_{x_2} F)^2\right) + \frac{1}{2}\,\partial^2_{x_2}\left((\partial_{x_1} F)^2 + (\partial_{x_2} F)^2\right).$$

Observe that

$$\frac{1}{2}\,\partial_{x_1}\left((\partial_{x_1} F)^2 + (\partial_{x_2} F)^2\right) = (\partial_{x_1} F)\left(\partial^2_{x_1} F\right) + (\partial_{x_2} F)\left(\partial_{x_1,x_2} F\right)$$

and

$$\begin{aligned}
&\tfrac{1}{2}\,\partial^2_{x_1}\left((\partial_{x_1} F)^2 + (\partial_{x_2} F)^2\right) \\
&= \partial_{x_1}\left[(\partial_{x_1} F)\left(\partial^2_{x_1} F\right) + (\partial_{x_2} F)\left(\partial_{x_1,x_2} F\right)\right] \\
&= \left(\partial^2_{x_1} F\right)^2 + \left(\partial_{x_1,x_2} F\right)^2 + (\partial_{x_1} F)\left(\partial^3_{x_1} F\right) + (\partial_{x_2} F)\left(\partial_{x_2}\partial^2_{x_1} F\right).
\end{aligned}$$

By symmetry arguments, we also have that

$$\begin{aligned}
&\tfrac{1}{2}\,\partial^2_{x_2}\left((\partial_{x_1} F)^2 + (\partial_{x_2} F)^2\right) \\
&= \partial_{x_2}\left[(\partial_{x_2} F)\left(\partial^2_{x_2} F\right) + (\partial_{x_1} F)\left(\partial_{x_1,x_2} F\right)\right] \\
&= \left(\partial^2_{x_2} F\right)^2 + \left(\partial_{x_1,x_2} F\right)^2 + (\partial_{x_2} F)\left(\partial^3_{x_2} F\right) + (\partial_{x_1} F)\left(\partial_{x_1}\partial^2_{x_2} F\right).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\tfrac{1}{2}\,\Delta\left(\|\nabla F\|^2\right) \\
&= \left(\partial^2_{x_2} F\right)^2 + 2\left(\partial_{x_1,x_2} F\right)^2 + \left(\partial^2_{x_1} F\right)^2 \\
&\quad + (\partial_{x_1} F)\left(\partial^3_{x_1} F\right) + (\partial_{x_2} F)\left(\partial_{x_2}\partial^2_{x_1} F\right) + (\partial_{x_2} F)\left(\partial^3_{x_2} F\right) + (\partial_{x_1} F)\left(\partial_{x_1}\partial^2_{x_2} F\right).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\nabla^2 F\ \nabla^2 F &= \begin{pmatrix} \partial^2_{x_1} F & \partial_{x_1,x_2} F \\ \partial_{x_2,x_1} F & \partial^2_{x_2} F \end{pmatrix}\begin{pmatrix} \partial^2_{x_1} F & \partial_{x_1,x_2} F \\ \partial_{x_2,x_1} F & \partial^2_{x_2} F \end{pmatrix} \\ &= \begin{pmatrix} (\partial^2_{x_1} F)^2 + (\partial_{x_1,x_2} F)^2 & (\partial^2_{x_1} F + \partial^2_{x_2} F)(\partial_{x_1,x_2} F) \\ (\partial^2_{x_1} F + \partial^2_{x_2} F)(\partial_{x_1,x_2} F) & (\partial^2_{x_2} F)^2 + (\partial_{x_1,x_2} F)^2 \end{pmatrix}.\end{aligned}$$

This implies that

$$\mathrm{tr}\left(\nabla^2 F\ \nabla^2 F\right) = (\partial^2_{x_1} F)^2 + 2(\partial_{x_1,x_2} F)^2 + (\partial^2_{x_2} F)^2.$$

Finally, we observe that

$$\begin{aligned}&(\partial_{x_1} F)\left(\partial^3_{x_1} F\right) + (\partial_{x_2} F)\left(\partial_{x_2}\partial^2_{x_1} F\right) + (\partial_{x_2} F)\left(\partial^3_{x_2} F\right) + (\partial_{x_1} F)\left(\partial_{x_1}\partial^2_{x_2} F\right) \\ &= \left\langle \begin{pmatrix} (\partial_{x_1} F) \\ (\partial_{x_2} F) \end{pmatrix}, \begin{pmatrix} \partial^3_{x_1} F + \partial_{x_1}\partial^2_{x_2} F \\ \partial^3_{x_2} F + \partial_{x_2}\partial^2_{x_1} F \end{pmatrix} \right\rangle = \tfrac{1}{2}\ \langle \nabla F, \nabla(\Delta F)\rangle.\end{aligned}$$

The last assertion follows from the fact that

$$\frac{1}{2}\ \nabla(\Delta F) \quad = \quad \frac{1}{2}\begin{pmatrix} \partial_{x_1}(\Delta F) \\ \partial_{x_2}(\Delta F) \end{pmatrix} = \begin{pmatrix} \partial^3_{x_1} F + \partial_{x_1}\partial^2_{x_2} F \\ \partial^3_{x_2} F + \partial_{x_2}\partial^2_{x_1} F \end{pmatrix}.$$

The end the proof of the exercise is now easily completed.

■

Solution to exercise 353:
We consider a first and second order generator L_1 and L_2 defined by

$$L = L_1 + L_2$$

with

$$L_1(f) = \partial_b(f) := \sum_{1\leq i\leq r} b^i\ \partial_i(f) \quad \text{and} \quad L_2(f) = \frac{1}{2}\ \partial^2_a(f) := \frac{1}{2}\sum_{1\leq i,j\leq r} a^{i,j}\ \partial_{i,j}(f)$$

for some drift function $b = (b^i)_{1\leq i\leq r}$ and some symmetric matrix functional $a = (a^{i,j})_{1\leq i,j\leq r}$ on $\mathbb{R}^r$. To simplify notation we use Einstein notation and we write $b^i\ \partial_i$ and $a^{i,j}\ \partial_{i,j}$ instead of $\sum_{1\leq i\leq r}\ b^i\ \partial_i$ and $\sum_{1\leq i,j\leq r}\ a^{i,j}\ \partial_{i,j}$.

We clearly have

$$\partial_i(fg) = f\partial_i g + g\partial_i f \ \Rightarrow\ L_1(fg) = f\ L_1(g) + g\ L_1(f).$$

In much the same way, we have

$$\partial_{i,j}(fg) = \partial_j\left(f\partial_i g + g\partial_i f\right) = f\partial_{i,j} g + g\partial_{i,j} f + \left(\partial_j f\partial_i g + \partial_i f\partial_j g\right),$$

from which we prove that

$$a^{i,j}\ \partial_{i,j}(fg) - f\ a^{i,j}\ \partial_{i,j}(g) - g\ a^{i,j}\ \partial_{i,j}(f) = 2\ a^{i,j}\ \partial_i f\ \partial_j g.$$

This clearly implies that

$$L_1(fg) = f\ L_1(g) + g\ L_1(f) + \Gamma_{L_1}(f,g) \quad \text{with} \quad \Gamma_{L_1}(f,g) := 0$$

and

$$L_2(fg) = f\ L_2(g) + g\ L_2(f) + \Gamma_{L_2}(f,g) \quad \text{with} \quad \Gamma_{L_2}(f,g) :=\ a^{i,j}\ \partial_i f\ \partial_j g.$$

Finally, we clearly have

$$\begin{aligned}\Gamma_L(f,g) &= L(fg) - fL(g) - gL(f) \\ &= (L_1(fg) - fL_1(g) - gL_1(f)) + (L_2(fg) - fL_2(g) - gL_2(f)) \\ &= \Gamma_{L_2}(f,g).\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 354:

Since the matrix a is symmetric, the operator

$$(f,g) \mapsto \Gamma_L(f,g) =\ a^{i,j}\ \partial_i f\ \partial_j g$$

is clearly a symmetric bilinear form. On the other hand $(f,g) \mapsto \Gamma_L(f,g)$ is a first order differential operator w.r.t. each coordinate. This yields

$$\Gamma_L(f,gh) = g\Gamma_L(f,h) + h\Gamma_L(f,g)$$

for any smooth functions (f,g,h).

We consider collections of smooth functions $f = (f^i)_{1\leq i\leq n}$ and $g = (g^j)_{1\leq j\leq m}$ on $\mathbb{R}^r$, and some smooth functions $F(x_1,\ldots,x_n)$ and $G(x_1,\ldots,x_m)$ on $\mathbb{R}^n$ and $\mathbb{R}^m$, for some $m,n\geq 1$.

For smooth functions of the form

$$F(f) = F(f^1,\ldots,f^n) \quad \text{and} \quad G(g) = G(g^1,\ldots,g^m)$$

we have

$$\begin{aligned}\partial_i(F(f)) &= (\partial_k F)(f)\ \partial_i(f^k) \ \left(= \sum_{1\leq k\leq n} (\partial_k F)(f)\ \partial_i(f^k)\right) \\ \partial_i(G(g)) &= (\partial_l G)(g)\ \partial_i(g^l) \ \left(= \sum_{1\leq l\leq m} (\partial_l G)(g)\ \partial_i(g^l)\right).\end{aligned}$$

This yields

$$\begin{aligned}\Gamma_L\left(F(f),G(g)\right) &= a^{i,j}\ \partial_i(F(f))\ \partial_i(G(g)) \\ &= (\partial_k F)(f)(\partial_l G)(g)\ \Gamma_L(f^k,g^l).\end{aligned}$$

In much the same way, we have

$$\begin{aligned}\partial_{i,j}(F(f)) &= \partial_j\left((\partial_k F)(f)\ \partial_i(f^k)\right) \\ &= (\partial_{k,l}F)(f)\ \partial_i(f^k)\partial_j(f^l) + (\partial_k F)(f)\ \partial_{i,j}(f^k).\end{aligned}$$

This implies that

$$\begin{aligned}L(F(f)) &= b^i\partial_i(F(f)) + \frac{1}{2}\ a^{i,j}\ \partial_{i,j}(F(f)) \\ &= (\partial_k F)(f)\ \left[b^i\ \partial_i(f^k) + \frac{1}{2}\ a^{i,j}\ \partial_{i,j}(f^k)\right] + (\partial_{k,l}F)(f)\ \frac{1}{2}\ a^{i,j}\ \partial_i(f^k)\partial_j(f^l) \\ &= (\partial_k F)(f)\ L(f^k) + \frac{1}{2}\ (\partial_{k,l}F)(f)\ \Gamma_L(f^k,f^l).\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 355:
We have

$$a^{i,j}\ \partial_i f\ a^{k,l}\ \partial_{k,l}(\partial_j g) = a^{k,l}\left(a^{i,j}\ \partial_i f\ \ \partial_j(\partial_{k,l}(g))\right) = a^{k,l}\ \Gamma_L(f,\partial_{k,l}g).$$

This yields

$$\begin{aligned} a^{i,j}\ \partial_i f\ L(\partial_j g) &= a^{i,j}\ \partial_i f\ b^k\partial_k\partial_j g + \frac{1}{2}\ a^{k,l}\ \Gamma_L(f,\partial_{k,l}g) \\ &= b^k\ \left(a^{i,j}\ \partial_i f\ \partial_j(\partial_k g)\right) + \frac{1}{2}\ a^{k,l}\ \Gamma_L(f,\partial_{k,l}g) \\ &= b^k\ \Gamma_L(f,\partial_k g) + \frac{1}{2}\ a^{k,l}\ \Gamma_L(f,\partial_{k,l}g). \end{aligned} \tag{30.44}$$

This ends the proof of the first assertion.

To check the second one, we observe that

$$\begin{aligned} \Gamma_L(f,L(g)) &= \Gamma_L(f,b^i\partial_i g) + \frac{1}{2}\ \Gamma_L(f,a^{i,j}\partial_{i,j}g) \\ &= b^i\Gamma_L(f,\partial_i g) + \frac{1}{2}\ a^{i,j}\ \Gamma_L(f,\partial_{i,j}g) \\ &\qquad + \Gamma_L(f,b^i)\ \partial_i g + \frac{1}{2}\ \Gamma_L(f,a^{i,j})\ \partial_{i,j}g. \end{aligned}$$

The last assertion follows from the formula

$$\Gamma_L(f,gh) = g\ \Gamma_L(f,h) + h\ \Gamma_L(f,g)$$

which we proved in exercise 354. Using (30.44) we check that

$$\Gamma_L(f,L(g)) = a^{i,j}\ \partial_i f\ L(\partial_j g) + \Gamma_L(f,b^i)\ \partial_i g + \frac{1}{2}\ \Gamma_L(f,a^{i,j})\ \partial_{i,j}g.$$

By symmetry arguments, we also have

$$\Gamma_L(g,L(f)) = a^{i,j}\ \partial_i g\ L(\partial_j f) + \Gamma_L(g,b^i)\ \partial_i f + \frac{1}{2}\ \Gamma_L(g,a^{i,j})\ \partial_{i,j}f.$$

Summing the two expressions yields

$$\begin{aligned} \Gamma_L(f,L(g)) + \Gamma_L(g,L(f)) &= a^{i,j}\ \overbrace{\left(\partial_i f\ L(\partial_j g) + \partial_i g\ L(\partial_j f)\right)}^{=L(\partial_i f\partial_j g)-\Gamma_L(\partial_i f,\partial_j g)} \\ &\qquad + \Gamma_L(f,b^i)\ \partial_i g + \frac{1}{2}\ \Gamma_L(f,a^{i,j})\ \partial_{i,j}g \\ &\qquad + \Gamma_L(g,b^i)\ \partial_i f + \frac{1}{2}\ \Gamma_L(g,a^{i,j})\ \partial_{i,j}f. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} L\left(\Gamma_L(f,g)\right) &= L\left(a^{i,j}\partial_i f\partial_j g\right) \\ &= a^{i,j}\ L\left(\partial_i f\partial_j g\right) + L\left(a^{i,j}\right)\ \partial_i f\partial_j g + \Gamma_L\left(a^{i,j},\partial_i f\partial_j g\right) \end{aligned}$$

and

$$\Gamma_L\left(a^{i,j},\partial_i f\partial_j g\right)=\Gamma_L\left(a^{i,j},\partial_j g\right)\ \partial_i f+\Gamma_L\left(a^{i,j},\partial_i f\right)\ \partial_j g.$$

We conclude that

$$\begin{aligned}
&\Gamma_{2,L}(f,g)\\
&=L\left(\Gamma_L(f,g)\right)-\Gamma_L(f,L(g))-\Gamma_L(g,L(f))\\
&=\left[L\left(a^{i,j}\right)\ \partial_i f\ \partial_j g-\Gamma_L(f,b^i)\ \partial_i g-\Gamma_L(g,b^i)\ \partial_i f\right]+a^{i,j}\ \Gamma_L(\partial_i f,\partial_j g)\\
&\quad+\left[\Gamma_L\left(a^{i,j},\partial_j g\right)\ \partial_i f+\Gamma_L\left(a^{i,j},\partial_i f\right)\partial_j g\right]-\tfrac{1}{2}\left[\Gamma_L(a^{i,j},f)\ \partial_{i,j}g+\Gamma_L(a^{i,j},g)\ \partial_{i,j}f\right].
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 356: Using the formulae

$$\Gamma_L(f,b^i)=a^{k,l}\ \partial_k f\ \partial_l b^i\qquad \Gamma_L\left(a^{i,j},\partial_i f\right)=a^{k,l}\partial_l a^{i,j}\ \partial_k\partial_i f$$

and

$$\Gamma_L(a^{i,j},f)=a^{k,l}\ \partial_k f\ \partial_l a^{i,j}$$

we find that

$$\begin{aligned}
\Gamma_{2,L}(f,g)=&\left(L\left(a^{i,j}\right)-\left\{a^{i,l}\ \partial_l b^j\ +a^{j,l}\ \partial_l b^i\ \right\}\right)\ \partial_i f\ \partial_j g\\
&+\left(a^{k,l}\ \partial_l a^{i,j}-\frac{1}{2}\ a^{i,l}\ \partial_l a^{k,j}\ \right)\ \left[\partial_i f\ \partial_{j,k} g+\partial_i g\ \partial_{j,k} f\right]+\ a^{i,j}\ a^{k,l}\ \partial_{i,k} f\ \partial_{j,l} g.
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 357:
By the definitions of the operators $\Gamma_{2,L}$ and Γ_L we have

$$\begin{aligned}
\Gamma_{2,L}(f,gh)&=L\left(\Gamma_L(f,gh)\right)-\Gamma_L\left(L(f),gh\right)-\Gamma_L\left(f,L(gh)\right)\\
L(gh)&=gL(h)+hL(g)+\Gamma_L(g,h).
\end{aligned}$$

Recalling that Γ_L is a bilinear form, we have

$$\begin{aligned}
\Gamma_L\left(f,L(gh)\right)&=\Gamma_L\left(f,gL(h)+hL(g)+\Gamma_L(g,h)\right)\\
&=\Gamma_L\left(f,gL(h)\right)+\Gamma_L\left(f,hL(g)\right)+\Gamma_L\left(f,\Gamma_L(g,h)\right),
\end{aligned}$$

as well as

$$\begin{aligned}
\Gamma_L\left(L(f),gh\right)&=g\ \Gamma_L\left(L(f),h\right)+h\ \Gamma_L\left(L(f),g\right)\\
\Gamma_L\left(f,gL(h)\right)&=g\ \Gamma_L\left(f,L(h)\right)+L(h)\ \Gamma_L\left(f,g\right)\\
\Gamma_L\left(f,hL(g)\right)&=h\ \Gamma_L\left(f,L(g)\right)+L(g)\ \Gamma_L\left(f,h\right).
\end{aligned}$$

On the other hand, we have

$$\Gamma_L(f,gh)\ =\ g\ \Gamma_L(f,h)+h\ \Gamma_L(f,g).$$

This yields

$$\begin{aligned}
L\left(\Gamma_L(f,gh)\right) &= L\left(g\ \Gamma_L(f,h) + h\ \Gamma_L(f,g)\right) \\
&= L\left(g\ \Gamma_L(f,h)\right) + L\left(h\ \Gamma_L(f,g)\right) \\
&= g\ L\left(\Gamma_L(f,h)\right) + L\left(g\right)\ \Gamma_L(f,h) + \Gamma_L\left(g, \Gamma_L(f,h)\right) \\
&\quad + h\ L\left(\Gamma_L(f,g)\right) + L\left(h\right)\ \Gamma_L(f,g) + \Gamma_L\left(h, \Gamma_L(f,g)\right).
\end{aligned}$$

Combining these formulae, we check directly that

$$\begin{aligned}
\Gamma_{2,L}(f,gh) &= \Gamma_L\left(g, \Gamma_L(f,h)\right) + \Gamma_L\left(h, \Gamma_L(f,g)\right) - \Gamma_L\left(f, \Gamma_L(g,h)\right) \\
&\quad + g\ L\left(\Gamma_L(f,h)\right) - g\ \Gamma_L\left(f, L(h)\right) - g\ \Gamma_L\left(L(f), h\right) \\
&\quad + h\ L\left(\Gamma_L(f,g)\right) - h\ \Gamma_L\left(f, L(g)\right) - h\ \Gamma_L\left(L(f), g\right).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\Gamma_{2,L}(f,gh) - h\Gamma_{2,L}(f,g) - g\Gamma_{2,L}(f,h) \\
&= \Gamma_L\left(g, \Gamma_L(f,h)\right) + \Gamma_L\left(h, \Gamma_L(f,g)\right) - \Gamma_L\left(f, \Gamma_L(g,h)\right).
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 358:

We use the differential formulae

$$\begin{aligned}
\partial_i f\ \partial_j(gh) &= g\ \partial_i f\ \partial_j(h) + h\ \partial_i f\ \partial_j(g) \\
\partial_i f\ \partial_{j,k}(gh) &= \partial_i f\ \partial_j\left(g\ \partial_k h + h\ \partial_k g\right) \\
&= \partial_i f\ \left(\partial_j g\ \partial_k h + \partial_j h\ \partial_k g + g\ \partial_{j,k} h + h\ \partial_{j,k} g\right) \\
&= g\ \left[\partial_i f\ \partial_{j,k} h\right] + h\ \left[\partial_i f\ \partial_{j,k} g\right] + \partial_i f\ \left(\partial_j g\ \partial_k h + \partial_j h\ \partial_k g\right),
\end{aligned}$$

as well as

$$\begin{aligned}
\partial_i(gh)\ \partial_{j,k}(f) &= g\ \partial_i(h)\ \partial_{j,k}(f) + h\ \partial_i(g)\ \partial_{j,k}(f) \\
\partial_{i,k} f\ \partial_{j,l}(gh) &= \partial_{i,k} f\ \left(\partial_j g\ \partial_l h + \partial_j h\ \partial_l g + g\ \partial_{j,l} h + h\ \partial_{j,l} g\right) \\
&= g\ \left(\partial_{i,k} f\ \partial_{j,l} h\right) + h\ \left(\partial_{i,k} f\ \partial_{j,l} g\right) + \partial_{i,k} f\ \left(\partial_j g\ \partial_l h + \partial_j h\ \partial_l g\right).
\end{aligned}$$

The terms in red correspond to the linear terms in the formula

$$\begin{aligned}
&\Gamma_{2,L}(f,gh) - h\ \Gamma_{2,L}(f,g) - g\ \Gamma_{2,L}(f,h) \\
&= \left(a^{k,l}\ \partial_l a^{i,j} - \frac{1}{2}\ a^{i,l}\ \partial_l a^{k,j}\right)\ \partial_i f\ \left(\partial_j g\ \partial_k h + \partial_j h\ \partial_k g\right) \\
&\qquad\qquad + a^{i,j}\ a^{k,l}\ \partial_{i,l} f\ \left(\partial_j g\ \partial_k h + \partial_j h\ \partial_k g\right).
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 359:

We clearly have $\mathcal{V} = \text{Vect}(U_1, U_2) \subset \mathbb{R}^3$ with the orthogonal vectors

$$U_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad U_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

The orthogonal projection on $\mathcal{V}$ is given for any vector $W = \begin{pmatrix} W^1 \\ W^2 \\ W^3 \end{pmatrix}$ by

$$\begin{aligned}
\pi_{\mathcal{V}}(W) &= \sum_{1 \le i \le 2} \langle U_i, W \rangle \; U_i \\
&= \frac{W^1 - W^2}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{W^1 + W^2 - 2W^3}{6} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{W^1 - W^2}{2} + \frac{W^1 + W^2 - 2W^3}{6} \\ \frac{W^2 - W^1}{2} + \frac{W^1 + W^2 - 2W^3}{6} \\ \frac{2W^3 - W^1 - W^2}{3} \end{pmatrix} \\
&= \frac{1}{3} \begin{pmatrix} \frac{3W^1 - 3W^2 + W^1 + W^2 - 2W^3}{2} \\ \frac{3W^2 - 3W^1 + W^1 + W^2 - 2W^3}{2} \\ 2W^3 - W^1 - W^2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2W^1 - W^2 - W^3 \\ 2W^2 - W^1 - W^3 \\ 2W^3 - W^1 - W^2 \end{pmatrix}.
\end{aligned}$$

We conclude that

$$\pi_{\mathcal{V}} = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

This ends the proof of the exercise.

■

Solution to exercise 360:

For any curve $t \in [0,1] \mapsto C(t) := \left(C^1(t), C^2(t), C^3(t)\right) \in S$, with $C(0) = x$ and $\frac{dC^i}{dt}(0) = V^i(x)$, with $i = 1, 2, 3$ we have

$$\varphi(C(t)) = 0 \Rightarrow 0 = \frac{d}{dt} \varphi(C(t))_{t=0} = \sum_{1 \le i \le 3} \left(\partial_{x_i} \varphi\right)(C(t))_{t=0} \frac{dC^i}{dt}(0).$$

This shows that

$$\langle \left(\partial \varphi\right)(x), V(x) \rangle = \langle \begin{pmatrix} \left(\partial_{x_1} \varphi\right)(x) \\ \left(\partial_{x_2} \varphi\right)(x) \\ \left(\partial_{x_3} \varphi\right)(x) \end{pmatrix}, \begin{pmatrix} V^1(x) \\ V^2(x) \\ V^3(x) \end{pmatrix} \rangle = 0.$$

We conclude that $\left(\partial \varphi\right)(x)$ is orthogonal to the tangent vectors $V(x)$ at $x \in S$. Thus, the equation of the tangent plane at x is given by the equation

$$\langle \begin{pmatrix} \left(\partial_{x_1} \varphi\right)(x) \\ \left(\partial_{x_2} \varphi\right)(x) \\ \left(\partial_{x_3} \varphi\right)(x) \end{pmatrix}, \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \\ y_3 - x_3 \end{pmatrix} \rangle = 0.$$

- For the hyperboloid $\varphi(x) = x_1^2 - x_2^2 - x_3^2 - 4 = 0$, we have

$$\begin{pmatrix} (\partial_{x_1}\varphi)(x) \\ (\partial_{x_2}\varphi)(x) \\ (\partial_{x_3}\varphi)(x) \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ -x_2 \\ -x_3 \end{pmatrix}.$$

- For the circular cone $\varphi(x) = x_1^2 + x_2^2 - x_3^2 = 0$, we have

$$\begin{pmatrix} (\partial_{x_1}\varphi)(x) \\ (\partial_{x_2}\varphi)(x) \\ (\partial_{x_3}\varphi)(x) \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ x_2 \\ -x_3 \end{pmatrix}.$$

This ends the proof of the exercise.

■

Solution to exercise 361:
For any parametric curve $t \in [0,1] \mapsto \psi(\theta(t)) := \psi(\theta_1(t), \theta_2(t))$, we have

$$\frac{d}{dt}\psi(\theta_1(t), \theta_2(t)) = (\partial_{\theta_1}\psi)(\theta)\ \frac{d\theta_1}{dt}(t) + (\partial_{\theta_2}\psi)(\theta)\ \frac{d\theta_2}{dt}(t).$$

This shows that the vector fields

$$(\partial_{\theta_1}\psi)(\theta) = \begin{pmatrix} -r\sin(\theta_1)\cos(\theta_2) \\ -r\sin(\theta_1)\sin(\theta_2) \\ r\cos(\theta_1) \end{pmatrix} \quad \text{and} \quad (\partial_{\theta_2}\psi)(\theta) = \begin{pmatrix} -(R + r\cos(\theta_1))\sin(\theta_2) \\ (R + r\cos(\theta_1))\cos(\theta_2) \\ 0 \end{pmatrix}$$

are tangent to the surface at the point $\psi(\theta)$. The normal vector $n(\theta)$ of the tangent plane at that point is given by the cross (vector) product

$$\begin{aligned} & n(\theta) \\ & = \begin{pmatrix} n^1(\theta) \\ n^2(\theta) \\ n^3(\theta) \end{pmatrix} \\ & = (\partial_{\theta_1}\psi)(\theta) \wedge (\partial_{\theta_2}\psi)(\theta) \\ & = \begin{pmatrix} -r\cos(\theta_1)(R + r\cos(\theta_1))\cos(\theta_2) \\ -r\cos(\theta_1)(R + r\cos(\theta_1))\sin(\theta_2) \\ -r\sin(\theta_1)\cos(\theta_2)(R + r\cos(\theta_1))\cos(\theta_2) - r\sin(\theta_1)\sin(\theta_2)(R + r\cos(\theta_1))\sin(\theta_2) \end{pmatrix}. \end{aligned}$$

The surface unit normal is given by the formula

$$\overline{n}(x,y) = \frac{n(\theta)}{\|n(\theta)\|} = \frac{n(\theta)}{\sqrt{\langle n(\theta), n(\theta)\rangle}}.$$

The equation of the tangent plane at $x = \psi(\theta)$ is given by the equation

$$\langle \begin{pmatrix} n^1(\theta) \\ n^2(\theta) \\ n^3(\theta) \end{pmatrix}, \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \\ y_3 - x_3 \end{pmatrix} \rangle = 0.$$

This ends the proof of the exercise.

■

Solution to exercise 362:
The tangent vector space at

$$x=\psi(\theta)=\begin{pmatrix}\psi^1(\theta)\\ \psi^2(\theta)\\ \psi^3(\theta)\end{pmatrix}=\begin{pmatrix}\theta_1\\ \theta_2\\ h(\theta)\end{pmatrix}\in S\Rightarrow\theta:=\phi(x)=\psi^{-1}(x)$$

is spanned by the vectors

$$V_1(x)=(\partial_{\theta_1}\psi)_{\phi(x)}=\begin{pmatrix}\left(\partial_{\theta_1}\psi^1\right)_{\phi(x)}\\ \left(\partial_{\theta_1}\psi^2\right)_{\phi(x)}\\ \left(\partial_{\theta_1}\psi^3\right)_{\phi(x)}\end{pmatrix}=\begin{pmatrix}1\\ 0\\ (\partial_{\theta_1}h)_{\phi(x)}\end{pmatrix}=\begin{pmatrix}1\\ 0\\ (\partial_{x_1}h)_x\end{pmatrix}$$

and

$$V_2(x)=(\partial_{\theta_2}\psi)_{\phi(x)}=\begin{pmatrix}\left(\partial_{\theta_2}\psi^1\right)_{\phi(x)}\\ \left(\partial_{\theta_2}\psi^2\right)_{\phi(x)}\\ \left(\partial_{\theta_2}\psi^3\right)_{\phi(x)}\end{pmatrix}=\begin{pmatrix}0\\ 1\\ (\partial_{\theta_2}h)_{\phi(x)}\end{pmatrix}=\begin{pmatrix}0\\ 1\\ (\partial_{x_2}h)_x\end{pmatrix}.$$

The orthogonal tangent vector space $T_x^{\perp}(S)$ is spanned by the vector

$$V_1^{\perp}(x)=(\partial\varphi)_x=\begin{pmatrix}(\partial_{x_1}\varphi)_x\\ (\partial_{x_2}\varphi)_x\\ (\partial_{x_3}\varphi)_x\end{pmatrix}=\begin{pmatrix}(\partial_{x_1}h)_x\\ (\partial_{x_2}h)_x\\ -1\end{pmatrix}. \tag{30.45}$$

The metric $g(x)$ on $T_x(S)$ is given by the matrix

$$\begin{aligned}g(x) &= \begin{pmatrix}g_{1,1}(x) & g_{1,2}(x)\\ g_{2,1}(x) & g_{2,2}(x)\end{pmatrix}=\begin{pmatrix}\langle V_1(x),V_1(x)\rangle & \langle V_1(x),V_2(x)\rangle\\ \langle V_2(x),V_1(x)\rangle & \langle V_2(x),V_2(x)\rangle\end{pmatrix}\\ &= \begin{pmatrix}1+(\partial_{x_1}h)_x^2 & (\partial_{x_1}h)_x\,(\partial_{x_2}h)_x\\ (\partial_{x_1}h)_x\,(\partial_{x_2}h)_x & 1+(\partial_{x_2}h)_x^2\end{pmatrix}.\end{aligned}$$

The metric $g_\perp(x)$ on $T_x(S)$ is given by the function

$$g_\perp(x)=\langle V_1^{\perp}(x),V_1^{\perp}(x)\rangle=1+(\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2\,.$$

This ends the proof of the exercise.

■

Solution to exercise 363:
The inverse of the matrix $g(x)$ is given by

$$\begin{aligned}g(x)^{-1} &= \begin{pmatrix}g^{1,1}(x) & g^{1,2}(x)\\ g^{2,1}(x) & g^{2,2}(x)\end{pmatrix}\\ &= \frac{1}{1+(\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2}\begin{pmatrix}1+(\partial_{x_2}h)_x^2 & -(\partial_{x_1}h)_x\,(\partial_{x_2}h)_x\\ -(\partial_{x_1}h)_x\,(\partial_{x_2}h)_x & 1+(\partial_{x_1}h)_x^2\end{pmatrix}.\end{aligned}$$

The orthogonal projection $\pi(x)$ of the vector field $W(x)=\begin{pmatrix}W^1(x)\\ W^2(x)\\ W^2(x)\end{pmatrix}$ is defined by

$$\pi(x)(W(x))=\sum_{i,j=1,2}g^{i,j}(x)\ \langle V_j(x),W(x)\rangle\ V_i(x)$$

and

$$\pi(x)(W(x)) = \frac{1}{1+(\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2}\ \langle V_1^{\perp}(x), W(x)\rangle\ V_1^{\perp}(x).$$

Notice that

$$\begin{array}{rcl}\langle V_1(x), W(x)\rangle & = & W^1(x)+(\partial_{x_1}h)_x\ W^3(x)\\ \langle V_2(x), W(x)\rangle & = & W^2(x)+(\partial_{x_2}h)_x\ W^3(x)\end{array}$$

and

$$\begin{aligned}&\left[g^{1,1}(x)\ \langle V_1(x), W(x)\rangle + g^{1,2}(x)\ \langle V_2(x), W(x)\rangle\right]\ V_1(x)\\ &= \frac{1}{1+(\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2}\ \left\{\left[1+(\partial_{x_2}h)_x^2\right]\left[W^1(x)+(\partial_{x_1}h)_x\ W^3(x)\right]\right.\\ &\qquad\left. -(\partial_{x_1}h)_x(\partial_{x_2}h)_x\left[W^2(x)+(\partial_{x_2}h)_x\ W^3(x)\right]\right\}\ \begin{pmatrix}1\\0\\(\partial_{x_1}h)_x\end{pmatrix}\\ &= \frac{1}{1+(\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2}\ \left\{\left[1+(\partial_{x_2}h)_x^2\right]W^1(x)-(\partial_{x_1}h)_x(\partial_{x_2}h)_x\ W^2(x)\right.\\ &\qquad\left. +(\partial_{x_1}h)_x\ W^3(x)\right\}\ \begin{pmatrix}1\\0\\(\partial_{x_1}h)_x\end{pmatrix}.\end{aligned}$$

In much the same way, we have

$$\begin{aligned}&\left[g^{2,1}(x)\ \langle V_1(x), W(x)\rangle + g^{2,2}(x)\ \langle V_2(x), W(x)\rangle\right]\ V_2(x)\\ &= \frac{1}{1+(\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2}\ \left\{-(\partial_{x_1}h)_x(\partial_{x_2}h)_x\left[W^1(x)+(\partial_{x_1}h)_x\ W^3(x)\right]\right.\\ &\qquad\left. +\left[1+(\partial_{x_1}h)_x^2\right]\left[W^2(x)+(\partial_{x_2}h)_x\ W^3(x)\right]\right\}\ \begin{pmatrix}0\\1\\(\partial_{x_2}h)_x\end{pmatrix}\\ &= \frac{1}{1+(\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2}\ \left\{-(\partial_{x_1}h)_x(\partial_{x_2}h)_x W^1(x)+\left[1+(\partial_{x_1}h)_x^2\right]\ W^2(x)\right.\\ &\qquad\left. +(\partial_{x_2}h)_x\ W^3(x)\right\}\ \begin{pmatrix}0\\1\\(\partial_{x_2}h)_x\end{pmatrix}.\end{aligned}$$

This implies that $\pi(x)(W(x)) = \begin{pmatrix}\pi(x)(W(x))^1\\ \pi(x)(W(x))^2\\ \pi(x)(W(x))^3\end{pmatrix}$ with

$$\begin{aligned}\pi(x)(W(x))^1 &= \frac{1}{1+(\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2}\\ &\quad\times\left\{\left[1+(\partial_{x_2}h)_x^2\right]W^1(x)-(\partial_{x_1}h)_x(\partial_{x_2}h)_x\ W^2(x)+(\partial_{x_1}h)_x\ W^3(x)\right\}\end{aligned}$$

and

$$\pi(x)(W(x))^2 = \frac{1}{1+(\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2}$$

$$\times\left\{-\left(\partial_{x_1}h\right)_x\left(\partial_{x_2}h\right)_x W^1(x)+\left[1+\left(\partial_{x_1}h\right)_x^2\right]\ W^2(x)+\left(\partial_{x_2}h\right)_x\ W^3(x)\right\},$$

as well as

$$\pi(x)(W(x))^3 = \frac{1}{1+(\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2}$$

$$\times\left[\left\{-\left(\partial_{x_1}h\right)_x\left(\partial_{x_2}h\right)_x^2 W^1(x)+\left[1+\left(\partial_{x_1}h\right)_x^2\right]\left(\partial_{x_2}h\right)_x\ W^2(x)+\left(\partial_{x_2}h\right)_x^2\ W^3(x)\right\}\right.$$

$$\left.+\left\{\left[1+\left(\partial_{x_2}h\right)_x^2\right]\left(\partial_{x_1}h\right)_x W^1(x)-\left(\partial_{x_1}h\right)_x^2\left(\partial_{x_2}h\right)_x\ W^2(x)+\left(\partial_{x_1}h\right)_x^2\ W^3(x)\right\}\right]$$

$$= \frac{1}{1+(\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2}$$

$$\times\left[\left\{\left(\partial_{x_1}h\right)_x W^1(x)+\left(\partial_{x_2}h\right)_x\ W^2(x)+\left(\left(\partial_{x_1}h\right)_x^2+\left(\partial_{x_2}h\right)_x^2\right)\ W^3(x)\right\}\right].$$

We conclude that

$$\begin{aligned}
&\pi(x)\\
&=\begin{pmatrix}\pi_1^1(x) & \pi_2^1(x) & \pi_3^1(x)\\ \pi_1^2(x) & \pi_2^2(x) & \pi_3^2(x)\\ \pi_1^3(x) & \pi_2^3(x) & \pi_3^3(x)\end{pmatrix}\\
&=\frac{1}{1+(\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2}\\
&\qquad\times\begin{pmatrix}\left[1+\left(\partial_{x_2}h\right)_x^2\right] & -\left(\partial_{x_1}h\right)_x\left(\partial_{x_2}h\right)_x & \left(\partial_{x_1}h\right)_x\\ -\left(\partial_{x_1}h\right)_x\left(\partial_{x_2}h\right)_x & \left[1+\left(\partial_{x_1}h\right)_x^2\right] & \left(\partial_{x_2}h\right)_x\\ \left(\partial_{x_1}h\right)_x & \left(\partial_{x_2}h\right)_x & \left(\partial_{x_1}h\right)_x^2+\left(\partial_{x_2}h\right)_x^2\end{pmatrix}.
\end{aligned} \tag{30.46}$$

Finally, the orthogonal projection $\pi_\perp(x)$ of a vector field $W(x)$ is given by

$$\begin{aligned}
\pi_\perp(x)(W(x)) &= g_\perp^{-1}(x)\ \langle V_1^\perp(x), W(x)\rangle\ V_1^\perp(x)\\
&= \frac{1}{1+(\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2}\\
&\qquad\left(\left(\partial_{x_1}h\right)_x W^1(x)+\left(\partial_{x_2}h\right)_x W^2(x)-W^3(x)\right)\begin{pmatrix}\left(\partial_{x_1}h\right)_x\\ \left(\partial_{x_2}h\right)_x\\ -1\end{pmatrix}.
\end{aligned}$$

This implies that

$$\pi_\perp(x) = \frac{1}{1+(\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2}$$

$$\times\begin{pmatrix}\left(\partial_{x_1}h\right)_x^2 & \left(\partial_{x_1}h\right)_x\left(\partial_{x_2}h\right)_x & -\left(\partial_{x_1}h\right)_x\\ \left(\partial_{x_1}h\right)_x\left(\partial_{x_2}h\right)_x & \left(\partial_{x_2}h\right)_x^2 & -\left(\partial_{x_2}h\right)_x\\ -\left(\partial_{x_1}h\right)_x & -\left(\partial_{x_2}h\right)_x & 1\end{pmatrix}.$$

Notice that

$$\pi_{\perp}(x) = \frac{1}{\|(\partial\varphi)_x\|^2}\ (\partial\varphi)_x\,(\partial\varphi)_x^T \quad \text{and} \quad \pi(x) = Id - \pi_{\perp}(x)$$

with the vector $(\partial\varphi)_x$ given in (30.45). This ends the proof of the exercise.

■

Solution to exercise 364:
Combining (19.24) with (30.46) we have

$$\begin{aligned}
&(\nabla F)(x)\\
&:= \pi(x)\,(\partial F)\,(x)\\
&= \frac{1}{1+(\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2}\\
&\quad\times \left(\begin{array}{ccc}
\left[1+(\partial_{x_2}h)_x^2\right] & -(\partial_{x_1}h)_x\,(\partial_{x_2}h)_x & (\partial_{x_1}h)_x\\
-(\partial_{x_1}h)_x\,(\partial_{x_2}h)_x & \left[1+(\partial_{x_1}h)_x^2\right] & (\partial_{x_2}h)_x\\
(\partial_{x_1}h)_x & (\partial_{x_2}h)_x & (\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2
\end{array}\right)\left(\begin{array}{c}
(\partial_{x_1}F)(x)\\
(\partial_{x_2}F)(x)\\
(\partial_{x_3}F)(x)
\end{array}\right).
\end{aligned}$$

Using (19.36) and (30.45), the mean curvature vector $\mathbb{H}(x)$ is given by

$$\mathbb{H}(x) = \sum_{1\leq i\leq 3} \partial_{x_i}\left(\overline{V}_1^{\perp,i}\right)(x)\ \overline{V}_1^{\perp}(x) \tag{30.47}$$

with

$$\overline{V}_1^{\perp}(x) = \left(\begin{array}{c}
\overline{V}_1^{\perp,1}(x)\\
\overline{V}_1^{\perp,2}(x)\\
\overline{V}_1^{\perp,3}(x)
\end{array}\right) = \frac{1}{\sqrt{1+(\partial_{x_1}h)_x^2+(\partial_{x_2}h)_x^2}}\ \left(\begin{array}{c}
(\partial_{x_1}h)_x\\
(\partial_{x_2}h)_x\\
-1
\end{array}\right).$$

Finally, using (19.71) we have

$$(\Delta F)(x) = \mathrm{tr}\left(\pi(x)(\partial^2 F)(x)\right) - \langle \mathbb{H}(x), (\partial F)(x)\rangle$$

with the projection matrix $\pi(x)$ defined in (30.46).
This ends the proof of the exercise.

■

Chapter 20

Solution to exercise 365:

The graph of the function h is defined as the null level set of the function

$$\varphi(x_1,x_2) = h(x_1) - x_2 \;\Rightarrow\; \partial\varphi(x) = \begin{pmatrix} \partial_{x_1} h \\ -1 \end{pmatrix} \perp \begin{pmatrix} 1 \\ \partial_{x_1} h \end{pmatrix}.$$

This implies that

$$T^{\perp}(S) = \mathrm{Vect}\left(V^{\perp}\right) \quad \text{and} \quad T(S) = \mathrm{Vect}\,(V)$$

with the unit vector fields

$$V^{\perp}(x) = \frac{\partial\varphi(x)}{\|\partial\varphi(x)\|} = \frac{1}{\sqrt{1+(\partial_{x_1}h)^2}} \begin{pmatrix} \partial_{x_1} h \\ -1 \end{pmatrix} \quad \text{and} \quad V(x) = \frac{1}{\sqrt{1+(\partial_{x_1}h)^2}} \begin{pmatrix} 1 \\ \partial_{x_1} h \end{pmatrix}.$$

In this situation, the mean curvature vector $\mathbb{H}$ is given by the formula

$$\begin{aligned}
\mathbb{H}(x) &= \left[\partial_{x_1}\left(\overline{V}^{\perp,1}(x)\right) + \overbrace{\partial_{x_2}\left(\overline{V}^{\perp,2}(x)\right)}^{=0}\right] \overline{V}^{\perp}(x) \\
&= \partial_{x_1}\left(\frac{\partial_{x_1}h}{\sqrt{1+(\partial_{x_1}h)^2}}\right) \frac{1}{\sqrt{1+(\partial_{x_1}h)^2}} \begin{pmatrix} \partial_{x_1} h \\ -1 \end{pmatrix}
\end{aligned}$$

Observe that

$$\begin{aligned}
\partial_{x_1}\left(\frac{\partial_{x_1}h}{\sqrt{1+(\partial_{x_1}h)^2}}\right) &= \frac{1}{\sqrt{1+(\partial_{x_1}h)^2}}\,\partial^2_{x_1}h - \frac{(\partial_{x_1}h)^2}{1+(\partial_{x_1}h)^2}\,\frac{\partial^2_{x_1}h}{\sqrt{1+(\partial_{x_1}h)^2}} \\
&= \frac{1}{(\partial_{x_1}h)^2}\,\frac{1}{\sqrt{1+(\partial_{x_1}h)^2}}\,\partial^2_{x_1}h.
\end{aligned}$$

This implies that

$$\mathbb{H}(x) = \frac{\partial^2_{x_1}h}{(\partial_{x_1}h)^2}\,\frac{1}{1+(\partial_{x_1}h)^2} \begin{pmatrix} \partial_{x_1} h \\ -1 \end{pmatrix}.$$

On the other hand, the projection matrix on $T_x(S)$ is defined by

$$\begin{aligned}
\pi(x) &= Id - V^{\perp}(x)V^{\perp}(x)^T \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{1+(\partial_{x_1}h)^2} \begin{pmatrix} (\partial_{x_1}h)^2 & -\partial_{x_1}h \\ -\partial_{x_1}h & 1 \end{pmatrix} \\
&= \frac{1}{1+(\partial_{x_1}h)^2} \begin{pmatrix} 1 & \partial_{x_1}h \\ \partial_{x_1}h & (\partial_{x_1}h)^2 \end{pmatrix}.
\end{aligned}$$

The diffusion equation of the Brownian motion on the manifold S is now defined by the formula (20.7) with the mean curvature vector and the projection matrix defined above. This ends the proof of the exercise.

■

Solution to exercise 366:
The ellipsoid can be interpreted as the hyper surface $S = \psi^{-1}(0)$ defined by the function

$$\varphi(x) := \|x\|_a - 1 \quad \text{with the norm} \quad \|x\|_a^2 := \left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_1}\right)^2 + \left(\frac{x_3}{a_1}\right)^2.$$

We have

$$\partial_{x_i}\varphi(x) = \frac{1}{a_i^2}\,\frac{x_i}{\|x\|_a} \Rightarrow \partial\varphi(x) = \frac{1}{\|x\|_a}\begin{bmatrix} x_1/a_1^2 \\ x_2/a_2^2 \\ x_3/a_3^2 \end{bmatrix}.$$

In this situation, the orthogonal space $T_x^\perp(S)$ is the one dimensional space spanned by the unit normal vector

$$\overline{V}^\perp(x) := \frac{\partial\varphi(x)}{\|\partial\varphi(x)\|} = \frac{1}{\sqrt{\sum_{1\leq i\leq 3}\ (x_i/a_i^2)^2}}\begin{bmatrix} x_1/a_1^2 \\ x_2/a_2^2 \\ x_3/a_3^2 \end{bmatrix}.$$

In addition the mean curvature vector $\mathbb{H}$ is defined by the formula

$$\mathbb{H}(x) = \left[\sum_{1\leq i\leq 3} \partial_{x_i}\left(\overline{V}^{\perp,i}(x)\right)\right]\ \overline{V}^\perp(x).$$

After some elementary computations we find that

$$\begin{aligned}
\partial_{x_i}\left(\overline{V}^{\perp,i}(x)\right) &= \frac{a_i^{-2}}{\sqrt{\sum_{1\leq i\leq 3}\ (x_i/a_i^2)^2}}\left(1 - \frac{(x_i/a_i^2)^2}{\sum_{1\leq i\leq 3}\ (x_i/a_i^2)^2}\right) \\
\Longrightarrow \mathbb{H}(x) &= \sum_{1\leq i\leq 3} a_i^{-2}\left(1 - \frac{(x_i/a_i^2)^2}{\sum_{1\leq i\leq 3}\ (x_i/a_i^2)^2}\right)\ \frac{1}{\sqrt{\sum_{1\leq i\leq 3}\ (x_i/a_i^2)^2}}\ \overline{V}^\perp(x) \\
&= \sum_{1\leq i\leq 3} a_i^{-2}\left(1 - \frac{(x_i/a_i^2)^2}{\sum_{1\leq i\leq 3}\ (x_i/a_i^2)^2}\right)\ \frac{1}{\sum_{1\leq i\leq 3}\ (x_i/a_i^2)^2}\begin{bmatrix} x_1/a_1^2 \\ x_2/a_2^2 \\ x_3/a_3^2 \end{bmatrix}.
\end{aligned}$$

The projection matrix on $T_x(S)$ is also defined by the matrix

$$\pi(x) = Id - \overline{V}^\perp(x)\overline{V}^\perp(x)^T.$$

The Brownian motion on the ellipsoid is now defined by the formula (20.7) with the mean curvature vector and the projection matrix defined above. This ends the proof of the exercise.

■

Solution to exercise 367:
The Brownian motion on the manifold $S = \varphi^{-1}(0)$ is given by (20.10) with the mean curvature vector $\mathbb{H}(x)$ defined in (30.47) and the projection matrix defined in (30.46). This ends the proof of the exercise.

■

Solution to exercise 368:
By (20.10), the Stratonovitch formulation of

$$dX_t = b(X_t)\ dt + \sigma(X_t)\ dW_t$$

is given by

$$\partial X_t = \left[b(X_t) - \frac{1}{2}\sigma(X_t)\sigma'(X_t)\right]\ \partial t + \sigma(X_t)\ \partial B_t.$$

This ends the proof of the exercise.

■

Solution to exercise 369:
By (20.10), the Stratonovitch formulation of

$$dX_t = a\ X_t\ dt + b\ X_t\ dW_t$$

is given by

$$\partial X_t = \left(a - \frac{b^2}{2}\right)\ X_t\ \partial t +\ b\ X_t\ \partial W_t.$$

Since the Stratonovitch calculus follows the standard rules of differential calculus, we have

$$\partial \log X_t = \frac{1}{X_t}\ \partial X_t = \left(a - \frac{b^2}{2}\right)\ \partial t +\ b\ \partial W_t$$

from which we conclude that

$$\log\left(X_t/X_0\right) = \int_0^t\ \frac{1}{X_s}\ \partial X_s =\ \left(a - \frac{b^2}{2}\right) t + b\ W_t.$$

This ends the proof of the exercise.

■

Solution to exercise 370:
We have

$$dX_t = aX_t + bX_t\ dW_t\ \Leftrightarrow\ \partial X_t = \left(a - \frac{b^2}{2}\right)\ X_t\ \partial t +\ b\ X_t\ \partial W_t.$$

Replacing a by $a + \frac{b^2}{2}$ we find that

$$dX_t = \left(a + \frac{b^2}{2}\right)\ X_t + bX_t\ dW_t \Leftrightarrow\ \partial X_t = a\ X_t\ \partial t +\ b\ X_t\ \partial W_t.$$

Arguing as in exercise 369 we have

$$\log\left(X_t/X_0\right) = \int_0^t\ \frac{1}{X_s}\ \partial X_s =\ a\ t + b\ W_t.$$

This ends the proof of the exercise.

■

Solution to exercise 371:
Using (20.11), the projection of Y_t into S is given by the diffusion

$$dX_t = \pi(X_t)\sigma(X_t)\ dB_t - \frac{1}{2}\ \mathbb{H}_\sigma(X_t)\ dt$$

with the curvature vector defined in (20.12), and the projection matrix defined in (30.46).
Using (20.13), we have

$$dF(X_t) = L(F)(X_t)\ dt + dM_t(F)$$

with the infinitesimal generator

$$L(F) = \frac{1}{2}\ \mathrm{tr}\left(\sigma^T\nabla\ \sigma^T\nabla F\right)$$

and the martingale

$$dM_t(F) = \langle \nabla F(X_t), \sigma(X_t)dB_t\rangle\,.$$

In the above displayed formula $(\nabla F)(x) = \pi(x)(\partial F)(x)$, with the gradient $(\partial F)(x)$ of F evaluated at x, and the projection matrix defined in (30.46).
This ends the proof of the exercise.

■

Solution to exercise 372:
We follow the developments outlined in the end of section 20.1. In this situation, the unit sphere $\mathbb{S}^p \subset \mathbb{R}^{p+1}$ is defined by the equation $\|x\| = 1$ we have

$$\varphi(x) = \|x\| - 1 \Rightarrow \partial\varphi(x) = x/\|x\| \quad \text{and} \quad \pi(x) = Id - \partial\varphi(x)\partial\varphi(x)^T = Id - \frac{xx^T}{x^Tx}.$$

In addition, using (20.4) the mean curvature vector is defined for any $x \neq 0$ by $\mathbb{H}(x) = p\ \frac{x}{x^Tx}$.
This yields the diffusion equation:

$$dX_t = -\frac{1}{2}\ \mathbb{H}(X_t)\ dt + \pi(X_t)\ dB_t = -\frac{p}{2}\ \frac{X_t}{X_t^TX_t}\ dt + \left(Id - \frac{X_tX_t^T}{X_t^TX_t}\right)\ dB_t. \tag{30.48}$$

This ends the proof of the exercise.

■

Solution to exercise 373:
We have from (30.48)

$$dX_t = -\frac{1}{2}\ \mathbb{H}(X_t)\ dt + \pi(X_t)\ dB_t = -\frac{X_t}{X_t^TX_t}\ dt + \left(Id - \frac{X_tX_t^T}{X_t^TX_t}\right)\ dB_t.$$

Using (20.7), this equation can be rewritten as follows

$$\left\{\begin{array}{rcl} dX_t^k &=& \sum_{1\leq j\leq 3}\left[\frac{1}{2}\ \partial_{\pi_j}(\pi_j^k)(X_t)\ dt + \pi_j^k(X_t)\ dB_t^j\right] \\ k &=& 1,2,3. \end{array}\right.$$

Using the rule (20.10) we conclude that

$$\partial X_t = \pi(X_t)\ \partial B_t.$$

This ends the proof of the exercise.

■

Solution to exercise 374:

Following the arguments provided in section 20.2.3, a discrete time numerical simulation is given by

$$X^{\epsilon}_{t_n} = \mathrm{proj}_{\mathbb{S}^2}\left(X^{\epsilon}_{t_{n-1}} - \frac{X_{t_{n-1}}}{X^T_{t_{n-1}}X_{t_{n-1}}}\,\epsilon + \left(Id - \frac{X_{t_{n-1}}X^T_{t_{n-1}}}{X^T_t X_t}\right)\sqrt{\epsilon}\,\overline{B}_n\right),$$

where $\overline{B}_n$ stands for a sequence of i.i.d. centered and normalized Gaussian r.v. on $\mathbb{R}^3$. Recall that the projection on the sphere is given by $\mathrm{proj}_{\mathbb{S}^2}(x_1,x_2,x_3) = \frac{1}{\sqrt{x_1^2+x_2^2+x_3^3}}(x_1,x_2,x_3)$.

This ends the proof of the exercise.

■

Chapter 21

Solution to exercise 375:

Recalling that

$$\Gamma_{\Delta_g}(f_1, f_2) = 2\,\langle \nabla_g f_1, \nabla_g f_2 \rangle_g$$

and using the differential rule

$$\nabla_g(f_2 f_3) \;=\; f_2\ \nabla_g f_3 \;+\; f_3\ \nabla_g f_2$$

we find that

$$\begin{array}{rcl}\Gamma_{\Delta_g}(f_1, f_2 f_3) & = & 2\,\langle \nabla_g f_1, \nabla_g (f_2 f_3) \rangle_g \\ & = & 2\ f_2\ \langle \nabla_g f_1, \nabla_g f_3 \rangle_g + 2\ f_3\ \langle \nabla_g f_1, \nabla_g f_2 \rangle_g \\ & = & f_2\ \Gamma_{\Delta_g}\,(f_1, f_3) + f_3 \Gamma_{\Delta_g}\,(f_1, f_2)\,.\end{array}$$

This ends the proof of the exercise.

∎

Solution to exercise 376:

Using the derivation formula discussed in exercise 375, the proof is the same as the algebraic proof given in exercise 357.

This ends the proof of the exercise.

∎

Solution to exercise 377:

- The elliptic paraboloid $\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = \frac{x_3}{c}$ can be parametrized for any $\theta = (\theta_1, \theta_2)$ by

$$\psi(\theta) = \left(\theta_1, \theta_2, c\ \left[\left(\frac{\theta_1}{a}\right)^2 + \left(\frac{\theta_2}{b}\right)^2\right]\right). \tag{30.49}$$

  Notice that

$$\phi = \psi^{-1} \Rightarrow \forall x \in S \quad \phi(x) = (\phi^1(x), \phi^2(x)) = (x_1, x_2)\,.$$

  The tangent plane at $x = \psi(\theta)$ is spanned by the vectors

$$(\partial_{\theta_1}\psi)_{\phi(x)} = \left(\begin{array}{c} 1 \\ 0 \\ \frac{2c}{a^2}\ x_1 \end{array}\right) \quad \text{and} \quad (\partial_{\theta_2}\psi)_{\phi(x)} = \left(\begin{array}{c} 0 \\ 1 \\ \frac{2c}{b^2}\ x_2 \end{array}\right).$$

- The hyperbolic paraboloid $\left(\frac{x_2}{a}\right)^2 - \left(\frac{x_1}{b}\right)^2 = \frac{x_3}{c}$ can be parametrized by

$$\psi(\theta_1, \theta_2) = \left(\theta_1, \theta_2, c\ \left[\left(\frac{\theta_2}{a}\right)^2 - \left(\frac{\theta_1}{b}\right)^2\right]\right). \tag{30.50}$$

Arguing as above, the tangent plane at $x = \psi(\theta)$ is spanned by the vectors

$$(\partial_{\theta_1}\psi)_{\phi(x)} = \left(\begin{array}{c} 1 \\ 0 \\ -\frac{2c}{b^2}\ x_1 \end{array}\right) \quad \text{and} \quad (\partial_{\theta_2}\psi)_{\phi(x)} = \left(\begin{array}{c} .0 \\ 1 \\ \frac{2c}{a^2}\ x_2 \end{array}\right).$$

- The sphere $x_1^2 + x_2^2 + x_3^2 = r^2$ can be parametrized by the spherical coordinates

$$\psi(\theta_1, \theta_2) = (r\ \sin(\theta_1)\cos(\theta_2), r\ \sin(\theta_1)\sin(\theta_2), r\ \cos(\theta_1)), \tag{30.51}$$

with the restrictions $S_\psi = \{(\theta_1, \theta_2)\ :\ \theta_1 \in [0, \pi],\ \theta_2 \in [0, 2\pi]\}$. Arguing as above, we notice that $\varphi^{-1}(x) := \phi(x) = (\phi_1(x), \phi_2(x))$ with

$$(r\ \sin(\theta_1)\cos(\theta_2), r\ \sin(\theta_1)\sin(\theta_2), r\ \cos(\theta_1)) = (x_1, x_2, x_3)$$

$$\Rightarrow \phi^1(x) = \arctan\left(\frac{\sqrt{x_1^2 + x_2^2}}{x_3}\right) = \theta_1 \quad \text{and} \quad \phi^2(x) = \arctan\left(\frac{x_2}{x_1}\right) = \theta_2.$$

Arguing as above, the tangent plane at $x = \psi(\theta)$ is spanned by the vectors

$$(\partial_{\theta_1}\psi)_{\phi(x)} = \left(\begin{array}{c} r\ \cos(\phi^1(x))\cos(\phi^2(x)) \\ r\ \cos(\phi^1(x))\sin(\phi^2(x)) \\ -r\ \sin(\phi^1(x)) \end{array}\right)$$

and

$$(\partial_{\theta_2}\psi)_{\phi(x)} = \left(\begin{array}{c} -r\ \sin(\phi^1(x))\sin(\phi^2(x)) \\ r\ \sin(\phi^1(x))\cos(\phi^2(x)) \\ 0 \end{array}\right).$$

- The cylinder $x_1^2 + x_2^2 = r^2$ and $x_3 \in \mathbb{R}$ can be parametrized by the coordinates

$$\psi(\theta_1, \theta_2) = (r\ \sin(\theta_1), r\ \cos(\theta_1), \theta_2), \tag{30.52}$$

with the restrictions $S_\psi = \{(\theta_1, \theta_2)\ :\ \theta_1 \in [0, 2\pi],\ \theta_2 \in \mathbb{R}\}$. Arguing as above, we notice that $\varphi^{-1}(x) := \phi(x) = (\phi_1(x), \phi_2(x))$ with

$$(r\sin(\theta_1), r\cos(\theta_1), \theta_2) = (x_1, x_2, x_3)$$

$$\Rightarrow \phi^1(x) = \arctan\left(\frac{x_1}{x_2}\right) = \theta_1 \quad \text{and} \quad \phi^2(x) = x_3 = \theta_2.$$

Arguing as above, the tangent plane at $x = \psi(\theta)$ is spanned by the vectors

$$(\partial_{\theta_1}\psi)_{\phi(x)} = \left(\begin{array}{c} r\cos(\phi^1(x)) \\ -r\sin(\phi^1(x)) \\ 0 \end{array}\right) \quad \text{and} \quad (\partial_{\theta_2}\psi)_{\phi(x)} = \left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right).$$

This ends the proof of the exercise.

■

Solution to exercise 378:

- The elliptic paraboloid $\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = \frac{x_3}{c}$ can be described as the null level set $S = \varphi^{-1}(0)$ of the function φ defined for any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ by

$$\varphi(x) := \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 - \frac{x_3}{c}.$$

  We have

$$\partial\varphi(x) = \begin{pmatrix} 2a^{-2}\ x_1 \\ 2b^{-2}\ x_2 \\ -c^{-1} \end{pmatrix} \quad \text{and} \quad \|\varphi(x)\| = 4a^{-2}\ (x_1/a)^2 + 4b^{-2}\ (x_2/b)^2 + c^{-2}.$$

  The unit normal at x is given by

$$n(x) = \frac{\partial\varphi(x)}{\|\varphi(x)\|} \Rightarrow \pi(x) = Id - n(x)n(x)^T.$$

- The hyperbolic paraboloid $\left(\frac{x_2}{a}\right)^2 - \left(\frac{x_1}{b}\right)^2 = \frac{x_3}{c}$ can be described as the null level set $S = \varphi^{-1}(0)$ of the function φ defined for any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ by

$$\varphi(x) := \left(\frac{x_2}{a}\right)^2 - \left(\frac{x_1}{b}\right)^2 - \frac{x_3}{c}.$$

  In this situation, we have

$$\partial\varphi(x) = \begin{pmatrix} -2b^{-2}\ x_1 \\ 2a^{-2}\ x_2 \\ -c^{-1} \end{pmatrix} \quad \text{and} \quad \|\varphi(x)\| = 4b^{-2}\ (x_1/b)^2 + 4a^{-2}\ (x_2/a)^2 + c^{-2}.$$

- The sphere $x_1^2 + x_2^2 + x_3^3 = r^2$ can be described as the null level set $S = \varphi^{-1}(0)$ of the function φ defined for any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ by

$$\varphi(x) := x_1^2 + x_2^2 + x_3^3 - r^2.$$

  In this situation, we have

$$\partial\varphi(x) = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad \|\varphi(x)\| = 4 \Rightarrow n(x) = x \Rightarrow \pi(x) = Id - xx^T.$$

- The sphere $x_1^2 + x_2^2 = r^2$ and $x_3 \in \mathbb{R}$ can be described as the null level set $S = \varphi^{-1}(0)$ of the function φ defined for any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ by

$$\varphi(x) := x_1^2 + x_2^2 - r^2.$$

  In this situation, we have

$$\partial\varphi(x) = 2 \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \quad \text{and} \quad \|\varphi(x)\| = 4\ (x_1^2 + x_2^2)$$

$$\Rightarrow n(x) = \frac{1}{\sqrt{x_1^2+x_2^2}} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \quad \text{and} \quad \pi(x) = Id - n(x)n(x)^T.$$

This ends the proof of the exercise.

■

Solution to exercise 379:

- The Riemannian scalar product g on the tangent spaces $T(S_\psi)$ associated with the parametrization ψ of the elliptic paraboloid defined in (30.49) is given by the matrices

$$\begin{aligned} g_{1,1}(\theta) &:= \langle \partial_{\theta_1}\psi(\theta), \partial_{\theta_1}\psi(\theta)\rangle = 1 + \left(\frac{2c}{a^2}\right)^2 \theta_1^2, \\ g_{2,2}(\theta) &= 1 + \left(\frac{2c}{b^2}\right)^2 \theta_1^2 \end{aligned}$$

and

$$\begin{aligned} g_{1,2}(\theta) &:= \langle \partial_{\theta_1}\psi(\theta), \partial_{\theta_2}\psi(\theta)\rangle \\ &= \left\langle \left(\begin{array}{c} 1 \\ 0 \\ \frac{2c}{a^2}\,\theta_i \end{array}\right), \left(\begin{array}{c} 0 \\ 1 \\ \frac{2c}{b^2}\,\theta_2 \end{array}\right)\right\rangle = \left(\frac{2c}{ab}\right)^2 \theta_1\theta_2. \end{aligned}$$

- For the hyperbolic paraboloid parametrization (30.50), we find that

$$\begin{aligned} g_{1,1}(\theta) &= 1 + \left(\frac{2c}{b^2}\right)^2 \theta_1^2, \\ g_{2,2}(\theta) &= 1 + \left(\frac{2c}{a^2}\right)^2 \theta_2^2 \quad \text{and} \quad g_{1,2}(\theta) = \left(\frac{2c}{ab}\right)^2 \theta_1\theta_2. \end{aligned}$$

- For the spherical parametrization (30.51), we find that

$$\begin{aligned} g_{1,1}(\theta) &:= \langle \partial_{\theta_1}\psi(\theta), \partial_{\theta_1}\psi(\theta)\rangle = r^2 \\ g_{2,2}(\theta) &= r^2 \sin^2(\theta_1) \end{aligned}$$

and $g_{1,2}(\theta) = 0$.

- For the cylindrical parametrization (30.52), we find that

$$\begin{aligned} g_{1,1}(\theta) &:= r^2, \\ g_{2,2}(\theta) &= 1 \quad \text{and} \quad g_{1,2}(\theta) = 0. \end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 380:

We recall (cf. for instance (21.9)) that the orthogonal projections on $T(S)$ are defined by the inverses $g^{-1} = (g^{i,j})_{1\leq i,j\leq 2}$ of the matrices $g = (g_{i,j})_{1\leq i,j\leq 2}$ associated with the Riemannian scalar product derived in exercise 379. The matrices g^{-1} can be computed using the formula

$$g = \left(\begin{array}{cc} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{array}\right) \Longrightarrow g^{-1} = \left(\begin{array}{cc} g^{1,1} & g^{1,2} \\ g^{2,1} & g^{2,2} \end{array}\right) = \frac{1}{g_{1,1}g_{2,2} - g_{1,2}g_{2,1}} \left(\begin{array}{cc} g_{2,2} & -g_{1,2} \\ -g_{2,1} & g_{1,1} \end{array}\right).$$

- For the elliptic paraboloid (30.49) we have

$$g = \begin{pmatrix} 1 + \left(\frac{2c}{a^2}\right)^2 \ \theta_1^2 & \left(\frac{2c}{ab}\right)^2 \ \theta_1\theta_2 \\ \left(\frac{2c}{ab}\right)^2 \ \theta_1\theta_2 & 1 + \left(\frac{2c}{b^2}\right)^2 \ \theta_1^2 \end{pmatrix}$$

  so that

$$\begin{aligned} g_{1,1}g_{2,2} - g_{1,2}g_{2,1} &= \left(1 + \left(\frac{2c}{a^2}\right)^2 \ \theta_1^2\right)\left(1 + \left(\frac{2c}{b^2}\right)^2 \ \theta_1^2\right) - \left(\frac{2c}{ab}\right)^4 \ \theta_1^2\theta_2^2 \\ &= 1 + \left(\frac{2c}{a^2}\right)^2 \ \theta_1^2 + \left(\frac{2c}{b^2}\right)^2 \ \theta_1^2. \end{aligned}$$

  This implies that

$$g^{-1} = \frac{1}{1 + \left(\frac{2c}{a^2}\right)^2 \ \theta_1^2 + \left(\frac{2c}{b^2}\right)^2 \ \theta_1^2} \begin{pmatrix} 1 + \left(\frac{2c}{b^2}\right)^2 \ \theta_1^2 & -\left(\frac{2c}{ab}\right)^2 \ \theta_1\theta_2 \\ -\left(\frac{2c}{ab}\right)^2 \ \theta_1\theta_2 & 1 + \left(\frac{2c}{a^2}\right)^2 \ \theta_1^2 \end{pmatrix}.$$

- The matrices g^{-1} for the hyperbolic paraboloid parametrization (30.50) are defined as the ones of the elliptic paraboloid (30.49) replacing (a, b) by (b, a).

- For the sphere (30.51) we have

$$g = r^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta_1) \end{pmatrix} \Longrightarrow g^{-1} = \frac{1}{r^2 \ \sin^2(\theta_1)} \begin{pmatrix} \sin^2(\theta_1) & 0 \\ 0 & 1 \end{pmatrix}$$

  up to the angles $\theta_1 \in \{0, \pi\}$.

- For the cylinder parametrization (30.52) we have

$$g = \begin{pmatrix} r^2 & 0 \\ 0 & 1 \end{pmatrix} \Longrightarrow g^{-1} = \begin{pmatrix} r^{-2} & 0 \\ 0 & 1 \end{pmatrix}.$$

  This ends the proof of the exercise.

■

Solution to exercise 381: Using the formula (21.21) the Riemannian gradient $\nabla_g f = g^{-1}\partial f$ is easily computed using the inverse matrix formula derived in exercise 380. For instance on the sphere (30.51) we have

$$\frac{1}{r^2} \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-2}(\theta_1) \end{pmatrix} \begin{pmatrix} \partial_{\theta_1} f \\ \partial_{\theta_2} f \end{pmatrix} = \frac{1}{r^2} \begin{pmatrix} \partial_{\theta_1} f \\ \sin^{-2}(\theta_1) \ \partial_{\theta_2} f \end{pmatrix}.$$

Solution to exercise 382:
Following the detailed calculation on the 2-sphere $\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1^2 + x_2^2 + x_3^2 = 1\}$ presented in section 24.1.2, the Christoffel symbols $C_{i,j}^k$ are given by

$$\begin{aligned} C_{1,1}^1 &= 0 = C_{1,2}^1 = C_{2,1}^1 = C_{1,1}^2 = C_{2,2}^2 \\ C_{2,2}^1(\theta) &= -\sin(\theta_1)\cos(\theta_1) \quad \text{and} \quad C_{1,2}^2(\theta) = C_{2,1}^2(\theta) = \frac{\cos(\theta_1)}{\sin(\theta_1)}. \end{aligned}$$

By (21.50), we have

$$\nabla_g^2(f) = g^{-1}\mathrm{Hess}_g(f) \quad \text{with} \quad (\mathrm{Hess}_g(f))_{m,m'} = \partial_{\theta_m,\theta_{m'}} f - \sum_{1 \le j \le p} C_{m,m'}^j \ \partial_{\theta_j} f.$$

In this situation, we have

$$\begin{array}{rcl}(\mathrm{Hess}_g(f))_{1,1} & = & \partial_{\theta_1,\theta_1}(f)\\ (\mathrm{Hess}_g(f))_{2,2} & = & \partial_{\theta_2,\theta_2} f + \sin(\theta_1)\cos(\theta_1)\ \partial_{\theta_1} f\\ (\mathrm{Hess}_g(f))_{1,2} & = & \partial_{\theta_1,\theta_2} f - \dfrac{\cos(\theta_1)}{\sin(\theta_1)}\ \partial_{\theta_2} f.\end{array}$$

This implies that

$$\begin{array}{rcl}\nabla^2_g(f) & = & \left(\begin{array}{cc}1 & 0\\ 0 & \sin^{-2}(\theta_1)\end{array}\right)\left(\begin{array}{cc}\partial^2_{\theta_1}(f) & \partial_{\theta_1,\theta_2} f - \cot(\theta_1)\ \partial_{\theta_2} f\\ \partial_{\theta_1,\theta_2} f - \cot(\theta_1)\ \partial_{\theta_2} f & \partial_{\theta_2,\theta_2} f + \sin(\theta_1)\cos(\theta_1)\ \partial_{\theta_1} f\end{array}\right)\\ & = & \left(\begin{array}{cc}\partial^2_{\theta_1}(f) & \partial_{\theta_1,\theta_2} f - \cot(\theta_1)\ \partial_{\theta_2} f\\ \sin^{-2}(\theta_1)\,(\partial_{\theta_1,\theta_2} f - \cot(\theta_1)\ \partial_{\theta_2} f) & \sin^{-2}(\theta_1)\ \partial^2_{\theta_2} f + \cot(\theta_1)\ \partial_{\theta_1} f\end{array}\right).\end{array}$$

Hence we can substitute to obtain

$$\begin{array}{rcl}\Delta_g(f) & = & \mathrm{tr}\left(\nabla^2_g(f)\right)\\ & = & \cot(\theta_1)\ \partial_{\theta_1} f + \partial^2_{\theta_1} f + \dfrac{1}{\sin^2(\theta_1)}\ \partial^2_{\theta_2} f\\ & = & \dfrac{1}{\sin(\theta_1)}\ \partial_{\theta_1}(\sin(\theta_1)\ \partial_{\theta_1} f) + \dfrac{1}{\sin^2(\theta_1)}\ \partial^2_{\theta_2} f.\end{array}$$

This ends the proof of the exercise.

■

Solution to exercise 383:
Each $(0,z)$-section of the cone S defined by $a\ z = \sqrt{x^2+y^2}$ is a circle of radius $a\ z$, for some $a>0$. The natural polar parametrization is given by the function

$$\psi\ :\ \theta = (\theta_1,\theta_2) \in [0,\infty[\times[0,2\pi] \mapsto \psi(\theta_1,\theta_2) = \left(\begin{array}{c}a\ \theta_1\cos(\theta_2)\\ a\ \theta_1\sin(\theta_2)\\ \theta_1\end{array}\right) \in S.$$

The tangent plane $T_{\psi(\theta)}(S)$ is spanned by the vectors

$$\partial_{\theta_1}\psi = \left(\begin{array}{c}a\cos(\theta_2)\\ a\sin(\theta_2)\\ 1\end{array}\right) \quad\perp\quad \partial_{\theta_2}\psi = a\ \theta_1\left(\begin{array}{c}-\sin(\theta_2)\\ \cos(\theta_2)\\ 0\end{array}\right).$$

The Riemannian metric is given by the diagonal (2×2)-matrix

$$g = \left(\begin{array}{cc}1+a^2 & 0\\ 0 & a^2\theta_1^2\end{array}\right) \Rightarrow \forall\theta_1>0 \quad g^{-1} = \left(\begin{array}{cc}(1+a^2)^{-1} & 0\\ 0 & a^{-2}\theta_1^{-2}\end{array}\right).$$

The Riemannian gradient on the cone is given by the formula

$$\nabla_g f \quad = \quad (1+a^2)^{-1}\ \partial_{\theta_1} f\ +\ a^{-2}\theta_1^{-2}\ \partial_{\theta_2} f.$$

To compute the second derivative, we observe that

$$\partial^2_{\theta_1}\psi = \left(\begin{array}{c}0\\ 0\\ 0\end{array}\right) \qquad \partial^2_{\theta_2}\psi = -a\ \theta_1\left(\begin{array}{c}\cos(\theta_2)\\ \sin(\theta_2)\\ 0\end{array}\right) \quad\text{and}\quad \partial_{\theta_1,\theta_2}\psi = a\left(\begin{array}{c}-\sin(\theta_2)\\ \cos(\theta_2)\\ 0\end{array}\right).$$

We clearly have

$$\partial^2_{\theta_2}\psi \perp \partial_{\theta_2}\psi \quad \text{and} \quad \partial_{\theta_1,\theta_2}\psi \perp \partial_{\theta_1}\psi$$

as well as

$$\left\langle \partial^2_{\theta_2}\psi, \partial_{\theta_1}\psi \right\rangle = -a^2\theta_1 \quad \text{and} \quad \langle \partial_{\theta_1,\theta_2}\psi, \partial_{\theta_2}\psi \rangle = a^2\theta_1.$$

This shows that the Christoffel symbols are all null except for

$$C^2_{1,2} = g^{2,2}\ \langle \partial_{\theta_2}\psi, \partial_{\theta_1,\theta_2}\psi \rangle = \frac{1}{\theta_1}$$

and

$$C^1_{2,2} = g^{1,1}\ \left\langle \partial_{\theta_1}\psi, \partial^2_{\theta_2}\psi \right\rangle = -\frac{a^2\theta_1}{1+a^2}.$$

By (21.50), the Hessian can be computed in terms of the Christoffel symbols with the formula

$$\nabla^2_g(f) = g^{-1}\mathrm{Hess}_g(f) \quad \text{with} \quad (\mathrm{Hess}_g(f))_{m,m'} = \partial_{\theta_m,\theta_{m'}} f - \sum_{1\leq j\leq p} C^j_{m,m'}\ \partial_{\theta_j} f.$$

In this situation, we have

$$\begin{array}{rcl}
(\mathrm{Hess}_g(f))_{1,1} & = & \partial^2_{\theta_1} f, \\
(\mathrm{Hess}_g(f))_{2,2} & = & \partial^2_{\theta_2} f + \dfrac{a^2\theta_1}{1+a^2}\ \partial_{\theta_1} f \quad \text{and} \quad (\mathrm{Hess}_g(f))_{1,2} = \partial_{\theta_1,\theta_2} f - \dfrac{1}{\theta_1}\ \partial_{\theta_2} f
\end{array}$$

and

$$\nabla^2_g f = \left(\begin{array}{cc} (1+a^2)^{-1}\partial^2_{\theta_1} f & (1+a^2)^{-1}\left[\partial_{\theta_1,\theta_2} f - \frac{1}{\theta_1}\ \partial_{\theta_2} f\right] \\ a^{-2}\theta_1^{-2}\left[\partial_{\theta_1,\theta_2} f - \frac{1}{\theta_1}\ \partial_{\theta_2} f\right] & a^{-2}\theta_1^{-2}\left[\partial^2_{\theta_2} f + \frac{a^2\theta_1}{1+a^2}\ \partial_{\theta_1} f\right] \end{array} \right).$$

This implies that

$$\Delta_g(f) = \mathrm{tr}\left(\nabla^2_g(f)\right) = (1+a^2)^{-1}\partial^2_{\theta_1} f + a^{-2}\theta_1^{-2}\partial^2_{\theta_2} f + (\theta_1(1+a^2))^{-1}\ \partial_{\theta_1} f.$$

This ends the proof of the exercise.

■

Solution to exercise 384:

The spherical parametrization of the ellipsoid S is defined by

$$\psi(\theta_1,\theta_2) = \left(\begin{array}{c} a_1\ \sin(\theta_1)\cos(\theta_2) \\ a_2\ \sin(\theta_1)\sin(\theta_2) \\ a_3\ \cos(\theta_1) \end{array} \right)$$

with the restrictions $S_\psi = \{(\theta_1,\theta_2)\ :\ \theta_1 \in [0,\pi],\ \theta_2 \in [0,2\pi]\}$. The tangent plane $T_{\psi(\theta)}(S)$ is spanned by the vectors

$$\partial_{\theta_1}\psi = \left(\begin{array}{c} a_1\ \cos(\theta_1)\cos(\theta_2) \\ a_2\ \cos(\theta_1)\sin(\theta_2) \\ -a_3\ \sin(\theta_1) \end{array} \right) \quad \perp \quad \partial_{\theta_2}\psi = \left(\begin{array}{c} -a_1\ \sin(\theta_1)\sin(\theta_2) \\ a_2\ \sin(\theta_1)\cos(\theta_2) \\ 0 \end{array} \right).$$

We have

$$\begin{array}{rcl}
\langle \partial_{\theta_1}\psi, \partial_{\theta_1}\psi \rangle & = & a_1^2\ \cos^2(\theta_1)\cos^2(\theta_2) + a_2^2\ \cos^2(\theta_1)\left(1-\cos^2(\theta_2)\right) + a_3^2\ \left(1-\cos^2(\theta_1)\right) \\
& = & \left[\left(a_1^2 - a_2^2\right)\ \cos^2(\theta_2) + \left(a_2^2 - a_3^2\right)\right]\cos^2(\theta_1) + a_3^2 \\
\langle \partial_{\theta_2}\psi, \partial_{\theta_2}\psi \rangle & = & a_1^2\ \sin^2(\theta_1)\sin^2(\theta_2) + a_2^2\ \sin^2(\theta_1)\left(1-\sin^2(\theta_2)\right) \\
& = & \left[\left(a_1^2 - a_2^2\right)\ \sin^2(\theta_2) + a_2^2\right]\ \sin^2(\theta_1).
\end{array}$$

This implies that

$$g = \begin{pmatrix} \left[\left(a_1^2 - a_2^2\right) \ \cos^2(\theta_2) + \left(a_2^2 - a_3^2\right)\right] \cos^2(\theta_1) + a_3^2 & 0 \\ 0 & \left[\left(a_1^2 - a_2^2\right) \ \sin^2(\theta_2) + a_2^2\right] \ \sin^2(\theta_1) \end{pmatrix}$$

from which we prove that

$$g^{-1} = \begin{pmatrix} \left(\left[\left(a_1^2 - a_2^2\right) \ \cos^2(\theta_2) + \left(a_2^2 - a_3^2\right)\right] \cos^2(\theta_1) + a_3^2\right)^{-1} & 0 \\ 0 & \left(\left[\left(a_1^2 - a_2^2\right) \ \sin^2(\theta_2) + a_2^2\right]\right)^{-1} \ \sin^{-2}(\theta_1) \end{pmatrix}$$

up to the angles $\theta_1 \in \{0, \pi\}$. The Riemannian gradient on the ellipsoid is given by the formula

$$\begin{aligned} \nabla_g f &= \left(\left[\left(a_1^2 - a_2^2\right) \ \cos^2(\theta_2) + \left(a_2^2 - a_3^2\right)\right] \cos^2(\theta_1) + a_3^2\right)^{-1} \ \partial_{\theta_1} f \\ &\quad + \left(\left[\left(a_1^2 - a_2^2\right) \ \sin^2(\theta_2) + a_2^2\right]\right)^{-1} \ \sin^{-2}(\theta_1) \ \partial_{\theta_2} f. \end{aligned}$$

Observe that

$$\partial_{\theta_1,\theta_1} \psi = - \begin{pmatrix} a_1 \ \sin(\theta_1)\cos(\theta_2) \\ a_2 \ \sin(\theta_1)\sin(\theta_2) \\ a_3 \ \cos(\theta_1) \end{pmatrix}.$$

This yields

$$\begin{aligned} \langle \partial_{\theta_1,\theta_1}\psi, \partial_{\theta_1}\psi\rangle &= -\left[(a_1^2 - a_3^2)\cos^2(\theta_2) + (a_2^2 - a_3^2)\sin^2(\theta_2)\right] \ \cos(\theta_1)\sin(\theta_1) \\ &= -\left[(a_1^2 - a_2^2)\cos^2(\theta_2) + (a_2^2 - a_3^2)\right] \ \cos(\theta_1)\sin(\theta_1) \\ &= -\frac{1}{2} \ \left[(a_1^2 - a_2^2)\cos^2(\theta_2) + (a_2^2 - a_3^2)\right] \ \sin(2\theta_1) \end{aligned}$$

and

$$\langle \partial_{\theta_1,\theta_1}\psi, \partial_{\theta_2}\psi\rangle = (a_1^2 \ - a_2^2) \ \sin^2(\theta_1) \ \sin(\theta_2)\cos(\theta_2) = \frac{1}{2} \ (a_1^2 \ - a_2^2) \ \sin^2(\theta_1) \ \sin(2\theta_2).$$

This implies that

$$\begin{aligned} C_{1,1}^1 &= g^{1,1} \ \langle \partial_{\theta_1}\psi, \partial_{\theta_1,\theta_1}\psi\rangle + g^{1,2} \ \langle \partial_{\theta_2}\psi, \partial_{\theta_1,\theta_1}\psi\rangle \\ &= g^{1,1} \ \langle \partial_{\theta_1}\psi, \partial_{\theta_1,\theta_1}\psi\rangle = -\frac{1}{2} \ \frac{\left[(a_1^2 - a_2^2)\cos^2(\theta_2) + (a_2^2 - a_3^2)\right] \ \sin(2\theta_1)}{\left[(a_1^2 - a_2^2) \ \cos^2(\theta_2) + (a_2^2 - a_3^2)\right]\cos^2(\theta_1) + a_3^2} \end{aligned}$$

and

$$\begin{aligned} C_{1,1}^2 &= g^{2,1} \ \langle \partial_{\theta_1}\psi, \partial_{\theta_1,\theta_1}\psi\rangle + g^{2,2} \ \langle \partial_{\theta_2}\psi, \partial_{\theta_1,\theta_1}\psi\rangle \\ &= g^{2,2} \ \langle \partial_{\theta_2}\psi, \partial_{\theta_1,\theta_1}\psi\rangle = \frac{1}{2} \ \frac{(a_1^2 \ - a_2^2) \ \sin(2\theta_2)}{\left[(a_1^2 - a_2^2) \ \sin^2(\theta_2) + a_2^2\right]}. \end{aligned}$$

In much the same way, we have

$$\partial_{\theta_2,\theta_2} \psi = - \begin{pmatrix} a_1 \ \sin(\theta_1)\cos(\theta_2) \\ a_2 \ \sin(\theta_1)\sin(\theta_2) \\ 0 \end{pmatrix}.$$

This yields

$$\begin{aligned} \langle \partial_{\theta_2,\theta_2}\psi, \partial_{\theta_1}\psi\rangle &= -\left((a_1^2 - a_2^2)\cos^2(\theta_2) + a_2^2\right) \ \sin(\theta_1) \ \cos(\theta_1) \\ &= -\frac{1}{2} \ \left((a_1^2 - a_2^2)\cos^2(\theta_2) + a_2^2\right) \ \sin(2\theta_1) \end{aligned}$$

and

$$\langle \partial_{\theta_2,\theta_2}\psi, \partial_{\theta_2}\psi\rangle = \left(a_1^2 - a_2^2\right) \sin^2(\theta_1) \cos(\theta_2)\sin(\theta_2) = \frac{1}{2} \left(a_1^2 - a_2^2\right) \sin^2(\theta_1) \sin(2\theta_2).$$

This yields

$$\begin{aligned} C_{2,2}^1 &= g^{1,1} \langle \partial_{\theta_2,\theta_2}\psi, \partial_{\theta_1}\psi\rangle + g^{1,2} \langle \partial_{\theta_2,\theta_2}\psi, \partial_{\theta_2}\psi\rangle \\ &= g^{1,1} \langle \partial_{\theta_2,\theta_2}\psi, \partial_{\theta_1}\psi\rangle = -\frac{1}{2} \frac{\left((a_1^2-a_2^2)\cos^2(\theta_2)+a_2^2\right) \sin(2\theta_1)}{[(a_1^2-a_2^2) \cos^2(\theta_2)+(a_2^2-a_3^2)]\cos^2(\theta_1)+a_3^2} \end{aligned}$$

and

$$\begin{aligned} C_{2,2}^2 &= g^{2,1} \langle \partial_{\theta_2,\theta_2}\psi, \partial_{\theta_1}\psi\rangle + g^{2,2} \langle \partial_{\theta_2,\theta_2}\psi, \partial_{\theta_2}\psi\rangle \\ &= g^{2,2} \langle \partial_{\theta_2,\theta_2}\psi, \partial_{\theta_2}\psi\rangle = \frac{1}{2} \frac{\left(a_1^2-a_2^2\right) \sin(2\theta_2)}{\left[(a_1^2-a_2^2) \sin^2(\theta_2)+a_2^2\right]}. \end{aligned}$$

Finally, we have

$$\partial_{\theta_1,\theta_2}\psi = \left(\begin{array}{c} -a_1 \cos(\theta_1)\sin(\theta_2) \\ a_2 \cos(\theta_1)\cos(\theta_2) \\ 0 \end{array}\right).$$

This yields

$$\begin{aligned} \langle \partial_{\theta_1,\theta_2}\psi, \partial_{\theta_1}\psi\rangle &= -\left(a_1^2-a_2^2\right) \cos^2(\theta_1) \cos(\theta_2)\sin(\theta_2) \\ &= -\frac{1}{2} \left(a_1^2-a_2^2\right) \cos^2(\theta_1) \sin(2\theta_2) \\ \langle \partial_{\theta_1,\theta_2}\psi, \partial_{\theta_2}\psi\rangle &= \left((a_1^2-a_2^2) \sin^2(\theta_2)+a_2^2\right) \sin(\theta_1)\cos(\theta_1) \\ &= \frac{1}{2}\left((a_1^2-a_2^2) \sin^2(\theta_2)+a_2^2\right) \sin(2\theta_1). \end{aligned}$$

This implies that

$$\begin{aligned} C_{1,2}^1 &= g^{1,1} \langle \partial_{\theta_1,\theta_2}\psi, \partial_{\theta_1}\psi\rangle + g^{1,2} \langle \partial_{\theta_1,\theta_2}\psi, \partial_{\theta_2}\psi\rangle \\ &= g^{1,1} \langle \partial_{\theta_1,\theta_2}\psi, \partial_{\theta_1}\psi\rangle = -\frac{1}{2} \frac{\left(a_1^2-a_2^2\right) \cos^2(\theta_1) \sin(2\theta_2)}{[(a_1^2-a_2^2) \cos^2(\theta_2)+(a_2^2-a_3^2)]\cos^2(\theta_1)+a_3^2} \end{aligned}$$

and

$$\begin{aligned} C_{1,2}^2 &= g^{2,1} \langle \partial_{\theta_1,\theta_2}\psi, \partial_{\theta_1}\psi\rangle + g^{2,2} \langle \partial_{\theta_1,\theta_2}\psi, \partial_{\theta_2}\psi\rangle \\ &= g^{2,2} \langle \partial_{\theta_1,\theta_2}\psi, \partial_{\theta_2}\psi\rangle = \frac{1}{2}\frac{\left((a_1^2-a_2^2) \sin^2(\theta_2)+a_2^2\right) \sin(2\theta_1)}{\left[(a_1^2-a_2^2) \sin^2(\theta_2)+a_2^2\right]\sin^2(\theta_1)}. \end{aligned}$$

By (21.50), the Hessian can be computed in terms of the Christoffel symbols with the formula

$$\nabla_g^2 f = g^{-1}\mathrm{Hess}_g(f) \quad \text{with} \quad (\mathrm{Hess}_g(f))_{m,m'} = \partial_{\theta_m,\theta_{m'}} f - \sum_{1\leq j\leq p} C_{m,m'}^j \partial_{\theta_j} f.$$

This ends the proof of the exercise.

■

Solution to exercise 385:

The tangent plane at $T_{\psi(\theta)}(S)$ is spanned by the vectors

$$\partial_{\theta_1}\psi = \begin{pmatrix} \partial_{\theta_1}u(\theta_1)\ \cos(\theta_2) \\ \partial_{\theta_1}u(\theta_1)\ \sin(\theta_2) \\ 1 \end{pmatrix} \quad \perp \quad \partial_{\theta_2}\psi = u(\theta_1)\begin{pmatrix} -\sin(\theta_2) \\ \cos(\theta_2) \\ 0 \end{pmatrix}.$$

This implies that

$$g = \begin{pmatrix} 1+(\partial_{\theta_1}u(\theta_1))^2 & 0 \\ 0 & u(\theta_1)^2 \end{pmatrix} \quad \text{and} \quad g^{-1} = \begin{pmatrix} \left(1+(\partial_{\theta_1}u(\theta_1))^2\right)^{-1} & 0 \\ 0 & u(\theta_1)^{-2} \end{pmatrix},$$

up to the parameters θ_1 s.t. $u(\theta_1) = 0$. The Riemannian gradient operator on the revolution surface is given by the formula

$$\nabla_g \quad = \quad \left(1+(\partial_{\theta_1}u(\theta_1))^2\right)^{-1}\ \partial_{\theta_1} + u(\theta_1)^{-2}\ \partial_{\theta_2}.$$

The second derivatives are clearly given by the formulae

$$\begin{aligned}
\partial^2_{\theta_1}\psi &= \partial^2_{\theta_1}u \begin{pmatrix} \cos(\theta_2) \\ \sin(\theta_2) \\ 0 \end{pmatrix} \perp\ \partial_{\theta_2}\psi \\
\partial_{\theta_1,\theta_2}\psi &= \partial_{\theta_1}u \begin{pmatrix} -\sin(\theta_2) \\ \cos(\theta_2) \\ 0 \end{pmatrix} = (\partial_{\theta_1}\log u)\quad \partial_{\theta_2}\psi \perp\ \partial_{\theta_1}\psi \\
\partial^2_{\theta_2}\psi &= -u \begin{pmatrix} \cos(\theta_2) \\ \sin(\theta_2) \\ 0 \end{pmatrix} \perp\ \partial_{\theta_2}\psi.
\end{aligned}$$

The Christoffel symbols are given by

$$\begin{aligned}
C^1_{1,1} &= g^{1,1}\ \left\langle\ \partial^2_{\theta_1}\psi,\ \partial_{\theta_1}\psi\right\rangle = \frac{\partial^2_{\theta_1}u\ \partial_{\theta_1}u}{1+(\partial_{\theta_1}u)^2} = \partial_{\theta_1}\log\sqrt{1+(\partial_{\theta_1}u)^2} \\
C^1_{2,2} &= g^{1,1}\ \left\langle\ \partial^2_{\theta_2}\psi,\ \partial_{\theta_1}\psi\right\rangle = -\frac{u\ \partial_{\theta_1}u}{1+(\partial_{\theta_1}u)^2} = -\frac{1}{2}\ \frac{\partial_{\theta_1}(u^2)}{1+(\partial_{\theta_1}u)^2} \\
C^2_{1,2} &= C^2_{2,1} = g^{2,2}\ \langle\ \partial_{\theta_1,\theta_2}\psi,\ \partial_{\theta_2}\psi\rangle = \frac{u\ \partial_{\theta_1}u}{u^2} = \partial_{\theta_1}(\log u)
\end{aligned}$$

and $C^1_{1,2} = C^2_{1,1} = C^2_{2,2} = 0$.

By (21.50), the Hessian can be computed in terms of the Christoffel symbols with the formula

$$\nabla^2_g(f) = g^{-1}\mathrm{Hess}_g(f) \quad \text{with} \quad (\mathrm{Hess}_g(f))_{m,m'} = \partial_{\theta_m,\theta_{m'}}f - \sum_{1\leq j\leq p} C^j_{m,m'}\ \partial_{\theta_j}f.$$

In this situation, we have

$$\begin{aligned}
(\mathrm{Hess}_g(f))_{1,1} &= \partial^2_{\theta_1}f - \partial_{\theta_1}\log\sqrt{1+(\partial_{\theta_1}u)^2}\ \partial_{\theta_1}f \\
(\mathrm{Hess}_g(f))_{2,2} &= \partial^2_{\theta_2}f + \frac{1}{2}\ \frac{\partial_{\theta_1}(u^2)}{1+(\partial_{\theta_1}u)^2}\ \partial_{\theta_1}f \\
(\mathrm{Hess}_g(f))_{1,2} &= \partial^2_{\theta_1,\theta_2}f - \partial_{\theta_1}(\log u)\ \partial_{\theta_2}f = (\mathrm{Hess}_g(f))_{2,1}
\end{aligned}$$

and therefore

$$\nabla_g^2 f = \begin{pmatrix} \left(1+(\partial_{\theta_1}u)^2\right)^{-1}\left(\partial^2_{\theta_1}f-\partial_{\theta_1}\log\sqrt{1+(\partial_{\theta_1}u)^2}\ \partial_{\theta_1}f\right) & \left(1+(\partial_{\theta_1}u)^2\right)^{-1}\left(\partial^2_{\theta_1,\theta_2}f-\partial_{\theta_1}(\log u)\ \partial_{\theta_2}f\right) \\ u^{-2}\left(\partial^2_{\theta_1,\theta_2}f-\partial_{\theta_1}(\log u)\ \partial_{\theta_2}f\right) & u^{-2}\left(\partial^2_{\theta_2}f+\frac{1}{2}\ \frac{\partial_{\theta_1}(u^2)}{1+(\partial_{\theta_1}u)^2}\ \partial_{\theta_1}f\right) \end{pmatrix}.$$

This also implies that

$$\Delta_g = \frac{1}{1+(\partial_{\theta_1}u)^2}\ \partial^2_{\theta_1}+\frac{1}{u^2}\ \partial^2_{\theta_2}+\frac{1}{1+(\partial_{\theta_1}u)^2}\ \partial_{\theta_1}\log\left(u/\sqrt{1+(\partial_{\theta_1}u)^2}\right)\ \partial_{\theta_1}.$$

This ends the proof of the exercise.

■

Solution to exercise 386:

When $u(\theta_1)=c+\cos(\theta_1)(>0)$ with $c>-1$ we have $u'(\theta_1)=-\sin(\theta_1)$ and $u''(\theta_1)=-\cos(\theta_1)$.

When $u(z)=c+\cos z$ with $c>-1$ we have

$$\partial_{\theta_1}\psi = \begin{pmatrix} -\sin(\theta_1)\ \cos(\theta_2) \\ -\sin(\theta_1)\ \sin(\theta_2) \\ 1 \end{pmatrix} \quad \perp \quad \partial_{\theta_2}\psi=(c+\cos(\theta_1))\begin{pmatrix} -\sin(\theta_2) \\ \cos(\theta_2) \\ 0\end{pmatrix}.$$

This implies that

$$g=\begin{pmatrix} 1+\sin^2(\theta_1) & 0 \\ 0 & (c+\cos(\theta_1))^2 \end{pmatrix} \quad \text{and} \quad g^{-1}=\begin{pmatrix} \left(1+\sin^2(\theta_1)\right)^{-1} & 0 \\ 0 & (c+\cos(\theta_1))^{-2} \end{pmatrix}$$

for any (θ_1,θ_2). The corresponding Riemannian gradient operator is given by the formula

$$\nabla_g \quad = \quad \left(1+\sin^2(\theta_1)\right)^{-1}\ \partial_{\theta_1}+(c+\cos(\theta_1))^{-2}\ \partial_{\theta_2}.$$

We also have that

$$\begin{aligned} C^1_{1,1} &= \frac{\sin(\theta_1)\cos(\theta_1)}{1+\sin^2(\theta_1)}=\frac{1}{2}\ \frac{\sin(2\theta_1)}{1+\sin^2(\theta_1)} \\ C^1_{2,2} &= \frac{(c+\cos(\theta_1))\ \sin(\theta_1)}{1+\sin^2(\theta_1)} \quad \text{and} \quad C^2_{1,2}=C^2_{2,1}=-\frac{\sin(\theta_1)}{(c+\cos(\theta_1))}. \end{aligned}$$

Therefore

$$\nabla_g^2 f=\begin{pmatrix} \left(1+\sin^2(\theta_1)\right)^{-1}\left(\partial^2_{\theta_1}f-\frac{1}{2}\ \frac{\sin(2\theta_1)}{1+\sin^2(\theta_1)}\ \partial_{\theta_1}f\right) & \left(1+\sin^2(\theta_1)\right)^{-1}\left(\partial^2_{\theta_1,\theta_2}f+\frac{\sin(\theta_1)}{(c+\cos(\theta_1))}\ \partial_{\theta_2}f\right) \\ (c+\cos(\theta_1))^{-2}\left(\partial^2_{\theta_1,\theta_2}f+\frac{\sin(\theta_1)}{(c+\cos(\theta_1))}\ \partial_{\theta_2}f\right) & (c+\cos(\theta_1))^{-2}\left(\partial^2_{\theta_2}f-\frac{(c+\cos(\theta_1))\ \sin(\theta_1)}{1+\sin^2(\theta_1)}\ \partial_{\theta_1}f\right) \end{pmatrix}.$$

This implies that

$$\begin{aligned} \Delta_g &= \frac{1}{1+\sin^2(\theta_1)}\ \partial^2_{\theta_1}+\frac{1}{(c+\cos(\theta_1))^2}\ \partial^2_{\theta_2} \\ &\qquad -\frac{\sin(\theta_1)}{1+\sin^2(\theta_1)}\left(\frac{\cos(\theta_1)}{1+\sin^2(\theta_1)}+\frac{1}{c+\cos(\theta_1)}\right)\ \partial_{\theta_1}. \end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 387:

When $u(z) = \cosh(z)$ we have $u'(\theta_1) = \sinh(\theta_1) = \frac{e^{\theta_1}-e^{-\theta_1}}{2}$ and $u'' = u$. In this situation, following the calculations provided in the proof of exercise 385 the tangent plane at $T_{\psi(\theta)}(S)$ is spanned by the vectors

$$\partial_{\theta_1}\psi = \left(\begin{array}{c} \sinh(\theta_1)\ \cos(\theta_2) \\ \sinh(\theta_1)\ \sin(\theta_2) \\ 1 \end{array}\right) \quad \perp \quad \partial_{\theta_2}\psi = \cosh(\theta_1)\left(\begin{array}{c} -\sin(\theta_2) \\ \cos(\theta_2) \\ 0 \end{array}\right).$$

Recalling that $\cosh^2(\theta_1) = \sinh^2(\theta_1) + 1$ this implies that

$$g = \cosh^2(\theta_1)\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right) \quad \text{and} \quad g^{-1} = \cosh^{-2}(\theta_1)\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right).$$

$$\nabla_g \;=\; \cosh^{-2}(\theta_1)\left(\partial_{\theta_1} + \partial_{\theta_2}\right).$$

The Christoffel symbols reduce to

$$\begin{array}{rcl} C^1_{1,1} & = & C^2_{1,2} = C^2_{2,1} = -C^1_{2,2} = \tanh(\theta_1) \\ C^1_{1,2} & = & C^1_{2,1} = C^2_{1,1} = C^2_{2,2} = 0. \end{array}$$

This yields

$$\nabla^2_g f = \cosh^{-2}(\theta_1)\left(\begin{array}{cc} \left(\partial^2_{\theta_1} f - \tanh(\theta_1)\ \partial_{\theta_1} f\right) & \left(\partial^2_{\theta_1,\theta_2} f + \tanh(\theta_1)\ \partial_{\theta_2} f\right) \\ \left(\partial^2_{\theta_1,\theta_2} f + \tanh(\theta_1)\ \partial_{\theta_2} f\right) & \left(\partial^2_{\theta_2} f + \tanh(\theta_1)\ \partial_{\theta_1} f\right) \end{array}\right).$$

Therefore

$$\Delta_g \;=\; \cosh^{-2}(\theta_1)\left(\partial^2_{\theta_1} + \partial^2_{\theta_2}\right).$$

This ends the proof of the exercise.

■

Solution to exercise 388:

By definition of the Christoffel symbols we have

$$\partial_{\theta_i,\theta_j}\psi = \sum_{k=1,2} C^k_{i,j}\ \partial_{\theta_k}\psi + \Omega^{i,j}\ n^{\perp}$$

with the orthogonal component

$$\Omega^{i,j} = \left\langle \partial_{\theta_i,\theta_j}\psi, n^{\perp}\right\rangle.$$

We clearly have

$$\begin{array}{rcl} c'(t) & = & \alpha_1'(t)\ (\partial_{\theta_1}\psi)_{\alpha(t)} + \alpha_2'(t)\ (\partial_{\theta_2}\psi)_{\alpha(t)} \\ c''(t) & = & \alpha_1''(t)\ (\partial_{\theta_1}\psi)_{\alpha(t)} + \alpha_2''(t)\ (\partial_{\theta_2}\psi)_{\alpha(t)} + \displaystyle\sum_{1\leq i,j\leq 2} \alpha_i'(t)\ \alpha_j'(t)\ \left(\partial_{\theta_i,\theta_j}\psi\right)_{\alpha(t)} \\ & = & c''_{tan}(t) + c''_{\perp}(t) \end{array}$$

with the tangential acceleration

$$c''_{tan}(t) := \sum_{k=1,2}\left(\alpha''_k(t) + \sum_{1\leq i,j\leq 2} C^k_{i,j}(\alpha(t))\ \alpha'_i(t)\ \alpha'_j(t)\right)\ (\partial_{\theta_k}\psi)_{\alpha(t)}\ \in\ T_{c(t)}(S)$$

and the normal component

$$c''_{\perp}(t) := \left(\sum_{1\leq i,j\leq 2} \Omega^{i,j}(\alpha(t))\ \alpha'_i(t)\ \alpha'_j(t)\right)\ n^{\perp}(\alpha(t))\ \in T^{\perp}_{c(t)}(S).$$

This ends the proof of the exercise.

■

Solution to exercise 389:
Recalling that $\omega'(t) = \|c'(t)\|$ we find that

$$\omega(\tau(s)) = s \Rightarrow \omega'(\tau(s))\tau'(s) = 1 \Rightarrow \tau'(s) = \frac{1}{\|c'(\tau(s))\|}.$$

This implies that

$$\overline{c}'(s) = \frac{c'(\tau(s))}{\|c'(\tau(s))\|} \in T_{\overline{c}(s)}(S) \Rightarrow T_{\overline{c}(s)}(S) = \text{Vect}\left(\overline{c}'(s), \overline{c}'(s)\wedge N^{\perp}_{\overline{c}(s)}\right).$$

Also observe that

$$\|\overline{c}'(s)\| = 1 \implies \langle \overline{c}''(s), \overline{c}'(s)\rangle = 0.$$

If we let let $\pi_{\overline{c}(s)}$ and $\pi^{\perp}_{\overline{c}(s)}$ the orthogonal projections on $T_{\overline{c}(s)}(S)$ and $T^{\perp}_{\overline{c}(s)}(S)$, then we have

$$\overline{c}''_{tan}(s) := \pi_{\overline{c}(s)}\left(\overline{c}''(s)\right) = \left\langle \overline{c}''(s), \overline{c}'(s)\wedge N^{\perp}_{\overline{c}(s)}\right\rangle\ \left(\overline{c}'(s)\wedge N^{\perp}_{\overline{c}(s)}\right).$$

On the other hand, we also have that

$$\begin{aligned}\overline{c}'(s) = c'(\tau(s))\ \tau'(s)\ \Rightarrow\ \overline{c}''(s)\ &=\ c''(\tau(s))\ (\tau'(s))^2 + c'(\tau(s))\ \tau''(s)\\ &=\ \frac{c''(\tau(s))}{\|c'(\tau(s))\|^2} + c'(\tau(s))\ \tau''(s).\end{aligned}$$

To compute $\tau''(s)$ we first observe that

$$\begin{aligned}\omega'(\tau(s))\tau'(s) = 1\ \Rightarrow\ \omega'(\tau(s))\ \tau''(s)\ &=\ -\omega''(\tau(s))\ (\tau'(s))^2\\ &=\ -\frac{\omega''(\tau(s))}{\|c'(\tau(s))\|^2}\end{aligned}$$

adn we also have

$$\begin{aligned}\omega'(t) = \|c'(t)\| = \sqrt{\langle c'(t), c'(t)\rangle} \Rightarrow\ \omega''(t)\ &=\ \frac{\langle c''(t), c'(t)\rangle}{\|c'(t)\|}\\ &=\ \left\langle c''(t), \frac{c'(t)}{\|c'(t)\|}\right\rangle\end{aligned}$$

from which we find that

$$\omega'(\tau(s))\ \tau''(s) = \|c'(\tau(s))\|\ \tau''(s)$$

and

$$\begin{aligned}
\omega'(\tau(s))\ \tau''(s) &= -\frac{\omega''(\tau(s))}{\|c'(\tau(s))\|^2} \\
&= -\frac{1}{\|c'(\tau(s))\|^2}\ \left\langle c''(\tau(s)), \frac{c'(\tau(s))}{\|c'(\tau(s))\|}\right\rangle = \|c'(\tau(s))\|\ \tau''(s).
\end{aligned}$$

We conclude that

$$\begin{aligned}
c'(\tau(s))\ \tau''(s) &= -\frac{1}{\|c'(\tau(s))\|^2}\ \left\langle c''(\tau(s)), \frac{c'(\tau(s))}{\|c'(\tau(s))\|}\right\rangle\ \frac{c'(\tau(s))}{\|c'(\tau(s))\|} \\
&= -\frac{1}{\|c'(\tau(s))\|^2}\ \langle c''(\tau(s)), \overline{c}'(s)\rangle\ \overline{c}'(s).
\end{aligned}$$

Finally we have the formula

$$\overline{c}''(s) = \frac{c''(\tau(s))}{\|c'(\tau(s))\|^2} \underbrace{-\frac{1}{\|c'(\tau(s))\|^2}\ \langle c''(\tau(s)), \overline{c}'(s)\rangle\ \overline{c}'(s)}_{\in T_{\overline{c}(s)}(S)}.$$

This clearly implies that

$$\begin{aligned}
\overline{c}''_{\perp}(s) &:= \pi^{\perp}_{\overline{c}(s)}\left(\overline{c}''(s)\right) \\
&= \frac{1}{\|c'(\tau(s))\|^2}\ \pi^{\perp}_{\overline{c}(s)}\left(c''(\tau(s))\right) \\
&= \frac{1}{\|c'(\tau(s))\|^2}\ \left\langle c''_{\perp}(\tau(s)), N^{\perp}(\overline{c}(s))\right\rangle\ N^{\perp}(\overline{c}(s)) \\
&= \frac{\sum_{1\leq i,j\leq 2}\ \Omega^{i,j}(\alpha(\tau(s)))\ \alpha'_i(\tau(s))\ \alpha'_j(\tau(s)}{\sum_{1\leq i,j\leq 2}\ g_{i,j}(\alpha(\tau(s)))\ \alpha'_i(\tau(s))\alpha'_j(\tau(s))}\ \ N^{\perp}(\overline{c}(s))
\end{aligned}$$

and

$$\begin{aligned}
\overline{c}'(s) \perp \overline{c}'(s)\wedge N^{\perp}_{\overline{c}(s)} \Rightarrow \left\langle \overline{c}''(s), \overline{c}'(s)\wedge N^{\perp}_{\overline{c}(s)}\right\rangle &= \frac{1}{\|c'(\tau(s))\|^2}\ \left\langle c''(\tau(s)), \overline{c}'(s)\wedge N^{\perp}_{\overline{c}(s)}\right\rangle \quad . \\
&= \frac{1}{\|c'(\tau(s))\|^2}\ \left\langle c''_{tan}(\tau(s)), \overline{c}'(s)\wedge N^{\perp}_{\overline{c}(s)}\right\rangle.
\end{aligned}$$

By construction we also have

$$\begin{aligned}
\|\overline{c}''(s)\|^2 &= \|\overline{c}''_{tan}(s)\|^2 + \|\overline{c}''_{\perp}(s)\|^2 \\
&= \left\|\frac{c''_{tan}(\tau(s)}{\|c'(\tau(s))\|^2}\right\|^2 + \left\|\frac{c''_{\perp}(\tau(s)}{\|c'(\tau(s))\|^2}\right\|^2 = \kappa^2_{tan}(\overline{c}(s)) + \kappa^2_{\perp}(\overline{c}(s)) := \kappa^2(\overline{c}(s))
\end{aligned}$$

with the tangential and the normal curvature

$$\begin{aligned}
&\kappa_{tan}(\overline{c}(s)) := \left\|\frac{c''_{tan}(\tau(s))}{\|c'(\tau(s))\|^2}\right\| \\
&= \frac{\left[\sum_{1\leq k,l\leq 2} g_{k,l}(\alpha(\tau(s)))\left\{\left(\alpha''_k(\tau(s)) + \sum_{1\leq i,j\leq 2} C^k_{i,j}(\alpha(\tau(s)))\ \alpha'_i(\tau(s))\ \alpha'_j(\tau(s))\right)\left(\alpha''_l(\tau(s)) + \sum_{1\leq i,j\leq 2} C^l_{i,j}(\alpha(\tau(s)))\ \alpha'_i(\tau(s))\ \alpha'_j(\tau(s))\right)\right\}\right]^{1/2}}{\sum_{1\leq i,j\leq 2} g_{i,j}(\alpha(\tau(s)))\ \alpha'_i(\tau(s))\ \alpha'_j(\tau(s))}
\end{aligned}$$

$$\kappa_\perp(\overline{c}(s)) := \left\| \frac{c''_\perp(\tau(s))}{\|c'(\tau(s))\|^2} \right\| = \frac{\sum_{1\leq i,j\leq 2} \ \Omega^{i,j}(\alpha(\tau(s)))\ \alpha'_i(\tau(s))\ \alpha'_j(\tau(s))}{\sum_{1\leq i,j\leq 2} g_{i,j}(\alpha(\tau(s)))\ \alpha'_i(\tau(s))\ \alpha'_j(\tau(s))}.$$

This ends the proof of the exercise.

■

Solution to exercise 390: The first assertion is proved in the solution of exercise 389. Notice that

$$R(v_1,v_2) := k \Rightarrow P(v_1,v_2) := \sum_{1\leq i,j\leq 2} \Omega^{i,j}(\phi(x))\ v_i\ v_j - k \sum_{1\leq i,j\leq 2} g_{i,j}(\phi(x))\ v_i\ v_j.$$

Taking the derivatives w.r.t. v_1 and v_2 we find that

$$\begin{array}{rcl}\partial_{v_1} P & = & 2\Omega^{1,1} v_1 + \left(\Omega^{1,2}+\Omega^{2,1}\right) v_2 - k\left(2g_{1,1}v_1 + (g_{1,2}+g_{2,1})v_2\right) = 0\\ \partial_{v_2} P & = & 2\Omega^{2,2} v_2 + \left(\Omega^{1,2}+\Omega^{2,1}\right) v_1 - k\left(2g_{2,2}v_2 + (g_{1,2}+g_{2,1})v_1\right) = 0.\end{array}$$

Or equivalently (Recalling that $g_{1,2}=g_{2,1}$ and $\Omega^{1,2}=\Omega^{2,1}$)

$$\left(\begin{array}{cc}\Omega^{1,1}-k\ g_{1,1} & \Omega^{1,2}-k\ g_{1,2}\\ \Omega^{1,2}-k\ g_{1,2} & \Omega^{2,2}-k\ g_{2,2}\end{array}\right)\left(\begin{array}{c}v_1\\ v_2\end{array}\right) = 0.$$

Notice that

$$\begin{aligned}&\det\left(\begin{array}{cc}\Omega^{1,1}-k\ g_{1,1} & \Omega^{1,2}-k\ g_{1,2}\\ \Omega^{1,2}-k\ g_{1,2} & \Omega^{2,2}-k\ g_{2,2}\end{array}\right)\\ &= \left(\Omega^{1,1}-k\ g_{1,1}\right)\left(\Omega^{2,2}-k\ g_{2,2}\right) - \left(\Omega^{1,2}-k\ g_{1,2}\right)^2\\ &= k^2\ \left(g_{1,1}g_{2,2}-g_{1,2}^2\right) - k\ \left(\Omega^{1,1}g_{2,2}+g_{1,1}\Omega^{2,2}-2\Omega^{1,2}g_{1,2}\right) + \left(\Omega^{1,1}\Omega^{2,2}-(\Omega^{1,2})^2\right)\\ &= \det(g)\ (k-k_1)\ (k-k_2) = \det(g)\ \left[k^2 - 2k\ \tfrac{(k_1+k_2)}{2} +\ k_1k_2\right] = 0\end{aligned}$$

with

$$\begin{array}{rcl}k_1 & = & \dfrac{k_1+k_2}{2} - \sqrt{\left(\dfrac{k_1+k_2}{2}\right)^2 - (k_1k_2)^2}\\ k_2 & = & \dfrac{k_1+k_2}{2} + \sqrt{\left(\dfrac{k_1+k_2}{2}\right)^2 - (k_1k_2)^2},\end{array}$$

and the sum and product given by

$$k_1\ k_2 = \det(\Omega)/\det(g) \quad \text{and} \quad \frac{(k_1+k_2)}{2} = \frac{1}{2}\left(\Omega^{1,1}g_{2,2}+g_{1,1}\Omega^{2,2}-2\Omega^{1,2}g_{1,2}\right)/\det(g).$$

Notice that

$$k_1+k_2 = \sum_{1\leq i,j\leq 2} g^{i,j}\Omega^{j,i} = \mathrm{tr}(g^{-1}\Omega).$$

The last assertion follows from the fact that

$$g = \left(\begin{array}{cc} g_{1,1} & g_{1,2}\\ g_{2,1} & g_{2,2}\end{array}\right) \Longrightarrow g^{-1} = \frac{1}{\det(g)}\left(\begin{array}{cc} g_{2,2} & -g_{1,2}\\ -g_{2,1} & g_{1,1}\end{array}\right).$$

This ends the proof of the exercise.

■

Solution to exercise 391:

By the definition of the unit normal field on the surface $S=\varphi^{-1}(0)$ we have

$$n^{\perp}(\theta)=N^{\perp}_{\psi(\theta)}=\frac{(\partial\varphi)_{\psi(\theta)}}{\left\|(\partial\varphi)_{\psi(\theta)}\right\|}.$$

Clearly

$$\begin{aligned}\left\|n^{\perp}(\theta)\right\|^2=\left\langle n^{\perp}(\theta),n^{\perp}(\theta)\right\rangle=1 \quad&\Rightarrow\quad \forall i=1,2\quad \left\langle\partial_{\theta_i}n^{\perp}(\theta),n^{\perp}(\theta)\right\rangle=0\\ &\Rightarrow\quad \forall i=1,2\quad \partial_{\theta_i}n^{\perp}(\theta)\in T_{\psi(\theta)}(S).\end{aligned}$$

This implies that

$$\partial_{\theta_i}n^{\perp}=\sum_{k,l=1,2}g^{k,l}\ \left\langle\partial_{\theta_k}\psi,\partial_{\theta_i}n^{\perp}\right\rangle\ \partial_{\theta_l}\psi.$$

On the other hand we also have that

$$\left\langle\partial_{\theta_k}\psi,n^{\perp}\right\rangle=0\quad\Rightarrow\quad\forall i=1,2\quad\overbrace{\left\langle\partial_{\theta_i,\theta_k}\psi,n^{\perp}\right\rangle}^{=\Omega^{i,k}}+\left\langle\partial_{\theta_k}\psi,\partial_{\theta_i}n^{\perp}\right\rangle=0.$$

This implies that

$$\partial_{\theta_i}n^{\perp}=-\sum_{l=1,2}\left(\sum_{k=1,2}g^{l,k}\ \Omega^{k,i}\right)\ \partial_{\theta_l}\psi.$$

Notice that

$$S_{l,i}=(g^{-1}\Omega)_{l,i}=(\Omega g^{-1})_{i,l}=\sum_{k=1,2}g^{l,k}\ \Omega^{k,i}=\sum_{k=1,2}\Omega^{i,k}g^{k,l}.$$

This yields

$$k_1+k_2\quad=\quad\mathrm{tr}(g^{-1}\Omega)=\sum_{1\leq i\leq 2}S_{i,i}=\mathrm{tr}(S).$$

This ends the proof of the exercise.

■

Solution to exercise 392:

With a slight abuse of notation, we write $C^k_{i,j}$ instead of $C^k_{i,j}\circ\psi$ for the Christoffel symbols in the parameter space S_ψ. We also recall that Ω is a symmetric matrix and hence $C^m_{i,j}=C^m_{j,i}$ holds.

Using the formulae derived in exercise 388 we have

$$\begin{aligned}\partial_{\theta_k,\theta_i,\theta_j}\psi\quad&=\quad\sum_{l=1,2}\left(C^l_{i,j}\ \partial_{\theta_k,\theta_l}\psi+\partial_{\theta_k}C^l_{i,j}\ \partial_{\theta_l}\psi\right)+\partial_{\theta_k}\Omega^{i,j}\ n^{\perp}+\Omega^{i,j}\ \partial_{\theta_k}n^{\perp}\\ &=\quad\sum_{m=1,2}\left[\partial_{\theta_k}C^m_{i,j}+\sum_{l=1,2}C^l_{i,j}C^m_{l,k}\right]\ \partial_{\theta_m}\psi\\ &\qquad\qquad+\left[\partial_{\theta_k}\Omega^{i,j}+\sum_{l=1,2}C^l_{i,j}\ \Omega^{l,k}\right]\ n^{\perp}+\Omega^{i,j}\ \partial_{\theta_k}n^{\perp}.\end{aligned}$$

The last formula follows from the fact that

$$\partial_{\theta_k,\theta_l}\psi = \sum_{m=1,2} C^m_{k,l}\ \partial_{\theta_m}\psi + \Omega^{k,l}\ n^{\perp}.$$

By exercise 391 we also have

$$\partial_{\theta_k} n^{\perp} = -\sum_{m=1,2} S_{m,k}\ \partial_{\theta_m}\psi.$$

This yields the formula

$$\begin{aligned}&\partial_{\theta_k,\theta_i,\theta_j}\psi\\ &= \sum_{m=1,2}\left[\partial_{\theta_k} C^m_{i,j} + \sum_{l=1,2} C^l_{i,j} C^m_{l,k} - \Omega^{i,j}\ S_{m,k}\right]\ \partial_{\theta_m}\psi + \left[\partial_{\theta_k}\Omega^{i,j} + \sum_{l=1,2} C^l_{i,j}\ \Omega^{l,k}\right]\ n^{\perp}\end{aligned}.$$

By symmetry arguments we also have

$$\begin{aligned}&\partial_{\theta_k,\theta_i,\theta_j}\psi = \partial_{\theta_j,\theta_i,\theta_k}\psi\\ &= \sum_{m=1,2}\left[\partial_{\theta_j} C^m_{i,k} + \sum_{l=1,2} C^l_{i,k} C^m_{l,j} - \Omega^{i,k}\ S_{m,j}\right]\ \partial_{\theta_m}\psi + \left[\partial_{\theta_j}\Omega^{i,k} + \sum_{l=1,2} C^l_{i,k}\ \Omega^{l,j}\right]\ n^{\perp}\end{aligned}.$$

This implies that

$$\partial_{\theta_j} C^m_{i,k} + \sum_{l=1,2} C^l_{i,k} C^m_{l,j} - \Omega^{i,k}\ S_{m,j} = \partial_{\theta_k} C^m_{i,j} + \sum_{l=1,2} C^l_{i,j} C^m_{l,k} - \Omega^{i,j}\ S_{m,k}$$

for any $m = 1,2$; or equivalently

$$R^m_{i,j,k} := \partial_{\theta_j} C^m_{k,i} - \partial_{\theta_k} C^m_{j,i} + \sum_{l=1,2}\left[C^l_{k,i} C^m_{j,l} - C^l_{j,i} C^m_{l,k}\right] = \Omega^{i,k}\ S_{m,j} - \Omega^{i,j}\ S_{m,k}.$$

In much the same way, identifying the normal components we find that

$$\partial_{\theta_j}\Omega^{i,k} - \partial_{\theta_k}\Omega^{i,j} = \sum_{l=1,2}\left[C^l_{i,j}\ \Omega^{l,k} - C^l_{i,k}\ \Omega^{l,j}\right].$$

Finally, recalling that $S = g^{-1}\Omega$ we have

$$R^m_{i,j,k} = \sum_{l=1,2}\left(\Omega^{i,k}\ \Omega^{l,j} - \Omega^{i,j}\ \Omega^{l,k}\right)\ g^{l,m}$$

and therefore

$$\begin{aligned}&\textstyle\sum_{m=1,2} g_{n,m}\ R^m_{i,j,k} = \left(\Omega^{i,k}\ \Omega^{n,j} - \Omega^{i,j}\ \Omega^{n,k}\right)\\ &\Longrightarrow \textstyle\sum_{m=1,2} g_{n,m}\ R^m_{i,n,i} = \Omega^{i,i}\ \Omega^{n,n} - \Omega^{i,n}\Omega^{n,i}\\ &\Longrightarrow \textstyle\sum_{m=1,2} g_{2,m}\ R^m_{1,2,1} = \sum_{m=1,2} g_{1,m}\ R^m_{2,1,2} = \Omega^{1,1}\ \Omega^{2,2} - \Omega^{1,2}\ \Omega^{2,1} = \det(\Omega).\end{aligned}$$

Since $S = g^{-1}\Omega$, we also have that

$$R^m_{i,m,k} = \Omega^{i,k}\ S_{m,m} - \Omega^{i,m}\ S_{m,k} \quad \Rightarrow \quad R_{i,k} = \sum_{1\leq m\leq 2} R^m_{i,m,k} = \Omega^{i,k}\ \mathrm{tr}(S) - (\Omega S)_{i,k}$$

$$\Rightarrow \quad \mathrm{tr}(g^{-1}R) = (\mathrm{tr}(S))^2 - \mathrm{tr}(S^2) = 2\mathrm{det}(S) = 2\ \kappa_{Gauss}.$$

The last assertion follows from the fact that for any (2×2)-matrix A we have

$$(\mathrm{tr}(A))^2 = \mathrm{tr}(A^2) + 2\ \mathrm{det}(A).$$

We check this claim by using a brute force calculation. If we set $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then we clearly have

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix}.$$

$$\Rightarrow (\mathrm{tr}(A))^2 - \mathrm{tr}(A^2) = a^2+d^2+2ad-a^2-bc-cb-d^2 = 2(ad-bc) = 2\ \mathrm{det}(A)$$

This ends the proof of the exercise.

∎

Chapter 22

Solution to exercise 393:

We use the formula (22.2). We consider the polar coordinates on $S = \mathbb{R}^2$ as discussed in (21.8). In this situation we have

$$\partial_{\theta_1}\psi = \begin{pmatrix} \cos(\theta_2) \\ \sin(\theta_2) \end{pmatrix} \perp \partial_{\theta_2}\psi = \begin{pmatrix} -\theta_1 \sin(\theta_2) \\ \theta_1 \cos(\theta_2) \end{pmatrix}.$$

The matrices g and g^{-1} are clearly given by

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \theta_1^2 \end{pmatrix} \quad \text{and} \quad g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \theta_1^{-2} \end{pmatrix}.$$

By (21.40), the mappings $(\Delta\phi^1)_\psi$ and $(\Delta\phi^2)_\psi$ reduce to

$$(\Delta\phi^1)_\psi = \frac{1}{2}\, g^{2,2}\, \partial_{\theta_1} g_{2,2} = \frac{1}{\theta_1} \quad \text{and} \quad (\Delta\phi^2)_\psi = 0.$$

The formula (22.2) reduces to the formulation of the Laplacian in polar coordinates

$$\Delta_g = \frac{1}{\theta_1}\, \partial_{\theta_1} + \partial^2_{\theta_1} + \frac{1}{\theta_1^2}\, \partial^2_{\theta_2}.$$

This ends the proof of the exercise.

■

Solution to exercise 394:

We use the same lines of arguments as in the proof of exercise 393. The spherical coordinates on $S = \mathbb{R}^3$ are given by

$$\psi(\theta) = \left\{ \begin{array}{rcl} \psi^1(\theta) & = & \theta_1 \sin(\theta_2)\cos(\theta_3) \\ \psi^2(\theta) & = & \theta_1 \sin(\theta_2)\sin(\theta_3) \\ \psi^3(\theta) & = & \theta_1 \cos(\theta_2) \end{array} \right.$$

In this situation we have the orthogonal tangent vector fields

$$\partial_{\theta_1}\psi = \begin{pmatrix} \sin(\theta_2)\cos(\theta_3) \\ \sin(\theta_2)\sin(\theta_3) \\ \cos(\theta_2) \end{pmatrix} \perp \partial_{\theta_2}\psi = \begin{pmatrix} \theta_1 \cos(\theta_2)\cos(\theta_3) \\ \theta_1 \cos(\theta_2)\sin(\theta_3) \\ -\theta_1 \sin(\theta_2) \end{pmatrix}$$
$$\perp \partial_{\theta_3}\psi = \begin{pmatrix} -\theta_1 \sin(\theta_2)\sin(\theta_3) \\ \theta_1 \sin(\theta_2)\cos(\theta_3) \\ 0 \end{pmatrix} \perp \partial_{\theta_2}\psi.$$

In this situation g and g^{-1} are clearly given by the diagonal matrices

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta_1^2 & 0 \\ 0 & 0 & \theta_1^2 \sin^2(\theta_2) \end{pmatrix} \quad \text{and} \quad g^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta_1^{-2} & 0 \\ 0 & 0 & \theta_1^{-2}\sin^{-2}(\theta_2) \end{pmatrix}.$$

By (21.40), the mappings $(\Delta\phi^i)_\psi$ $i=1,2,3$ reduce to

$$\begin{aligned}(\Delta\phi^1)_\psi &= \frac{1}{2}\left(g^{2,2}\partial_{\theta_1}g_{2,2}+g^{3,3}\partial_{\theta_1}g_{3,3}\right)=\frac{2}{\theta_1}\\ (\Delta\phi^2)_\psi &= \frac{1}{2}\,g^{2,2}\,g^{3,3}\partial_{\theta_2}g_{3,3}=\frac{1}{\theta_1^2}\,\frac{\cos(\theta_2)}{\sin(\theta_2)}\quad\text{and}\quad(\Delta\phi^3)_\psi=0.\end{aligned}$$

The formula (22.2) reduces to the formulation of the Laplacian in spherical coordinates

$$\Delta_g=\frac{2}{\theta_1}\,\partial_{\theta_1}+\frac{1}{\theta_1^2}\,\frac{\cos(\theta_2)}{\sin(\theta_2)}\,\partial_{\theta_2}+\partial^2_{\theta_1}+\frac{1}{\theta_1^2}\,\partial^2_{\theta_2}+\frac{1}{\theta_1^2\sin^2(\theta_2)}\,\partial^2_{\theta_3}.$$

This ends the proof of the exercise.

■

Solution to exercise 395:
Solving exercise 384, we get the Christoffel symbols $C^k_{i,j}$ associated with the spherical coordinates. We then set

$$C^1:=\begin{pmatrix}C^1_{1,1} & C^1_{1,2}\\ C^1_{1,2} & C^1_{2,2}\end{pmatrix}\quad\text{and}\quad C^2:=\begin{pmatrix}C^2_{1,1} & C^2_{1,2}\\ C^2_{1,2} & C^2_{2,2}\end{pmatrix}.$$

Using (21.43) and (21.10) for $i=1,2$ we have

$$\left(\Delta\phi^i\right)_\psi=-\mathrm{tr}\left(g^{-1}C^i\right)\quad\text{and}\quad\left(\nabla\phi^i\right)_\psi=g^{i,i}\,\left(\partial_{\theta_1}\psi\right).$$

By (22.6) the Brownian motion on the ellipsoid equipped with the spherical coordinates is defined by the equations

$$d\Theta^i_t=\frac{1}{2}\,\left(\Delta\phi^i\right)_\psi(\Theta_t)\,dt+\left(\nabla\phi^i\right)^T_\psi(\Theta_t)\,dB_t$$

where B_t stands for a standard r-dimensional Brownian motion on $\mathbb{R}^3$.
This ends the proof of the exercise.

■

Solution to exercise 396: Following the solution of exercise 383, the Christoffel symbols associated with the spherical coordinates are given by the matrices

$$C^1=\begin{pmatrix}C^1_{1,1} & C^1_{1,2}\\ C^1_{1,2} & C^1_{2,2}\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & -\frac{a^2\theta_1}{1+a^2}\end{pmatrix}$$

and

$$C^2=\begin{pmatrix}C^2_{1,1} & C^2_{1,2}\\ C^2_{1,2} & C^2_{2,2}\end{pmatrix}=\begin{pmatrix}0 & \frac{1}{\theta_1}\\ \frac{1}{\theta_1} & 0\end{pmatrix}.$$

In addition we have proved in exercise 383 that

$$\partial_{\theta_1}\psi=\begin{pmatrix}a\cos(\theta_2)\\ a\sin(\theta_2)\\ 1\end{pmatrix}\quad\perp\quad\partial_{\theta_2}\psi=a\;\theta_1\begin{pmatrix}-\sin(\theta_2)\\ \cos(\theta_2)\\ 0\end{pmatrix}$$

and

$$g^{-1}=\begin{pmatrix}(1+a^2)^{-1} & 0\\ 0 & a^{-2}\theta_1^{-2}\end{pmatrix}.$$

A simple calculation shows that

$$g^{-1}C^1 = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{\theta_1(1+a^2)} \end{pmatrix} \quad \text{and} \quad g^{-1}C^2 = \begin{pmatrix} 0 & \frac{1}{\theta_1(1+a^2)} \\ \frac{1}{\theta_1^3 a^2} & 0 \end{pmatrix}.$$

Using (21.43) we find that

$$\left(\Delta\phi^1\right)_\psi = -\sum_{1\leq j,k\leq 2} g^{j,k}\; C^1_{k,j} = -\mathrm{tr}\left(g^{-1}C^1\right) = \frac{1}{\theta_1(1+a^2)}$$

as well as

$$\left(\Delta\phi^2\right)_\psi = -\sum_{1\leq j,k\leq 2} g^{j,k}\; C^2_{k,j} = -\mathrm{tr}\left(g^{-1}C^2\right) = 0.$$

On the other hand using (21.10) we have

$$\left(\nabla\phi^1\right)_\psi = g^{1,1}\;\partial_{\theta_1}\psi = \frac{1}{1+a^2}\begin{pmatrix} a\cos(\theta_2) \\ a\sin(\theta_2) \\ 1 \end{pmatrix}$$

and

$$\left(\nabla\phi^2\right)_\psi = g^{2,2}\;\partial_{\theta_2}\psi = \frac{1}{a\;\theta_1}\begin{pmatrix} -\sin(\theta_2) \\ \cos(\theta_2) \\ 0 \end{pmatrix}.$$

By (22.6) the Brownian motion on the cone is defined by the equations

$$\begin{aligned} d\Theta^1_t &= \frac{1}{2}\;\left(\Delta\phi^1\right)_\psi(\Theta_t)\;dt + \left(\nabla\phi^1\right)^T_\psi(\Theta_t)\;dB_t \\ &= \frac{1}{2\Theta^1_t(1+a^2)}\;dt + \frac{1}{1+a^2}\;\left[a\cos(\Theta^2_t)\;\;dB^1_t + a\sin(\Theta^2_t)\;\;dB^2_t + dB^3_t\right] \\ d\Theta^2_t &= \frac{1}{2}\;\left(\Delta\phi^2\right)_\psi(\Theta_t)\;dt + \left(\nabla\phi^2\right)^T_\psi(\Theta_t)\;dB_t \\ &= \frac{1}{a\Theta^1_t}\;\left[-\sin(\Theta^2_t)\;\;dB^1_t + \cos(\Theta^2_t)\;\;dB^2_t\right] \end{aligned}$$

where B_t stands for a standard Brownian motion on $\mathbb{R}^3$.

Next we check that the generator L_g of the above diffusion coincides with half of the Laplacian operator presented in exercise 383. To this end we simply notice that

$$d\Theta^1_t d\Theta^2_t = 0, \qquad d\Theta^1_t d\Theta^1_t = \frac{dt}{1+a^2} \quad \text{and} \quad d\Theta^2_t d\Theta^2_t = \frac{dt}{a^2(\Theta^1_t)^2}.$$

Using Doeblin-Itō fomula, this implies that

$$L_g = \frac{1}{2(1+a^2)}\;\partial^2_{\theta_1} + \frac{1}{2a^2\theta_1^2}\;\partial^2_{\theta_2} + \frac{1}{2\theta_1(1+a^2)}\;\partial_{\theta_1}.$$

Notice that L_g is also the generator of the diffusion process

$$\begin{aligned} d\Theta^1_t &= \frac{1}{2\Theta^1_t(1+a^2)}\;dt + \frac{1}{\sqrt{1+a^2}}\;dB^1_t \\ d\Theta^2_t &= \frac{1}{a\Theta^1_t}\;dB^2_t \end{aligned}$$

with a couple (B_t^1, B_t^2) of independent Brownian motions on $\mathbb{R}$. This ends the proof of the exercise.

■

Solution to exercise 397:

In view of the Laplacian generator derived in exercise 385 a Brownian motion on S is defined by the diffusion

$$\begin{aligned}
d\Theta_t^1 &= \frac{1}{4}\,\partial_{\theta_1}\left(\log\left(\frac{u^2}{1+(\partial_{\theta_1}u)^2}\right)\right)(\Theta_t^1)\;dt + \frac{1}{\sqrt{1+(\partial_{\theta_1}u)^2\,(\Theta_t^1)}}\;dB_t^1\\
d\Theta_t^2 &= \frac{1}{u\,(\Theta_t^1)}\;dB_t^2,
\end{aligned}$$

with a couple (B_t^1, B_t^2) of independent Brownian motions on $\mathbb{R}$.

This ends the proof of the exercise.

■

Solution to exercise 398:

When $u(\theta_1) = c + \cos(\theta_1)(>0)$ with $c > -1$ we have

$$\begin{aligned}
d\Theta_t^1 &= -\frac{1}{2}\left(\frac{\sin(2\Theta_t^1)}{2\left(1+\sin^2(\Theta_t^1)\right)} + \frac{\sin(\Theta_t^1)}{c+\cos(\Theta_t^1)}\right)\;dt + \frac{1}{\sqrt{1+\sin^2(\Theta_t^1)}}\;dB_t^1\\
d\Theta_t^2 &= \frac{1}{c+\cos(\Theta_t^1)}\;dB_t^2.
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 399:

The tangent space $T_{\psi(\theta)}(S)$ is spanned by the vector fields

$$\partial_{\theta_1}\psi = \begin{pmatrix}\cos(\theta_2)\\ \sin(\theta_2)\\ 0\end{pmatrix} \quad\perp\quad \partial_{\theta_2}\psi = \begin{pmatrix}-\theta_1\,\sin(\theta_2)\\ \theta_1\,\cos(\theta_2)\\ 1\end{pmatrix}.$$

The normal vector field is defined by

$$\partial_{\theta_1}\psi\wedge\partial_{\theta_2}\psi = \begin{pmatrix}\cos(\theta_2)\\ \sin(\theta_2)\\ 0\end{pmatrix}\wedge\begin{pmatrix}-\theta_1\,\sin(\theta_2)\\ \theta_1\,\cos(\theta_2)\\ 1\end{pmatrix} = \begin{pmatrix}\sin(\theta_2)\\ -\cos(\theta_2)\\ \theta_1\end{pmatrix}.$$

Thus, the unit normal vector field $n^\perp$ is given by

$$n^\perp = \frac{\partial_{\theta_1}\psi\wedge\partial_{\theta_2}\psi}{\|\partial_{\theta_1}\psi\wedge\partial_{\theta_2}\psi\|} = \frac{1}{\sqrt{1+\theta_1^2}}\begin{pmatrix}\sin(\theta_2)\\ -\cos(\theta_2)\\ \theta_1\end{pmatrix}.$$

The Riemannian metric is given by the matrix

$$g = \begin{pmatrix}1 & 0\\ 0 & 1+\theta_1^2\end{pmatrix} \Rightarrow g^{-1} = \begin{pmatrix}1 & 0\\ 0 & (1+\theta_1^2)^{-1}\end{pmatrix}.$$

We also have $\partial^2_{\theta_1}\psi = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and

$$\partial_{\theta_1}\psi \perp \partial_{\theta_1,\theta_2}\psi = \begin{pmatrix} -\sin(\theta_2) \\ \cos(\theta_2) \\ 0 \end{pmatrix} \perp \partial^2_{\theta_2}\psi = -\theta_1 \begin{pmatrix} \cos(\theta_2) \\ \sin(\theta_2) \\ 0 \end{pmatrix} = -\theta_1\ \partial_{\theta_1}\psi.$$

This implies that $C^1_{1,1} = C^2_{1,1} = C^1_{1,2} = C^2_{2,2} = 0$ and

$$\begin{array}{rcl} C^1_{2,2} & := & g^{1,1}\ \left\langle \partial_{\theta_1}\psi, \partial^2_{\theta_2}\psi \right\rangle = -\theta_1 \\ C^2_{1,2} & = & g^{2,2}\ \left\langle \partial_{\theta_2}\psi, \partial_{\theta_1,\theta_2}\psi \right\rangle = \dfrac{\theta_1}{1+\theta_1^2}. \end{array}$$

In other words the Christoffel symbols are defined by the matrices

$$C^1 = \begin{pmatrix} 0 & 0 \\ 0 & -\theta_1 \end{pmatrix} \quad \text{and} \quad C^2 = \frac{\theta_1}{1+\theta_1^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This yields

$$g^{-1}C^1 = \begin{pmatrix} 1 & 0 \\ 0 & (1+\theta_1^2)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -\theta_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\theta_1}{1+\theta_1^2} \end{pmatrix}$$

and

$$g^{-1}C^2 = \frac{\theta_1}{1+\theta_1^2} \begin{pmatrix} 1 & 0 \\ 0 & (1+\theta_1^2)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\theta_1}{1+\theta_1^2} \begin{pmatrix} 0 & 1 \\ \frac{1}{1+\theta_1^2} & 0 \end{pmatrix}.$$

Using (21.43) we find that

$$\begin{array}{rcl} \left(\Delta\phi^1\right)_\psi & = & -\displaystyle\sum_{1\leq j,k\leq 2} g^{j,k}\ C^1_{k,j} = -\mathrm{tr}\left(g^{-1}C^1\right) = \frac{\theta_1}{1+\theta_1^2} \\ \left(\Delta\phi^2\right)_\psi & = & -\displaystyle\sum_{1\leq j,k\leq 2} g^{j,k}\ C^2_{k,j} = -\mathrm{tr}\left(g^{-1}C^2\right) = 0. \end{array}$$

On the other hand using (21.10) we have

$$\left(\nabla\phi^1\right)_\psi = g^{1,1}\ \partial_{\theta_1}\psi = \begin{pmatrix} \cos(\theta_2) \\ \sin(\theta_2) \\ 0 \end{pmatrix}$$

and

$$\left(\nabla\phi^2\right)_\psi = g^{2,2}\ \partial_{\theta_2}\psi = \frac{1}{1+\theta_1^2} \begin{pmatrix} -\theta_1\ \sin(\theta_2) \\ \theta_1\ \cos(\theta_2) \\ 1 \end{pmatrix}.$$

Therefore, by (22.6) the Brownian motion on the helicoid is defined by the diffusion

$$\begin{array}{rcl} d\Theta^1_t & = & \dfrac{1}{2}\ \left(\Delta\phi^1\right)_\psi(\Theta_t)\ dt + \left(\nabla\phi^1\right)^T_\psi(\Theta_t)\ dB_t \\ & = & \dfrac{1}{2}\ \dfrac{\Theta^1_t}{1+(\Theta^1_t)^2}\ dt + \left(\cos(\Theta^2_t)\ dB^1_t + \sin(\Theta^2_t)\ dB^2_t\right) \\ d\Theta^2_t & = & \dfrac{1}{2}\ \left(\Delta\phi^2\right)_\psi(\Theta_t)\ dt + \left(\nabla\phi^2\right)^T_\psi(\Theta_t)\ dB_t \\ & = & \dfrac{1}{1+(\Theta^1_t)^2}\ \left(-\Theta^1_t\ \sin(\Theta^2_t)\ dB^1_t + \Theta^1_t\ \cos(\Theta^2_t)\ dB^2_t + dB^3_t\right), \end{array}$$

where $B_t = (B_t^1, B_t^2, B_t^3)$ stands for a standard 3-dimensional Brownian motion on $\mathbb{R}^3$. Notice that

$$d\Theta_t^1 d\Theta_t^1 = dt \qquad d\Theta_t^2 d\Theta_t^2 = \frac{dt}{1+(\Theta_t^1)^2} \quad \text{and} \quad d\Theta_t^1 d\Theta_t^2 = 0.$$

Using Doeblin-Itō formula we check that the generator of the diffusion on the helicoid is given by the operator

$$L_g = \frac{1}{2}\,\partial_{\theta_1}^2 + \frac{1}{2}\,\frac{1}{1+\theta_1^2}\,\partial_{\theta_2}^2 + \frac{1}{2}\,\frac{\theta_1}{1+\theta_1^2}\,\partial_{\theta_1}.$$

Finally we observe that L_g is also the generator of the diffusion process

$$\begin{array}{rcl} d\Theta_t^1 & = & \dfrac{1}{2}\,\dfrac{\Theta_t^1}{1+(\Theta_t^1)^2}\,dt + dB_t^1 \\ d\Theta_t^2 & = & \dfrac{1}{\sqrt{1+(\Theta_t^1)^2}}\,dB_t^2 \end{array}$$

with a couple (B_t^1, B_t^2) of independent Brownian motions on $\mathbb{R}$. This ends the proof of the exercise.

■

Solution to exercise 400:
We observe that

$$\varphi(x_1, x_2, x_2) = x_2 \cos(x_3) - x_1 \sin(x_3)$$

$$\Longrightarrow (\varphi \circ \psi)(\theta) = \theta_1\ \sin(\theta_2)\ \cos(\theta_2) - \theta_1\ \cos(\theta_2)\ \sin(\theta_2) = 0.$$

This shows that ψ is a parametrization of the helicoid defined by the null level set $S = \varphi^{-1}(0)$ of the smooth function φ. The unit normal vector field on S is given by

$$N^\perp := \frac{\partial\varphi}{\|\partial\varphi\|} = -\frac{1}{\sqrt{1+(x_2\sin(x_3)+x_1\cos(x_3))^2}} \begin{pmatrix} \sin(x_3) \\ -\cos(x_3) \\ (x_2\sin(x_3)+x_1\cos(x_3)) \end{pmatrix}.$$

$$\begin{array}{rcl} \pi(x) & = & Id - \dfrac{1}{\|\partial\varphi\|^2}\,\partial\varphi\partial\varphi^T \\ & = & Id + \dfrac{1}{1+(x_2\sin(x_3)+x_1\cos(x_3))^2} \\ & & \times \begin{pmatrix} \sin^2(x_3) & -\cos(x_3)\sin(x_3) & \sin(x_3)(x_2\sin(x_3)+x_1\cos(x_3)) \\ -\cos(x_3)\sin(x_3) & \cos^2(x_3) & -\cos(x_3)(x_2\sin(x_3)+x_1\cos(x_3)) \\ (x_2\sin(x_3)+x_1\cos(x_3))\sin(x_3) & -(x_2\sin(x_3)+x_1\cos(x_3))\cos(x_3) & (x_2\sin(x_3)+x_1\cos(x_3))^2 \end{pmatrix}. \end{array}$$

By (20.2) the mean curvature vector $\mathbb{H}$ is given by

$$\mathbb{H} = \left[\sum_{1\leq i\leq 3} \partial_{x_i}\left(\frac{\partial_{x_i}\varphi}{\|\partial\varphi\|}\right)\right]\,\frac{\partial\varphi}{\|\partial\varphi\|}.$$

We set $A(x) := x_2\sin(x_3) + x_1\cos(x_3)$ and we observe that

$$\begin{array}{rcl} \partial_{x_i}\left(\dfrac{1}{\sqrt{1+A^2}}\right) & = & -\dfrac{1}{\sqrt{1+A^2}}\,\dfrac{A}{1+A^2}\,\partial_{x_i}A \\ & = & -\dfrac{1}{2}\,\dfrac{1}{\sqrt{1+A^2}}\,\partial_{x_i}\log(1+A^2) \end{array}$$

as well as

$$\partial_{x_1} A = \cos(x_3) \qquad \partial_{x_2} A = \sin(x_3) \quad \text{and} \quad \partial_{x_3} A = x_2 \cos(x_3) - x_1 \sin(x_3).$$

$$\sum_{1\leq i\leq 3} \partial_{x_i}\left(\frac{\partial_{x_i}\varphi}{\|\partial\varphi\|}\right)$$

$$= \overbrace{\sin(x_3)\ \partial_{x_1}\left(\frac{1}{\sqrt{1+A^2}}\right) - \cos(x_3)\ \partial_{x_2}\left(\frac{1}{\sqrt{1+A^2}}\right)}^{=0} + A\ \partial_{x_3}\left(\frac{1}{\sqrt{1+A^2}}\right) + \frac{\partial_{x_3} A}{\sqrt{1+A^2}}$$

$$= \left(1 - \tfrac{A^2}{1+A^2}\right)\ \frac{\partial_{x_3} A}{\sqrt{1+A^2}} = \frac{x_2 \cos(x_3) - x_1 \sin(x_3)}{\left(1 + [x_2 \sin(x_3) + x_1 \cos(x_3)]^2\right)^{3/2}}.$$

This implies that

$$\mathbb{H}(x) = -\frac{x_2 \cos(x_3) - x_1 \sin(x_3)}{\left(1 + [x_2 \sin(x_3) + x_1 \cos(x_3)]^2\right)^2} \begin{pmatrix} \sin(x_3) \\ -\cos(x_3) \\ (x_2 \sin(x_3) + x_1 \cos(x_3)) \end{pmatrix}.$$

By (20.7) the Brownian motion on the helicoid in the ambient space $\mathbb{R}^3$ is defined by the diffusion equation

$$dX_t = -\frac{1}{2}\ \mathbb{H}(X_t)\ dt + \pi(X_t)\ dB_t.$$

This ends the proof of the exercise.

■

Solution to exercise 401:
Solving exercise 387 we have seen that

$$\partial_{\theta_1}\psi = \begin{pmatrix} \sinh(\theta_1)\ \cos(\theta_2) \\ \sinh(\theta_1)\ \sin(\theta_2) \\ 1 \end{pmatrix} \quad \perp \quad \partial_{\theta_2}\psi = \cosh(\theta_1) \begin{pmatrix} -\sin(\theta_2) \\ \cos(\theta_2) \\ 0 \end{pmatrix}$$

as well as

$$g^{-1} = \cosh^{-2}(\theta_1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In addition, the Christoffel symbols are defined by the matrices

$$C^1 = \tanh(\theta_1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad C^2 = \tanh(\theta_1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This yields

$$\begin{aligned} g^{-1}C^1 &= \tanh(\theta_1)\cosh^{-2}(\theta_1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ g^{-1}C^2 &= \tanh(\theta_1)\cosh^{-2}(\theta_1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Using (21.43) we find that

$$\left(\Delta\phi^1\right)_\psi = \left(\Delta\phi^2\right)_\psi = 0.$$

On the other hand using (21.10) we have

$$\left(\nabla\phi^1\right)_\psi = g^{1,1}\ \partial_{\theta_1}\psi = \frac{1}{\cosh^2(\theta_1)}\begin{pmatrix}\sinh(\theta_1)\ \cos(\theta_2)\\ \sinh(\theta_1)\ \sin(\theta_2)\\ 1\end{pmatrix}$$

and

$$\left(\nabla\phi^2\right)_\psi = g^{2,2}\ \partial_{\theta_2}\psi = \frac{1}{\cosh^2(\theta_1)}\begin{pmatrix}-\cosh(\theta_1)\sin(\theta_2)\\ \cosh(\theta_1)\cos(\theta_2)\\ 0\end{pmatrix}.$$

Therefore, by (22.6) the Brownian motion on the catenoid is defined by the diffusion

$$\begin{aligned}
d\Theta^1_t &= \left(\nabla\phi^1\right)^T_\psi(\Theta_t)\ dB_t\\
&= \frac{1}{\cosh^2(\Theta^1_t)}\left(\sinh(\Theta^1_t)\ \cos(\Theta^2_t)\ dB^1_t + \sinh(\Theta^1_t)\ \sin(\Theta^2_t)\ dB^2_t + dB^3_t\right)\\
d\Theta^2_t &= \left(\nabla\phi^2\right)^T_\psi(\Theta_t)\ dB_t\\
&= \frac{1}{\cosh^2(\Theta^1_t)}\left(-\cosh(\Theta^1_t)\ \sin(\Theta^2_t)\ dB^1_t + \cosh(\Theta^1_t)\ \cos(\Theta^2_t)\ dB^2_t\right),
\end{aligned}$$

where $B_t = (B^1_t, B^2_t, B^3_t)$ stands for a standard 3-dimensional Brownian motion on $\mathbb{R}^3$. Notice that

$$d\Theta^1_t d\Theta^1_t = \frac{dt}{1+\sinh^2(\Theta^1_t)} \qquad d\Theta^2_t d\Theta^2_t = \frac{dt}{\cosh^2(\Theta^1_t)} \quad \text{and} \quad d\Theta^1_t d\Theta^2_t = 0.$$

Using Doeblin-Itō fomula we check that the generator of the diffusion on the helicoid is given by the operator

$$L_g = \frac{1}{2}\ \frac{1}{\cosh^2(\theta_1)}\ \partial^2_{\theta_1} + \frac{1}{2}\ \frac{1}{\cosh^2(\theta_1)}\ \partial^2_{\theta_2}.$$

Finally we observe that L_g is also the generator of the diffusion process

$$d\Theta^1_t = \frac{1}{\cosh(\Theta^1_t)}\ dB^1_t \quad \text{and} \quad d\Theta^2_t = \frac{1}{\cosh(\Theta^1_t)}\ dB^2_t$$

with a couple (B^1_t, B^2_t) of independent Brownian motions on $\mathbb{R}$.

This ends the proof of the exercise.

■

Solution to exercise 402:

The detailed construction of the Brownian motion on the unit circle equipped with the polar coordinates is provided in section 22.3.1.

■

Solution to exercise 403:

The detailed construction of the Brownian motion on the unit 2-sphere equipped with the spherical coordinates is provided in section 22.3.2.

■

Solution to exercise 404:

A construction of the Brownian motion on the unit p-sphere in the ambient space (in terms of orthogonal projections on the tangent space) is presented in (20.8).

■

Solution to exercise 405:

A construction of the Brownian motion on the cylinder in the ambient space (in terms of orthogonal projections on the tangent space) is presented in (20.9).

■

Solution to exercise 406:

The detailed construction of the Brownian motion on the 2-Torus equipped with the polar coordinates is provided in section 22.4

■

Solution to exercise 407:

The detailed construction of the Brownian motion on the p-simplex. is provided in section 22.5.

■

Chapter 23

Solution to exercise 408:
We have

$$\begin{aligned}\frac{1}{2}\partial_t\|\nabla F(C_x(t))\|^2 &= \partial_t \langle \nabla F(C_x(t)), \nabla F(C_x(t))\rangle \\ &= \langle \partial_t [\nabla F(C_x(t))], \nabla F(C_x(t))\rangle\end{aligned}$$

with

$$\partial_t [\nabla F(C_x(t))] = \left[\begin{array}{c}\partial_t [\nabla F(C_x(t))]_1 \\ \vdots \\ \partial_t [\nabla F(C_x(t))]_r\end{array}\right].$$

We recall that

$$\nabla F = \left[\begin{array}{c}[\nabla F]_1 \\ \vdots \\ [\nabla F]_r\end{array}\right] = \pi\partial F = \left[\begin{array}{c}\partial_{\pi_1} F \\ \vdots \\ \partial_{\pi_r} F\end{array}\right] \quad \text{and} \quad \partial_W \nabla F = \left[\begin{array}{c}\partial_W\partial_{\pi_1} F \\ \vdots \\ \partial_W\partial_{\pi_r} F\end{array}\right],$$

with the orthogonal projection matrix π on $T(S)$. For each $1 \leq i \leq r$ we have

$$\partial_t [\partial_{\pi_i} F(C_x(t))] = \sum_{1\leq j\leq r} (\partial_{x_j}\partial_\pi F)(C_x(t))\ W^j(C_x(t)) = (\partial_W\partial_{\pi_i})(C_x(t)).$$

Therefore

$$\partial_t [\nabla F(C_x(t))] = (\partial_W \nabla F)(C_x(t)).$$

This implies that

$$\frac{1}{2}\partial_t\|\nabla F(C_x(t))\|^2 = \langle (\partial_W \nabla F)(C_x(t)), \nabla F(C_x(t))\rangle = \langle (\nabla_W \nabla F)(C_x(t)), \nabla F(C_x(t))\rangle$$

with $\nabla_W = \pi\partial_W$. By (19.55), (19.57) and (19.51) we have

$$\langle \nabla_W \nabla F, \nabla F\rangle = \frac{1}{2}\ \partial_W \langle \nabla F, \nabla F\rangle = \langle \nabla_{\nabla F}\nabla F, W\rangle = \nabla^2 F(\nabla F, W) := \left\langle \nabla F, (\nabla^2 F) W\right\rangle.$$

This ends the proof of the first assertion. If we choose

$$W = -\nabla F \quad \text{with} \quad \nabla^2 F \geq \lambda\ Id$$

then we find that

$$\left\langle \nabla F, (\nabla^2 F) W\right\rangle = -\left\langle \nabla F, (\nabla^2 F)\nabla F\right\rangle \leq -\lambda\ \|\nabla F\|^2.$$

This clearly implies that

$$\frac{1}{2}\partial_t\|\nabla F(C_x(t))\|^2 \leq -\lambda\ \|\nabla F(C_x(t))\|^2.$$

By Gronwall's lemma we conclude that

$$\|\nabla F(C_x(t))\| \leq e^{-\lambda t}\ \|\nabla F(x)\|.$$

Using (23.5) we have

$$\begin{aligned}
&\dot{C_x}\ (t) = -\nabla F(C_x(t)) \\
&\Longrightarrow\ 0 \leq F(x) - F(C_x(t)) \ = \ \int_0^t\ \|\nabla F(C_x(s))\|^2\ ds \\
&\qquad\qquad \leq\ \|\nabla F(x)\|^2 \int_0^t e^{-2\lambda s}\ ds \leq \frac{1}{2\lambda}\ \|\nabla F(x)\|^2.
\end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 409:

We consider a couple of r-dimensional vectors $U = \left(U^i\right)_{1\leq i\leq r}$ and $V = \left(V^i\right)_{1\leq i\leq r} \in \mathbb{R}^r$. Using the decomposition of U and V in the unit basis vectors e_i of $\mathbb{R}^r$

$$U = \sum_{1\leq i\leq r} U^i\ e_i \quad \text{and} \quad V = \sum_{1\leq j\leq r} V^j\ e_j$$

and recalling the rules $e_i \wedge e_i = 0$ and $(e_i \wedge e_j) = -(e_j \wedge e_i)$ we check that

$$\begin{aligned}
U \wedge V &= \sum_{1\leq i,j\leq r} U^i V^j\ (e_i \wedge e_j) \\
&= \sum_{1\leq i<j\leq r} U^i V^j\ (e_i \wedge e_j) + \sum_{1\leq i>j\leq r} U^i V^j\ (e_i \wedge e_j) \\
&= \sum_{1\leq i<j\leq r} \left(U^i V^j - U^j V^i\right)\ (e_i \wedge e_j)\,.
\end{aligned}$$

On the other hand $(e_i \wedge e_j)$ are mutually orthogonal so that

$$\langle U \wedge V, U \wedge V\rangle = \sum_{1\leq i<j\leq r} \left(U^i V^j - U^j V^i\right)^2 = \|U \wedge V\|^2\,.$$

We are now in position to check the Lagrange identity. Using elementary manipulations, we have

$$\begin{aligned}
\|U\|^2 \times \|V\|^2 - |\langle U, V\rangle|^2 &= \left(\sum_{1\leq i\leq r} (U^i)^2\right)\left(\sum_{1\leq i\leq r} (V^i)^2\right) - \left(\sum_{1\leq i\leq r} U^i V^i\right)^2 \\
&= \sum_{1\leq i,j\leq r} (U^i)^2 (V^j)^2 - \sum_{1\leq i,j\leq r} U^i U^j\ V^i V^j \\
&= \sum_{1\leq i\leq r} (U^i V^i)^2 + \sum_{1\leq i<j\leq r} \left[(U^i)^2 (V^j)^2 + (U^j)^2 (V^i)^2\right] \\
&\qquad - \sum_{1\leq i\leq r} (U^i V^i)^2 - 2 \sum_{1\leq i<j\leq r} U^i U^j\ V^i V^j \\
&= \sum_{1\leq i<j\leq r} \left[(U^i V^j)^2 - 2(U^i V^j)(U^j V^i) + (U^j V^i)^2\right].
\end{aligned}$$

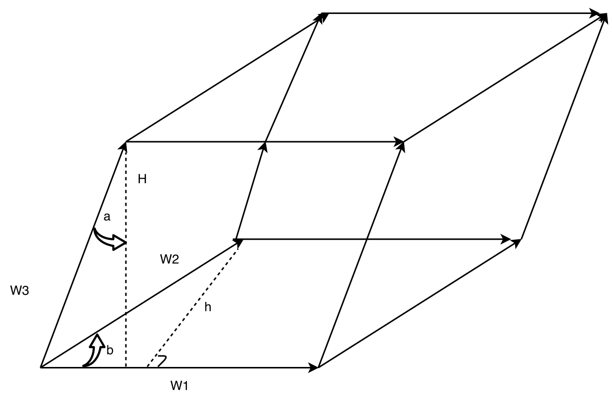
This implies that

$$\|U\|^2 \times \|V\|^2 - |\langle U, V\rangle|^2 \quad = \quad \sum_{1\leq i<j\leq r} \left[U^i V^j - U^j V^i\right]^2 = \|U \wedge V\|^2 .$$

This ends the proof of the exercise. ■

Solution to exercise 410:

Recall that the volume of a parallepiped $\mathcal{P}(W_1, W_2, W_3)$ in $\mathbb{R}^3$ formed by three independent vectors (W_1, W_2, W_3) is the surface of the base-parallelogram $\mathcal{P}(W_1, W_2)$ in $\mathbb{R}^2$ formed by the vectors (W_1, W_2) multiplied by the height H.



On the other hand, we have

$$\text{Surface}\left(\mathcal{P}(W_1, W_2)\right) = h \times \|W_1\| \quad \text{with} \quad \sin(b) = \frac{h}{\|W_2\|} \quad \text{and} \quad \cos(a) = \frac{H}{\|W_3\|}.$$

This implies that

$$\text{Volume}\left(\mathcal{P}(W_1, W_2, W_3)\right) = H \ h \times \|W_1\| = \cos(a)\sin(b) \ \|W_1\| \ \|W_2\| \ \|W_3\| \ .$$

It is also well known that

$$\|W_1 \wedge W_2\| = \sin(b) \ \|W_1\| \ \|W_2\|.$$

We conclude that

$$\begin{aligned}\text{Volume}\left(\mathcal{P}(W_1, W_2, W_3)\right) &= \cos(a) \ \|W_1 \wedge W_2\| \ \|W_3\| \\ &= |\langle W_1 \wedge W_2, W_3\rangle| = |\det(W_1, W_2, W_3)\rangle| .\end{aligned}$$

The last assertion follows from (23.14). This ends the proof of the exercise. ■

Solution to exercise 411:

The geodesics of the unit sphere have been computed in some details in the end of section 24.1.2. Next, we provide a more detailed discussion. Section 24.1.2 contains the derivation of the Christofell symbols associated with spherical coordinates $\psi\ :\ \theta=(\theta_1,\theta_2)\in (]0,\pi[\times]0,2\pi[)\mapsto S$

$$\psi(\theta)=\left(\begin{array}{c}\sin(\theta_1)\cos(\theta_2)\\ \sin(\theta_1)\sin(\theta_2)\\ \cos(\theta_1)\end{array}\right)=(\partial\varphi)_{\psi}(\theta).$$

These parameters are given by

$$C^1_{1,1}=0=C^2_{1,1}=0=C^2_{2,2}\quad,\quad C^1_{1,2}=0=C^1_{2,1}$$

and

$$C^2_{1,2}(\theta)=C^2_{2,1}(\theta)=\frac{\cos(\theta_1)}{\sin(\theta_1)}\quad,\quad C^1_{2,2}(\theta)=-\sin(\theta_1)\cos(\theta_1).$$

This ends the proof of the exercise. By (23.2), the geodesics curves $c(t)=\left(c^1(t),c^2(t)\right)\in S_\psi$ starting at $c_0=(\theta_1,\theta_2)$ with some initial velocity $\dot{c}_0=(\dot{\theta}_1,\dot{\theta}_2)$ are defined by

$$\left\{\begin{array}{rcl}\ddot{c}^{\,1}_t&=&\sin(c^1_t)\cos(c^1_t)\left(\dot{c}^{\,2}_t\right)^2\\ \ddot{c}^{\,2}_t&=&-2\,\frac{\cos(c^1_t)}{\sin(c^1_t)}\ \dot{c}^{\,1}_t\ \ \dot{c}^{\,2}_t\end{array}\right.\quad\text{with}\quad\left\{\begin{array}{rclrcl}c^1_0&=&\theta_1&\dot{c}^{\,1}_0&=&\dot{\theta}_1\\ c^2_0&=&\theta_2&\dot{c}^{\,2}_0&=&\dot{\theta}_2\,.\end{array}\right.$$

These equations are rather complex to solve numerically. Nevertheless, we notice that rotations are isometries on the sphere so we can rotate the sphere so that the initial starting point is $c_0=(\theta_1,\theta_2)=\left(\frac{\pi}{2},0\right)$. Now, we rotate the sphere w.r.t. the $(0,x_1)$ axis so that $\dot{\theta}_1=0$. The differential equations of the geodesic curve remain the same and the solution is now given by

$$c(t)=\left(\begin{array}{c}\frac{\pi}{2}\\ \dot{\theta}_2\ t\end{array}\right)\quad\left(\Rightarrow\ddot{c}^{\,1}_t=\ddot{c}^{\,2}_t=\dot{c}^{\,1}_t=\cos(c^1_t)=0\right).$$

Finally, we observe that

$$C(t)=\psi(c(t))=\left(\begin{array}{c}\sin\left(\frac{\pi}{2}\right)\cos\left(\dot{\theta}_2\ t\right)\\ \sin\left(\frac{\pi}{2}\right)\sin\left(\dot{\theta}_2\ t\right)\\ \cos(\frac{\pi}{2})\end{array}\right)=\left(\begin{array}{c}\cos\left(\dot{\theta}_2\ t\right)\\ \sin\left(\dot{\theta}_2\ t\right)\\ 0\end{array}\right).$$

This shows that this geodesic reduces to the equator of the sphere, that is the intersection of $\mathbb{S}^2$ with the plane $\{x=(x_1,x_2,x_3)\ :\ x_3=0\}$. Since all rotations transform the equator into great circles we conclude that the great circles are the geodesics of the sphere.

This ends the proof of the exercise. ■

Solution to exercise 412: Notice that

$$C(t)=\psi(c(t))=\psi(c_1(t),c_2(t))\Rightarrow\frac{d}{dt}C(t)=(\partial_{\theta_1}\psi)_{c(t)}\ \ \dot{c}_1\ (t)+(\partial_{\theta_2}\psi)_{c(t)}\ \ \dot{c}_2\ (t)$$

for the curve $c\ :\ t\in[a,b]\mapsto c(t)\in S_\psi$. This implies that

$$\begin{array}{rcl}\|C(t)\|^2&=&\left\langle\left[(\partial_{\theta_1}\psi)_{c(t)}\ \ \dot{c}_1\ (t)+(\partial_{\theta_2}\psi)_{c(t)}\ \ \dot{c}_2\ (t)\right],\left[(\partial_{\theta_1}\psi)_{c(t)}\ \ \dot{c}_1\ (t)+(\partial_{\theta_2}\psi)_{c(t)}\ \ \dot{c}_2\ (t)\right]\right\rangle\\ &=&\left\langle(\partial_{\theta_1}\psi)_{c(t)},(\partial_{\theta_1}\psi)_{c(t)}\right\rangle\ \ \dot{c}_1^{\ 2}\ (t)\ +2\ \left\langle(\partial_{\theta_1}\psi)_{c(t)},(\partial_{\theta_2}\psi)_{c(t)}\right\rangle\ \dot{c}_1\ (t)\ \dot{c}_2\ (t)\\ &&\qquad+\left\langle(\partial_{\theta_2}\psi)_{c(t)},(\partial_{\theta_2}\psi)_{c(t)}\right\rangle\ \ \dot{c}_2^{\ 2}\ (t)\\ &=&g_{1,1}(c(t))\ \dot{c}_1^{\ 2}\ (t)+2\ g_{1,2}(c(t))\ \dot{c}_1^{\ 2}\ (t)+g_{2,2}(c(t))\ \dot{c}_2^{\ 2}\ (t).\end{array}$$

This shows that

$$s = \int_a^b \quad \|C(t)\| \ dt = \int_a^b \left\|\dot{c}\ (t)\right\|_{g(c(t))} \ dt.$$

We consider the cylinder S defined by $x_1^2 + x_2^2 = r$ and $x_3 \in \mathbb{R}$, equipped with the parametrization

$$\theta = (\theta_1, \theta_2) \mapsto \psi(\theta) = \left\{ \begin{array}{lcl} \psi^1(\theta) & = & r\cos(\theta_1) \\ \psi^2(\theta) & = & r\sin(\theta_1) \\ \psi^3(\theta) & = & \theta_2. \end{array} \right.$$

In this situation, we have

$$\partial_{\theta_1}\psi = r \left(\begin{array}{c} -\sin(\theta_1) \\ \cos(\theta_1) \\ 0 \end{array} \right) \perp \partial_{\theta_2}\psi = \left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \Rightarrow g_{1,1} = r^2 \quad g_{1,2} = 0 = g_{2,1} \quad \text{and} \quad g_{2,2} = 1.$$

This implies that

$$\mathcal{L}(C, [a,b]) = \int_a^b \sqrt{r^2 \ \dot{c_1}^2 \ (t) + \dot{c_2}^2 \ (t)} \ dt.$$

Finally, we have

$$C_1(t) = \psi(\alpha t, \beta) \ \Rightarrow \ \dot{c_1} \ (t) = \alpha \quad \text{and} \quad \dot{c_2} \ (t) = 0.$$

This implies that

$$\mathcal{L}(C_1, [a,b]) = r\alpha \ \int_a^b \ 1 \ dt = \alpha r \ (b-a).$$

In the same way, we prove that

$$\mathcal{L}(C_0, [a,b]) = r\alpha \ \int_a^b \ 1 \ dt = \alpha r \ (b-a) = \mathcal{L}(C_1, [a,b]).$$

Also

$$\mathcal{L}(C_2, [a,b]) = r\alpha \ \int_a^b \ 2t \ dt = \alpha r \ (b^2 - a^2) = (a+b) \ \mathcal{L}(C_1, [a,b]).$$

This ends the proof of the exercise. ∎

Solution to exercise 413:
We consider the disk parametrization (23.17). We also recall (cf. (21.14)) that

$$W = \langle W, \nabla\phi^1 \rangle \ \left(\partial_{\theta_1}\psi\right)_\phi + \langle W, \nabla\phi^2 \rangle \ \left(\partial_{\theta_2}\psi\right)_\phi$$

with the inverse mapping $\phi = \psi^{-1}$ and is covariant derivatives

$$\nabla\phi^i = g_\phi^{i,1} \ \left(\partial_{\theta_1}\psi\right)_\phi + g_\phi^{i,2} \ \left(\partial_{\theta_2}\psi\right)_\phi.$$

In the above display we have used the notation $g_\phi^{i,j} = g^{i,j} \circ \phi$ with the entries $g^{i,j}$ of g^{-1}, and $\left(\partial_{\theta_2}\psi\right)_\phi = (\partial_{\theta_2}\psi) \circ \phi$. In this situation we have

$$\langle \psi, \partial_{\theta_1}\psi \rangle = -R^2(1-\theta_1) \quad \text{and} \quad \langle \psi, \partial_{\theta_2}\psi \rangle = 0$$

and

$$g^{-1} = \left(\begin{array}{cc} R^{-2} & 0 \\ 0 & R^{-2}(1-\theta_1)^{-2} \end{array} \right).$$

This implies that

$$W\circ\psi=\frac{1}{2}\ \psi\Rightarrow\langle W\circ\psi,(\nabla\phi^1)\circ\psi\rangle=-\frac{1}{2}(1-\theta_1)\quad\text{and}\quad\langle W\circ\psi,(\nabla\phi^2)\circ\psi\rangle=0.$$

Therefore

$$W\circ\psi=V^1\ \partial_{\theta_1}\psi+V^2\ \partial_{\theta_1}\psi\quad\text{with}\quad V^1=-\frac{1}{2}(1-\theta_1)\quad\text{and}\quad V^2=0.$$

We conclude that

$$\begin{aligned}\int_S \operatorname{div}(W)\ d\mu_S &= \int_{]0,1[\times]0,2\pi[}\operatorname{div}_g(V)\ d\mu_g\\ &= \int_{]0,1[\times]0,2\pi[}\left[\partial_{\theta_1}\left(\sqrt{\det}(g)\ V^1\right)+\partial_{\theta_2}\left(\sqrt{\det}(g)\ V^2\right)\right]\ d\theta_1 d\theta_2\\ &= R^2\int_0^{2\pi}\left[\int_0^1\partial_{\theta_1}\left(-\frac{1}{2}(1-\theta_1)^2\right)\ d\theta_1\right]\ d\theta_2=\pi\ R^2.\end{aligned}$$

This result coincides with the one obtained in (23.18). In the present exercise, we have computed the integral without using the divergence theorem.

This ends the proof of the exercise. ■

Solution to exercise 414:

We parametrize the sphere $\mathbb{S}$ with the spherical coordinates

$$\psi(\theta_1,\theta_2)=\begin{pmatrix} r\ \sin(\theta_1)\cos(\theta_2)\\ r\ \sin(\theta_1)\sin(\theta_2)\\ r\ \cos(\theta_1)\end{pmatrix}.$$

We have

$$\partial_{\theta_1}\psi=\begin{pmatrix} r\ \cos(\theta_1)\cos(\theta_2)\\ r\ \cos(\theta_1)\sin(\theta_2)\\ -r\ \sin(\theta_1)\end{pmatrix}\perp\partial_{\theta_2}\psi=\begin{pmatrix} -r\ \sin(\theta_1)\sin(\theta_2)\\ r\ \sin(\theta_1)\cos(\theta_2)\\ 0\end{pmatrix}.$$

In this situation, the surface Riemannian metric is given by

$$g=r^2\ \begin{pmatrix} 1 & 0\\ 0 & \sin^2(\theta_1)\end{pmatrix}\Rightarrow\det(g)((\theta_1,\theta_2))=r^4\ \sin^2(\theta_1)$$

as soon as $\theta_1\not\in\{0,\pi\}$. The corresponding Riemannian surface measure is defined by

$$\mu_g(d(\theta_1,\theta_2))=\sqrt{\det(g)(\theta_1,\theta_2)}\ d\theta_1 d\theta_2=r^2\ \sin(\theta_1)\ d\theta_1 d\theta_2$$

where $d\theta_1 d\theta_2$ stands for an infinitesimal neighborhood of some point $(\theta_1,\theta_2)\in([0,\pi]\times[0,2\pi[) := \mathbb{S}_\psi$ Observe that the outward-pointing normal at some point $x=\psi(\theta)$ is given by

$$n^\perp=N^\perp(\psi(\theta))=\frac{1}{r}\ \psi(\theta_1,\theta_2).$$

On the other hand, we have

$$W(x)=\frac{1}{3}\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}\Longrightarrow W\circ\psi=\frac{1}{3}\ \psi\Longrightarrow\langle W\circ\psi,N^\perp\circ\psi\rangle=\frac{1}{3}\ \langle\psi,\psi\rangle=\frac{r}{3}.$$

This implies that

$$\begin{aligned}\int_{\mathbb{S}} \langle W, N^{\perp}\rangle \, d\mu_{\mathbb{S}} &= \int_{\mathbb{S}_\psi} \langle W\circ\psi, N^{\perp}\circ\psi\rangle \, d\mu_g \\ &= \frac{r^3}{3}\int_0^{2\pi} \underbrace{\left(\int_0^{\pi} \sin(\theta_1)\, d\theta_1\right)}_{[\cos(\theta_1)]_\pi^0=2} d\theta_2 = \frac{4r^3\pi}{3}.\end{aligned}$$

Using the fact that $\mathrm{div}(W)=1$ we check that

$$\mu_{\mathbb{B}}(\mathbb{B}) = \int_{\mathbb{B}} \mathrm{div}(W)\, d\mu_{\mathbb{B}} = \int_{\partial\mathbb{B}} \langle W, N^{\perp}\rangle \, d\mu_{\partial\mathbb{B}} = \frac{4r^3\pi}{3}.$$

For any $x\in\mathbb{S}$, we also have

$$N^{\perp}(x) = \frac{x}{\|x\|} = \frac{x}{r} \Rightarrow \langle W, N^{\perp}\rangle = \frac{1}{3r}\langle x, x\rangle = \frac{r}{3}$$

from which we find that

$$\int_{\partial\mathbb{B}} \langle W, N^{\perp}\rangle \, d\mu_{\partial\mathbb{B}} = \frac{r}{3}\,\mu_{\partial\mathbb{B}}(\partial\mathbb{B}) = \frac{4r^3\pi}{3} \Rightarrow \mu_{\partial\mathbb{B}}(\partial\mathbb{B}) = 4r^2\pi.$$

This ends the proof of the exercise. ■

Solution to exercise 415: We parametrize the 3-Ball $\mathbb{B}$ with the coordinates

$$\psi_0(\theta_0,\theta_1,\theta_2) = \begin{pmatrix} r(1-\theta_0)\ \sin(\theta_1)\cos(\theta_2) \\ r(1-\theta_0)\ \sin(\theta_1)\sin(\theta_2) \\ r(1-\theta_0)\ \cos(\theta_1)\end{pmatrix}$$

with $(\theta_0,\theta_1,\theta_2)\in([0,1]\times[0,\pi]\times[0,2\pi[) := \mathbb{B}_{\psi_0}$. We have

$$\psi_0\left(\{0\}\times([0,\pi]\times[0,2\pi[)\right) = \ = \partial\mathbb{B}.$$

In addition, the parametrization of the boundary $\partial\mathbb{B}$ is given by the spherical coordinates discussed in exercise 414 and defined by the trace mapping

$$(\theta_1,\theta_2)\in(\partial\mathbb{B})_{\psi_0} = [0,\pi]\times[0,2\pi[= \mathbb{S}_\psi \mapsto \psi(\theta_1,\theta_2) = \psi_0(0,\theta_1,\theta_2).$$

On the other hand, we have

$$\partial_{\theta_1}\psi_0 = r(1-\theta_0)\begin{pmatrix}\cos(\theta_1)\cos(\theta_2)\\ \cos(\theta_1)\sin(\theta_2)\\ -\sin(\theta_1)\end{pmatrix} \perp \partial_{\theta_2}\psi_0 = r(1-\theta_0)\begin{pmatrix}-\sin(\theta_1)\sin(\theta_2)\\ \sin(\theta_1)\cos(\theta_2)\\ 0\end{pmatrix}$$

as well as

$$\partial_{\theta_1}\psi_0 \perp \partial_{\theta_0}\psi_0 = -r\ n^{\perp} \perp \partial_{\theta_2}\psi_0$$

with the unit outward pointing normal on the sphere

$$n^{\perp}(\theta_1,\theta_2) := \begin{pmatrix}\sin(\theta_1)\cos(\theta_2)\\ \sin(\theta_1)\sin(\theta_2)\\ \cos(\theta_1)\end{pmatrix}.$$

After some elementary manipulations, we have

$$\partial_{\theta_1}\psi_0 \wedge \partial_{\theta_2}\psi_0 = r^2(1-\theta_0)^2\ \sin(\theta_1)\begin{pmatrix}\sin(\theta_1)\cos(\theta_2)\\ \sin(\theta_1)\sin(\theta_2)\\ \cos(\theta_1)\end{pmatrix} = r^2(1-\theta_0)^2\ \sin(\theta_1)\ n^{\perp}(\theta_1,\theta_2).$$

This implies that

$$\langle(\partial_{\theta_1}\psi_0 \wedge \partial_{\theta_2}\psi_0)\,, -\partial_{\theta_0}\psi_0\rangle = r^3(1-\theta_0)^2\ \sin(\theta_1)$$

and

$$\|(\partial_{\theta_1}\psi_0 \wedge \partial_{\theta_2}\psi_0)\| = r^2(1-\theta_0)^2\ \sin(\theta_1).$$

In this situation, the Riemannian metric on the 3-Ball is given by the (3×3)-diagonal matrix

$$\begin{aligned} g &= \begin{pmatrix} \langle\partial_{\theta_0}\psi_0,\partial_{\theta_0}\psi_0\rangle & \langle\partial_{\theta_0}\psi_0,\partial_{\theta_1}\psi_0\rangle & \langle\partial_{\theta_0}\psi_0,\partial_{\theta_2}\psi_0\rangle\\ \langle\partial_{\theta_1}\psi_0,\partial_{\theta_0}\psi_0\rangle & \langle\partial_{\theta_1}\psi_0,\partial_{\theta_1}\psi_0\rangle & \langle\partial_{\theta_1}\psi_0,\partial_{\theta_2}\psi_0\rangle\\ \langle\partial_{\theta_2}\psi_0,\partial_{\theta_0}\psi_0\rangle & \langle\partial_{\theta_2}\psi_0,\partial_{\theta_1}\psi_0\rangle & \langle\partial_{\theta_2}\psi_0,\partial_{\theta_2}\psi_0\rangle \end{pmatrix}\\ &= \begin{pmatrix} r^2 & 0 & 0\\ 0 & r^2(1-\theta_0)^2 & 0\\ 0 & 0 & r^2(1-\theta_0)^2\sin^2(\theta_1)\end{pmatrix}. \end{aligned}$$

In much the same way, the Riemannian metric on the 2-sphere $\mathbb{S}=\partial\mathbb{B}$ is given by the (2×2)-diagonal matrix

$$g_\partial = \begin{pmatrix} \langle\partial_{\theta_1}\psi,\partial_{\theta_1}\psi\rangle & \langle\partial_{\theta_1}\psi,\partial_{\theta_2}\psi\rangle\\ \langle\partial_{\theta_2}\psi,\partial_{\theta_1}\psi\rangle & \langle\partial_{\theta_2}\psi,\partial_{\theta_2}\psi\rangle\end{pmatrix} = \begin{pmatrix} r^2 & 0\\ 0 & r^2\sin^2(\theta_1)\end{pmatrix}.$$

The corresponding volume and surface measures on $\mathbb{B}$ and $\partial\mathbb{B}=\mathbb{S}$ are given by

$$\mu_g(d(\theta_0,\theta_1,\theta_2)) = r\ (1-\theta_0)^2\ d\theta_0\ \mu_{g_\partial}(d(\theta_1,\theta_2))\ \ \text{with}\ \ \mu_{g_\partial}(d(\theta_1,\theta_2)) := r^2\ \sin(\theta_1)d\theta_1 d\theta_2.$$

From previous calculations, we have

$$\begin{aligned}\mu_g(d(\theta_0,\theta_1,\theta_2)) &= \langle(\partial_{\theta_1}\psi_0 \wedge \partial_{\theta_2}\psi_0)\,, -\partial_{\theta_0}\psi_0\rangle\ d\theta_0 d\theta_1 d\theta_2\\ &= \left\langle -\partial_{\theta_0}\psi_0, n^{\perp}\right\rangle\ d\theta_0 \times \|(\partial_{\theta_1}\psi_0 \wedge \partial_{\theta_2}\psi_0)\|\, d\theta_1 d\theta_2.\end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 416:
We use the same notation as in the solution of exercise 415. Expressed in the local coordinate ψ_0 any vector field W on $\mathbb{B}$ takes the form

$$W\circ\psi_0 = V^0\ \partial_{\theta_0}\psi_0 + V^1\ \partial_{\theta_1}\psi_0 + V^2\ \partial_{\theta_2}\psi_0$$

for some vector field $V=\left(V^i\right)_{0\le i\le 2}$ which can be computed using the formula (21.14). Recalling that

$$\begin{aligned}(\partial_{\theta_1}\psi_0)(\theta)\perp(\partial_{\theta_0}\psi_0)(\theta) &= -r\ n^{\perp}(\theta_1,\theta_2)\\ &= -r\ \left(N^{\perp}\circ\psi_0\right)(0,\theta_1,\theta_2)\\ &= -r\ \left(N^{\perp}\circ\psi\right)(\theta_1,\theta_2)\perp(\partial_{\theta_2}\psi_0)(\theta)\perp(\partial_{\theta_1}\psi_0)(\theta)\end{aligned}$$

for any $\theta = (\theta_0, \theta_1, \theta_2)$ we have

$$\left\langle (W \circ \psi)(\theta_1, \theta_2), \left(N^{\perp} \circ \psi\right)(\theta_1, \theta_2) \right\rangle = -r\ V^0(0, \theta_1, \theta_2) \tag{30.53}$$

so that

$$\begin{aligned} \int_{\partial\mathbb{B}} \left\langle W, N^{\perp} \right\rangle\ d\mu_{\partial\mathbb{B}} &= \int_{\mathbb{S}_\psi} -r\ V^0(0, \theta_1, \theta_2)\ \mu_{g_\partial}(d(\theta_1, \theta_2)) \\ &= -r^3 \int_0^{2\pi} \left[\int_0^{\pi} V^0(0, \theta_1, \theta_2)\ \sin(\theta_1)\ d\theta_1\right]\ d\theta_2. \end{aligned}$$

Also,

$$\psi_0(\theta_0, \theta_1, 0) = (1 - \theta_0) \begin{pmatrix} r\ \sin(\theta_1) \\ 0 \\ \cos(\theta_1) \end{pmatrix} = \psi_0(\theta_0, \theta_1, 2\pi)$$

$$\Rightarrow (W \circ \psi_0)(\theta_0, \theta_1, 0) = (W \circ \psi_0)(\theta_0, \theta_1, 2\pi)$$

which implies that

$$\begin{aligned} &\langle (W \circ \psi_0)(\theta_0, \theta_1, 0), (\partial_{\theta_0}\psi_0)(\theta_0, \theta_1, 0) \rangle \\ &= r^2\ V^0(\theta_0, \theta_1, 0) \\ &= r^2\ V^0(\theta_0, \theta_1, 2\pi) = \langle (W \circ \psi_0)(\theta_0, \theta_1, 2\pi), (\partial_{\theta_0}\psi_0)(\theta_0, \theta_1, 2\pi) \rangle. \end{aligned}$$

In much the same way we find that

$$r^2\ (1 - \theta_0)^2\ V^1(\theta_0, \theta_1, 0) = r^2\ \ (1 - \theta_0)^2\ V^1(\theta_0, \theta_1, 2\pi)$$

and

$$r^2\ (1 - \theta_0)^2\ \sin^2(\theta_1)\ V^2(\theta_0, \theta_1, 0) = r^2\ \ (1 - \theta_0)^2\ \sin^2(\theta_1)\ V^2(\theta_0, \theta_1, 2\pi).$$

We conclude that

$$\forall i = 0, 1, 2 \qquad V^i(\theta_0, \theta_1, 0) = V^i(\theta_0, \theta_1, 2\pi) \tag{30.54}$$

for any $\theta_0 \in [0, 1[$ and $\theta_1 \in]0, \pi[$.

We are now in position to compute the divergence integral

$$\begin{aligned} &\textstyle\int_{\mathbb{B}} \operatorname{div}(W)\ d\mu_{\mathbb{B}} = \int_{\mathbb{B}_{\psi_0}} \operatorname{div}_g(V)\ d\mu_g \\ &= \textstyle\int_0^{2\pi} \left[\int_0^{\pi} \left[\int_0^1 \left[\partial_{\theta_0}\left(V^0\sqrt{\det(g)}\right) + \partial_{\theta_1}\left(V^1\sqrt{\det(g)}\right) + \partial_{\theta_2}\left(V^2\sqrt{\det(g)}\right)\right]\ d\theta_0\right] d\theta_1\right] d\theta_2. \end{aligned}$$

Recalling that $\sqrt{\det(g)} = r^3\ (1 - \theta_0)^2\ \sin(\theta_1)$, the first integral is given by

$$\begin{aligned} &\int_0^{2\pi} \left[\int_0^{\pi} \left[\int_0^1 \partial_{\theta_0}\left(V^0\sqrt{\det(g)}\right)\ d\theta_0\right] d\theta_1\right] d\theta_2 \\ &= r^3 \int_0^{2\pi} \left[\int_0^{\pi} \left[\int_0^1 \partial_{\theta_0}\left(V^0(\theta_0, \theta_1, \theta_2)\ (1 - \theta_0)^2\right)\ d\theta_0\right] \sin(\theta_1) d\theta_1\right] d\theta_2 \\ &= \int_0^{2\pi} \left[\int_0^{\pi} (-rV^0(0, \theta_1, \theta_2))\ r^2 \sin(\theta_1) d\theta_1\right] d\theta_2 \\ &= \int_{]0,\pi[\times]0,2\pi[} \left\langle (W \circ \psi), \left(N^{\perp} \circ \psi\right) \right\rangle\ d\mu_{g_\partial}. \end{aligned}$$

The last assertion follows from (30.53).

The second integral is equal to zero due to

$$\int_0^{2\pi}\left[\int_0^1\left[\int_0^{\pi}\partial_{\theta_1}\left(V^1\sqrt{\det(g)}\right)\,d\theta_1\right]\,d\theta_0\right]d\theta_2$$

$$= r^3\int_0^{2\pi}\left[\int_0^1\underbrace{\left[\sin(\theta_1)\;V^1(\theta_0,\theta_1,\theta_2)\right]_0^{\pi}}_{=0}d\theta_0\right]d\theta_2 = 0.$$

and by (30.54) the third one is also equal to zero. We conclude that

$$\int_{\mathbb{B}}\operatorname{div}(W)\,d\mu_{\mathbb{B}} = \int_{\mathbb{S}_\psi}\left\langle (W\circ\psi)\,,\left(N^{\perp}\circ\psi\right)\right\rangle\,d\mu_{g_\partial} = \int_{\partial\mathbb{B}}\langle W,N^{\perp}\rangle\,d\mu_{\partial\mathbb{B}}.$$

This ends the proof of the exercise. ■

Solution to exercise 417:

The Langevin diffusion reduces to the linear Ornstein-Uhlenbeck process given by

$$X_t \;=\; e^{-tb/m}\,X_0+\frac{\sigma}{m}\int_0^t e^{-(t-s)b/m}\,dW_s.$$

It is readily checked that X_t is a Gaussian random variable with mean and variance given by:

$$\begin{aligned}
\mathbb{E}(X_t) &= e^{-tb/m}\,\mathbb{E}(X_0),\\
\operatorname{Var}(X_t) &= e^{-2tb/m}\,\operatorname{Var}(X_0)+\left(\frac{\sigma}{m}\right)^2\int_0^t e^{-(2b/m)s}\,ds\\
&= e^{-2tb/m}\,\operatorname{Var}(X_0)+\frac{\sigma^2}{2mb}\left(1-e^{-(2b/m)t}\right).
\end{aligned}$$

We have

$$dX_t = -\beta\,\partial_x V(X_t)\,dt+\overline{\sigma}\,dW_t \quad\text{with}\qquad V(x)=x^2/2\quad \beta=b/m\quad\text{and}\quad \overline{\sigma}=\sigma/m.$$

Notice that

$$\frac{2\beta}{\overline{\sigma}^2}\,\frac{x^2}{2} = 2\,\frac{b}{m}\,\frac{m^2}{\sigma^2}\,\frac{x^2}{2} = \frac{bm}{\sigma^2}\,x^2.$$

Using (23.23) the invariant measure is given by the Gaussian

$$\pi(dx)\propto\exp\left(-\frac{bm}{\sigma^2}\,x^2\right)\,dx.$$

From previous calculations, we have

$$\begin{aligned}
X_t &= e^{-tb/m}\,X_0+\frac{\sigma}{m}\left(\int_0^t e^{-(2b/m)s}\,ds\right)^{1/2}\frac{\int_0^t e^{-(t-s)b/m}\,dW_s}{\left(\int_0^t e^{-(2b/m)s}\,ds\right)^{1/2}}\\
&= e^{-tb/m}\,X_0+\frac{\sigma}{\sqrt{2mb}}\left(1-e^{-(2b/m)t}\right)^{1/2}\,V_s^{(t)}
\end{aligned}$$

with the centered Gaussian random variable $V_s^{(t)}$ with unit variance given by

$$V_s^{(t)} = \frac{\int_0^t e^{-(t-s)b/m}\, dW_s}{\left(\int_0^t e^{-(2b/m)s}\, ds\right)^{1/2}} \Rightarrow \text{Law}\left(\frac{\sigma}{\sqrt{2mb}}\, V_s^{(t)}\right) = \pi.$$

Using the estimates provided in example 8.3.10, we find that

$$\mathbb{W}\left(\text{Law}(X_t), \pi\right) \leq e^{-tb/m} \left(\mathbb{E}(X_0) + \frac{\sigma^2}{2mb}\, e^{-(b/m)t}\right).$$

This ends the proof of the exercise. ■

Solution to exercise 418:

By (20.3) and (20.4) the orthogonal projection and the mean curvature vector on $\mathbb{S}^2$ are given by

$$\pi(x) = Id - \frac{xx^T}{x^T x} \quad \text{and} \quad \mathbb{H}(x) = 2\, \frac{x}{x^T x}.$$

On the other hand we have

$$V(x) = x^T A x = \sum_{1\leq i\leq 3} x_i \sum_{1\leq j\leq 3} a_{i,j} x_j \Rightarrow (\partial V)(x) = (A + A^T)x.$$

Thus, using (23.28) we prove that

$$\begin{aligned} dX_t &= \left(Id - \frac{X_t X_t^T}{X_t^T X_t}\right) \left(-(A+A^T)(X_t)dt + dB_t\right) - \frac{X_t}{X_t^T X_t}\, dt \\ &= \left(Id - X_t X_t^T\right) \left(-(A+A^T)(X_t)dt + dB_t\right) - X_t\, dt \quad (\Leftarrow \|X_t\| = 1) \end{aligned}$$

has the desired reversible measure π.

This ends the proof of the exercise. ■

Solution to exercise 419:

Using the spherical coordinates ψ defined in (30.51), the Riemannian scalar product is given by the matrix

$$g(\theta) = \begin{pmatrix} g_{1,1}(\theta) & g_{1,2}\theta) \\ g_{1,2}\theta) & g_{2,2}\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta_1) \end{pmatrix}.$$

$$\Rightarrow \sqrt{\det(g(\theta))} = \sin(\theta_1) \Rightarrow \mu_g(d\theta) = \sin(\theta_1) d\theta_1 d\theta_2$$

with $\theta = (\theta_1, \theta_2) \in S_\psi = ([0,\pi] \times [0, 2\pi])$. On the other hand, we have

$$V(x) = x^T A x \Rightarrow U(\theta) := V(\psi(\theta)) = \psi(\theta)^T A \psi(\theta).$$

Using the fact that

$$g^{-1}(\theta) = \begin{pmatrix} g^{1,1}(\theta) & g^{1,2}\theta) \\ g^{1,2}\theta) & g^{2,2}\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-2}(\theta_1) \end{pmatrix} \Rightarrow \sqrt{g^{-1}(\theta)} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-1}(\theta_1) \end{pmatrix}.$$

By (23.29), the Brownian motion $\overline{B}_t$ on the Riemannian manifold S_ψ is defined for any $1 \leq i \leq p$ by

$$d\overline{B}_t^1 = dB_t^1 + \frac{1}{2}\cot(\Theta^1(t))\, dt.$$

This ends the proof of the exercise. ■

Solution to exercise 420:

The generator L of the diffusion $\mathcal{X}^\epsilon_t=(X_t,V_t)$ is given for any smooth function $f(x,v)$ by

$$\begin{aligned}L^\epsilon(f)(x,v) &= \epsilon\ \partial_v U^V(v)\ \partial_x f(x,v)-\left[\partial_v U^V(v)+\epsilon\partial_x U^X(x)\right]\ \partial_v f(x,v)+\partial_v^2 f(x,v)\\ &= \epsilon\left[\partial_v U^V(v)\ \partial_x f(x,v)-\partial_x U^X(x)\ \partial_v f(x,v)\right]-\partial_v U^V(v)\ \partial_v f(x,v)+\partial_v^2 f(x,v).\end{aligned}$$

For any smooth functions $f(x,v)$ and $g(x,v)$ with compact support we have

$$\begin{aligned}\pi(gL^\epsilon(f)) &\propto \int e^{-(U^X(x)+U^V(v))}\ g(x,v)\\ &\quad\times\left[\epsilon\partial_v U^V(v)\ \partial_x f(x,v)-\left[\partial_v U^V(v)+\epsilon\partial_x U^X(x)\right]\ \partial_v f(x,v)+\partial_v^2 f(x,v)\right]\ dxdv.\end{aligned}$$

By integration by parts, we have

$$\begin{aligned}&\int e^{-(U^X(x)+U^V(v))}\ g(x,v)\ \partial_v U^V(v)\ \partial_x f(x,v)\ dxdv\\ &= -\int e^{-(U^X(x)+U^V(v))}\ \left[\partial_v U^V(v)\ \underbrace{e^{U^X(x)}\ \partial_x\left(g(x,v)\ e^{-U^X(x)}\right)}_{=\underline{-g(x,v)\partial_x U^X(x)}+\partial_x g(x,v)}\right] f(x,v)\ dxdv\end{aligned}$$

and

$$\begin{aligned}&-\int e^{-(U^X(x)+U^V(v))}\ g(x,v)\ \partial_v U^V(v)\ \partial_v f(x,v)\ dxdv\\ &=\int e^{-(U^X(x)+U^V(v))}\ \underline{\left[e^{U^V(v)}\partial_v\left(\ g(x,v)\ e^{-U^V(v)}\ \partial_v U^V(v)\right)\right]} f(x,v)\ dxdv,\end{aligned}$$

as well as

$$\begin{aligned}&-\int e^{-(U^X(x)+U^V(v))}\ g(x,v)\ \partial_x U^X(x)\ \partial_v f(x,v)\ dxdv\\ &=\int e^{-(U^X(x)+U^V(v))}\ \left[\partial_x U^X(x)\ \underbrace{e^{U^V(v)}\partial_v\left(g(x,v)e^{-U^V(v)}\right)}_{=\underline{-g(x,v)\partial_v U^V(v)}+\partial_v g(x,v)}\right] f(x,v)\ dxdv.\end{aligned}$$

Finally we have

$$\begin{aligned}&\int e^{-(U^X(x)+U^V(v))}\ g(x,v)\ \ \partial_v^2 f(x,v)\ dxdv\\ &=\int e^{-(U^X(x)+U^V(v))}\ \left[e^{U^V(v)}\ \underbrace{\partial_v^2\left(\ g(x,v)\ e^{-U^V(v)}\right)}_{=\underline{-\partial_v\left(g(x,v)e^{-U^V(v)}\partial_v(U^V)(v)\right)}+\partial_v\left(e^{-U^V(v)}\partial_v g(x,y)\right)}\right] f(x,v)\ dxdv.\end{aligned}$$

Summing these terms we find that

$$\begin{aligned}\pi(gL^\epsilon(f)) &\propto \int e^{-(U^X(x)+U^V(v))}\ f(x,v)\\ &\quad\times\left[\epsilon\left(\partial_x U^X(x)\partial_v g(x,v)-\partial_v U^V(v)\partial_x g(x,v)\right)-\partial_v U^v(v)\partial_v g(x,v)+\partial_v^2 g(x,v)\right]\ dxdv\\ &= \pi\left(fL^{\epsilon,\star}(g)\right),\end{aligned}$$

with

$$L^{\epsilon,\star}(g)(x,v) = -\epsilon\left[\partial_v U^V(v)\ \partial_x g(x,v) - \partial_x U^X(x)\ \partial_v g(x,v)\right] - \partial_v U^V(v)\ \partial_v g(x,v) + \partial_v^2 g(x,v).$$

This shows that $L^{\epsilon,\star} = L^{-\epsilon}$.

This ends the proof of the exercise. ■

Chapter 25

Solution to exercise 421:

Using the fact that $(-1)^{\epsilon} \stackrel{in\ law}{=} U$, we check that

$$\Delta Y_n \quad := \quad Y_n - Y_{n-1} = v\ (-1)^{N_n} = v\ \ (U_1 \dots U_n)\,.$$

We set $X_n = U_1 \dots U_n$ The sequence (X_n, Y_n) is a Markov chain with transitions defined by the equation

$$\begin{cases} X_{n+1} &= X_n U_{n+1} \\ \Delta Y_{n+1} &= v X_{n+1}. \end{cases}$$

$$\begin{aligned} K(f)(x,y) &= \mathbb{E}(f(X_{n+1}, Y_{n+1}) \mid (X_n, Y_n) = (x,y)) \\ &= f(x, y+vx)\ e^{-a} + f(-x, y-vx)\ \left(1 - e^{-a}\right). \end{aligned}$$

We clearly have

$$a = \log 2 \quad \text{and} \quad v = v_0 t \Rightarrow w_t(y) = \mathbb{E}(g(Y_{n+1}) \mid (X_n, Y_n) = (1,y)) = \frac{g(y+v_0 t) + g(y - v_0 t)}{2}$$

and

$$\partial_t^2 w_t(y) = v^2\ \frac{\partial_y^2 g(y+v_0 t) + \partial_y^2 g(y - v_0 t)}{2} = v_0^2 \partial_y^2 w_t(y)$$

with the initial condition

$$w_0 = g \quad \text{and} \quad \partial_t w_{|t=0} = 0.$$

When $v = bh$ and $a = \lambda h$ we have

$$\begin{aligned} & h^{-1}\left[K_h(f)(x,y) - f(x,y)\right] \\ &= h^{-1}[f(x, y+bh) - f(x,y)]\ e^{-\alpha h} + [f(-x, y-bh) - f(x,y)]\ h^{-1}\left(1 - e^{-\alpha h}\right) \\ &\longrightarrow_{h \downarrow 0}\ b\ \partial_y f(x,y) + \lambda\ [f(-x,y) - f(x,y)] := L(f)(x,y). \end{aligned}$$

The generator L is a jump process with a first order term corresponding to a deterministic transport given the first variable. The Markov process with generator L is defined by

$$\begin{cases} \mathcal{X}_t &= (-1)^{N_t} \\ d\mathcal{Y}_t &= \mathcal{X}_t\ b\ dt \end{cases}$$

where N_t denotes a Poisson process with intensity λ. Notice that $\mathcal{X}_t$ changes its sign at a rate λ. In addition, given the "sign process" $\mathcal{X}_t$, the second component $\mathcal{Y}_t$ is a deterministic process with drift function $(\mathcal{X}_t \times b)$.

The process $(\mathcal{X}_t, \mathcal{Y}_t)$ coincides with the random 2-velocity process discussed in exercise 209 when $b(x) = b$, $a(x) = 0$ and $\lambda(+1) = \lambda(-1) := \lambda$.

This ends the proof of the exercise.

■

Solution to exercise 422:

Following the solution of exercise 209 we have

$$\partial_t p_t(1,y) = \lambda\,(\,p_t(-1,y) - p_t(1,y)) - b\partial_y p_t(1,y)$$

and

$$\partial_t p_t(-1,y) = \lambda\,(p_t(1,y) - p_t(-1,y)) + b\partial_y p_t(-1,y).$$

This implies that

$$\partial_t q_t^+(y) = -b\;\partial_y q_t^-(y) \quad \text{and} \quad \partial_t q_t^-(y) = -2\lambda\; q_t^-(y) - b\;\partial_y q_t^+(y).$$

Taking the partial derivative of the l.h.s. w.r.t. t and the r.h.s. w.r.t. y we find that

$$\partial_t^2 q_t^+(y) = -b\;\partial_{t,y} q_t^-(y) \quad \text{and} \quad \partial_{y,t} q_t^-(y) = -2\lambda\; \underbrace{\partial_y q_t^-(y)}_{=-\frac{1}{b}\;\partial_t q_t^+(y)}\; -b\;\partial_y^2 q_t^+(y)$$

from which we conclude that

$$\partial_t^2 q_t^+(y) + 2\lambda\;\partial_t q_t^+(y) = b^2\;\partial_y^2 q_t^+(y).$$

We also have

$$\mathbb{E}(\mathcal{Y}_t^2) = \int y^2\;\left(p_t(1,y) + p_t(-1,y)\right)\;dy = \int y^2\; q_t^+(y)\;dy$$

and

$$\begin{aligned}\int y^2\;\partial_t^2 q_t^+(y)\;dy + 2\lambda \int y^2\;\partial_t q_t^+(y) &= b^2\int y^2 \partial_y^2 q_t^+(y)\;dy \\ &= b^2\int \partial_y^2(y^2)\;q_t^+(y)\;dy = 2b^2.\end{aligned}$$

This yields

$$\partial_t^2 \mathbb{E}(Y_t^2) + 2\lambda\;\partial_t \mathbb{E}(Y_t^2) = 2b^2.$$

The solution associated with the initial conditions $\mathbb{E}(Y_0^2) = m_0$ and $m_0' = \partial_t \mathbb{E}(Y_t^2)_{t=0} > -1$ is given by

$$\mathbb{E}(Y_t^2) = \frac{b^2}{\lambda}\;\frac{1}{1+m_0'}\;\left(m_0 + (1+m_0')t - \frac{1}{2\lambda}\left(1 - e^{-2\lambda t}\right)\right).$$

To check this result we notice that

$$\begin{aligned}2\lambda\;\partial_t \mathbb{E}(Y_t^2) &= 2b^2\;\frac{1}{1+m_0'}\;\left((1+m_0') - e^{-2\lambda t}\right) \\ \partial_t^2 \mathbb{E}(Y_t^2) &= 2b^2\;\frac{1}{1+m_0'}\;e^{-2\lambda t}.\end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 423:

Following the developments of section 14.4.3 the random state $X_{T_{\mathbb{S}(x,\rho)}}$ is uniformly distributed on the sphere $\mathbb{S}(x,\rho)$. See also the exercise 240 on the rotational invariance of the 2-dimensional Brownian motion. We also refer the reader to the discussion on 444.

The Dirichlet-Poisson problem has the form (15.38) with $(V,g)=(0,0)$. Using (15.42) we have

$$v(x)=\mathbb{E}\left(h(x+W_{T_D})\right)$$

This ends the proof of the exercise. ■

Solution to exercise 424: When $c=1$ we recall that $B_n+W_n=N+n$ with $N=b+w$. Thus the Markov chain model is defined by

$$B_{n+1}=B_n+1_{U_{n+1}\in[0,B_n/(N_0+n)]}$$

with a sequence of i.i.d. uniform random variables U_n on $[0,1]$.

This ends the proof of the exercise.

■

Solution to exercise 425:

By construction, we have

$$\begin{aligned}\mathbb{E}\left(f(\overline{X}_{n+1})\mid X_n\right) &= \frac{1}{2}\,f(\overline{X}_n)+\left(\frac{1}{2}\sum_{i:\;X_n^i=1}\mu(i)\right)f(\overline{X}_n-1)\\ &\qquad+\left(\frac{1}{2}\sum_{i:\;X_n^i=0}\mu(i)\right)f(\overline{X}_n+1).\end{aligned}$$

When μ is the uniform probability, for any function f on $\{1,\ldots,d\}$ we find that

$$\mathbb{E}\left(f(\overline{X}_{n+1})\mid X_n\right)=\frac{1}{2}\,f(\overline{X}_n)+\frac{\overline{X}_n}{2d}\,f(\overline{X}_n-1)+\frac{d-\overline{X}_n}{2d}\,f(\overline{X}_n+1).$$

This shows that $\overline{X}_n$ is a Markov chain with transitions

$$\overline{M}(k,l)=\frac{1}{2}\,1_k(l)+\frac{k}{2d}\,1_{k-1}(l)+\frac{d-k}{2d}\,1_{k+1}(l)$$

$$\begin{aligned}&\left(\begin{array}{c}b\\k\end{array}\right)\,M(k,l)\\ &=(b-1)!\,\tfrac{1}{2}\left(\tfrac{b}{k!(b-k)!}1_k(l)+\tfrac{1}{(b-l)!(l-1)!}\,1_k(l-1)+\,\tfrac{1}{(b-(l+1))!l!}\,1_k(l+1)\right).\end{aligned}$$

This yields

$$\sum_{0\leq k\leq b}\left(\begin{array}{c}b\\k\end{array}\right)\,M(k,l)=(b-1)!\,\frac{1}{2}\left(\frac{b+l+(b-l)}{(b-l)!\,l!}\right)=\frac{b!}{(b-l)!l!}=\left(\begin{array}{c}b\\l\end{array}\right).$$

The lazy chain being aperiodic, the invariant measure is unique. Using Kac's formula, we have

$$\mathbb{E}(T_k\mid X_0=k)=\frac{1}{\pi(x)}=\frac{k!(b-k)!}{b!}\;2^k.$$

When $k=0$, we the expected return time is given by $\mathbb{E}(T_0\mid X_0=0)=2^{100}\simeq 2.27\;10^{30}$. This ends the proof of the exercise.

■

Solution to exercise 426: When $d = 2$, the number of steps up/down should be the same. This also holds for the number of steps left/right. We find that the number of paths with k steps up, and k steps down, $(n-k)$ up and $(n-k)$ down is given by

$$\begin{pmatrix} & & 2n & \\ k & k & (n-k) & (n-k) \end{pmatrix} = \frac{(2n)!}{k!^2(n-k)!^2}.$$

Recalling that $\sum_{0\leq k\leq n} \begin{pmatrix} n \\ k \end{pmatrix}^2 = \begin{pmatrix} 2n \\ n \end{pmatrix}$, this implies that

$$\begin{aligned}
&\mathbb{P}\left(X_{2n} = (0,0) \mid X_0 = (0,0)\right) \\
&= \sum_{0\leq k\leq n} \tfrac{(2n)!}{k!^2(n-k)!^2} \left(\tfrac{1}{4}\right)^{2n} \\
&= \tfrac{(2n)!\ 2^{-4n}}{n!^2} \sum_{0\leq k\leq n} \tfrac{n!^2}{k!^2(n-k)!^2} = \tfrac{(2n)!\ 2^{-4n}}{n!^2} \sum_{0\leq k\leq n} \begin{pmatrix} n \\ k \end{pmatrix}^2 \\
&= \left(2^{-2n} \tfrac{(2n)!}{n!^2}\right)^2 \simeq \tfrac{1}{\pi n}.
\end{aligned}$$

For $d = 3$, we observe that the chain needs to do the same number k of steps left/right, the same number l of steps up/down, and the same number $(n-k-l)$ of steps forward/backward. The number of such paths is given by

$$\begin{pmatrix} & & & 2n & & \\ k & k & l & l & (n-k-l) & (n-k-l) \end{pmatrix} = \frac{(2n)!}{k!^2 l!^2 (n-k-l)!^2}.$$

This implies that

$$\begin{aligned}
&\mathbb{P}\left(X_{2n} = (0,0,0) \mid X_0 = (0,0,0)\right) \\
&= \sum_{0\leq k+l\leq n} \frac{2n!}{k!^2 l!^2 (n-k-l)!^2} \left(\frac{1}{6}\right)^{2n} \\
&= \frac{(2n)!}{n!^2}\ 2^{-2n} \sum_{0\leq k+l\leq n} \frac{n!^2}{k!^2 l!^2 (n-k-l)!^2} \left(\frac{1}{3}\right)^{2n} \\
&= \frac{(2n)!}{n!^2}\ 2^{-2n} \\
&\times \textstyle\sum_{0\leq k+l\leq n} \mathbb{P}\left(\text{placing } n \text{ balls in three boxes with } k, l, (n-k-l) \text{ balls}\right)^2.
\end{aligned}$$

Since the probability of placing these n balls in three boxes with $k, l, (n-k-l)$ balls is maximal when $k \simeq l \simeq n-k-l \simeq \frac{n}{3}$ (use Stirling's formula to convince yourself), we have

$$\begin{aligned}
&\mathbb{P}\left(X_{2n} = (0,0,0) \mid X_0 = (0,0,0)\right) \\
&\leq \frac{(2n)!}{n!^2}\ 2^{-2n} \times \left(\frac{n!}{3^n \lfloor n/3 \rfloor!^3}\right) \\
&\times \textstyle\sum_{0\leq k+l\leq n} \underbrace{\mathbb{P}\left(\text{placing } n \text{ balls in three boxes with } k, l, (n-k-l) \text{ balls}\right)}_{=1}.
\end{aligned}$$

Now, by Stirling's formula, we have

$$\frac{n!}{3^n \lfloor n/3 \rfloor!^3} \simeq \frac{\sqrt{2\pi n}\ n^n\ e^{-n}}{3^n 2\pi(n/3)\ \ \sqrt{2\pi n/3}\ (n/3)^n\ e^{-n}} = \frac{1}{2\pi(n/(3\sqrt{3}))}.$$

This result combined with (25.1) implies that

$$\mathbb{P}\left(X_{2n} = (0,0,0) \mid X_0 = (0,0,0)\right) \leq c/n^{3/2}$$

with some positive constant c.

This ends the proof of the exercise.

■

Solution to exercise 427:

Using the same line of reasoning as the one we used in section 25.6.2, we find that

$$\begin{aligned}
&\mathbb{E}\left(f(X_1,\ldots,X_n) \mid (B_0,W_0)=(b,w)\right) \\
&= \sum_{0\leq k\leq n}\ \sum_{x_1+\ldots+x_n=k} f(x_1,\ldots,x_n) \\
&\qquad \times \frac{C\left(\frac{b}{c}+\sum_{1\leq i\leq n} x_i\right)\Gamma\left(\frac{w}{c}+(n-\sum_{1\leq i\leq n} x_i)\right)}{\Gamma\left(\frac{b+w}{c}+n\right)} \times \left(\frac{\Gamma\left(\frac{b}{c}\right)\Gamma\left(\frac{w}{c}\right)}{\Gamma\left(\frac{b+w}{c}\right)}\right)^{-1} \\
&= \int_0^1 \sum_{0\leq k\leq n}\ \sum_{x_1+\ldots+x_n=k} f(x_1,\ldots,x_n) \\
&\qquad\qquad \times u^{\sum_{1\leq i\leq n} x_i}\ \ (1-u)^{(n-\sum_{1\leq i\leq n} x_i)}\ p_{\left(\frac{b}{c},\frac{w}{c}\right)}(u)\ du.
\end{aligned}$$

This ends the proof of the first assertion. On the other hand, we readily check that $B_n = B_{n-1} + c\ X_n$ and $B_n + W_n = B_{n-1} + W_{n-1} + c$, from which we prove that

$$\begin{aligned}
\mathbb{E}\left(\frac{B_n}{B_n+W_n} \mid \mathcal{F}_{n-1}\right) &= \frac{1}{B_{n-1}+W_{n-1}+c}\left(B_{n-1}+c\ \frac{B_{n-1}}{B_{n-1}+W_{n-1}}\right) \\
&= \frac{B_{n-1}}{B_{n-1}+W_{n-1}}
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\left(e^{it\frac{B_n}{B_n+W_n}} \mid (B_0,W_0)=(b,w)\right) \\
&= \int_0^1 \sum_{0\leq k\leq n} \binom{n}{k} e^{it\frac{b+kc}{b+w+nc}}\ u^k\ (1-u)^{(n-k)}\ p_{\left(\frac{b}{c},\frac{w}{c}\right)}(u)\ du \\
&= e^{\frac{itb}{b+w+nc}} \int_0^1 \left\{\sum_{0\leq k\leq n} \binom{n}{k} e^{it\frac{kc}{b+w+nc}}\ u^k\ (1-u)^{(n-k)}\right\} p_{\left(\frac{b}{c},\frac{w}{c}\right)}(u)\ du \\
&= e^{\frac{itb}{b+w+nc}} \int_0^1 \left(\underbrace{e^{\frac{itc}{b+w+nc}}\ u+(1-u)}_{1+itu\frac{c}{b+w+nc}}\right)^n p_{\left(\frac{b}{c},\frac{w}{c}\right)}(u)\ du.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \mathbb{E}\left(e^{it\frac{B_n}{B_n+W_n}} \mid (B_0, W_0) = (b, w)\right) = \int_0^1 e^{itu}\; p_{(b/c,w/c)}(u)\; du$$

from which we conclude that $\frac{B_n}{B_n+W_n}$ converges in law, as $n \to \infty$, to $U_{(b/c,w/c)}$. This ends the proof of the exercise.

■

Solution to exercise 428:

In exercise 427, we proved that $P_n := \frac{B_n}{N_0+n}$ converges in law, as $n \uparrow$, to a random variable P_∞ with $\mathtt{Beta}(b, w)$ distribution. We consider the re-scaled continuous process

$$X_t = P_{\lfloor Nt \rfloor} = \frac{B_{\lfloor Nt \rfloor}}{N + \lfloor Nt \rfloor}.$$

When $h \downarrow 0$ and $N \uparrow \infty$ have

$$1 \leftarrow \frac{1 + (h/t)}{1 + (2/(Nt))} \leq \frac{\lfloor N(t+h) \rfloor}{\lfloor Nt \rfloor + 1} < \frac{N(t+h)+1}{Nt} = 1 + (h/t) + 1/(Nt) \to\; 1.$$

This shows that $\lfloor N(t+h) \rfloor = \lfloor Nt \rfloor + 1$ and

$$\Delta_h B_{\lfloor Nt \rfloor} = B_{\lfloor N(t+h) \rfloor} - B_{\lfloor Nt \rfloor} = B_{\lfloor Nt \rfloor + 1} - B_{\lfloor Nt \rfloor}.$$

By construction, we have

$$\mathbb{P}\left(\Delta_h B_{\lfloor Nt \rfloor} = 1 \mid X_t\right) = X_t = 1 - \mathbb{P}\left(\Delta_h B_{\lfloor Nt \rfloor} = 0 \mid X_t\right) \Rightarrow \mathbb{E}\left(\Delta_h B_{\lfloor Nt \rfloor} \mid X_t\right) = X_t.$$

Also

$$\begin{aligned}
\mathrm{Var}\left(\Delta_h B_{\lfloor Nt \rfloor} \mid X_t\right) &= \mathbb{E}\left(\left(\Delta_h B_{\lfloor Nt \rfloor} - X_t\right)^2 \mid X_t\right) \\
&= \mathbb{E}\left(\left(\Delta_h B_{\lfloor Nt \rfloor}\right)^2 \mid X_t\right) + X_t^2 - 2X_t \mathbb{E}\left(\Delta_h B_{\lfloor Nt \rfloor} \mid X_t\right) \\
&= 1 \times X_t - X_t^2 = X_t(1 - X_t).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\Delta_h X_t &:= X_{t+h} - X_t = \frac{B_{\lfloor N(t+h) \rfloor}}{N + \lfloor N(t+h) \rfloor} - X_t \\
&= \frac{B_{\lfloor Nt \rfloor + 1}}{N + \lfloor Nt \rfloor + 1} - X_t = \frac{B_{\lfloor Nt \rfloor} + \Delta_h B_{\lfloor Nt \rfloor}}{N + \lfloor Nt \rfloor + 1} - X_t \\
&= \frac{(N + \lfloor Nt \rfloor)\; X_t + \Delta_h B_{\lfloor Nt \rfloor}}{N + \lfloor Nt \rfloor + 1} - X_t = \frac{\Delta_h B_{\lfloor Nt \rfloor} - X_t}{N + \lfloor Nt \rfloor + 1}.
\end{aligned}$$

From previous calculations, we prove that

$$\begin{aligned}
\mathbb{E}\left(\Delta_h X_t \mid X_t\right) &= \frac{\mathbb{E}\left(\Delta_h B_{\lfloor Nt \rfloor} \mid X_t\right) - X_t}{N + \lfloor Nt \rfloor + 1} = 0 \\
\mathbb{E}\left(\left(\Delta_h X_t\right)^2 \mid X_t\right) &= \frac{1}{(N + \lfloor Nt \rfloor + 1)^2}\, \mathbb{E}\left(\left(\Delta_h B_{\lfloor Nt \rfloor} - X_t\right)^2 \mid X_t\right) \\
&= \frac{X_t(1 - X_t)}{(N + \lfloor Nt \rfloor + 1)^2}.
\end{aligned}$$

Finally observe that

$$\begin{aligned}\frac{1}{(N+\lfloor Nt\rfloor+1)} &= \frac{1}{N}\left(\frac{1}{1+t}-\left[\frac{1}{1+t}-\frac{1}{(1+\lfloor Nt\rfloor/N+1/N)}\right]\right)\\ &= \frac{1}{N}\,\frac{1}{1+t}\,\underbrace{\left(1-\frac{(\lfloor Nt\rfloor/N-t)+1/N}{1+\lfloor Nt\rfloor/N+1/N}\right)}_{\in[1-2/N,1]}.\end{aligned}$$

The last assertion is due to

$$0\leq \lfloor Nt\rfloor/N-t\leq 1/N \Rightarrow 0\leq \frac{(\lfloor Nt\rfloor/N-t)+1/N}{1+\lfloor Nt\rfloor/N+1/N}\leq 2/N.$$

For $h=1/N$, this implies that

$$\mathbb{E}\left((\Delta_h X_t)^2 \;\mid X_t\right)=\frac{1}{N}\,\frac{X_t(1-X_t)}{(1+t)^2}\;h\;\left(1-\epsilon_t(N)\right)\quad \text{with}\quad \epsilon_t(N)\in[0,2/N].$$

The Wright-Fisher diffusion approximation on some time mesh $t_{n-1}<t_n$ s.t. $(t_n-t_{n-1})=h=1/N$ is given by the increments

$$Y_{t_n+h}-Y_{t_n}=\frac{1}{1+t_n}\sqrt{\frac{Y_{t_n}(1-Y_{t_n})}{N}}\;\sqrt{h}\;\mathcal{V}_n$$

with some centered Gaussian random variable $\mathcal{V}_n$ with unit mean. We then check immediately that

$$\mathbb{E}(Y_{t_n+h}-Y_{t_n}\mid Y_{t_n})=0$$

and

$$\mathbb{E}((Y_{t_n+h}-Y_{t_n})^2\mid Y_{t_n})=\frac{1}{(1+t_n)^2}\;\frac{Y_{t_n}(1-Y_{t_n})}{N}\;h.$$

This shows that the processes X_t and Y_t follow the same evolution as $h=1/N$ and $N\uparrow\infty$. This ends the proof of the exercise.

■

Solution to exercise 429: We have

$$\forall i\in S\quad \mathbb{P}\left(A_n(i)\right)=\prod_{1\leq p\leq n}\mathbb{P}\left(X_p\neq i\right)=\left(1-\frac{1}{d}\right)^n=\mathbb{P}\left(A_n(1)\right).$$

Using the fact that

$$\begin{aligned}(T>n) &= \{\exists i\in S\;:\;\forall 1\leq p\leq n\;X_p\in S-\{i\}\}\\ &= \cup_{i\in S}\{\forall 1\leq p\leq n\;X_p\in S-\{i\}\}=\cup_{i\in S}A_n(i)\end{aligned}$$

we prove that

$$\mathbb{P}\left(T>n\right)\leq\sum_{i\in S}\mathbb{P}\left(A_n(i)\right)=d\;\left(1-\frac{1}{d}\right)^n\leq d\;e^{-n/d}.$$

Hence we conclude that

$$\mathbb{P}\left(T>d\log(d)+md\;\right)\leq d\;\exp\left(-\frac{d\log(d)+md}{d}\right)=e^{-m}.$$

This ends the proof of the exercise.

■

Solution to exercise 430: We have that

$$\psi \ : \ \theta \in \mathbb{T} := \mathbb{R}/\left(2\pi\mathbb{Z}\right) \ \mapsto \ \psi(\theta) = e^{iW_t} \in \mathcal{C} = \{x = (x_1, x_2) \in \mathbb{R}^2 \ : \ x_1^2 + x_2^2 = 1\}.$$

For any function F on $\mathcal{C}$ we set $f = F \circ \psi$. We also denote by $\overline{W}_t = \{W_t + 2k\pi \ : \ k \in \mathbb{Z}\}$ the class of equivalence of the state W_t in the 1-dimensional Torus $\mathbb{R}/\left(2\pi\mathbb{Z}\right)$. In this notation we have

$$\forall k \in \mathbb{Z} \qquad \psi(W_t + 2k\pi) = \psi(W_t) := \psi\left(\overline{W}_t\right).$$

Furthermore we have

$$\begin{aligned}
\mathbb{E}\left(F\left(\psi(W_t)\right) \mid W_0 = w_0\right) &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} F(\psi(w))\ e^{-\frac{1}{2t}\ (w-w_0)^2}\ dw \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2\pi t}} \int_{2k\pi}^{2(k+1)\pi} F(\psi(w))\ e^{-\frac{1}{2t}\ (w-w_0)^2}\ dw \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2\pi t}} \int_{0}^{2\pi} F(\psi(v))\ e^{-\frac{1}{2t}\ (v-w_0+2k\pi)^2}\ dv \\
&= \int_0^{2\pi} F(\psi(v))\ p_t(v-w_0)\ dv = \int_0^{2\pi} f(v)\ p_t(v-w_0)\ dv \\
&:= \mathbb{E}(f(\overline{W}_t) \mid \overline{W}_0 = w_0) = \mathbb{E}(f(w_0 + \overline{W}_t) \mid \overline{W}_0 = 0)
\end{aligned}$$

with

$$p_t(v) \ := \ \frac{1}{\sqrt{2\pi t}} \sum_{n\in\mathbb{Z}} e^{-\frac{1}{2t}\ (v+2n\pi)^2} = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \mathbb{E}\left(e^{-inW_t}\right)\ e^{inv}.$$

The r.h.s. equality comes from the Poisson summation formula applied to the 2π-periodic function

$$h(v) = \sum_{n\in\mathbb{Z}} p(v + 2n\pi) = \sum_{n\in\mathbb{Z}} \widehat{p}(n)\ e^{inv}$$

with the Fourier coefficient

$$\widehat{p}(n) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} p(v)\ e^{-inv}\ dv.$$

We check this claim taking the Fourier transform of h

$$\widehat{h}(n) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \int_0^{2\pi} p(v+2n\pi)\ e^{-inv}\ dv = \frac{1}{2\pi} \int_{-\infty}^{+\infty} p(v)\ e^{-inv}\ dv = \widehat{p}(n).$$

On the other hand, we have

$$\mathbb{E}\left(e^{-inW_t}\right) = e^{-n^2t/2} \Rightarrow p_t(v) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} e^{-n^2t/2}\ e^{inv}.$$

We conclude that

$$p_t(v) = \frac{1}{2\pi} + \frac{1}{2\pi} \sum_{n\in\mathbb{Z}-\{0\}} e^{-n^2t/2}\ e^{inv} \Rightarrow \left|p_t(v) - \frac{1}{2\pi}\right| \leq \frac{1}{\pi} \sum_{n\geq 1} e^{-n^2t/2}.$$

This implies that

$$\|P_t(f) - \mu(f)\| \leq \frac{1}{\pi} \sum_{n \geq 1} e^{-n^2 t/2} \ \to_{t \to \infty} \ 0,$$

which ends the proof of the exercise.

■

Chapter 26

Solution to exercise 431:

We let $T_{q\to(q-1)}$ be the random time needed to move from state q to $q-1$. By construction, $T_{q\to(q-1)}$ is a geometric random variable with success probability $\left(1-\frac{(d)_q}{d^q}\right)$, and we have the decomposition

$$T' = T_{R_0\to(R_0-1)} + T_{(R_0-1)\to(R_0-2)} + \ldots + T_{2\to 1}.$$

Given $R_0 = q$, we have

$$T' - (R_0-1) = \left(T_{R_0\to(R_0-1)} - 1\right) + \left(T_{(R_0-1)\to(R_0-2)} - 1\right) + \ldots + \left(T_{2\to 1} - 1\right).$$

$$\mathbb{E}\left(e^{t\ \frac{T'-(R_0-1)}{d}}\ \mid R_0\right) = \prod_{1\leq q<R_0} \mathbb{E}\left(e^{t\ \frac{T_{q+1\to q}-1}{d}}\right).$$

For any geometric random variable N with success probability $\alpha \in]0,1[$, and for any $0 \leq t < -d\log(1-\alpha)$ we have

$$\mathbb{E}(e^{t(N-1)/d}) = \alpha \sum_{n\geq 1} (1-\alpha)^{n-1} e^{t(n-1)/d} = \frac{\alpha}{1-(1-\alpha)e^{t/d}}.$$

Applying this formula to

$$1-\alpha = \beta_q := \frac{(d)_q}{d^q} \leq e^{-\frac{q(q-1)}{2d}} \quad \text{and} \quad 0 \leq t < \frac{q(q-1)}{2}\ (\leq -d\log(1-\alpha))$$

we find that

$$\mathbb{E}\left(e^{t\ \frac{T_{q\to(q-1)}-1}{d}}\right) = \frac{1-\beta_q}{1-\beta_q\ e^{t/d}}.$$

We consider for any $x>0$ the function

$$f\ :\ x \in \left[0, e^{-t/d}\right] \mapsto \frac{1-x}{1-x\ e^{t/d}}.$$

An elementary calculation shows that

$$f'(x) = \frac{-(1-x\ e^{t/d}) - (1-x)(-e^{t/d})}{(1-x\ e^{t/d})^2} = \frac{e^{t/d}-1}{(1-x\ e^{t/d})^2} \geq 0.$$

Therefore we conclude that f is increasing and

$$\beta_q \leq e^{-\frac{q(q-1)}{2d}} \Rightarrow \frac{1-\beta_q}{1-\beta_q\ e^{t/d}} \leq \frac{1-e^{-\frac{q(q-1)}{2d}}}{1-e^{-\frac{q(q-1)}{2d}}\ e^{t/d}}.$$

We consider for any $x>0$ the function

$$g\ :\ y \in [0,x] \mapsto g(y) = x\left(1-e^{y-x}\right) + (y-x)\ \left(1-e^{-x}\right).$$

Notice that $g(0) = g(x) = 0$ and

$$g'(y) = -x\ e^{y-x} + (1 - e^{-x}) \geq 0 \Leftrightarrow \frac{e^x - e^0}{x - 0} = e^{\theta(x)} \geq e^y$$

with the state $\theta(x) \in [0, x]$ given by the mean value theorem. This shows that $g(y)$ is increasing for $y \in [0, \theta(x)]$ and decreasing for $y \in [\theta(x), x]$. Since $g(0) = g(x) = 0$, we conclude that

$$\forall y \in [0, x] \qquad g(y) \geq 0$$

$$\Downarrow$$

$$\forall y \in [0, x[\qquad \frac{1 - e^{-x}}{1 - e^{y-x}} = \frac{1 - e^{-x}}{1 - e^{-x}/e^{-y}} \leq \frac{x}{x - y} = \frac{1}{1 - (y/x)}.$$

Applying this inequality to $x = \frac{q(q-1)}{2d} \geq y = t/d$, we find that

$$\forall 0 \leq t \leq \frac{q(q-1)}{2} \qquad \frac{1 - e^{-\frac{q(q-1)}{2d}}}{1 - e^{-\frac{q(q-1)}{2d}}\ e^{t/d}} \leq \frac{1}{1 - \frac{t}{q(q-1)/2}}.$$

Hence

$$\mathbb{E}\left(e^{t\ \frac{T_{q\to(q-1)}-1}{d}}\right) \leq \frac{1}{1 - \frac{t}{q(q-1)/2}}$$

for any $0 \leq t < 1 \left(\leq \frac{q(q-1)}{2}\right)$.

This implies that on the event $R_0 > 1$

$$\forall t \in [0, 1[\qquad \mathbb{E}\left(e^{t\ \frac{T'-(R_0-1)}{d}}\ \mid R_0\right) = \prod_{1\leq q<R_0} \frac{1}{1 - \frac{t}{q(q+1)/2}}.$$

On the other hand, for any exponential random variable E_q with parameter $\lambda_q = q(q+1)/2 (> 1)$ we have

$$\forall 0 \leq t < 1 (\leq \lambda_q) \qquad \mathbb{E}\left(e^{t\ E_q}\right) = \frac{1}{1 - t/\lambda_q}.$$

This implies that for any $0 \leq t < 1$

$$\begin{aligned} &\mathbb{E}\left(e^{t\ \frac{T'-(R_0-1)}{d}}\ \mid R_0\right) \\ &\leq \mathbb{E}\left(e^{t\ \sum_{1\leq q<R_0} \frac{2}{q(q+1)}\ \mathcal{E}_q}\ \mid R_0\right) \\ &\leq h(t) := \mathbb{E}\left(e^{t\sum_{1\leq q<\infty} \frac{2}{q(q+1)}\ \mathcal{E}_q}\ \right) = \prod_{q\geq 1} \frac{1}{1 - \frac{t}{q(q+1)/2}} \end{aligned}$$

where $\mathcal{E}_q$ stands for a sequence of independent exponential random variables with unit parameter.

$$\begin{aligned} &\mathbb{P}\left(\frac{T'-(R_0-1)}{d} \geq n \mid R_0\right) \\ &= \mathbb{E}\left(1_{\frac{T'-(R_0-1)}{d}\geq n}\ e^{-t\ \frac{T'-(R_0-1)}{d}} e^{t\ \frac{T'-(R_0-1)}{d}}\ \mid R_0\right) \leq e^{-tn}\ h(t). \end{aligned}$$

This implies that

$$\mathbb{P}(\ T' \geq m \ \mid R_0\,) \leq e^{-t\left(\frac{m-(R_0-1)}{d}\right)}\ h(t) \leq e^{-mt/d}\ e^t\ h(t) \leq \frac{ae}{1-t}\ e^{-mt/d}$$

with the finite constant

$$a := \prod_{q\geq 2} \frac{1}{1-\frac{1}{q(q+1)/2}}.$$

The proof of the last estimate follows from the fact that

$$\forall 0 \leq t < 1 \qquad e^t h(t) = \frac{e^t}{1-t} \prod_{q\geq 2} \frac{1}{1-\frac{t}{q(q+1)/2}} \ \leq \ \frac{e}{1-t}\ a.$$

We conclude that

$$\mathbb{P}(\ T' \geq m \ \mid R_0\,) \leq ae \inf_{t\in[0,1[} \frac{e^{-mt/d}}{1-t}.$$

By zeroing out the derivative on the r.h.s., we see directly that the infimum is attained when $t = 1 - d/m$ since

$$\frac{\partial}{\partial t} \frac{e^{-mt/d}}{1-t} = \frac{e^{-mt/d}}{(1-t)^2}\ \left(1 - \frac{m}{d}(1-t)\right) = 0 \Leftrightarrow t = 1 - d/m.$$

This shows that for any $m \geq 2d$

$$\sup_{1\leq q\leq d} \mathbb{P}(\ T' \geq m \ \mid R_0 = q\,) \leq ae\ \frac{e^{-m(1-d/m)/d}}{1-(1-d/m)} = a\ \frac{m}{d}\ e^{-(m/d-1)}.$$

This ends the proof of (26.19).

Since $q(q+1)/2 \geq 3 \geq 2$ for any $q \geq 2$, using the given estimate of the logarithm we prove that

$$\begin{aligned}
\log a \quad &:= \quad -\sum_{q\geq 2} \log\left(1 - \frac{1}{q(q+1)/2}\right) \\
&\leq \quad 2 \sum_{q\geq 2} \frac{1}{q(q+1)} + 2 \sum_{q\geq 2} \frac{1}{q(q+1)}\ \underbrace{\frac{1}{q(q+1)/2}}_{\geq 2} \\
&\leq \quad 3 \sum_{q\geq 2} \frac{1}{q(q+1)} = 3 \sum_{q\geq 2} \left[\frac{1}{q} - \frac{1}{(q+1)}\right] = 3/2.
\end{aligned}$$

This implies that

$$\sup_{1\leq q\leq d} \mathbb{P}(\ T' \geq m \ \mid R_0 = q\,) \leq \frac{m}{d}\ \exp\left[-\left(\frac{m}{d} - \frac{5}{2}\right)\right].$$

This ends the proof of the exercise.

■

Solution to exercise 432:
Recalling that

$$Y \sim \mathrm{Geo}(p) \Rightarrow \mathbb{E}(Y) = \frac{1}{p} \quad \text{and} \quad \mathrm{Var}(Y) = \mathbb{E}((Y - \mathbb{E}(Y))^2) = \frac{1-p}{p^2}$$

for any success parameter $p \in]0,1]$, we readily check that

$$\mathbb{E}(T) = \sum_{1\leq i\leq d} \mathbb{E}(T_i) = d \sum_{1\leq i\leq d} \frac{1}{i} \simeq d \ \log d.$$

We can use the upper bounds (30.55) to estimate more precisely these expectations. On the other hand, we also have that

$$\begin{aligned}
\mathrm{Var}(T) &= \sum_{1\leq i\leq d} \mathbb{E}((T_i - \mathbb{E}(T_i))^2) \\
&= \sum_{1\leq i\leq d} \frac{1-(i/d)}{(i/d)^2} \\
&= d^2 \sum_{1\leq i\leq d} \left(\frac{1}{i^2} - \frac{1}{d\ i}\right) = d^2 \left(\sum_{1\leq i\leq d} \frac{1}{i^2} - \sum_{1\leq i\leq d} \frac{1}{d\ i}\right) \leq 2d^2.
\end{aligned}$$

We note that $Var(T)$ stabilizes asymptotically to $\frac{\pi^2}{6}d^2$ and does not exceed $2d^2$ for any d.
This ends the proof of the exercise.

■

Solution to exercise 433:

The following analysis follows the book of D. A. Levin and Y. Peres [180]. We let T be the stopping time associated with the first time all cards are marked in the transposition shuffle introduced on page 715. We denote by L_n and R_n the card chosen by the left hand and the one chosen by the right hand. We denote by $\mathcal{M}_n \in \{1,\ldots,d\}$ the set of cards marked up to time n (included); and we let $\mathcal{P}_n \in \{1,\ldots,d\}$ be the set of positions occupied by the cards $\mathcal{M}_n$ after the n-th transposition.

Given the triplet $(n, \mathcal{M}_n, \mathcal{P}_n)$, all the permutations of the cards in $\mathcal{M}_n$ on the positions $\mathcal{P}_n$ are equally likely. We prove this assertion by induction w.r.t. the time parameter n. For $n=1$, the result is immediate since at the original cards are all unmarked (we mark a single R_1 for any pair of chosen cards (L_1, R_1)). We assume that the assertion is true at some rank n. We choose the cards L_{n+1} and R_{n+1}:

- When no card is marked we have $\mathcal{M}_{n+1} = \mathcal{M}_n$:
  - (L_{n+1}, R_{n+1}) are already marked, thus $\mathcal{P}_{n+1} = \mathcal{P}_n$. In this case, the shuffle produces a uniform random transposition in $\mathcal{M}_n$, all permutations of $\mathcal{M}_n$ remain equally likely (by the induction hypothesis).
  - L_{n+1} is unmarked but R_{n+1} was already marked. In this case, after the shuffle, $\mathcal{P}_{n+1}$ is deduced from $\mathcal{P}_n$ by deleting the position of R_{n+1} and adding the one of L_{n+1}. For a given set of positions $\mathcal{P}_n$ the choices of $R_{n+1} \in \mathcal{P}_n$ are equally likely. The permutations of $\mathcal{M}_n$ on $\mathcal{P}_n$ being equally likely (by the induction hypothesis), the permutations of the new set $\mathcal{M}_n$ on $\mathcal{P}_{n+1}$ remain equally likely.
- If R_{n+1} is marked, then L_{n+1} is equally likely to be any element of $\mathcal{M}_{n+1} = \mathcal{M}_n \cup \{R_{n+1}\}$. We also have $\mathcal{P}_{n+1} = \mathcal{P}_n \cup \{\text{the position of } L_{n+1} \text{ at time } n\}$. Any permutation of $\mathcal{M}_n$ on $\mathcal{P}_n \cup \{\text{the position of } L_{n+1} \text{ at time } n\}$ uniquely determines a permutation of $\mathcal{M}_{n+1}$ on $\mathcal{P}_{n+1}$. Thus, all such permutations are equally likely.

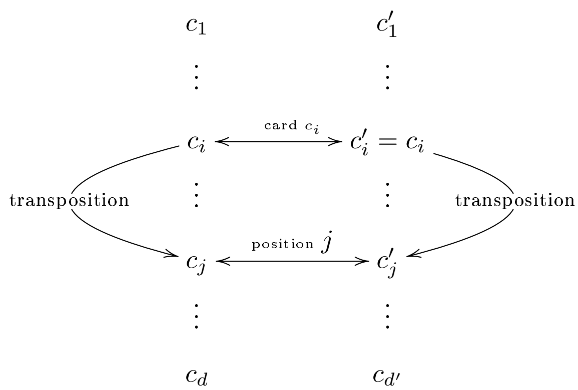
In all cases, the set of permutations of $\mathcal{M}_n$ on $\mathcal{P}_n$ makes equal contributions to all possible permutations of $\mathcal{M}_{n+1}$ on $\mathcal{P}_{n+1}$. Summing over all the configurations discussed above, we prove that all permutations of $\mathcal{M}_{n+1}$ on $\mathcal{P}_{n+1}$ are equally likely.

This ends the proof of the exercise.

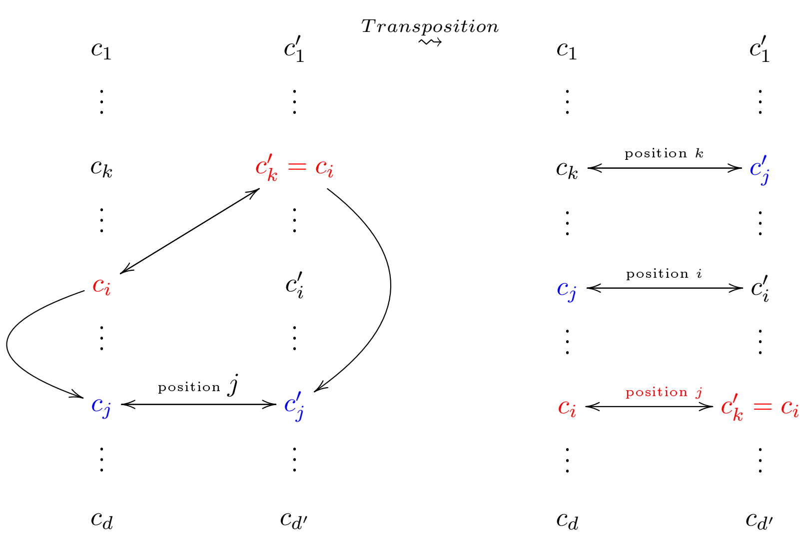
■

Solution to exercise 434:

1. When the selected card $c_i = c'_i$ has the same position in both decks, the transposition does not affect *the number* of cards $N_{n+1} = N_n$ that occupy the same position (only their position level is changed).



2. The selected card c_i is not in the same position, say $c'_k = c_i$. In this case the transposition diagram below shows that *after the transposition* the cards at the j-th position $c'_k = c_i$ become equal. We notice that in this case, we have $\underline{c_i \neq c'_i}$ by construction, but also $\underline{c_k \neq c'_k}$ since we have $c_i \neq c_k$ and $c'_i \neq c'_k$ (otherwise if $c_k = c'_k$ we would have $c_i \neq c_k \neq c'_k \Rightarrow c_i \neq c'_k$ and we arrive at contradiction to our assumption).



In this situation, three cases may occur

(a) $c_j = c'_j$: In this situation, we lose this alignment but we have also created a new one since the cards at the j-th position $c'_k = c_i$ becomes equal. In this case, $N_{n+1} = N_n$.

(b) $c_j \neq c'_j$: In this case, as argued above at least one alignment has been made at the j-th position.

i. If $c_k \neq c'_j$ and $c_j \neq c'_i$ then $N_{n+1} = N_n + 1$.

ii. If $c_k \neq c'_j$ but $c_j = c'_i$ then $N_{n+1} = N_n + 2$.

iii. If $c_k = c'_j$ but $c_j \neq c'_i$ then $N_{n+1} = N_n + 2$.

iv. If $c_k = c'_j$ but $c_j = c'_i$ then $N_{n+1} = N_n + 3$.

This ends the proof of (26.11). The proof of (26.12) follows from the fact that

$$\mathbb{E}\left(N_{n+1} \mid N_n\right) \geq N_n + \left(1 - \frac{N_n}{d}\right) = \left(1 - \frac{1}{d}\right) \; N_n + 1.$$

$$\begin{aligned}
\mathbb{E}\left(N_n \mid N_0\right) &\geq \left(1 - \frac{1}{d}\right) \; \mathbb{E}(N_{n-1} \mid N_0) + 1 \\
&= \left(1 - \frac{1}{d}\right)^n \; N_0 + \sum_{0 \leq p < n} \left(1 - \frac{1}{d}\right)^p \\
&= \left(1 - \frac{1}{d}\right)^n \; N_0 + d \; \left(1 - \left(1 - \frac{1}{d}\right)^n\right) \\
&= d - \left(1 - \frac{1}{d}\right)^n \; (d - N_0).
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 435:
We have

$$\begin{aligned}
\mathbb{E}(T) &= 1 + d^2 \sum_{1 \leq i < d} \frac{1}{i \times (d - (i-1))} \\
&= 1 + \frac{d^2}{d+1} \sum_{1 \leq i < d} \left(\frac{1}{i} + \frac{1}{d - (i-1)}\right) \simeq 2d \log{(d)}.
\end{aligned}$$

We can use the upper bound

$$\begin{aligned}
\left(\sum_{1 \leq i < d} \int_i^{i+1} \frac{1}{t} \; dt\right) &= \log{(d+1)} \\
&\leq \sum_{1 \leq i \leq d} \frac{1}{i} \leq \left(1 + \sum_{2 \leq i \leq d} \int_{i-1}^{i} \frac{1}{t} \; dt\right) = (1 + \log d)
\end{aligned} \tag{30.55}$$

to check that

$$\begin{aligned}
\mathbb{E}(T) &= 1 + \frac{d^2}{d+1} \left(2 \sum_{1 \leq i \leq d} \frac{1}{i} - \left(1 + \frac{1}{d}\right)\right) \\
&\leq 1 + \frac{d^2}{d+1} \left(2 \log d + \left(1 - \frac{1}{d}\right)\right) \leq 2d \log d + d + 1 \leq 2d(1 + \log d).
\end{aligned}$$

The last assertion is a direct consequence of theorem 8.3.18. This ends the proof of the exercise. ∎

Solution to exercise 436:

A each time step, say n, of have $d^2 = d(d-1)+d$ possible choices of the pair $(L_n, R_n) = (i,j) \in \{1,\ldots,d\}^d$

- After the first card, say m_1, has been marked, we have $1\times(d-1)+1 = 1\times d$ possible pairs (m_1, j) with $j \in \{1,\ldots,d\} - \{m_1\}$ and (m_1, m_1), for which the right hand card will be marked; This random time is a Geometric r.v. with success probability
$$p_1 = \frac{1\times d}{d^2}$$
- After the second card, say m_2, has been marked, we have two marked cards m_1 and m_2. In this situation, we have $2\times(d-2)+2 = 2\times(d-1)$ possible pairs (m,j) with $m \in \{m_1, m_2\}$ and $j \in \{1,\ldots,d\} - \{m_1, m_2\}$, and $(m_i, m_i)_{1\leq i\leq 2}$ for which a new right hand card will be marked. Therefore, the random time T_2 is a Geometric r.v. with success probability
$$p_2 = \frac{2\times(d-1)}{d^2}$$
- After the third card, say m_3, has been marked, we have three marked cards m_1, m_2 and m_3. In this situation, we have $3\times(d-3)+3 = 3\times(d-2)$ possible pairs (m,j) with $m \in \{m_1, m_2, m_3\}$ and $j \in \{1,\ldots,d\} - \{m_1, m_2, m_3\}$, and $(m_i, m_i)_{1\leq i\leq 3}$ for which a new right hand card will be marked. Therefore, the random time T_2 is a Geometric r.v. with success probability
$$p_2 = \frac{3\times(d-2)}{d^2}$$
- More generally, after the i-th card, say m_i, has been marked, we have i marked cards $(m_k)_{1:leqk\leq i}$. In this situation, we have $i\times(d-i)+i = i\times(d-(i-1))$ possible pairs (m,j) with $m \in \{m_1,\ldots,m_i\}$ and $j \in \{1,\ldots,d\} - \{m_1,\ldots,m_i\}$, and $(m_k, m_k)_{1\leq k\leq i}$ for which a new right hand card will be marked. Therefore, the random time T_i is a Geometric r.v. with success probability
$$p_i = \frac{i\times(d-(i-1))}{d^2}$$

This ends the proof of (26.14).

The proof of the exercise is completed.

∎

Solution to exercise 437:

We have

$$\begin{aligned}
\mathbb{E}(T) &= \sum_{n\geq 1} \mathbb{P}(T > n) = \sum_{n=1}^{\log_2 d^2} \left(1 - e^{-\frac{d^2}{2^n}}\right) + \sum_{n>\log_2 d^2} \left(1 - e^{-\frac{d^2}{2^n}}\right) \\
&\leq \log_2 d^2 + \sum_{n=1+\log_2 d^2}^{\infty} \frac{d^2}{2^n} \\
&= \log_2 d^2 + \sum_{n\geq 1} d^2/2^{n+\log_2 d^2} = \log_2 d^2 + \sum_{n\geq 1} 2^{-n} = 1 + \log_2 d^2.
\end{aligned}$$

In the r.h.s. of the first line we have used the fact that $1 - e^{-x} \leq x$, for any $x \geq 0$.

This ends the proof of the exercise.

■

Solution to exercise 438:

We observe that

$$\begin{aligned}
\|X_n - Y_n\|_{\mathcal{F}} &= \left(\frac{1}{2}\right)^n \sum_{k\geq 1} 2^{-k} \ \|x\|_{K_k} \\
&= \left(\frac{1}{2}\right)^n \sum_{k\geq 1} 2^{-k} \ \ k \\
&= \left(\frac{1}{2}\right)^{n+1} \sum_{k\geq 1} k \ 2^{-(k-1)} = \left(\frac{1}{2}\right)^{n-1}
\end{aligned} \tag{30.56}$$

with the constant (random) function

$$Y_n(x) = \sum_{0\leq p<n} \frac{1}{2^p} \ I_{n-p} \stackrel{Law}{=} \sum_{0\leq p<n} \frac{1}{2^p} \ I_{p+1} = Z_n.$$

The r.h.s. bound in (30.56) is due to the fact that

$$\sum_{k\geq 1} k \ 2^{-(k-1)} = \left(\frac{\partial}{\partial\lambda} \sum_{k\geq 0} x^k\right)_{x=1/2} = (1/(1-1/2)^2) = 4.$$

Using the inequalities

$$\|Y_{n+1} - F\|_{\mathcal{F}} - 2^{-n} \leq \|X_{n+1} - F\|_{\mathcal{F}} \leq \|Y_{n+1} - F\|_{\mathcal{F}} + 2^{-n}$$

whenever the limits exists we also check that

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(\|X_{n+1} - F\|_{\mathcal{F}} \leq \epsilon\right) &= \lim_{n\to\infty} \mathbb{P}\left(\|Y_{n+1} - F\|_{\mathcal{F}} \leq \epsilon\right) \\
&= \lim_{n\to\infty} \mathbb{P}\left(\|Z_{n+1} - F\|_{\mathcal{F}} \leq \epsilon\right).
\end{aligned} \tag{30.57}$$

To take the final step, we have the almost sure convergence

$$Z_{n+1} = \sum_{0\leq p\leq n} \frac{1}{2^p} \ I_{p+1} \ \to_{n\to\infty} \sum_{p\geq 0} \frac{1}{2^p} \ I_{p+1} = Z_\infty$$

and

$$\begin{aligned}
Z_\infty - Z_{n+1} &= \sum_{p\geq n+1} \frac{1}{2^p} \ I_{p+1} = \frac{1}{2^{n+1}} \ I_{n+2} + \frac{1}{2^{n+2}} \ I_{n+3} + \ldots \\
&= \frac{1}{2^{n+1}} \underbrace{\left(I_{n+2} + \frac{1}{2} \ I_{n+3} + \ldots\right)}_{:= Z'_{n,\infty} \stackrel{Law}{=} Z_\infty}.
\end{aligned}$$

$$\begin{aligned}
\|Z_\infty - F\|_{\mathcal{F}} - \frac{1}{2^{n+1}} \left\|Z'_{n,\infty}\right\|_{\mathcal{F}} &= \|Z_\infty - F\|_{\mathcal{F}} - \|Z_{n+1} - Z_\infty\|_{\mathcal{F}} \\
&\leq \|Z_{n+1} - F\|_{\mathcal{F}} \\
&\leq \|Z_{n+1} - Z_\infty\|_{\mathcal{F}} + \|Z_\infty - F\|_{\mathcal{F}} \\
&= \|Z_\infty - F\|_{\mathcal{F}} + \frac{1}{2^{n+1}} \left\|Z'_{n,\infty}\right\|_{\mathcal{F}}.
\end{aligned}$$

Thus, we have the almost sure convergence

$$\lim_{n\to\infty} \|Z_{n+1} - F\|_{\mathcal{F}} = \|Z_\infty - F\|_{\mathcal{F}}.$$

By (30.57) we conclude that

$$\lim_{n\to\infty} \mathbb{P}\left(\|X_{n+1} - F\|_{\mathcal{F}} \leq \epsilon\right) = \mathbb{P}\left(\|Z_\infty - F\|_{\mathcal{F}} \leq \epsilon\right).$$

This ends the proof of the exercise.

■

Solution to exercise 439:
These functions are clearly contractions of the unit interval

$$|f_0(x) - f_0(y)| = |f_1(x) - f_1(y)| = \frac{1}{2}\ |x-y| < |x-y|.$$

In both cases S is reduced to a half:

$$S = (S_0 \cup S_1) \quad \text{with} \quad S_0 = f_0(S) = \left[0, \frac{1}{2}\right] \quad \text{and} \quad S_1 = f_1(S) = \left[\frac{1}{2}, 1\right].$$

Arguing as in section 26.2, we have

$$X_n(x) = \frac{x}{2^n} + \frac{1}{2} \sum_{0\leq p<n} \frac{\epsilon_{n-p}}{2^p} \stackrel{Law}{=} \frac{x}{2^n} + \sum_{1\leq p\leq n} \frac{\epsilon_p}{2^p} \ \rightarrow_{n\uparrow\infty}\ X_\infty := \sum_{n\geq 1} \frac{\epsilon_n}{2^n}$$

and

$$\frac{1}{2}\ X_\infty + \frac{\epsilon_0}{2} \ = \sum_{n\geq 0} \frac{\epsilon_n}{2^{n+1}} \stackrel{law}{=} \sum_{n\geq 0} \frac{\epsilon_{n+1}}{2^{n+1}} = X_\infty.$$

Our next objective is to prove that X_∞ is a conversion to base 2 of a random number

$$U = \epsilon_1\ \frac{1}{2} + \epsilon_2\ \frac{1}{2^2} + \ldots + \epsilon_n\ \frac{1}{2^n} + \ldots \tag{30.58}$$

that is uniformly chosen in $[0,1]$. We check the first claim using the binary decomposition

$$2U := \underbrace{\lfloor 2U \rfloor}_{=\epsilon_1} + \underbrace{\{2U\}}_{=U_1} \rightsquigarrow 2U_1 := \underbrace{\lfloor 2U_1 \rfloor}_{=\epsilon_2} + \underbrace{\{2U_1\}}_{=U_2}.$$

Now, we observe that

$$\begin{array}{rcl}
\mathbb{P}\left(\epsilon_1 = 0\ ,\ U_1 \leq u_1\right) & = & \mathbb{P}\left(0 \leq 2U < 1\ ,\ 2U \leq u_1\right) \\
& = & \dfrac{1}{2} \times u_1 \\
& = & \mathbb{P}\left(\epsilon_1 = 0\right)\ \mathbb{P}\left(U_1 \leq u_1 \mid \epsilon_1 = 0\right).
\end{array}$$

This shows that ϵ_1 is a Bernoulli random variable with parameter $1/2$, and given ϵ_1 the random variable U_1 can be seen as an independent uniform on $[0,1]$. Iterating this reasoning, we check that $U \stackrel{law}{=} X_\infty$.

Recalling that $U_{a,b} = a + (b-a)U$ is uniform on $[a,b] \subset [0,1]$ and

$$U \stackrel{law}{=} \epsilon\ U_{0,a} + (1-\epsilon)\ U_{a,1} \tag{30.59}$$

with a Bernoulli random variable $\mathbb{P}(\epsilon = 1) = 1 - \mathbb{P}(\epsilon = 0) = a$ (cf.exercice 439), we can also notice that

$$\begin{aligned} U &\overset{law}{=} \epsilon_1 \, U_{0,1/2} + (1-\epsilon_1)\, U_{1/2,1} \\ &= \epsilon_1 \underbrace{\frac{U}{2}}_{U_{0,1/2}} + (1-\epsilon_1) \underbrace{\left[\frac{1}{2} + \left(1 - \frac{1}{2}\right) U\right]}_{U_{1/2,1}} = f_{\epsilon_1}(U). \end{aligned}$$

To prove (30.59) we use the conditioning formulae

$$\begin{aligned} \mathbb{P}\left(\epsilon \, U_{0,a} + (1-\epsilon)\, U_{a,1} \leq u \mid \epsilon = 1\right) &= \mathbb{P}\left(U_{0,a} = aU \leq u\right) \\ &= \frac{u}{a} \, 1_{[0,a]}(u) + 1_{[a,1]}(u) \\ \mathbb{P}\left(\epsilon \, U_{0,a} + (1-\epsilon)\, U_{a,1} \leq u \mid \epsilon = 0\right) &= \mathbb{P}\left(U_{a,1} = a + (1-a)U \leq u\right) \\ &= \frac{(u-a)}{(1-a)} \, 1_{[a,1]}(u). \end{aligned}$$

We get

$$\begin{aligned} &\mathbb{P}\left(\epsilon \, U_{0,a} + (1-\epsilon)\, U_{a,1} \leq u\right) \\ &= u \, 1_{[0,a]}(u) + a \, 1_{[a,1]}(u) + (u-a) \, 1_{[a,1]}(u) \\ &= u - a \, 1_{[a,1]}(u) + a 1_{[a,1]}(u) = u. \end{aligned}$$

Finally, using the fact that $\mathrm{lip}(f_i) = 1/2$ we prove that

$$\mathbb{W}(\mathrm{Law}(X_n(x)), \mathrm{Law}(X_n(y))) \leq 2^{-n} \, |x-y|.$$

Due to proposition 8.3.13 we also readily have that

$$\mathbb{W}(\mathrm{Law}(X_n(x)), \pi) = \mathbb{W}(\delta_x M^n, \pi M^n) \leq 2^{-n} \int \pi(dy) \, |x-y|$$

with the Markov transition M of the chain and its invariant uniform distribution π on $[0,1]$. This implies that

$$\sup_{x \in [0,1]} \mathbb{W}(\mathrm{Law}(X_n(x)), \pi) \leq 2^{-n}.$$

This ends the proof of the exercise.

■

Solution to exercise 440:
We have

$$\begin{aligned} \text{Full-length}(J_n) &= \text{Full-length}\left(\cup_{k=1}^n \left(\cup_{l=1}^{2^{k-1}} J_{k,l}\right)\right) = \sum_{k=1}^n \sum_{l=1}^{2^{k-1}} \text{Full-length}(J_{k,l}) \\ &= \sum_{k=1}^n 2^{k-1} \frac{1}{3^k} = \frac{1}{3} \times \frac{1 - \left(\frac{2}{3}\right)^n}{1 - \frac{2}{3}} \\ &= 1 - \left(\frac{2}{3}\right)^n \uparrow \text{Full-length}(J_\infty) = 1 \quad \text{lorsque } n \uparrow \infty \\ &\qquad \Longrightarrow \text{Full-length}(I_n) \downarrow \text{Full-length}(I_\infty) = 0. \end{aligned}$$

This ends the proof of (26.17).

We use the fact that

$$\begin{aligned} X_n &= \frac{1}{3}\ X_{n-1} + \frac{2}{3}\ \epsilon_n \\ &= \frac{1}{3^2}\ X_{n-2} + \frac{2}{3}\left(\frac{1}{3}\ \epsilon_{n-1} + \epsilon_n\right) \\ &= \frac{1}{3^3}\ X_{n-3} + \frac{2}{3}\ \left(\frac{1}{3^2}\ \epsilon_{n-2} + \frac{1}{3}\ \epsilon_{n-1} + \epsilon_n\right). \end{aligned}$$

Iterating this procedure we prove (26.18).

We prove the last assertion by induction w.r.t. the time parameter. The result is immediate for $n = 0$. We further assume that the result has been verified at rank $(n-1)$. For any function φ on $[0,1]$ we have

$$\begin{aligned} \mathbb{E}(\varphi(X_n)) &= \mathbb{E}(\varphi(f_{\epsilon_n}(X_{n-1}))) = \mathbb{E}(\mathbb{E}(\varphi(f_{\epsilon_n}(X_{n-1})) \mid X_{n-1})) \\ &= \frac{1}{2} \times \mathbb{E}\left(\varphi\left(f_0(X_{n-1})\right) + \varphi\left(f_1(X_{n-1})\right)\right). \end{aligned}$$

Under our induction hypothesis, we have

$$\begin{aligned} &\mathbb{E}\left(\varphi\left(f_0(X_{n-1}) + \varphi(f_1(X_{n-1}))\right)\right) \\ &= \left(\tfrac{3}{2}\right)^{n-1}\left[\int_{S_{n-1}}\ \varphi(f_0(x))\ dx + \int_{S_{n-1}}\ \varphi(f_1(x))\ dx\right] \\ &= 3\ \left(\tfrac{3}{2}\right)^{n-1}\ \int_{f(S_{n-1})}\ \varphi(x)\ dx = 3\ \left(\tfrac{3}{2}\right)^{n-1}\ \int_{S_n}\ \varphi(x)\ dx. \end{aligned}$$

We conclude that

$$\mathbb{E}(\varphi(X_n)) = \left(\frac{3}{2}\right)^n \int_{S_n} \varphi(x)\ dx = \int \varphi(x)\ \underbrace{\left(\frac{3}{2}\right)^n\ 1_{S_n}(x)\ dx}_{=\mathbb{P}(X_n \in dx)}.$$

This ends the proof of the exercise.

■

Chapter 27

Solution to exercise 441:

We have

$$\left\{\begin{array}{rcl} dY_t^1 &=& b_t^1(Y_t)\ dt := Y_t^2\ dt \\ dY_t^2 &=& \underbrace{-\left(\alpha\ Y_t^2 + m^{-1}\ \partial_x U_t(Y_t^1)\right)}_{b_t^2(Y_t)}\ dt + \sigma\ dW_t. \end{array}\right.$$

The Fokker-Planck equation is given by

$$\begin{aligned} \partial_t p_t(y) &= -\partial_{y_1}\left(b_t^1\ p_t\right)(y) - \partial_{y_2}\left(b_t^2\ p_t\right)(y) + \frac{1}{2}\ \sigma^2 \partial_{y_2}^2 p_t(y) \\ &= -y_2\ \partial_{y_1}\left(p_t\right)(y) + m^{-1}\ \partial_x U_t(y_1)\ \partial_{y_2}\left(p_t\right)(y) + \alpha\ \partial_{y_2}\left(y_2\ p_t\right)(y) + \frac{1}{2}\ \sigma^2 \partial_{y_2}^2 p_t(y). \end{aligned}$$

Rewritten in terms of the variables $(x, v) = (y_1, y_2)$ the equation takes the form

$$\partial_t p_t = -v\ \partial_x p_t + m^{-1}\ \partial_x U_t\ \partial_v p_t + \alpha\ \partial_v\left(v\ p_t\right) + \frac{1}{2}\ \sigma^2 \partial_v^2 p_t.$$

Letting $2^{-1}\sigma^2/\alpha = \kappa T/m$ we obtain the desired equation.

This ends the proof of the exercise.

■

Solution to exercise 442:

Following the arguments developed in exercise 441, the desired equation resumes to the traditional Fokker-Planck equation associated with the diffusion

$$dX_t = b_t(X_t)\ dt + \tau\ dW_t$$

with

$$b_t(x) := (m\alpha)^{-1}\ \partial_x U_t \quad \text{and} \quad \tau := \sigma/\alpha.$$

In this situation, we have

$$\begin{aligned} m\alpha\ \partial_t p_t &= \partial_x\left(\partial_x U_t\ p_t\right) + \frac{m\alpha}{2}\ \sigma^2 \partial_x^2 p_t \\ &= \partial_x\left(\partial_x U_t\ p_t\right)(y) + \kappa\ T\ \partial_x^2 p_t. \end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 443:

Notice that

$$\frac{\sum_{i\geq 0}\ e^{-tE_i}\ \langle f, \varphi_i\rangle\ \varphi_i(x)}{\sum_{i\geq 0}\ e^{-tE_i}\ \langle 1, \varphi_i\rangle\ \varphi_i(x)} = \frac{\langle f, \varphi_0\rangle\ \varphi_0(x) + \sum_{i\geq 1}\ e^{-t(E_i - E_0)}\ \langle f, \varphi_i\rangle\ \varphi_i(x)}{\langle 1, \varphi_0\rangle\ \varphi_0(x) + \sum_{i\geq 1}\ e^{-t(E_i - E_0)}\ \langle 1, \varphi_i\rangle\ \varphi_i(x)}.$$

This implies that

$$e^{t(E_1-E_0)}\ \left[\frac{Q_t(f)(x)}{Q_t(1)(x)}-\frac{\langle f,\varphi_0\rangle}{\langle 1,\varphi_0\rangle}\right]$$

$$=\sum_{i\geq 1}\ e^{-t(E_i-E_1)}\ \frac{\langle 1,\varphi_i\rangle\ \varphi_i(x)}{\langle 1,\varphi_0\rangle\ \varphi_0(x)+\ \sum_{j\geq 1}\ e^{-t(E_j-E_0)}\ \langle 1,\varphi_j\rangle\ \varphi_j(x)}\ \left[\frac{\langle f,\varphi_i\rangle}{\langle 1,\varphi_i\rangle}-\frac{\langle f,\varphi_0\rangle}{\langle 1,\varphi_0\rangle}\right].$$

We conclude that

$$\frac{Q_t(f)(x)}{Q_t(1)(x)}-\frac{\langle f,\varphi_0\rangle}{\langle 1,\varphi_0\rangle}=\mathrm{O}\left(e^{-t(E_1-E_0)}\right).$$

This ends the proof of the exercise.

∎

Solution to exercise 444:
We have

$$L^V(\varphi)=-E_i\varphi\Rightarrow\partial_t Q_t(\varphi_i)=Q_t(L^V(\varphi))=-E_i\ Q_t(\varphi)\Rightarrow Q_t(\varphi_i)=e^{-E_it}\ \varphi_i$$

and

$$f=\sum_{i\geq 0}\langle f,\varphi_i\rangle\ \varphi_i(x)\Rightarrow Q_t(f)=\sum_{i\geq 0}\ e^{-E_it}\ \langle f,\varphi_i\rangle\ \varphi_i.$$

Observe that

$$\mu(f^2)=\sum_{i\geq 0}\langle f,\varphi_i\rangle^2.$$

This implies that

$$e^{E_0t}\ Q_t(f)=\ \left[\langle f,\varphi_0\rangle\ \varphi_0+\sum_{i\geq 1}\ e^{-(E_i-E_0)t}\ \langle f,\varphi_i\rangle\ \varphi_i\right]$$

from which we conclude that

$$e^{2E_0t}\ \mu\left(Q_t(f)^2\right)=\langle f,\varphi_0\rangle^2+\sum_{i\geq 1}\ e^{-(E_i-E_0)t}\ \langle f,\varphi_i\rangle^2\leq\mu(f^2).$$

For finite spaces S with cardinality r we have the crude estimate

$$e^{E_0t}\ \|Q_t(f)\|\leq\sup_{1\leq i\leq r}\left[|\langle f,\varphi_i\rangle|\ \ \|\varphi_i\|\right]\left[1+\sum_{1\leq i\leq r}\ e^{-(E_i-E_0)t}\right]\leq r\ \ \|f\|\ \sup_{1\leq i\leq r}\left[\|\varphi_i\|^2\right].$$

For $f=1$ we also have

$$e^{-\|V\|t}\leq Q_t(1)(x)=\mathbb{E}\left(e^{-\int_0^t V(X_s)ds}\mid X_0=x\right)\leq c\ e^{-E_0t}\ .$$

This ends the proof of the exercise.

∎

Solution to exercise 445:

We notice that

$$L^V(\varphi_0) = L(\varphi_0) - V\varphi_0 = E_0\varphi_0 \Rightarrow V = \varphi_0^{-1}L(\varphi_0) - E_0.$$

Using the exponential change of probability measures discussed in section 18.3, this implies that

$$\begin{aligned}
\gamma_t(f) &:= \mathbb{E}\left(f(X_t)\ e^{-\int_0^t V(X_s)ds}\right) = e^{E_0 t}\ \mathbb{E}\left(f(X_t)\ e^{-\int_0^t [\varphi_0^{-1}L(\varphi_0)](X_s)ds}\right) \\
&= e^{E_0 t}\ \mathbb{E}\left(\frac{\varphi_0(X_0)}{\varphi_0(X_t)}\ f(X_t)\ \frac{\varphi_0(X_t)}{\varphi_0(X_0)}\ e^{-\int_0^t [\varphi_0^{-1}L(\varphi_0)](X_s)ds}\right) \\
&= e^{E_0 t}\ \eta_0(\varphi_0)\ \mathbb{E}\left(\varphi_0^{-1}(X_t^{\varphi_0})\ f(X_t^{\varphi_0})\right),
\end{aligned}$$

with a Markov process $X_t^{\varphi_0}$ with initial distribution $\eta_0^{[\varphi_0]} = \Psi_{\varphi_0}(\eta_0)$ and an infinitesimal generator

$$L^{[\varphi_0]}(f) = L(f) + \varphi_0^{-1}\Gamma_L(\varphi_0, f).$$

For $d = 1$ and $L = \frac{1}{2}\partial_x^2$ we have

$$2\ \Gamma_L(\varphi_0, f) = \partial_x(\varphi_0\partial_x f + f\partial\varphi_0) - \varphi_0\partial_x^2(f) - f\partial_x^2(\varphi_0) = 2\partial\varphi_0\partial f$$

and

$$L^{[\varphi_0]}(f) = \frac{1}{2}\partial_x^2 f + \varphi_0^{-1}\partial_x\varphi_0\partial_x f = \partial_x(\log\varphi_0)\ \partial_x(f) + \frac{1}{2}\partial_x^2 f.$$

This ends the proof of the exercise.

■

Solution to exercise 446:

The first assertion is proved in section 16.1.3. To check the second claim, we observe that

$$\mu\left(f_1 L(f_2)\right) = \mu\left(L(f_1)f_2\right) \implies \mu\left(f_1\mathcal{H}(f_2)\right) = \mu\left(\mathcal{H}(f_1)f_2\right) \tag{30.60}$$

for any couple of smooth functions f_1, f_2. This shows that $\mathcal{H}$ is reversible w.r.t. μ.

On the other hand, we have

$$\mathcal{H}(\varphi_0) = E_0\varphi_0 \quad \Rightarrow \quad \mu(\mathcal{H}(\varphi_0)) = -\overbrace{\mu(L(\varphi_0))}^{=\mu(L(1)\varphi_0)=0} + \mu(V\varphi_0) = E_0\ \mu(\varphi_0) \Rightarrow \Psi_{\varphi_0}(\mu)(V) = E_0.$$

This yields

$$\begin{aligned}
\Psi_{\varphi_0}(\mu)(\mathcal{H}(f)) - \Psi_{\varphi_0}(\mu)(V)\Psi_{\varphi_0}(\mu)(f) &= \frac{1}{\mu(\varphi_0)}\ \left[\mu(\varphi_0\mathcal{H}(f)) - \Psi_{\varphi_0}(\mu)(V)\mu(\varphi_0 f)\right] \\
&= \frac{1}{\mu(\varphi_0)}\ \left[\mu(\mathcal{H}(\varphi_0)f) - \Psi_{\varphi_0}(\mu)(V)\mu(\varphi_0 f)\right] \\
&= \Psi_{\varphi_0}(\mu)(f)\ \left[E_0 - \Psi_{\varphi_0}(\mu)(V)\right] = 0.
\end{aligned}$$

We conclude that

$$\begin{aligned}
\eta_\infty(L_{\eta_\infty}(f)) &= \eta_\infty(L(f)) - \eta_\infty(fV) + \eta_\infty(f)\eta_\infty(V) \\
&= -\left[\eta_\infty(\mathcal{H}(f)) - \eta_\infty(f)\eta_\infty(V)\right] = 0.
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 447:

When $V(0)=V(1)$ we have

$$\mathbb{P}(X_t=0)=e^{-bt}\ \eta_0(0)+\frac{c}{b}\ \left(1-e^{-bt}\right)\ \longrightarrow_{t\to\infty}\ \mu(0):=\frac{c}{b}=1-\mu(1)$$

with $b=\lambda(0)+\lambda(1)$ and $c=\lambda(1)$. It is readily checked that

$$\mu(0)=\frac{\lambda(1)}{\lambda(0)+\lambda(1)}\Rightarrow\mu(0)\lambda(0)=\frac{\lambda(1)\lambda(0)}{\lambda(0)+\lambda(1)}=\mu(1)\lambda(1).$$

This yields

$$\begin{array}{rcl}\mu(fL(g)) & = & \mu(0)f(0)\lambda(0)\ (g(1)-g(0))+\mu(1)f(1)\lambda(1)\ (g(0)-g(1)),\\ & = & -\dfrac{\lambda(1)\lambda(0)}{\lambda(0)+\lambda(1)}\ ([g(1)-g(0)][f(1)-f(0)])=\mu(gL(f)).\end{array}$$

In the further development we assume that $V(0)>V(1)$.

We observe that

$$\begin{array}{rcl}L^V(\varphi)(0) & = & \lambda(0)\ (\varphi(1)-\varphi(0))-V(0)\varphi(0)=-\left(\lambda(0)+V(0)\right)\varphi(0)+\lambda(0)\ \varphi(1),\\ L^V(\varphi)(1) & = & \lambda(1)\ (\varphi(0)-\varphi(1))-V(1)\varphi(1)=\lambda(1)\ \varphi(0)-\left(\lambda(1)+V(1)\right)\varphi(1).\end{array}$$

This yields

$$L^V(\varphi)=-E\ \varphi\Leftrightarrow\left(\begin{array}{cc}E-(\lambda(0)+V(0)) & \lambda(0)\\ \lambda(1) & E-(\lambda(1)+V(1))\end{array}\right)\left(\begin{array}{c}\varphi(0)\\ \varphi(1)\end{array}\right)=\left(\begin{array}{c}0\\ 0\end{array}\right).$$

The eigenvalues E_0,E_1 are the roots of the determinant

$$\begin{aligned}&\det\left(\begin{array}{cc}E-(\lambda(0)+V(0)) & \lambda(0)\\ \lambda(1) & E-(\lambda(1)+V(1))\end{array}\right)\\ &=(E-E_0)(E-E_1)\\ &=\left[E-(\lambda(0)+V(0))\right]\left[E-(\lambda(1)+V(1))\right]-\lambda(0)\lambda(1)\\ &=\left[E-\left(\tfrac{\lambda(0)+V(0)+\lambda(1)+V(1)}{2}\right)\right]^2\\ &\qquad\qquad-\left[\left(\tfrac{\lambda(0)+V(0)+\lambda(1)+V(1)}{2}\right)^2-\left(\lambda(0)+V(0)\right)\left(\lambda(1)+V(1)\right)+\lambda(0)\lambda(1)\right].\end{aligned}\tag{30.61}$$

On the other hand, we have

$$\begin{aligned}
&\left(\frac{\lambda(0)+V(0)+\lambda(1)+V(1)}{2}\right)^2\\
&=\left(\frac{V(0)-V(1)+\lambda(0)+\lambda(1)}{2}+V(1)\right)^2\\
&=\left(\frac{V(0)-V(1)+\lambda(0)+\lambda(1)}{2}\right)^2+V(1)\left(V(0)-V(1)+\lambda(0)+\lambda(1)\right)+V(1)^2\\
&=\left(\frac{b}{2}\right)^2+V(1)\left(V(0)+\lambda(0)\right)+V(1)\lambda(1)\\
&=\left(\frac{b}{2}\right)^2+V(1)\left(V(0)+\lambda(0)\right)+V(0)\lambda(1)-\underbrace{\lambda(1)(V(0)-V(1))}_{=ac}.
\end{aligned}$$

We conclude that

$$\det\begin{pmatrix} E-(\lambda(0)+V(0)) & \lambda(0)\\ \lambda(1) & E-(\lambda(1)+V(1))\end{pmatrix}=\left[E-\left(\tfrac{\lambda(0)+V(0)+\lambda(1)+V(1)}{2}\right)\right]^2-\left[\left(\tfrac{b}{2}\right)^2-ac\right].$$

Notice that

$$\frac{\lambda(0)+V(0)+\lambda(1)+V(1)}{2}=V(1)+\frac{V(0)-V(1)+\lambda(0)+\lambda(1)}{2}=V(1)+\frac{b}{2}.$$

This yields the roots

$$\begin{aligned}
E_0 &= V(1)+\frac{b}{2}-\sqrt{\left(\frac{b}{2}\right)^2-ac}=V(1)-z_2,\\
E_1 &= V(1)+\frac{b}{2}+\sqrt{\left(\frac{b}{2}\right)^2-ac}=V(1)-z_1\geq E_0.
\end{aligned}$$

By (30.61) we also have the formula

$$\begin{aligned}
E_0E_1 &= \left(\lambda(0)+V(0)\right)\left(\lambda(1)+V(1)\right)-\lambda(0)\lambda(1),\\
E_0+E_1 &= \left(\lambda(0)+V(0)\right)+\left(\lambda(1)+V(1)\right).
\end{aligned}\tag{30.62}$$

- When $c=\lambda(1)=0$, and $(b=a+\lambda(0)\geq)a=V(0)-V(1)>0$, we have $\mu(1)=1$ and the eigenvalues are given by

  $$E_0=V(1)<E_1=V(0)+\lambda(0).$$

  In addition, we have

  $$\eta_t(0)=e^{-bt}\ \frac{\eta_0(0)}{1-\frac{a}{b}\eta_0(0)\left(1-e^{-bt}\right)}\ \longrightarrow_{t\to\infty}\ \eta_\infty(0):=0$$

  as well as $\eta_t(1)\ \longrightarrow_{t\to\infty}\ \eta_\infty(1):=1$. This implies that

  $$\eta_t(0)=-\frac{1}{a}\ \partial_t\left(\log\left(1-\frac{a}{b}\eta_0(0)\left(1-e^{-bt}\right)\right)\right)$$

and therefore

$$\begin{aligned}\int_0^t \eta_s(V)ds &= V(1)\int_0^t (1-\eta_s(0))ds + V(0)\int_0^t \eta_s(0)ds\\ &= V(1)t + a\int_0^t \eta_s(0)ds\\ &= V(1)t + \left[\log\left(1-\frac{a}{b}\eta_0(0)\left(1-e^{-bs}\right)\right)\right]_{s=0}^{s=t}\\ &= V(1)t + \log\left[1-\frac{a}{b}\eta_0(0)\left(1-e^{-bt}\right)\right].\end{aligned}$$

We conclude that

$$\tfrac{1}{t}\log\gamma(1) = \frac{1}{t}\int_0^t \eta_s(V)ds$$

$$= \eta_\infty(V) + \frac{1}{t}\log\left[1-\frac{V(0)-V(1)}{[V(0)-V(1)]+\lambda(0)}\eta_0(0)\left(1-e^{-\{[V(0)-V(1)]+\lambda(0)\}t}\right)\right].$$

By definition of φ_0 we have

$$L^V(\varphi_0)(0) = L(\varphi_0)(0) - V(0)\varphi_0(0) = -E_0\ \varphi_0(0) = -V(1)\ \varphi_0(0)$$

$$\Leftrightarrow L(\varphi_0)(0) = (V(0)-V(1))\ \varphi_0(0)$$

$$\Leftrightarrow \lambda(0)\ (\varphi_0(1)-\varphi_0(0)) = (V(0)-V(1))\ \varphi_0(0)$$

$$\Leftrightarrow \lambda(0)\ \varphi_0(1) = ((V(0)-V(1))+\lambda(0))\ \varphi_0(0).$$

We also notice that

$$\lambda(1)=0 \Rightarrow \mu(0)=\pi_0(0)=0=1-\mu(1)=1-\pi_0(1)\Rightarrow \mu=\pi_0=\eta_\infty.$$

We have then

$$\begin{aligned}L_{\pi_0}(f)(0) &= \lambda(0)\ (f(1)-f(0)) + V(0)\ (f(1)-f(0))\ \pi_0(1)\\ &= (f(1)-f(0))\ \ (\lambda(0)+V(0))\\ L_{\pi_0}(f)(1) &= 0.\end{aligned}$$

This implies that

$$\pi_0 L_{\pi_0}(f) \quad = \quad \pi_0(0)\ L_{\pi_0}(f)(0) + \pi_0(1)\ L_{\pi_0}(f)(1) = 0.$$

- When $b^2 = 4ac > 0$, we recall that $a=c=b/2>0 \Leftrightarrow a = V(0)-V(1) = \lambda(1) > 0$ and $\lambda(0)=0$, so that

$$\mu(0)=1 \quad \text{and} \quad E_0 = E_1 = V(0) := E.$$

In addition, we have

$$\eta_t(1) = 1-\eta_t(0) = \frac{\eta_0(1)}{1+a\eta_0(1)\ t} = \frac{1}{a}\ \partial_t\left(\log\left(1+a\eta_0(1)\ t\right)\right) \ \longrightarrow_{t\to\infty}\ \eta_\infty(1) := 0,$$

as well as $\eta_t(0) \longrightarrow_{t\to\infty} \eta_\infty(0) := 1$. This shows that

$$\begin{aligned}
\int_0^t \eta_s(V)ds &= V(1) \int_0^t \eta_s(1)ds + V(0) \int_0^t (1-\eta_s(1))ds \\
&= V(0)t - (V(0)-V(1)) \int_0^t \eta_s(1)ds \\
&= V(0)t - [\log(1+a\ \eta_0(1)\ s)]_{s=0}^{s=t} = V(0)t - \log(1+a\ \eta_0(1)\ t).
\end{aligned}$$

We conclude that

$$\frac{1}{t}\int_0^t \eta_s(V)ds = \overbrace{\eta_\infty(V)}^{=V(0)} - \frac{1}{t}\ \log(1+a\eta_0(1)\ t).$$

For any function φ we have

$$L^V(\varphi)(0) = \overbrace{L(\varphi)(0)}^{=0} - V(0)\varphi(0) = -E\ \varphi(0).$$

and

$$\begin{aligned}
&L^V(\varphi)(1) = L(\varphi)(1) - V(1)\varphi(1) = -E\ \varphi(1) \\
&\Leftrightarrow \lambda(1)\ \varphi(0) = \underbrace{[\lambda(1) + (V(1)-V(0))]}_{=0}\ \varphi(1) \Leftrightarrow \varphi(0) = 0.
\end{aligned}$$

In this situation we have $\eta_\infty(0) := \mu(0) = 1$ and $\mu(\varphi_0) = 0$.

- When $b^2 > 4ac$ and $a = V(0) - V(1) > 0$ (i.e. $V(0) > V(1)$, $\lambda(0), \lambda(1) > 0$) we have checked that

$$\eta_t(0) + \frac{z_2}{a} = \left(\eta_0(0) + \frac{z_2}{a}\right)\ \frac{(z_2 - z_1)\ e^{-(z_2-z_1)t}}{(a\eta_0(0)+z_2)\ e^{-(z_2-z_1)t} - (a\eta_0(0)+z_1)} \longrightarrow_{t\to\infty} 0.$$

This shows that

$$\eta_t(0) \longrightarrow_{t\to\infty} \eta_\infty(0) = 1 - \eta_\infty(1) = -\frac{z_2}{a}$$

with

$$-\frac{z_1}{a} > -\frac{z_2}{a} = \left(\frac{b}{2a} - \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}\right) \in [0,1].$$

In this situation, we recall that

$$b > a + c \Rightarrow -z_1 > a \Rightarrow -(a\eta_0(0) + z_1) > -(a + z_1) > 0$$

and

$$(a\eta_0(0)+z_2)\ e^{-(z_2-z_1)t} - (a\eta_0(0)+z_1) > 0.$$

After some elementary computations, we find that

$$\eta_t(0) = -\frac{1}{a}\ \partial_t\left(\log\left[\left(\eta_0(0)+\frac{z_2}{a}\right)\ e^{z_1 t} - \left(\eta_0(0)+\frac{z_1}{a}\right)\ e^{z_2 t}\right]\right).$$

This yields

$$\begin{aligned}\int_0^t \eta_s(V)ds &= V(1)t + a \int_0^t \eta_s(0)\ ds \\ &= V(1)t - \left[\log\left(\left(\eta_0(0) + \frac{z_2}{a}\right)\ e^{z_1 s} - \left(\eta_0(0) + \frac{z_1}{a}\right)\ e^{z_2 s}\right)\right]_{s=0}^{s=t} \\ &= (V(1) - z_2)\,t + \log\left[\frac{(z_2 - z_1)}{(a\eta_0(0) + z_2)\ e^{-(z_2 - z_1)t} - (a\eta_0(0) + z_1)}\right].\end{aligned}$$

Observe that

$$\eta_\infty(V) = \eta_\infty(0)\ V(0) + \eta_\infty(1)\ V(1) = -\frac{z_2}{a}\ (V(0) - V(1)) + V(1) = V(1) - z_2.$$

This implies that

$$\begin{aligned}&\frac{1}{t}\int_0^t \eta_s(V)ds \\ &= \eta_\infty(V) + \frac{1}{t}\log\left[\frac{(z_2 - z_1)}{(a\eta_0(0) + z_2)\ e^{-(z_2 - z_1)t} - (a\eta_0(0) + z_1)}\right].\end{aligned}$$

Using the fact that

$$E_1 - E_0 = z_2 - z_1 > 0 \quad \text{and} \quad a = V(0) - V(1) > 0$$

we have

$$\begin{aligned}&(a\eta_0(0) + z_2)\ e^{-(z_2 - z_1)t} - (a\eta_0(0) + z_1) \\ &= a\left\{(\eta_0(0) + z_2/a)\ e^{-(E_1 - E_0)t} - (\eta_0(0) + z_2/a - (E_1 - E_0)/a)\right\} \\ &= -a\left\{(\eta_0(0) - \eta_\infty(0))\ \left(1 - e^{-(E_1 - E_0)t}\right) + (E_1 - E_0)/(V(1) - V(0))\right\}.\end{aligned}$$

This yields the formula

$$\begin{aligned}&\frac{1}{t}\int_0^t \eta_s(V)ds \\ &= \eta_\infty(V) + \frac{1}{t}\log\left[\frac{(E_1 - E_0)/(V(1) - V(0))}{(\eta_0(0) - \eta_\infty(0))\ \left(1 - e^{-(E_1 - E_0)t}\right) + (E_1 - E_0)/(V(1) - V(0))}\right].\end{aligned}$$

By the definition of φ_0 we have

$$\begin{aligned}&L^V(\varphi_0)(0) = L(\varphi_0)(0) - V(0)\varphi_0(0) = -(V(1) - z_2)\ \varphi_0(0) \\ &\Leftrightarrow \lambda(0)(\varphi_0(1) - \varphi_0(0)) = (a + z_2)\ \varphi_0(0) \\ &\Leftrightarrow \lambda(0)\varphi_0(1) = (a + z_2 + \lambda(0))\ \varphi_0(0) = (b + z_2 - \lambda(1))\ \varphi_0(0).\end{aligned}$$

In much the same way, we have

$$\begin{aligned}&L^V(\varphi_0)(1) = L(\varphi_0)(1) - V(1)\varphi_0(1) = -(V(1) - z_2)\ \varphi_0(1) \\ &\Leftrightarrow \lambda(1)(\varphi_0(0) - \varphi_0(1)) = z_2\ \varphi_0(1) \Leftrightarrow \lambda(1)\varphi_0(0) = (z_2 + \lambda(1))\ \varphi_0(1).\end{aligned}$$

This shows that
$$\frac{\varphi_0(1)}{\varphi_0(0)} = \frac{a + z_2 + \lambda(0)}{\lambda(0)} = \frac{\lambda(1)}{z_2 + \lambda(1)}.$$
Notice that
$$\frac{a + z_2 + \lambda(0)}{\lambda(0)} = \frac{\lambda(1)}{z_2 + \lambda(1)} \Longleftrightarrow z_2^2 + bz_2 + ac = 0$$
with $b := a + \lambda(0) + \lambda(1)$ and $c := \lambda(1)$.

Consider the probability measure $\pi_0 = \Psi_{\varphi_0}(\mu)$ on S defined by
$$\pi_0(0) := \frac{\varphi_0(0)}{\mu(\varphi_0)} \ \mu(0) = 1 - \pi_0(1)$$
with the L-reversible measure μ defined above. We have
$$\begin{aligned} L^V(\varphi_0) = L(\varphi_0) - V\varphi_0 = -\eta_\infty(V) \ \varphi_0 \quad &\Longleftrightarrow \quad \mu(V\varphi_0) = \eta_\infty(V) \ \mu(\varphi_0) \\ &\Longleftrightarrow \quad \pi_0(V) = \eta_\infty(V). \end{aligned}$$

We recall that
$$L_{\pi_0}(f)(x) = L(f)(x) - V(x) \ \ (\pi_0(f) - f(x))$$
$$\Rightarrow \pi_0\left(L_{\pi_0}(f)\right) = \pi_0\left(L(f)\right) + \pi_0(fV) - \pi_0(V)\pi_0(f).$$
Using the L-reversibility property of μ we have
$$\begin{aligned} \pi_0\left(L(f)\right) &= \frac{\mu(\varphi_0 L(f))}{\mu(\varphi_0)} = \frac{\mu(fL(\varphi_0))}{\mu(\varphi_0)} = \frac{\mu(f\varphi_0 V)}{\mu(\varphi_0)} - \eta_\infty(V) \ \frac{\mu(f\varphi_0)}{\mu(\varphi_0)} \\ &= \pi_0(fV) - \pi_0(V)\pi_0(f). \end{aligned}$$
We conclude that
$$\pi_0\left(L_{\pi_0}(f)\right) = 0 \Rightarrow \pi_0 = \eta_\infty.$$
We can also check directly that $\pi_0 = \eta_\infty$ using the formula
$$\pi_0(0) = \frac{\varphi_0(0)\mu(0)}{\varphi_0(0)\mu(0) + \varphi_0(1)\mu(1)} = \frac{\lambda(1)}{\lambda(1) + \lambda(0) \ \varphi_0(1)/\varphi_0(0)}$$
and
$$\frac{\varphi_0(1)}{\varphi_0(0)} = \frac{a + z_2 + \lambda(0)}{\lambda(0)} \Rightarrow \pi_0(0) = \frac{1}{1 + \frac{a+z_2+\lambda(0)}{\lambda(0)\lambda(1)}}$$
so that
$$\begin{aligned} \pi_0(0) = \eta_\infty(0) = -z_2/a \quad &\Leftrightarrow \quad \frac{a}{z_2} + 1 + \frac{a + z_2 + \lambda(0)}{\lambda(0)\lambda(1)} = 0 \\ &\Leftrightarrow \quad z_2^2 + bz_2 + ac = 0. \end{aligned}$$

In much the same way we check that
$$\frac{\varphi_1(1)}{\varphi_1(0)} = \frac{\lambda(0) + (V(0) - V(1)) + z_1}{\lambda(0)} = \frac{\lambda(1)}{\lambda(1) + z_1}.$$

Our next objective is to check that $\mu(\varphi_0\varphi_1) = 0$. Observe that

$$\begin{aligned}
\mu(\varphi_0\varphi_1) &= \mu(0)\varphi_0(0)\varphi_1(0) + \mu(1)\varphi_0(1)\varphi_1(1) \\
&\propto \lambda(1)\varphi_0(0)\varphi_1(0) + \lambda(0)\ \varphi_0(0)\varphi_1(0)\ \frac{\lambda(1)}{\lambda(1)+z_1}\ \frac{\lambda(1)}{\lambda(1)+z_2} \\
&= \lambda(1)\varphi_0(0)\varphi_1(0)\left(1 + \frac{\lambda(0)\lambda(1)}{(\lambda(1)+z_1)(\lambda(1)+z_2)}\right).
\end{aligned}$$

On the other hand, recalling that $z_1 = V(1) - E_1$ and $z_2 = V(1) - E_0$ we have

$$\lambda(0)\lambda(1) + (\lambda(1)+z_1)(\lambda(1)+z_2)$$

$$= \lambda(0)\lambda(1) + (\lambda(1)+V(1)-E_1)(\lambda(1)+V(1)-E_0)$$

$$= E_0E_1 - (\lambda(1)+V(1))\,(E_0+E_1) + (\lambda(1)+V(1))^2 + \lambda(0)\lambda(1) = 0.$$

The last assertion is checked using (30.62).

This ends the proof of the exercise.

■

Solution to exercise 448:

We have

$$\partial_t\alpha(t) = -\frac{i}{2}\sqrt{\frac{k}{m}}\ \alpha(t)\ \Rightarrow\ i\hbar\partial_t\psi(t,x) = \frac{\hbar}{2}\sqrt{\frac{k}{m}}\ \psi(t,x).$$

In much the same way, we have

$$\begin{aligned}
\partial_x^2\psi_0(x) &= -c\ \sqrt{\frac{km}{\hbar^2}}\ \partial_x\left(x\ \exp\left(-\frac{1}{2}\sqrt{\frac{km}{\hbar^2}}\ x^2\right)\right) \\
&= -c\ \sqrt{\frac{km}{\hbar^2}}\ \psi_0(x) + c\ \frac{km}{\hbar^2}\ x^2\ \psi_0(x).
\end{aligned}$$

This implies that

$$-\frac{\hbar^2}{2m}\partial_x^2\psi(t,x) = \left(\frac{\hbar}{2}\ \sqrt{\frac{k}{m}} - \frac{km}{2}\ x^2\right)\ \psi(t,x) = i\hbar\partial_t\psi(t,x) - \frac{km}{2}\ x^2\ \psi(t,x).$$

This ends the proof of the exercise.

■

Solution to exercise 449:

The Feynman-Kac formula is a direct consequence of the exponential change of probability measures discussed in section 18.3.

Using the fact that L is reversible with respect to some non negative measure μ, for any couple of smooth functions f_1 and f_2 we have

$$\begin{aligned}
\mu\left(\varphi_{\mathcal{T}}^2\ f_1\ L^{[\varphi_{\mathcal{T}}]}(f_2)\right) &= \mu\left(\varphi_{\mathcal{T}}^2\ f_1\ L(f_2)\right) + \mu\left(\varphi_{\mathcal{T}}^2\ f_1\ \left[\varphi_{\mathcal{T}}^{-1}\Gamma_L(\varphi_{\mathcal{T}}, f_2)\right]\right) \\
&= \mu\left(\varphi_{\mathcal{T}}^2\ f_1\ L(f_2)\right) + \mu\left(\varphi_{\mathcal{T}}^2\ f_1\ \left[\varphi_{\mathcal{T}}^{-1}L(\varphi_{\mathcal{T}}f_2) - L(f_2) - \varphi_{\mathcal{T}}^{-1}f_2L(\varphi_{\mathcal{T}})\right]\right) \\
&= \underbrace{\mu\left(\varphi_{\mathcal{T}}\ f_1\ L(\varphi_{\mathcal{T}}f_2)\right)}_{=\mu(\varphi_{\mathcal{T}}\ f_2\ L(\varphi_{\mathcal{T}}f_1))} - \mu\left(\varphi_{\mathcal{T}}\ f_1\ f_2L(\varphi_{\mathcal{T}})\right) = \mu\left(\varphi_{\mathcal{T}}^2\ f_2\ L^{[\varphi_{\mathcal{T}}]}(f_1)\right).
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 450:

Expressing any normalized function $\varphi \in \mathbb{L}_2(\mu)$ by using the basis of orthonormal eigenfunctions φ_i, $i \geq 0$, we find that

$$\varphi = \sum_{i\geq 0} \langle \varphi, \varphi_i \rangle \; \varphi_i \Rightarrow \quad \begin{array}{rcl} \langle \varphi, \mathcal{H}(\varphi) \rangle & = & \sum_{i\geq 0} E_i \; \langle \varphi, \varphi_i \rangle^2 \; \langle \varphi_i, \varphi_i \rangle \\ & = & \sum_{i\geq 0} E_i \; \langle \varphi, \varphi_i \rangle^2 \geq E_0 \; \sum_{i\geq 0} \; \langle \varphi, \varphi_i \rangle^2 = E_0 \; \langle \varphi, \varphi \rangle. \end{array} \tag{30.63}$$

The last assertion follows from

$$\langle \varphi, \varphi \rangle = \sum_{i,j\geq 0} \langle \varphi, \varphi_i \rangle \; \langle \varphi, \varphi_j \rangle \; \underbrace{\langle \varphi_i, \varphi_j \rangle}_{1_{i=j}} = \sum_{i\geq 0} \; \langle \varphi, \varphi_i \rangle^2.$$

This yields the variational principle

$$\frac{\langle \varphi, \mathcal{H}(\varphi) \rangle}{\langle \varphi, \varphi \rangle} \geq E_0. \tag{30.64}$$

Replacing φ by φ_0 in (30.63) we have

$$\sum_{i\geq 0} E_i \; \underbrace{\langle \varphi_0, \varphi_i \rangle^2}_{1_{i=0}} = E_0 = E_0 \; \langle \varphi_0, \varphi_0 \rangle.$$

This implies that

$$E_0 = \frac{\langle \varphi_0, \mathcal{H}(\varphi_0) \rangle}{\langle \varphi_0, \varphi_0 \rangle} = \inf_{\varphi \in \mathbb{L}_2(\mu)} \frac{\langle \varphi, \mathcal{H}(\varphi) \rangle}{\langle \varphi, \varphi \rangle}.$$

The last assertion is a direct consequence of:

$$V_{\mathcal{T}} = \varphi_{\mathcal{T}}^{-1} \mathcal{H}(\varphi_{\mathcal{T}}) \Rightarrow \Psi_{\varphi_{\mathcal{T}}^2}(\mu)(V_{\mathcal{T}}) = \frac{\mu(\varphi_{\mathcal{T}}^2 \; V_{\mathcal{T}})}{\mu(\varphi_{\mathcal{T}}^2)} = \frac{\langle \varphi_{\mathcal{T}}, \mathcal{H}(\varphi_{\mathcal{T}}) \rangle}{\langle \varphi, \varphi \rangle} \geq E_0.$$

This ends the proof of the exercise.

■

Solution to exercise 451:

For any given state $q \in \Lambda$ we set

$$f_q \; : \; x \in \{0,1\}^{\Lambda} \mapsto f_q(x) = x(q).$$

We have

$$L(f_q)\,(x) = \sum_{p\in\Lambda} \; x(p) \; r_d \; \left(x^{p,0}(q) - x(q)\right) + \sum_{p\in\Lambda} \; (1 - x(p)) \; r_b \; \sum_{s\sim p} x(s) \; \left(x^{p,1}(q) - x(q)\right).$$

Recalling that

$$x^{p,e}(q) = \left\{ \begin{array}{rcl} x(q) & \text{if} & q \neq p \\ e & \text{if} & q = p. \end{array} \right.$$

we find that

$$\begin{aligned}
L(f_q)\,(x) &= x(q)\ r_d\ \left(x^{q,0}(q) - x(q)\right) + \ (1 - x(q))\ r_b\ \sum_{s\sim q} x(s)\ \left(x^{q,1}(q) - x(q)\right) \\
&= -r_d\ x(q)^2 + r_b\ \sum_{s\sim q} x(s)\ \left(1 - x(q)\right)^2 \\
&= -r_d\ x(q) + r_b\ \sum_{s\sim q} x(s)\ \left(1 - x(q)\right) = -r_d\ f_q(x) + r_b\ \sum_{p\sim q} f_p(x)\ \left(1 - f_q(x)\right).
\end{aligned}$$

This yields

$$\partial_t \eta_t(f_q) = \eta_t(L(f_q)) = -r_d\ \eta_t(f_q) + r_b\ \sum_{p\sim q} \int\ \eta_t(dx)\ \ f_p(x)\ \left(1 - f_q(x)\right)$$

from which we conclude that

$$\mu_t(q) = \eta_t(f_q) \Rightarrow \partial_t \mu_t(q) = -r_d\ \mu_t(q) + r_b\ \sum_{p\sim q}\ \mu_t(p)(1 - \mu_t(q)) - \sum_{p\sim q} \mathrm{Cov}_t(p,q),$$

with

$$\mathrm{Cov}_t(p,q) := \sum_{p\sim q} \int\ \eta_t(dx)\ \ (f_p(x) - \eta_t(f_p))\ (f_q(x) - \eta_t(f_q)).$$

For regular homogeneous lattices we have

$$\forall p,q \in \Lambda \quad \eta_t(f_q) = \eta_t(f_p) := z_t \quad \text{and} \quad \|\{s \in \Lambda\ :\ s \sim p\}\| = \|\{s \in \Lambda\ :\ s \sim q\}\| := n.$$

In this situation, we find that

$$\dot{z_t} = -r_d\ z_t + r_b\ n\ z_t(1 - z_t) - n\mathrm{Var}_t(p) \quad \text{with} \quad \mathrm{Var}_t(p) := \mathrm{Cov}_t(p,p).$$

This ends the proof of the exercise.

■

Solution to exercise 452:

$$\begin{aligned}
\dot{z_t} &= -r_d\ z_t + r_b\ n\ z_t(1 - z_t) = -r_b\ n\ z_t^2 + (r_b\ n - r_d)\ z \\
&= a\ z_t^2 + b\ z = z\ (a\ z_t + b) \quad \text{with} \quad a = -r_b\ n \quad \text{and} \quad b = (r_b\ n - r_d).
\end{aligned}$$

There are two stationary solutions

$$z_t = 0 \quad \text{and} \quad z_t = -b/a$$

corresponding to the roots of the characteristic polynomial $p(z) = z\ (a\ z + b)$. If we set $\lambda := r_b/r_d$, then we observe that

$$\frac{a}{b} = \frac{1}{\frac{1}{n\lambda} - 1} > 0 \Leftrightarrow \lambda < \frac{1}{n}.$$

In this situation, using the same arguments as in the proof of exercise 308, the solution is given by

$$\begin{aligned}
z_t &= e^{bt}\ \frac{z_0}{1 - \frac{a}{b}\ z_0\,(1 - e^{bt})} = \frac{z_0}{e^{-bt} + \frac{a}{b}\ z_0\,(1 - e^{-bt})} \\
&= \frac{b}{a} + \left(z_0 - \frac{b}{a}\right)\ \frac{e^{-bt}}{e^{-bt} + \frac{a}{b}\ z_0\,(1 - e^{-bt})}.
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 453:
We let $P_t(x,y) = \mathbb{P}(X_t = y \mid X_0 = x)$ be the probability transitions of X_t. By construction, we have

$$f = 1_y \;\Rightarrow\; P_t(f)(x_0) = \mathbb{E}\left(f(X_t) \mid X_0 = x\right) = P_t(x,y).$$

We also have the forward equation

$$f = 1_y \;\Rightarrow\; \partial_t P_t(x,y) = \partial_t P_t(f)(x) = L\left(P_t(f)\right)(x) = \sum_{p\in\Lambda} \lambda(p,x)\;\left(P_t(x^p,y) - P_t(x,y)\right).$$

We choose

$$\lambda(p,x) = \frac{e^{-\beta v(p,x) x(p)}}{e^{\beta v(p,x)} + e^{-\beta v(p,x)}} \quad \text{with} \quad v(p,x) = h(p,x) + \sum_{q\in\Lambda-\{p\}} j(p-q)\; x(q).$$

Since j is symmetric and $j(0) = 0$, for any fixed $p \in \Lambda$ we have

$$\begin{aligned}
&\frac{1}{2}\sum_{r\in\Lambda}\sum_{s\in\Lambda} j(r-s)\; x(r)x(s) \\
&\frac{1}{2}\; x(p) \sum_{s\in\Lambda} j(p-s)\; x(s) + \frac{1}{2}\sum_{r\in\Lambda-\{p\}}\sum_{s\in\Lambda} j(r-s)\; x(r)x(s) \\
&= x(p) \sum_{q\in\Lambda-\{p\}} j(p-q)\; x(q) + \frac{1}{2}\sum_{r\in\Lambda-\{p\}}\sum_{s\in\Lambda-\{p\}} j(r-s)\; x(r)x(s).
\end{aligned}$$

Also notice that $\lambda(p,x)$ stands for the rate at which x jumps to $y = x^p$. In addition, starting from $y = x^p$, the rate at which y comes back to x is given by $\lambda(p,x^p)$. Thus, to check the reversibility property it suffices to check that

$$\forall p\in\Lambda\; \forall x\in S \qquad \pi(x)\;\lambda(p,x) = \pi(x^p)\;\lambda(p,x^p). \tag{30.65}$$

To be more precise, suppose that the above property is satisfied. In this case for any functions f and g on S we have

$$\sum_{(p,x)\in(\Lambda\times S)} \pi(x)\lambda(p,x)\; f(x)\; g(x^p) \quad = \quad \sum_{(p,x)\in(\Lambda\times S)} \pi(x^p)\lambda(p,x^p)\; f(x)\; g(x^p).$$

Since

$$y = x^p \;\Leftrightarrow\; y^p = x,$$

by using the change of variable $y = x^p$ we readily check that

$$\begin{aligned}
\sum_{(p,x)\in(\Lambda\times S)} \pi(x^p)\lambda(p,x^p)\; f(x)\; g(x^p) \quad &= \quad \sum_{(p,y)\in(\Lambda\times S)} \pi(y)\lambda(p,y)\; f(y^p)\; g(y) \\
&= \quad \sum_{(p,x)\in(\Lambda\times S)} \pi(x)\lambda(p,x)\; f(x^p)\; g(x).
\end{aligned}$$

This yields the formula

$$\sum_{(p,x)\in(\Lambda\times S)} \pi(x)\lambda(p,x)\; f(x)\; g(x^p) = \sum_{(p,x)\in(\Lambda\times S)} \pi(x)\lambda(p,x)\; f(x^p)\; g(x).$$

This implies that

$$\begin{aligned}\pi(fL(g)) &= \sum_{(p,x)\in(\Lambda\times S)} \pi(x)\lambda(p,x)\ f(x)\ (g(x^p)-g(x))\\ &= \sum_{(p,x)\in(\Lambda\times S)} \pi(x)\lambda(p,x)\ g(x)\ (f(x^p)-f(x)) = \pi(L(f)g).\end{aligned}$$

We conclude that π is L-reversible.

To check (30.65) we use the balance equation

$$\begin{aligned}&\left(e^{\beta v(p,x)}+e^{-\beta v(p,x)}\right)\pi(x)\ \lambda(p,x)\\ &= \exp\left\{-\beta\left[h(p)+\sum_{q\in\Lambda-\{p\}} j(p-q)\ x(q)\right]x(p)\right.\\ &\qquad\left.+\frac{\beta}{2}\sum_{(r,s)\in\Lambda^2} j(r-s)\ x(r)x(s)+\beta\sum_{q\in\Lambda} h(q)\ x(q)\right\}\\ &= \exp\left(\frac{\beta}{2}\sum_{(r,s)\in(\Lambda-\{p\})^2} j(r-s)\ x(r)x(s)+\beta\sum_{q\in\Lambda-\{p\}} h(q)\ x(q)\right)\\ &= \left(e^{\beta v(p,x^p)}+e^{-\beta v(p,x^p)}\right)\ \pi(x^p)\ \lambda(p,x^p).\end{aligned}$$

The last assertion comes from the fact that

$$v(p,x^p)=h(p)+\textstyle\sum_{q\in\Lambda-\{p\}} j(p-q)\ x^p(q)=h(p)+\sum_{q\in\Lambda-\{p\}} j(p-q)\ x(q)=v(p,x)$$

$$\Rightarrow \lambda(p,x^p)=\frac{e^{-\beta v(p,x^p)x^p(p)}}{e^{\beta v(p,x^p)}+e^{-\beta v(p,x^p)}}=\frac{e^{\beta v(p)x(p)}}{e^{\beta v(p)}+e^{-\beta v(p)}},$$

and

$$\begin{aligned}H(x^p) &= -\frac{1}{2}\sum_{(r,s)\in\Lambda^2} j(r-s)\ x^p(r)x^p(s)-\sum_{r\in\Lambda} h(r)\ x^p(r)\\ &= -\frac{1}{2}\sum_{(r,s)\in(\Lambda-\{p\})^2} j(r-s)\ x(r)x(s)-\sum_{r\in\Lambda-\{p\}} h(r)\ x(r)\\ &\qquad + x(p)\ \underbrace{\left[\sum_{r\in\Lambda-\{p\}} j(r-s)\ x(r)+h(p)\right]}_{=v(p)}.\end{aligned}$$

This shows that

$$\begin{aligned}&\beta\left(-H(x^p)-v(p,x^p)x^p(p)\right)\\ &= \frac{\beta}{2}\sum_{(r,s)\in(\Lambda\times\Lambda-\{p\})} j(r-s)\ x(r)x(s)+\beta\sum_{r\in\Lambda-\{p\}} h(r)\ x(r)-\beta x(p)v(p).\end{aligned}$$

This ends the proof of the exercise. ■

Chapter 28

Solution to exercise 454:

The idea is to look for solutions of the form

$$X_t = \frac{Y_t}{1+\int_0^t Y_s ds} \quad \text{with} \quad dY_t = aY_t dt + bY_t dW_t \Rightarrow Y_t = Y_0 e^{(a-\frac{b^2}{2})t+bW_t}.$$

We also set $Y_0 = X_0$. Applying The Doeblin-Itō formula, we have

$$\begin{aligned} dX_t &= \frac{dY_t}{1+\int_0^t Y_s ds} + Y_t \ d\left(\frac{1}{1+\int_0^t Y_s ds}\right) \\ &= \frac{aY_t dt + bY_t dW_t}{1+\int_0^t Y_s ds} - \left(\frac{Y_t}{1+\int_0^t Y_s ds}\right)^2 dt = X_t(a - X_t)\ dt + bX_t dW_t. \end{aligned}$$

Choosing $a = \lambda$ and $b = \sigma$ we obtain the solution

$$X_t = X_0 \ \frac{e^{(\lambda-\frac{\sigma^2}{2})t+\sigma W_t}}{1 + X_0 \int_0^t e^{(\lambda-\frac{\sigma^2}{2})s+\sigma W_s} ds} = \frac{Y_t}{1+\int_0^t Y_s ds}$$

with $Y_t = X_0 e^{(\lambda-\frac{\sigma^2}{2})t+\sigma W_t}$. This ends the proof of the exercise.

■

Solution to exercise 455:

We have

$$X_t = \frac{Y_t}{1+\frac{a}{b}\ \int_0^t Y_s ds} \quad \text{with} \quad dY_t = aY_t dt + \sigma Y_t dW_t \Rightarrow Y_t = X_0 e^{(a-\frac{\sigma^2}{2})t+\sigma W_t}.$$

We also set $Y_0 = X_0$. Applying The Doeblin-Itō formula, we have

$$\begin{aligned} dX_t &= \frac{dY_t}{1+\frac{a}{b}\ \int_0^t Y_s ds} + Y_t \ d\left(\frac{1}{1+\frac{a}{b}\ \int_0^t Y_s ds}\right) \\ &= \frac{aY_t dt + \sigma Y_t dW_t}{1+\frac{a}{b}\int_0^t Y_s ds} - \frac{a}{b}\left(\frac{Y_t}{1+\frac{a}{b}\int_0^t Y_s ds}\right)^2 dt = a\ X_t\ \left(1 - \frac{X_t}{b}\right)\ dt + \sigma X_t dW_t. \end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 456:

Recall that

$$\overline{N}_t^1 := N^1\left(\int_0^t \ \lambda_1\ X(s)\ \left(1 - \frac{X(s)}{N}\right)\ ds\right)$$

is a Poisson process with stochastic intensity

$$\lambda_t^1 = \int_0^t \lambda_1 \ X(s) \ \left(1 - \frac{X(s)}{N}\right) \ ds.$$

In the same way, the process

$$\overline{N}_t^2 := N^2 \left(\int_0^t \lambda_2 \ x(s) \ \left(1 + \alpha_2 \ \frac{x(s)}{N}\right) \ ds\right)$$

is a Poisson process with stochastic intensity

$$\lambda_t^2 = \int_0^t \lambda_2 \ x(s) \ \left(1 + \alpha_2 \ \frac{x(s)}{N}\right) \ ds.$$

Therefore the process X_t evolves according to two transitions. The birth type transitions $X_t \rightsquigarrow X_t + 1$ arise at rate λ_t^1. The death type transitions $X_t \rightsquigarrow X_t - 1$ arise at rate λ_t^2. When the process hits 0 it remains at 0. We conclude that the generator of the process X_t coincides with the birth and death generator (28.6).

This ends the proof of the exercise.

■

Solution to exercise 457:

We have

$$\begin{aligned} dX_t^2 &= 2X_t dX_t + \sigma^2 X_t^2 dt \\ &= 2a \ X_t^2 \ \left(\left(1 + \frac{\sigma^2}{2a}\right) - \frac{X_t^2}{b}\right) \ dt + 2\sigma \ X_t^2 \ dW_t \\ &= \overline{a} \ X_t^2 \ \left(1 - \frac{X_t^2}{\overline{b}}\right) \ dt + \overline{\sigma} \ X_t^2 \ dW_t \end{aligned}$$

with

$$\overline{a} = 2a \left(1 + \frac{\sigma^2}{2a}\right) = 2a + \sigma^2 \qquad \overline{b} = b \ \left(1 + \frac{\sigma^2}{2a}\right) \quad \text{and} \quad \overline{\sigma} = 2\sigma.$$

Using exercise (455) we have

$$X_t^2 = \frac{Y_t}{1 + \frac{\overline{a}}{\overline{b}} \ \int_0^t Y_s ds} \quad \text{with} \quad Y_t = X_0^2 \exp\left[(\overline{a} - \frac{\overline{\sigma}^2}{2}) \ t + \overline{\sigma} W_t\right].$$

Notice that

$$\frac{\overline{a}}{\overline{b}} = 2 \ \frac{a}{b} \quad \text{and} \quad \overline{a} - \frac{\overline{\sigma}^2}{2} = 2a + \sigma^2 - 2\sigma^2 = 2a - \sigma^2.$$

This ends the proof of the exercise.

■

Solution to exercise 454:

A commonly used stochastic version is given by the 2-d diffusion

$$\left\{ \begin{array}{rcl} dX_t &=& X_t \left(a_1 - b_{1,1} X_t + b_{1,2} Y_t\right) \ dt + \left(\sigma_{1,1} X_t + \sigma_{1,2} Y_t\right) \ dW_t^1 \\ dY_t &=& Y_t \left(a_2 - b_{2,2} Y_t + b_{2,1} X_t\right) \ dt + \left(\sigma_{2,1} X_t + \sigma_{2,2} Y_t\right) \ dW_t^2 \end{array} \right.$$

with a 2-dimensional Brownian motion, and some positive parameters $\sigma_{i,j}$. These diffusions can be approximated using Euler type schemes on some time mesh. This ends the proof of the exercise.

■

Solution to exercise 459:

We readily check that

$$x'_t + y'_t = 0 \Rightarrow x_t + y_t = x_0 + y_0 = N.$$

The equilibrium states of the system (28.18) are defined by

$$\frac{\lambda_c}{N}\, xy = (\lambda_b + \lambda_r)\, y \Rightarrow (x,y) = (N,0) \quad \text{or} \quad (x,y) = \left(N\, \frac{\lambda_b + \lambda_r}{\lambda_c}, N\, \left(1 - \frac{\lambda_b + \lambda_r}{\lambda_c}\right)\right).$$

The first equilibrium state occurs when $\lambda_b + \lambda_r \leq \lambda_c$, while the second one occurs when $\lambda_b + \lambda_r > \lambda_c$.

Recalling that $x_t = N - y_t$, the equation (28.18) reduces to a single evolution model

$$y'_t = \frac{\lambda_c}{N}\, y(N-y) - (\lambda_b + \lambda_r)\, y.$$

The stochastic version is often defined by

$$dY_t = \left(\frac{\lambda_c}{N}\, Y_t(N-Y_t) - (\lambda_b + \lambda_r)\, Y_t\right)\, dt \; + \left(\frac{\lambda_c}{N}\, Y_t(N-Y_t) - (\lambda_b + \lambda_r)\, Y_t\right)^{1/2}\, dW_t.$$

This ends the proof of the exercise.

■

Solution to exercise 460:

The first assertion is immediate. We use the same arguments as in the proof of exercise 456. Recall that

$$N^i\left[\int_0^t \lambda_i(X(s), Y_s))\, ds\right]$$

are Poisson processes with stochastic intensities $\lambda_i(X(s))$. At rate $\lambda_1(X(t), Y(t))) := a\, X(t)\, dt$ the process jumps from $\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$ to $\begin{pmatrix} X(t)+1 \\ Y(t) \end{pmatrix}$. In much the same way, at rate $\lambda_2(X(t), Y(t))) := b\, X(t)\, dt$ the process jumps from $\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$ to $\begin{pmatrix} X(t)-1 \\ Y(t) \end{pmatrix}$. At rate $\lambda_3(X(t), Y(t))) := c\, Y(t)\, dt$ the process jumps from $\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$ to $\begin{pmatrix} X(t) \\ Y(t)-1 \end{pmatrix}$; and finally at rate $\lambda_4(X(t), Y(t))) := d\, X(t)Y(t)\, dt$ the process jumps from $\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$ to $\begin{pmatrix} X(t) \\ Y(t)+1 \end{pmatrix}$. We conclude that the generator of the process $\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$ is given by

$$\begin{aligned} L(f)(x,y) &= \lambda_1(x,y)\, (f(x+1,y) - f(x,y)) + \lambda_2(x,y)\, (f(x-1,y) - f(x,y)) \\ &\quad + \lambda_3(x,y)\, (f(x,y-1) - f(x,y)) + \lambda_4(x,y)\, (f(x,y+1) - f(x,y)). \end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 461:

For any function f on the state space $S=\mathbb{N}$, we set $p_t(f)=\sum_{x\in\mathbb{N}} f(x)p_t(x)$. In this notation, we have

$$\begin{aligned}
&\frac{d}{dt}p_t(f)\\
&=\sum_{x\geq 1}\left[(a_1\ (x-1)+b)\ p_t(x-1)+a_2\ (x+1)\ p_t(x+1)-[b+(a_1+a_2)\ x]\ \ p_t(x)\right]\ f(x)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -f(0)\left[bf(0)p_t(0)-a_2\ p_t(1)\right]\\
&=\sum_{x\geq 0}(a_1\ x+b)\ p_t(x)\ f(x+1)+\sum_{x\geq 2}a_2\ x\ p_t(x)\ f(x-1)\\
&\qquad\qquad -\sum_{x\geq 1}\left[(a_1x+b)+a_2\ x\ \right]\ p_t(x)\ f(x)-f(0)\left[bf(0)p_t(0)-a_2\ p_t(1)\right].
\end{aligned}$$

This yields

$$\begin{aligned}
\frac{d}{dt}p_t(f)&=\sum_{x\geq 0}(a_1\ x+b)\ p_t(x)\ f(x+1)+\sum_{x\geq 0}a_2\ x\ p_t(x)\ f(x-1)\\
&\qquad\qquad -\sum_{x\geq 0}\left[(a_1x+b)+a_2\ x\ \right]\ p_t(x)\ f(x)\\
&=\sum_{x\geq 0}\left\{(a_1\ x+b)\ \ \ [f(x+1)-f(x)]+a_2\ x\ \ [f(x-1)-f(x)]\right\}\ p_t(x)=p_t(L(f)),
\end{aligned}$$

with the generator

$$L(f)(x)=(a_1\ x+b)\ \ [f(x+1)-f(x)]+a_2\ x\ \ [f(x-1)-f(x)]\,.$$

We conclude that X_t is a birth and death process with linear birth rate $\lambda_{\text{birth}}(x)=(a_1\ x+b)$ and linear death rate $\lambda_{\text{death}}(x)=a_2\ x$.

The equilibrium distribution π satisfies

$$\forall x\geq 1\qquad (a_1\ (x-1)+b)\ \pi(x-1)+a_2\ (x+1)\ \pi(x+1)-[b+(a_1+a_2)\ x]\ \ \pi(x)=0$$

and

$$-b\pi(0)+a_2\pi(1)=0\ \Rightarrow\ \pi(1)=\frac{b}{a_2}\ \pi(0).$$

Choosing $x=1$ in the first equation we find that that

$$\begin{aligned}
0&=b\ \pi(0)+2a_2\ \pi(2)-[b+(a_1+a_2)]\ \ \pi(1)\\
&=b\ \pi(0)+2a_2\ \pi(2)-[b+(a_1+a_2)]\ \ \frac{b}{a_2}\ \pi(0)\\
&=2a_2\ \pi(2)-\frac{b}{a_2}\,[b+a_1]\ \ \pi(0)\ \ \Rightarrow\ \ \pi(2)=\frac{b}{2a_2^2}\ \ [b+a_1]\ \ \pi(0).
\end{aligned}$$

Substituting $x = 2$ in the same equation, we have

$$\begin{aligned}
0 &= (a_1 + b)\ \pi(1) + 3a_2\ \pi(3) - [b + 2(a_1 + a_2)]\ \pi(2) \\
&= 3a_2\ \pi(3) - \frac{b}{a_2^2}\ \left[\frac{1}{2}\,[b + 2(a_1 + a_2)\ ] - a_2\right][b + a_1]\ \pi(0) \\
&= 3a_2\ \pi(3) - \frac{b}{2a_2^2}\ [b + 2a_1]\,[b + a_1]\ \pi(0)\ \Rightarrow\ \pi(3) = \frac{b}{3!a_2^3}\ [b + 2a_1]\,[b + a_1]\ \pi(0).
\end{aligned}$$

We further assume that

$$\pi(x) = \frac{1}{x!\ a_2^x}\ \left\{\prod_{0 \le k < x} [b + k\ a_1]\right\}\ \pi(0) = \frac{1}{x!}\ \left(\frac{a_1}{a_2}\right)^x\ \left\{\prod_{0 \le k < x} \left[\frac{b}{a_1} + k\right]\right\}\ \pi(0) \quad (30.66)$$

and will prove (30.66) by mathematical induction. Assume that the formula is true for for any $x \in \{0, \ldots, n\}$. Choosing $x = n$ in the Kolmogorov equation we have

$$\begin{aligned}
0 &= (a_1\ (n-1) + b)\ \pi(n-1) + a_2\ (n+1)\ \pi(n+1) - [b + (a_1 + a_2)\ n]\ \pi(n) \\
&= a_2\ (n+1)\ \pi(n+1) - [[b + (a_1 + a_2)\ n]\ \pi(n) - (a_1\ (n-1) + b)\ \pi(n-1)]\,.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&[[b + (a_1 + a_2)\ n]\ \pi(n) - (a_1\ (n-1) + b)\ \pi(n-1)] \\
&= \left[[b + na_1 + a_2 n]\ \frac{1}{n!}\ - \frac{1}{(n-1)!}\ a_2\right]\pi(0)\ \frac{b}{a_2^n}\ \prod_{1 \le k \le (n-1)} [b + k\ a_1] \\
&= \pi(0)\ \frac{1}{n!a_2^n}\ \prod_{0 \le k \le n} [b + k\ a_1]\,.
\end{aligned}$$

After substituting back in the Kolmogorov equation we get

$$\pi(n+1) = \pi(0)\ \frac{1}{(n+1)!a_2^{n+1}}\ \prod_{0 \le k \le n} [b + k\ a_1]$$

and this finalises the proof by induction.

Notice that (30.66) can be rewritten in terms of generalized binomial coefficients

$$\forall x \ge 0 \qquad \pi(x) = \pi(0)\ \left(\frac{a_1}{a_2}\right)^x\ \left(\begin{array}{c} \frac{b}{a_1} + (x-1) \\ x \end{array}\right)$$

with $\pi(0) = (1 - a_1/a_2)^{b/a_1}$. The last assertion follows from the fact the generalized hypergeometric series formula applied to $c = b/a_1$ and $\beta = a_1/a_2$ gives

$$(c \in \mathbb{R} \quad \alpha = 1 - \beta \in [0,1]) \Rightarrow\ \ 1 = \alpha^c\ \frac{1}{(1-\beta)^c}\ \ =\ \ \alpha^c\ \sum_{x \ge 0} \left(\begin{array}{c} c + (x-1) \\ x \end{array}\right) \beta^x.$$

This ends the proof of the exercise.

■

Solution to exercise 462:

Let us look into the chance that $\alpha(x)$ hits the origin provided that it starts from a certain state $X_0 = x \in \mathbb{N} = S$; that is, into the probability of extinction of the population starting with x individuals.

If we denote by T the first time the chain hits the origin, then we have

$$\begin{aligned}
\alpha(x) &:= \mathbb{P}(T < \infty \mid X_0 = x) \\
&= \sum_{y \in S} \mathbb{P}(T < \infty \mid X_1 = y, X_0 = x)\ \mathbb{P}(X_1 = y \mid X_0 = x) \\
&= p(x)\ \alpha(x+1)\ + q(x)\alpha(x-1)\ + \alpha(x)(1 - p(x) - q(x)).
\end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned}
(\alpha(x+1) - \alpha(x)) &= \frac{q(x)}{p(x)}(\alpha(x) - \alpha(x-1)) \\
&= \ldots = \left\{\prod_{y=1}^{x} \frac{q(y)}{p(y)}\right\}\ (\alpha(1) - \alpha(0)).
\end{aligned}$$

This implies that

$$\begin{aligned}
\alpha(y+1) &= \alpha(0) + \sum_{x=0}^{y} (\alpha(x+1) - \alpha(x)) \\
&= 1 - (1 - \alpha(1))\ \sum_{x=0}^{y} \left\{\prod_{y=1}^{x} \frac{q(y)}{p(y)}\right\}.
\end{aligned}$$

The case $W := \sum_{x \geq 0} \left\{\prod_{y=1}^{x} \frac{q(y)}{p(y)}\right\} = \infty$ forces $\alpha(1) = 1$, and therefore $\alpha(y) = 1$ for any $y \in \mathbb{N}$. Hence the chain is recurrent.

In the opposite case, any choice of $\alpha(1) < 0$ such that

$$1 - (1 - \alpha(1))\ \sum_{x \geq 0} \left\{\prod_{y=1}^{x} \frac{q(y)}{p(y)}\right\} = 1 - (1 - \alpha(1))\ W \geq 0$$

satisfies the equation. The case where the l.h.s. is null provides the minimal solution

$$\begin{aligned}
1 + (\alpha(1) - 1)\ W = 0 \quad &\Rightarrow \quad \alpha(1) = 1 - \frac{1}{W} \\
&\Rightarrow \quad \alpha(y+1) = 1 - \frac{1}{W} \sum_{x=0}^{y} \left\{\prod_{y=1}^{x} \frac{q(y)}{p(y)}\right\} \\
&\qquad = \sum_{x > y} \left\{\prod_{y=1}^{x} \frac{q(y)}{p(y)}\right\} / \sum_{x \geq 0} \left\{\prod_{y=1}^{x} \frac{q(y)}{p(y)}\right\}.
\end{aligned}$$

It remains to check that the extinction probability $\alpha(x)$ coincides with the minimal solution. For any solution $\beta(x)$ (28.19) such that $\beta(0) = 1$ we have for any $x > 0$

$$\begin{aligned}
\beta(x) &:= \sum_{y > 0} M(x,y)\beta(y) + M(x,0)\beta(0) \\
&= M(x,0) + \sum_{y_1 \geq 0} M(x,y_1) 1_{\neq 0}(y_1)\beta(y_1).
\end{aligned}$$

By induction w.r.t. the number of summations we prove that

$$\begin{aligned}
\beta(x) &= M(x,0) + \sum_{y_1\geq 0} M(x,y_1)1_{\neq 0}(y_1)M(y_1,0)\\
&\qquad + \sum_{y_1\geq 0}\sum_{y_2\geq 0} M(x,y_1)1_{\neq 0}(y_1)M(y_1,y_2)1_{\neq 0}(y_2)\beta(y_2)\\
&= \ldots\\
&= \sum_{1\leq p\leq n} \mathbb{P}\left(T=p \mid X_0=x\right)\\
&\qquad + \sum_{y_1,\ldots,y_n\geq 0}\left\{\prod_{1\leq p\leq n} M(x,y_p)1_{\neq 0}(y_p)\right\}\beta(y_n).
\end{aligned}$$

This shows that

$$\begin{aligned}
\beta(x) &= \mathbb{P}\left(T\leq n \mid X_0=x\right) + \mathbb{E}\left(1_{T>n}\ \beta(X_n) \mid X_0=x\right)\\
&\geq \mathbb{P}\left(T\leq n \mid X_0=x\right)\uparrow_{n\uparrow\infty}\alpha(x).
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 463:
The proof of the exercise is a simple combination of the proofs of the exercises 317 and 318, thus it is omitted and left to the reader. See also (15.31) and the developments of section 16.1.3 and section 16.2.
This ends the proof of the exercise.

■

Solution to exercise 464:
We have

$$\partial_t\gamma_t(1) = \gamma_t(W_t) + \mu(1) = \gamma_t(1)\ \eta_t(W_t) + \mu(1) \quad \text{with} \quad W_t = U_t - V_t.$$

This implies that

$$\gamma_t(1) = e^{\int_0^t \eta_s(W_s)ds}\ \gamma_0(1) + \left[\int_0^t e^{\int_s^t \eta_r(W_r)dr}\ ds\right]\ \mu(1).$$

Observe that

$$\begin{aligned}
\partial_t\eta_t(f) &= \frac{1}{\gamma_t(1)}\ \partial_t\gamma_t(f) - \frac{\gamma_t(f)}{\gamma_t(1)}\ \frac{1}{\gamma_t(1)}\ \partial_t\gamma_t(1)\\
&= \frac{1}{\gamma_t(1)}\left[\gamma_t(L_t(f)) + \gamma_t(W_t f) + \mu(f)\right] - \frac{\gamma_t(f)}{\gamma_t(1)}\frac{1}{\gamma_t(1)}\left[\gamma_t(1)\ \eta_t(W_t) + \mu(1)\right]\\
&= \eta_t(L_t(f)) + \eta_t(W_t f) + \frac{\mu(1)}{\gamma_t(1)}\ \overline{\mu}(f) - \eta_t(f)\left[\eta_t(W_t) + \frac{\mu(1)}{\gamma_t(1)}\right].
\end{aligned}$$

This yields

$$\partial_t\eta_t(f) = \eta_t(L_t(f)) + \eta_t(W_t f) - \eta_t(f)\eta_t(W_t) + \frac{\mu(1)}{\gamma_t(1)}\ \left(\overline{\mu}(f) - \eta_t(f)\right).$$

We conclude that

$$\partial_t \eta_t(f) = \eta_t\left(\overline{L}_{t,\gamma_t(1),\eta_t}(f)\right) \quad \text{with} \quad \overline{L}_{t,\gamma_t(1),\eta_t} = L_{t,\eta_t}(f) + L^0_{t,\gamma_t(1),\eta_t}.$$

The mean field approximation $\eta_t^N = \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi_t^i}$ associated with these model is defined in terms of a Markov chain $\xi_t = (\xi_t^i)_{1\leq i\leq N}$ with *fixed* population size with a generator $\overline{L}_{t,\gamma_t^N(1),\eta_t^N}$ that depends on the total mass approximating equation

$$\partial_t \gamma_t^N(1) = \gamma_t^N(1)\ \eta_t^N(W_t) + \mu(1).$$

This ends the proof of the exercise.

■

Solution to exercise 465:

By definition of the branching process, the occupation measures $\mathcal{X}_t^i := \sum_{1\leq i\leq N_t}\ \delta_{\xi_t^{i,j}}$ are conditionally independent given the initial configuration $\xi_0 = \left(\xi_0^i\right)_{1\leq i\leq N_0}$. This implies that

$$\forall 1\leq i\leq N_0 \qquad \mathbb{E}\left(\mathcal{X}_t^i(\varphi)\mid \xi_0^i = x_0^i\right) = Q_{0,t}(\varphi)(x_0^i)$$

as well as

$$\mathcal{X}_t(\varphi) = \sum_{1\leq i\leq N_0}\mathcal{X}_t^i(\varphi) \Rightarrow \overline{\mathcal{X}}_t = \frac{1}{N_0}\sum_{1\leq i\leq N_0}\mathcal{X}_t^i = \frac{1}{N_0}\ \mathcal{X}_t \Rightarrow \mathbb{E}\left(\overline{\mathcal{X}}_t(\varphi)\right) = \gamma_t(\varphi).$$

We also have the variance formula

$$\mathbb{E}\left(\left[\frac{1}{N_0}\sum_{1\leq i\leq N_0}\mathcal{X}_t^i(\varphi) - \gamma_t(\varphi)\right]^2\right) = \frac{1}{N_0}\ \mathbb{E}\left(\left[\mathcal{X}_t^1(\varphi) - \gamma_t(\varphi)\right]^2\right).$$

We further assume that $N_0 = 1$. Using the analysis developed in section 28.4.3.2 we have

$$d\mathcal{X}_t(\varphi) = \mathcal{X}_t\left(L_t^W(\varphi)\right)\ dt + dM_t^{(1)}(\varphi)$$

with a martingale $M_t^{(1)}(\varphi)$ w.r.t. $\mathcal{F}_t = \sigma(\xi_s,\ s\leq t)$ with angle bracket

$$\partial_t\left\langle M^{(1)}(\varphi), M^{(1)}(\varphi)\right\rangle_t = \mathcal{X}_t\left[\Gamma_{L_t}(\varphi,\varphi) + (U_t+V_t)\ \varphi^2\right].$$

Choosing $\varphi = 1$, this shows that

$$N_t = \mathcal{X}_t(1) \implies dN_t = \mathcal{X}_t\left(W_t\right)\ dt + dM_t^{(1)}(1)$$

with a martingale $M_t^{(1)}(1)$ w.r.t. $\mathcal{F}_t = \sigma(\xi_s,\ s\leq t)$ with angle bracket

$$\partial_t\left\langle M^{(1)}(1), M^{(1)}(1)\right\rangle_t = \mathcal{X}_t\left[(U_t+V_t)\right].$$

This implies that

$$\mathbb{E}(N_t) = \gamma_t(1) = 1 + \int_0^t \gamma_s(W_s)\ ds \ \Leftrightarrow\ \partial_t\mathbb{E}(N_t) = \gamma_t(W_t).$$

When $W_t = 0$ we clearly have $\mathbb{E}(N_t) = N_0 = 1$. When $W_s \geq \epsilon$ we have

$$\partial_t\mathbb{E}(N_t) = \gamma_t(W_t) \geq \epsilon\ \mathbb{E}(N_t) \Rightarrow \partial_t \log\mathbb{E}(N_t) \geq \epsilon \Rightarrow \log\mathbb{E}(N_t) \geq \epsilon t \Rightarrow \mathbb{E}(N_t) \geq e^{\epsilon t}.$$

In much the same way, we check that

$$W_s \leq -\epsilon \Rightarrow \mathbb{E}(N_t) \leq e^{-\epsilon t}.$$

Applying the Doeblin-Itō formula to the function

$$f(s,x) = \sum_{1\leq i\leq N} Q_{s,t}(\varphi)(x_i)$$

w.r.t. time parameter $s \in [0,t]$ we obtain the formula

$$d\mathcal{X}_s(Q_{s,t}(\varphi)) \quad = \quad \mathcal{X}_s\left(\overbrace{\partial_s(Q_{s,t}(\varphi)) + L_s^W(Q_{s,t}(\varphi))}^{=0}\right)\ dt + dM_s^{(2)}(\varphi) = dM_s^{(2)}(\varphi)$$

with a martingale $M_s^{(2)}(\varphi)$, $s \in [0,t]$, with angle bracket

$$\partial_s \left\langle M^{(2)}(\varphi), M^{(2)}(\varphi)\right\rangle_s = \mathcal{X}_s\left[\Gamma_{L_s}(Q_{s,t}(\varphi), Q_{s,t}(\varphi)) + (U_s+V_s)\ \ (Q_{s,t}(\varphi))^2\right].$$

We conclude that

$$\begin{aligned}
&\mathbb{E}\left([\mathcal{X}_t(\varphi) - \mathcal{X}_0(Q_{0,t}(\varphi))]^2\right)\\
&= \mathbb{E}\left([\mathcal{X}_t(Q_{t,t}(\varphi)) - \mathcal{X}_0(Q_{0,t}(\varphi))]^2\right) = \mathbb{E}\left(\left[M_t^{(2)}(\varphi) - M_0^{(2)}(\varphi)\right]^2\right)\\
&= \mathbb{E}\left(\left\langle M^{(2)}(\varphi), M^{(2)}(\varphi)\right\rangle_t\right) = \int_0^t \gamma_s\left[\Gamma_{L_s}(Q_{s,t}(\varphi), Q_{s,t}(\varphi)) + (U_s+V_s)\ \ (Q_{s,t}(\varphi))^2\right]\ ds.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\partial_s\left(\gamma_s\left([Q_{s,t}(\varphi)]^2\right)\right) &= \gamma_s\left(\partial_s\left[Q_{s,t}(\varphi)\right]^2\right) + (\partial_s\gamma_s)\left([Q_{s,t}(\varphi)]^2\right)\\
&= -2\gamma_s\left(Q_{s,t}(\varphi)\ L_s^W Q_{s,t}(\varphi)\right) + \gamma_s L_s^W\left([Q_{s,t}(\varphi)]^2\right)\\
&= \gamma_s\left(\Gamma_{L_s}(Q_{s,t}(\varphi), Q_{s,t}(\varphi)) + (U_s - V_s)\ \ (Q_{s,t}(\varphi))^2\right)\\
&= \partial_s\mathbb{E}\left(\left\langle M^{(2)}(\varphi), M^{(2)}(\varphi)\right\rangle_s\right) - 2\ \gamma_s\left(V_s\left(Q_{s,t}(\varphi)\right)^2\right).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\mathbb{E}\left([\mathcal{X}_t(\varphi) - \mathcal{X}_0(Q_{0,t}(\varphi))]^2\right)\\
&= \mathbb{E}\left(\left\langle M^{(2)}(\varphi), M^{(2)}(\varphi)\right\rangle_t\right) = \gamma_t\left(\varphi^2\right) - \eta_0\left([Q_{0,t}(\varphi)]^2\right) + 2\int_0^t \gamma_s\left(V_s\left(Q_{s,t}(\varphi)\right)^2\right)\ ds,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left([\mathcal{X}_t(\varphi) - \eta_0(Q_{0,t}(\varphi))]^2\right)\\
&= \mathbb{E}\left([\mathcal{X}_t(\varphi) - \mathcal{X}_0(Q_{0,t}(\varphi))]^2\right) + \mathbb{E}\left([\mathcal{X}_0(Q_{0,t}(\varphi)) - \eta_0(Q_{0,t}(\varphi))]^2\right)\\
&= \gamma_t\left(\varphi^2\right) - \eta_0\left([Q_{0,t}(\varphi)]^2\right) + \eta_0\left[(Q_{0,t}(\varphi) - \eta_0(Q_{0,t}(\varphi)))^2\right] + 2\int_0^t \gamma_s\left(V_s\left(Q_{s,t}(\varphi)\right)^2\right)\ ds.
\end{aligned}$$

We conclude that

$$\begin{aligned}
&\mathbb{E}\left([\mathcal{X}_t(\varphi)-\gamma_t(\varphi)]^2\right)\\
&=\mathbb{E}\left([\mathcal{X}_t(\varphi)-\eta_0(Q_{0,t}(\varphi))]^2\right)\\
&=\mathbb{E}\left([\mathcal{X}_t(\varphi)-\mathcal{X}_0(Q_{0,t}(\varphi))]^2\right)+\mathbb{E}\left([\mathcal{X}_0(Q_{0,t}(\varphi))-\eta_0(Q_{0,t}(\varphi))]^2\right)\\
&=\gamma_t\left(\varphi^2\right)-\gamma_t\left(\varphi\right)^2+2\int_0^t\gamma_s\left(V_s\left(Q_{s,t}(\varphi)\right)^2\right)\,ds.
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 466 :

We follow the developments of section 28.4.3.1 and section 28.4.3.2. For functions of the form

$$F(x_1,\ldots,x_p)=\sum_{1\leq i\leq p}f(x_i)=p\ m(x)(f)\quad\text{with}\quad m(x)=\frac{1}{p}\sum_{1\leq i\leq p}\delta_{x_i}$$

for any $x=(x_1,\ldots,x_p)\in S^p$, for some $p\geq 1$, we have

$$\mathcal{L}^m(F)(x)=pm(x)(L(f)).$$

In much the same way, we have

$$\begin{aligned}
\mathcal{K}(F)(x)-F(x)&=\int\mathcal{K}(x,dy)\ (F(y)-F(x))\\
&=\frac{1}{p}\sum_{1\leq i\leq p}\sum_{k\geq 1}\alpha_k\ [F(\theta_i^k(x))-F(x)]
\end{aligned}$$

with

$$F(\theta_i^k(x))-F(x)=(k-1)f(x_i).$$

This implies that

$$\lambda_t(x)\ [\mathcal{K}_t(F)(x)-F(x)]=p\ \lambda\ m(x)(f)\ \left[\sum_{k\geq 1}\alpha_k\ k-1\right]=p\ \lambda\ m(x)(f)\ \left(\overline{\alpha}-1\right).$$

We conclude that

$$\mathcal{L}(F)(x)=pm(x)(L_t(f))+p\ \lambda\ m(x)(f)\ \left(\overline{\alpha}_t-1\right).$$

In much the same way, we have

$$\begin{aligned}
p\ \lambda\int\mathcal{K}_t(x,dy)\ (F(y)-F(x))^2&=\lambda\sum_{1\leq i\leq p}\sum_{k\geq 1}\alpha_{t,k}\ (k-1)^2f^2(x_i)\\
&=\lambda\ p\ m(x)(f^2)\sum_{k\geq 1}\alpha_k\ (k-1)^2.
\end{aligned}$$

The first moment of the branching process is given by

$$\gamma_t(f) = \mathbb{E}\left(\sum_{1\leq i\leq N_t} f(\xi_t^i)\right) = \mathbb{E}\left(\mathcal{X}_t(f)\right) \quad \text{with} \quad \mathcal{X}_t = \sum_{1\leq i\leq N_t} \delta_{\xi_t^i}.$$

Applying the Doeblin-Itō formula (28.9) we find that

$$d\mathcal{X}_t(f) = \mathcal{X}_t(L(f)) + \lambda\ \mathcal{X}_t(f)\ \ (\overline{\alpha}-1)\ \ dt + dM_t(f)$$

with a collection of martingales $M_t(f)$ with angle-bracket given by the formulae

$$\partial_t \left\langle M(f), M(f)\right\rangle_t = \left[\mathcal{X}_t\left(\Gamma_L(f,f)\right) + \sigma^2\ \mathcal{X}_t(f^2)\right].$$

Choosing $f=1$ we find that

$$dN_t = \lambda\ N_t\ \ (\overline{\alpha}-1)\ \ dt + dM_t$$

with a martingale M_t with angle-bracket given by the formulae

$$\partial_t \left\langle M, M\right\rangle_t = \sigma^2\ N_t.$$

This also implies that

$$\partial_t\gamma_t(f) = \gamma_t(L(f)) + \lambda\ \mathcal{X}_t(f)\ \ (\overline{\alpha}-1)$$

and now we can easily check that

$$\gamma_t(f) := N_0\ \mathbb{E}\left[f(X_t)\right]\ \ \exp\left\{\lambda\left(\overline{\alpha}-1\right)t\right\}.$$

By choosing $f=1$ we conclude that

$$\mathbb{E}(N_t/N_0) = e^{\lambda(\overline{\alpha}-1)t}.$$

If we consider the model (28.14) we have $\alpha_k = 1_{k=2}$, $\lambda=1$. In this situation, we have

$$\overline{\alpha}-1 := \sum_{k\geq 1} \alpha_k\ k - 1 = 1 \quad \text{and} \quad \sigma^2 := \lambda \sum_{k\geq 1} \alpha_k\ (k-1)^2 = 1.$$

In this case,

$$N_t - N_0 = \int_0^t N_s\ ds + M_t \Rightarrow \mathbb{E}(N_t/N_0) = e^t$$

with a martingale M_t with angle-bracket given by the formulae

$$\partial_t \left\langle M, M\right\rangle_t = \ N_t.$$

This ends the proof of the exercise.

■

Solution to exercise 467 :
We consider regular polynomial/product test functions f on $\boldsymbol{S} = \cup_{p\geq 0} S^p$ of the form

$$\forall p\geq 0 \quad \forall x = (x_1,\ldots,x_p)\in S^p \qquad f(x) = \prod_{0\leq i\leq p} f(x_i).$$

Using the developments of section 28.4.3.4, the function

$$x \in S \mapsto v_t(x) := \mathcal{P}_{0,t}(f)(x) = \mathbb{E}(f(\xi_t)) = \mathbb{E}\left(\prod_{1\leq i\leq N_t} f\left(x+\xi_t^i\right)\right)$$

satisfies the Kolmogorov-Petrovskii-Piskunov equations (28.14) with the initial condition $v_0 = \mathcal{P}_{0,0}(f) = f$. This ends the proof of the exercise.

■

Solution to exercise 468 : Using exercise 467, the solution of the Kolmogorov-Petrovskii-Piskunov equation (28.14) , with the initial condition $v_0 = 1_{[0,\infty[}$ is given by

$$v_t(x) = \mathbb{E}\left(\prod_{1\leq i\leq N_t} 1_{[0,\infty[}\left(x+\xi_t^i\right)\right) = \mathbb{P}\left(\inf_{1\leq i\leq N_t} \xi_t^i \geq -x\right).$$

By symmetry arguments, we have

$$\mathbb{P}\left(\sup_{1\leq i\leq N_t} -\xi_t^i \leq x\right) = \mathbb{P}\left(\sup_{1\leq i\leq N_t} \xi_t^i \leq x\right).$$

This shows that

$$v_t(x) \in [0,1] \qquad \lim_{x\to\infty} u_t(x) \quad \text{and} \quad \lim_{x\to-\infty} u_t(x) = 0.$$

In addition, for any $y \geq x$ also have

$$\begin{aligned} v_t(y) - v_t(x) &= \mathbb{P}\left(\sup_{1\leq i\leq N_t} \xi_t^i \leq y\right) - \mathbb{P}\left(\sup_{1\leq i\leq N_t} \xi_t^i \leq x\right) \\ &= \mathbb{P}\left(x \leq \sup_{1\leq i\leq N_t} \xi_t^i \leq y\right) > 0. \end{aligned}$$

We conclude that the function v_t is strictly increasing from 0 to 1. Therefore, for any $\epsilon \in]0,1[$ there exists some $x_\epsilon(t)$ such that

$$v_t(x_\epsilon(t)) = \epsilon \iff x_\epsilon(t) = v_t^{-1}(\epsilon).$$

This ends the proof of the exercise.

■

Solution to exercise 469:

Choosing a constant function $x \in \mathbb{R} \mapsto f(x) = r$, for some $r \in]0,1[$ we find that

$$v_t(x) = \mathcal{P}_{0,t}(f)(x) = \mathbb{E}\left(r^{N_t}\right) =: g_t(r)$$

satisfies the equation

$$\partial_t g_t(r) = g_t(r)^2 - g_t(r) = g_t(r)\left(g_t(r) - 1\right)$$

with the initial condition

$$N_0 = 1 \Longrightarrow \mathbb{E}\left(r^{N_0}\right) =: g_0(r) = r.$$

The solution is given by

$$\frac{1}{g_t(r)} = e^t \ \frac{1}{g_0(r)} - \int_0^t \ e^{t-s} \ ds = e^t \ \frac{1}{r} - \left(e^t - 1\right).$$

We check this claim using the fact that

$$\partial_t g_t(r) = g_t(r)^2 - g_t(r) \iff \partial_t \frac{1}{g_t(r)} = \frac{1}{g_t(r)} - 1.$$

This implies that

$$g_t(r) = \frac{re^{-t}}{1 - r\,(1 - e^{-t})} = \frac{re^{-t}}{(1 - r) + re^{-t}}.$$

When $r > 1$ the solution blows up at

$$re^{-t_b} = r - 1 \Leftrightarrow e^{-t_b} = 1 - 1/r \Leftrightarrow t_b = -\log(1 - 1/r).$$

This ends the proof of the exercise.

∎

Chapter 29

Solution to exercise 470:

The generator $L_{u,s}$ of the controlled diffusion

$$dX_s = (X_s - u)\ a_s\ ds\ +\ u\ [b_s\ ds + \sigma_t\ dW_s] = [(X_s - u)\ a_s\ + u\ b_s]\ ds\ +\ u\ \sigma_s\ dW_s$$

is given by

$$L_{u,s}(f)(x) = [(x-u)\ a_s\ + u\ b_s]\ f'(x) + \frac{1}{2}\ (u\ \sigma_s)^2\ f''(x).$$

Using (29.19) we have

$$\begin{aligned} -\partial_s V_s(x) &= \sup_{u\in[0,\infty[} [L_{u,s}(V_s)(x)] \\ &= \sup_{u\in[0,\infty[} \left[[(x-u)\ a_s\ + u\ b_s]\ V_s'(x) + \frac{1}{2}\ (u\ \sigma_s)^2\ V_s''(x) \right] \end{aligned}$$

for any $x > 0$ and $s \in [0,t]$. We further assume that the value function has the form

$$V_s(x) = \beta_s\ x^\alpha\ 1_{x\geq 0}$$

for some functions $s \in [0,t] \mapsto \beta_s$. This assertion is clearly met at the final time horizon $s = t$ since we have

$$V_t(x) = x^\alpha\ 1_{x\geq 0}.$$

The HJB equation is given for any $x > 0$ by

$$\begin{aligned} -\partial_s V_s(x) &= -\partial_s \beta_s\ x^\alpha\ 1_{x\geq 0} \\ &= \sup_{u\in[0,\infty[} \left[[(x-u)\ a_s\ + u\ b_s]\ \beta_s\ f'(x) + \frac{1}{2}\ (u\ \sigma_s)^2\ \beta_s\ f''(x) \right] \\ &= \sup_{u\in[0,\infty[} \left[[(x-u)\ a_s\ + u\ b_s]\ \beta_s\ \alpha\ x^{\alpha-1} + \frac{1}{2}\ (u\ \sigma_s)^2\ \beta_s\ \alpha(\alpha-1)\ x^{\alpha-2} \right] \\ &= \sup_{u\in[0,\infty[} \left[[(1-u/x)\ a_s\ + (u/x)\ b_s]\ \beta_s\ \alpha\ x^\alpha + \frac{1}{2}\ \sigma_s^2\ (u/x)^2 \beta_s\ \alpha(\alpha-1)\ x^\alpha \right] \\ &= \alpha\ x^\alpha \sup_{u\in[0,\infty[} \left[[(1-u)\ a_s\ + u\ b_s]\ \beta_s + \frac{1}{2}\ \sigma_s^2\ u^2 \beta_s\ (\alpha-1) \right]. \end{aligned}$$

We observe that

$$\begin{aligned} &[(1-u)\ a_s\ + u\ b_s]\ \beta_s + \tfrac{1}{2}\ \sigma_s^2\ u^2 \beta_s\ (\alpha-1) \\ &= a_s\ \beta_s + \sigma_s^2 \beta_s\ (\alpha-1)\ \left\{ -u\ \tfrac{[b_s-a_s]}{\sigma_s^2(1-\alpha)}\ + \tfrac{1}{2}\ u^2 \right\} \\ &= a_s\ \beta_s + \tfrac{1}{2}\ \sigma_s^2 \beta_s\ (1-\alpha)\ \left\{ \left[\tfrac{[b_s-a_s]}{\sigma_s^2(1-\alpha)} \right]^2 - \left[u - \tfrac{[b_s-a_s]}{\sigma_s^2(1-\alpha)} \right]^2 \right\}, \end{aligned}$$

from which we conclude that

$$\partial_s \log \beta_s = -\alpha \left[a_s + \frac{1}{2} \frac{[b_s - a_s]^2}{\sigma_s^2(1-\alpha)} \right]$$

with the terminal condition $\beta_t = 1$. This yields

$$\forall s \in [0,t] \qquad \beta_s = \exp\left(\alpha \int_s^t \left[a_r + \frac{1}{2} \frac{[b_r - a_r]^2}{\sigma_r^2(1-\alpha)} \right] dr \right).$$

The optimal control is given by

$$v_s(x) = \frac{[b_s - a_s]}{\sigma_s^2(1-\alpha)} \, x.$$

This ends the proof of the exercise.

■

Solution to exercise 471:
To check this claim, we observe that

$$\begin{aligned} \left[u + R^{-1}Sx\right]' R \left[u + R^{-1}Sx\right] &= u'Ru + \left[R^{-1}Sx\right]' Ru + u'Sx + \left[R^{-1}Sx\right]' Sx \\ &= u'Ru + x'S'u + u'Sx + x'S'R^{-1}Sx \\ &= u'Ru + 2\ u'Sx + x'S'R^{-1}Sx. \end{aligned}$$

In the last assertion we have used the fact that R^{-1} is a symmetric matrix and $x'S'u = (u'Sx)' = u'Sx$. When R is definite negative we clearly have

$$\sup_{u \in U} [u'Ru + 2\ u'Sx] = -x'S'R^{-1}Sx$$

and the supremum is attained for

$$u = v(x) = -R^{-1}Sx.$$

We check that

$$\forall 0 \leq k \leq n \qquad V_k(x) := x'P_kx + \alpha_k$$

by using a backward induction w.r.t. the parameter k. The result is immediate for $k = n$ since we have

$$\alpha_n = 0 \Longrightarrow V_n(x) := x'P_nx = f_n.$$

We further assume that the result has been checked at rank $(k+1)$. The Bellman equation (29.12) has the form

$$\begin{aligned} V_k(x) &= \sup_{u \in U} \left[x'Q_kx + u'R_ku + \alpha_{k+1} + \mathbb{E}_u \left(X'_{k+1}P_{k+1}X_{k+1} \mid X_k = x \right) \right] \\ &= x'Q_kx + \alpha_{k+1} + \sup_{u \in U} \left[u'R_ku + \mathbb{E}_u \left(X'_{k+1}P_{k+1}X_{k+1} \mid X_k = x \right) \right]. \end{aligned}$$

On the other hand, given $X_k = x$ and the control $U_k = u$ we have

$$\begin{aligned} X'_{k+1}P_{k+1}X_{k+1} &= (A_{k+1}x + B_{k+1}u + C_{k+1}W_{k+1})' P_{k+1} (A_{k+1}x + B_{k+1}u + C_{k+1}W_{k+1}) \\ &= (A_{k+1}x + B_{k+1}u)' P_{k+1} (A_{k+1}x + B_{k+1}u) \\ &\quad + 2 (A_{k+1}x + B_{k+1}u)' P_{k+1}C_{k+1}W_{k+1} + W'_{k+1}C'_{k+1}P_{k+1}C_{k+1}W_{k+1}. \end{aligned}$$

The last assertion follows from the fact that P_{k+1} is symmetric. Taking the expectations w.r.t. the distribution of W_{k+1}, we have

$$\begin{aligned}
&\mathbb{E}_u\left(X'_{k+1}P_{k+1}X_{k+1} \mid X_k = x\right) - \mathrm{tr}\left(C'_{k+1}P_{k+1}C_{k+1}\right) \\
&= \left(A_{k+1}x + B_{k+1}u\right)' P_{k+1}\left(A_{k+1}x + B_{k+1}u\right) \\
&= x'\left(A'_{k+1}P_{k+1}A_{k+1}\right)x + 2u'B'_{k+1}P_{k+1}A_{k+1}x + u'\left(B'_{k+1}P_{k+1}B_{k+1}\right)u.
\end{aligned}$$

This implies that

$$\begin{aligned}
V_k(x) &= x'\left[Q_k + \left(A'_{k+1}P_{k+1}A_{k+1}\right)\right]x + \alpha_{k+1} + \mathrm{tr}\left(C'_{k+1}P_{k+1}C_{k+1}\right) \\
&\qquad + \sup_{u\in U}\left[u'\left[R_k + \left(B'_{k+1}P_{k+1}B_{k+1}\right)\right]u + 2u'\left(B'_{k+1}P_{k+1}A_{k+1}\right)x\right].
\end{aligned}$$

The supremum is attained for

$$u = v_k(x) := -\left[R_k + \left(B'_{k+1}P_{k+1}B_{k+1}\right)\right]^{-1}\left(B'_{k+1}P_{k+1}A_{k+1}\right)x$$

and we have

$$\begin{aligned}
V_k(x) &= x'\left[Q_k + \left(A'_{k+1}P_{k+1}A_{k+1}\right)\right]x + \alpha_{k+1} + \mathrm{tr}\left(C'_{k+1}P_{k+1}C_{k+1}\right) \\
&\qquad - x'\left(B'_{k+1}P_{k+1}A_{k+1}\right)'\left[R_k + \left(B'_{k+1}P_{k+1}B_{k+1}\right)\right]^{-1}\left(B'_{k+1}P_{k+1}A_{k+1}\right)x.
\end{aligned}$$

This shows that

$$P_k = Q_k + A'_{k+1}P_{k+1}A_{k+1} - A'_{k+1}P_{k+1}B_{k+1}\left[R_k + \left(B'_{k+1}P_{k+1}B_{k+1}\right)\right]^{-1}\left(B'_{k+1}P_{k+1}A_{k+1}\right)$$

and

$$\alpha_k = \alpha_{k+1} + \mathrm{tr}\left(C'_{k+1}P_{k+1}C_{k+1}\right).$$

This ends the proof of the induction, and the proof of the exercise is now completed.

Solution to exercise 472:

The exercise is a direct consequence of the martingale optimality principle (29.13).

■

Solution to exercise 473:

Let $W_t = (W_t^i)_{1\leq i\leq p}$ be an p-dimensional Brownian motion. Consider the linear controlled $\mathbb{R}^q$-valued diffusion

$$dX_t = \left(A_tX_t + B_tu_t\right)\ dt + C_t dW_t$$

with $u_t \in U := \mathbb{R}^r$, and matrices (A_t, B_t, C_t) with appropriate dimensions. We consider the stochastic control problem (29.15) with

$$f_t(x) = x'P_tx \quad \text{and} \quad g_s(x,u) = x'Q_sx + u'R_su$$

for some definite negative and symmetric square matrices (P_t, Q_s, R_s) with appropriate dimensions.

The collection of generators associated with the controlled diffusion are given by

$$L_{t,u}(f)(x) = (\partial f)(x)'(A_tx + B_tu) + \frac{1}{2}\mathrm{tr}\left(C'_t\partial^2 fC_t\right)$$

for any $(x,u) \in (\mathbb{R}^q \times \mathbb{R}^r)$, with the column gradient vector $\partial f = (\partial_{x_i} f)_{1\leq i\leq q}$ and the Hessian matrix $\partial^2 f = \left(\partial_{x_i,x_j} f\right)_{1\leq i,j\leq q}$.

Let us check that

$$\forall 0 \leq s \leq t \qquad V_s(x) := x' P_s x + \alpha_s,$$

with the boundary terminal condition $\alpha_t = 0$, some symmetric matrices P_s and some parameters α_s. Notice that

$$\partial_s V_s(x) = x' \partial_s P_s x + \partial_s \alpha_s,$$

as well as

$$\partial V_s = 2P_s\ x \quad \text{and} \quad \frac{1}{2}\,\partial^2 V_s = P_s.$$

The Bellman equation (29.19) takes the form

$$\begin{array}{rcl} -\partial_s V_s(x) & = & -x'\partial_s P_s x - \partial_s \alpha_s \\ & = & x' Q_s x + \sup\limits_{u\in U}\left[u' R_s u + (\partial V_s)(x)'(A_s x + B_s u) + \dfrac{1}{2}\mathrm{tr}\left(C_s' \partial^2 V_s C_s\right)\right] \\ & = & x' Q_s x + \mathrm{tr}\left(C_s' P_s C_s\right) + 2x' P_s\ A_s x + \sup\limits_{u\in U}\left[u' R_s u + 2x' P_s\ B_s u\right]. \end{array}$$

Observe that

$$x' P_s\ A_s x = \left(x' P_s\ A_s x\right)' = x' A_s' P_s x$$

and

$$x' P_s\ B_s u = \left(x' P_s\ B_s u\right)' = u' B_s' P_s x.$$

This implies that

$$\begin{array}{rcl} -\partial_s V_s(x) & = & -x'\partial_s P_s x - \partial_s \alpha_s \\ & = & x' Q_s x + \mathrm{tr}\left(C_s' P_s C_s\right) + x'\left(A_s' P_s + P_s A_s\right)x + \sup\limits_{u\in U}\left[u' R_s u + 2u'\left(B_s' P_s\right)x\right]. \end{array}$$

Using the first part of exercise 471, we prove that the supremum is attained for

$$u = v_s(x) = -R_s^{-1}\left(B_s' P_s\right)x$$

and

$$\sup_{u\in U}\left[u' R_s u + 2u'\left(B_s' P_s\right)x\right] = -x'\left(B_s' P_s\right)' R_s^{-1}\left(B_s' P_s\right)x = -x' P_s B_s R_s^{-1} B_s' P_s x.$$

This yields the formula

$$-x'\partial_s P_s x - \partial_s \alpha_s \quad = \quad \mathrm{tr}\left(C_s' P_s C_s\right) + x'\left(Q_s + A_s' P_s + P_s A_s - P_s B_s R_s^{-1} B_s' P_s\right)x +$$

and we conclude that

$$\begin{array}{rcl} -\partial_s P_s & = & Q_s + A_s' P_s + P_s A_s - P_s B_s R_s^{-1} B_s' P_s \\ -\partial_s \alpha_s & = & \mathrm{tr}\left(C_s' P_s C_s\right). \end{array}$$

This ends the proof of the exercise.

■

Solution to exercise 474:

Next, we examine the 1-dimensional situation with $A_s = B_s = C_s = Q_s = R_s = 1$, and a null terminal condition $P_t = 0$. In this situation, the parameters P_s and α_s satisfy the backward equations

$$\begin{aligned} -\partial_s P_s &= 1 + 2P_s - P_s^2 \\ -\partial_s \alpha_s &= P_s \end{aligned}$$

with the boundary terminal conditions $\alpha_t = P_t = 0$. The optimal policy is given by the feedback control

$$v_s(x) = -P_s x.$$

Finally, we assume that $A_s = Q_s = 0$, $B_s = C_s = 1$, $P_t = P(< 0)$, $R_s = R(< 0)$ (and the boundary condition $\alpha_t = 0$). In this situation, we have

$$\begin{aligned} \partial_s P_s &= R^{-1}\, P_s^2 \quad \Leftrightarrow \quad \partial_s P_s^{-1} = R \quad \Leftrightarrow \quad P_s^{-1} = P^{-1} + R^{-1}(t-s) \\ -\partial_s \alpha_s &= P_s \quad \Leftrightarrow \quad \alpha_s = \int_s^t P_\tau d\tau. \end{aligned}$$

We conclude that

$$P_s = \frac{RP}{R + P(t-s)} \quad \text{and} \quad \alpha_s = -R\, \log\left(\frac{R}{R+P(t-s)}\right).$$

The optimal policy is given by the feedback control

$$v_s(x) = -R^{-1} P_s x = -\underbrace{\frac{P}{R+P(t-s)}}_{>0}\ x.$$

Notice that $s \in [0,t] \mapsto \frac{P}{R+P(t-s)} \in]0,\infty[$ is an increasing function. For any starting state X_0 the optimal controlled diffusion is given for any $s \in [0,t]$ by

$$dX_s = v_s(X_s)\ ds\ +\ dW_s = -\frac{P}{R+P(t-s)}\ X_s\ ds +\ dW_s.$$

Finally, using the martingale optimality principle (29.22),

$$\begin{aligned} \mathbb{V}_s(v) &:= \alpha_s + X_s' P_s X_s + \int_0^s \left[X_r' Q_r X_r + v_r\left(X_r\right)' R_r v_r\left(X_r\right)\right]\ dr \\ &= -R\ \log\left(\frac{R}{R+P(t-s)}\right) + \frac{RP\ X_s^2}{R+P(t-s)} + R\int_0^s \left[\frac{PX_r}{R+P(t-r)}\right]^2 dr \end{aligned}$$

is a martingale w.r.t. to $\mathcal{F}_s = \sigma\left(X_r,\ r \leq s\right)$ ending at $\mathbb{V}_t(v) = X_t' P_t X_t$. Thus, the end of the exercise is a direct consequence of the martingale optimality principle (29.22). This ends the proof of the exercise.

■

Solution to exercise 475:
The exercise is a direct consequence of the martingale optimality principle (29.22).

■

Solution to exercise 476:

The generator of the controlled diffusion is given by

$$L_{t,u}(f)(x) = (A_t x + B_t u + C_t)\ \partial_x f(x) + \frac{1}{2}\ (a_t x + b_t u + c_t)^2\ \ \partial_x^2 f(x).$$

Let us check that

$$\forall 0 \leq s \leq t \qquad V_s(x) := P_s x^2 + \beta_s x + \alpha_s$$

with the boundary terminal condition $\alpha_t = \beta_t = 0$, some negative parameter P_s and some $(\alpha_s, \beta_s) \in \mathbb{R}^2$.

Notice that

$$\partial_s V_s(x) = \partial_s P_s\ x^2 + \partial_s \beta_s\ x + \partial_s \alpha_s$$

as well as

$$\partial V_s = 2P_s\ x + \beta_s \quad \text{and} \quad \frac{1}{2}\ \partial^2 V_s = P_s.$$

The Bellman equation (29.19) takes the form

$$\begin{aligned}
-\partial_s V_s(x) &= -\partial_s P_s\ x^2 - \partial_s \beta_s\ x - \partial_s \alpha_s \\
&= Q_s x^2 + \sup_{u \in U} \Big[S_s\ x\ u + R_s u^2 + (A_s x + B_s u + C_s) \partial V_s(x) \\
&\qquad\qquad + \frac{1}{2} (a_s x + b_s u + c_s)^2\, \partial_x^2 V_s(x) \Big] \\
&= Q_s x^2 + \sup_{u \in U} \Big[S_s\ x\ u + R_s u^2 + (A_s x + B_s u + C_s)\,(2P_s x + \beta_s) \\
&\qquad\qquad + (a_s x + b_s u + c_s)^2\, P_s \Big] \\
&= Q_s x^2 + (A_s x + C_s)\,(2P_s x + \beta_s) + (a_s x + c_s)^2\, P_s \\
&\quad + \sup_{u \in U} \left[u^2 (R_s + b_s^2 P_s) + 2u\ [S_s x/2 + (P_s x + \beta_s/2)\, B_s + (a_s x + c_s)\, b_s P_s] \right].
\end{aligned}$$

Observe that

$$\begin{aligned}
& S_s x/2 + (P_s x + \beta_s/2)\, B_s + (a_s x + c_s)\, b_s P_s \\
&= x\ \ [S_s/2 + (a_s b_s + B_s) P_s] + [B_s \beta_s/2 + b_s c_s P_s]
\end{aligned}$$

and

$$\begin{aligned}
& u^2 (R_s + b_s^2 P_s) + 2u\ \ (x\,[S_s/2 + P_s B_s + a_s b_s P_s] + [B_s \beta_s/2 + b_s c_s P_s]) \\
&= \underbrace{(R_s + b_s^2 P_s)}_{<0}\ \left(u + \frac{x\,[S_s/2 + (a_s b_s + B_s) P_s] + [B_s \beta_s/2 + b_s c_s P_s]}{R_s + b_s^2 P_s} \right)^2 \\
&\qquad - \frac{(x\,[S_s/2 + (a_s b_s + B_s) P_s] + [B_s \beta_s/2 + b_s c_s P_s])^2}{R_s + b_s^2 P_s}.
\end{aligned}$$

We readily check that the supremum is attained for

$$u = v_s(x) = -x\ \frac{(a_s b_s + B_s) P_s + S_s/2}{R_s + b_s^2 P_s} - \frac{b_s c_s P_s + B_s \beta_s/2}{R_s + b_s^2 P_s}$$

and

$$\begin{aligned}
-\partial_s V_s(x) &= -\partial_s P_s\ x^2 - \partial_s \beta_s\ x - \partial_s \alpha_s \\
&= Q_s x^2 + (A_s x + C_s)(2P_s x + \beta_s) + (a_s x + c_s)^2 P_s \\
&\quad - \frac{(x\,[S_s/2 + (a_s b_s + B_s)P_s] + [b_s c_s P_s + B_s \beta_s/2])^2}{R_s + b_s^2 P_s}.
\end{aligned}$$

This implies that

$$-\partial_s P_s = Q_s + \left(a_s^2 + 2A_s\right) P_s - \left(R_s + b_s^2 P_s\right)^{-1} \left((a_s b_s + B_s)P_s + S_s/2\right)^2$$

and

$$\begin{aligned}
-\partial_s \beta_s &= A_s \beta_s + 2\,(a_s c_s + C_s)\, P_s \\
&\qquad -2\left(R_s + b_s^2 P_s\right)^{-1} \left((a_s b_s + B_s)P_s + S_s/2\right)[b_s c_s P_s + B_s \beta_s/2]\,,
\end{aligned}$$

as well as

$$-\partial_s \alpha_s = C_s \beta_s + c_s^2 P_s - \left(R_s + b_s^2 P_s\right)^{-1} [b_s c_s P_s + B_s \beta_s/2]^2\,.$$

This ends the proof of the exercise.

■

Solution to exercise 477:
By construction, we have

$$\begin{aligned}
F_{k,n}(X,v) &:= \sum_{k \le l < n} Z_{k,l}(X,\nu)\ g_l(X_l, v_l) + Z_{k,n}(X,\nu)\ f_n(X_n) \\
&= g_k(X_k, v_k) + z_k(v_k, X_k) \\
&\qquad \times \left[\sum_{k+1 \le l < n} Z_{k+1,l}(X,\nu)\ g_l(X_l, v_l) + Z_{k+1,n}(X,\nu)\ f_n(X_n) \right\} \\
&= g_k(X_k, v_k) + z_k(v_k, X_k)\ F_{k+1,n}(X,v).
\end{aligned}$$

The value function (29.11) associated with the payoff function $F_n(X,v)$ takes the form

$$V_k(x_k) \quad := \quad \sup_{v \in \mathcal{V}_{k,n-1}} \mathbb{E}_v\left(F_{k,n}(X,v) \mid X_k = x_k\right).$$

For $k = n$, we have

$$V_n(x_n) = f_n(X_n).$$

For $k = (n-1)$, we have

$$F_{n-1,n}(X,v) := g_{n-1}(v_{n-1}, X_{n-1}) + z_{n-1}(v_{n-1}, X_{n-1})\ f_n(X_n).$$

$$V_{n-1}(x) \quad := \quad \sup_{u \in U_{n-1}} \left[g_{n-1}(u,x) + z_{n-1}(u,x)\ \mathbb{E}_u(f_n(X_n) \mid X_{n-1} = x)\right].$$

For $k = (n-2)$, we have

$$F_{n-2,n}(X,v) := g_{n-2}(v_{n-2}, X_{n-2}) + z_{n-2}(v_{n-2}, X_{n-2})\ F_{n-1,n}(X,v).$$

Arguing as in the proof of the Bellman equation (29.12) we check that

$$\begin{aligned}
V_{n-2}(x) &= \sup_{(u,v)\in\mathcal{V}_{n-2,n-1}} \left[g_{n-2}(u,x) + z_{n-2}(u,x)\ \mathbb{E}_{(u,v)}(F_{n-1,n}(X,v) \mid X_{n-2}=x)\right] \\
&= \sup_{u\in U_{n-2}} \left[g_{n-2}(u,x) + z_{n-2}(u,x) \sup_{v\in\mathcal{V}_{n-1}} \mathbb{E}_{(u,v)}(F_{n-1,n}(X,v) \mid X_{n-2}=x)\right] \\
&= \sup_{u\in U_{n-2}} \left[g_{n-2}(u,x) + z_{n-2}(u,x) \sup_{v\in\mathcal{V}_{n-1}} \mathbb{E}_{v}(V_{n-1}(X_{n-1}) \mid X_{n-2}=x)\right].
\end{aligned}$$

Iterating this reasoning we conclude that

$$\begin{aligned}
V_l(x_l) &= \sup_{u\in U_l} \left[g_l(x_l,u) + z_l(u,x_l)\ \mathbb{E}_u(V_{l+1}(X_{l+1}) \mid X_l = x_l)\right] \\
&= \sup_{u\in U_l} \left[g_l(x_l,u) + z_l(u,x_l)\ M_{u,l+1}\left(V_{l+1}\right)(x_l)\right],
\end{aligned}$$

with $0 \le l < n$ and the terminal (a.k.a. boundary) condition $V_n = f_n$.

This ends the proof of the exercise.

∎

Solution to exercise 478:

The discrete time approximation of the value functions are given by

$$F^h_{t_n}(X^h,u) := \sum_{0\le k<n} Z^h_{t_k}(X,\nu)\ g_{t_k}(X^h_{t_k},u_{t_k})\ h + Z^h_{t_n}(X,\nu)\ f_{t_n}(X^h_{t_n}),$$

with

$$Z^h_{t_l}(X,\nu) := \prod_{0\le k<l} z^h_{t_k}(u_{t_k},X^h_{t_k}) \quad \text{and} \quad z^h_{t_k}(u_{t_k},X^h_{t_k}) = \exp\left(H_{t_k}(u_{t_k},X_{t_k})\ h\right).$$

We have the first order approximations

$$\begin{aligned}
z^h_{t_k}(u,x) &= 1 + H_{t_k}(u,x)\ h + \mathrm{O}(h^2) \\
P^h_{u,t_n}(\varphi) &= \varphi + L_{t_n,u}(\varphi)\ h\ + \mathrm{O}\left(h^2\right).
\end{aligned}$$

This yields

$$\begin{aligned}
V^h_{t_l}(x) &= \sup_{u\in U}\left[g_{t_l}(x,u)\ h + \ z^h_{t_l}(u_{t_l},x)\ P^h_{u,t_{l+1}}\left(V^h_{t_{l+1}}\right)(x)\right] \\
&= \sup_{u\in U}\left[g_{t_l}(x,u)\ h + (1+H_{t_k}(u,x)\ h)\ \ \left(V^h_{t_{l+1}} + L_{t_n,u}(V^h_{t_{l+1}})\ h\right)(x) + \mathrm{O}\left(h^2\right)\right].
\end{aligned}$$

This implies that

$$-h^{-1}\left[V^h_{t_{l+1}}(x) - V^h_{t_l}(x)\right] = \sup_{u\in U}\left[g_{t_l}(x,u) + H_{t_k}(u,x)\ V^h_{t_{l+1}} + L_{t_n,u}(V^h_{t_{l+1}})(x) + \mathrm{O}\left(h\right)\right].$$

Taking formally the limit as $h \downarrow 0$ with $t_l \downarrow s$ we find that the value function

$$\begin{aligned}
\lim_{h\downarrow 0} V^h_{t_l}(x) &:= V_s(x) \\
&= \sup_{v\in\mathcal{V}_{s,t}} \mathbb{E}_v\left(\int_s^t Z_{s,r}(X,u)\ g_r(X_r,v_r(X_r))\ dr + Z_{s,t}(X,u)\ f_t(X_t) \mid X_s = x\right)
\end{aligned}$$

satisfies the equation

$$-\partial_s V_s(x) = \sup_{u\in U}\left[g_s(x,u) + L^H_{s,u}\left(V_s\right)(x)\right]$$

with terminal condition $V_t = f_t$ and the Schrödinger operator

$$L^H_{s,u}(\varphi)(x) = L_{s,u}(\varphi)(x) + H_s(u,x)\ \varphi(x).$$

As in the discrete time case, the optimal strategy is obtained by applying the optimal control charts $x \mapsto v_s(x)$ computed in the one step backward recursion (29.19).

This ends the proof of the exercise.

■

Solution to exercise 479:

Using the Doeblin-Itō formula we have

$$V_{s_2}(X^{(v)}_{s_2}) = V_{s_1}(X^{(v)}_{s_1}) + \int_{s_1}^{s_2}\ \left[\partial_s V_s(X^{(v)}_s) + L_{v_s,s}(V_s)(X^{(v)}_s)\right]\ ds + M_{s_2}(V) - M_{s_1}(V)$$

for some $\mathcal{F}^{(v)}_s$-martingale $M_s(V)$ and for any $0 \leq s_1 \leq s_2 \leq t$. This implies that

$$V_{s_1}(X^{(v)}_{s_1}) = \mathbb{E}\left[V_{s_2}(X^{(v)}_{s_2}) - \int_{s_1}^{s_2}\ \left[\partial_s V_s(X^{(v)}_s) + L_{v_s,s}(V_s)(X^{(v)}_s)\right]\ ds \mid \mathcal{F}^{(v)}_{s_1}\ \right].$$

Using (29.20) we find that

$$V_{s_1}(x) \geq \mathbb{E}_v\left(\int_{s_1}^{s_2}\ g_s(X_s, v_s(X_s))\ ds + V_{s_2}\left(X_{s_2}\right)\ \mid X_{s_1} = x\right)$$

with the equality on optimal control policies. This implies that

$$\mathbb{E}\left[V_{s_2}(X_{s_2}) - \textstyle\int_{s_1}^{s_2}\ \left[\partial_s V_s(X_s) + L_{v_s,s}(V_s)(X_s)\right]\ ds \mid X_{s_1} = x\ \right]$$

$$\geq \mathbb{E}_v\left(\textstyle\int_{s_1}^{s_2}\ g_s(X_s, v_s(X_s)\ ds + V_{s_2}\left(X_{s_2}\right)\ \mid X_{s_1} = x\right)$$

$$\Rightarrow\ (s_2 - s_1)^{-1}\mathbb{E}_v\left(\textstyle\int_{s_1}^{s_2}\ \left[\partial_s V_s(X_s) + g_s(X_s, v_s(X_s)) + L_{v_s,s}(V_s)(X_s)\right]\ ds \mid X_{s_1} = x\right) \leq 0,$$

with the equality on optimal control policies. Taking the limit $s_2 \to s_1$ we find that

$$-\partial_s V_s(x) \geq g_s(x,u) + L_{u,s}(V_s)(x)$$

for any $u \in U$, with the equality on optimal ones; that is, we have that

$$-\partial_s V_s(x) = \sup_{u\in U}\left(g_s(x,u) + L_{u,s}(V_s)(x)\right).$$

This ends the proof of the exercise.

■

Solution to exercise 480:

The collection of generators associated with the controlled diffusion are given by

$$L_{u,t}(f)(x) = (\partial f)(x)'\ \left(b_t(x) + \sigma_t(x)\ u\right) + \frac{1}{2}\ \mathrm{tr}\left(\sigma_t(x)'\partial^2 f(x)\sigma_t(x)\right)$$

for any $(x, u) \in (\mathbb{R}^r \times \mathbb{R}^r)$, with the column gradient vector $\partial f = (\partial_{x_i} f)_{1 \leq i \leq r}$ and the Hessian matrix $\partial^2 f = \left(\partial_{x_i, x_j} f\right)_{1 \leq i, j \leq r}$. The Bellman equation (29.19) takes the form

$$\begin{aligned}
-\partial_s V_s(x) &= \sup_{u \in U} \Big[h_s(x) + \frac{1}{2}\ u' R_s u + (\partial V_s)(x)'\ \ (b_s(x) + \sigma_s(x)\ u) \\
&\qquad\qquad + \frac{1}{2}\ \mathrm{tr}\left(\sigma_s(x)' \partial^2 V_s(x) \sigma_s(x)\right) \Big] \\
&= h_s(x) + (\partial V_s)(x)'\ b_s(x) + \frac{1}{2}\ \mathrm{tr}\left(\sigma_s(x)' \partial^2 V_s(x) \sigma(x)\right) \\
&\qquad\qquad + \frac{1}{2} \sup_{u \in U} [\ u' R_s u + 2\ u' \sigma_s(x)' (\partial V_s)(x)].
\end{aligned}$$

Using the first part of exercise 471 (replacing the vector Sx by the vector $\sigma_s(x)'(\partial V_s)(x)$), we prove that the supremum is attained for

$$u = v_s(x) := -R_s^{-1} \sigma_s(x)' (\partial V_s)(x)$$

and

$$\sup_{u \in U} [\ u' R_s u + 2\ u' \sigma_s(x)' (\partial V_s)(x)] = -(\partial V_s)(x)' \sigma_s(x) R_s^{-1} \sigma_s(x)' (\partial V_s)(x).$$

This yields the Hamilton-Jacobi-Bellman equation

$$-\partial_s V_s = h_s + (\partial V_s)'\ b_s - \frac{1}{2} (\partial V_s)' \sigma_s R_s^{-1} \sigma_s' (\partial V_s) + \frac{1}{2}\ \mathrm{tr}\left(\sigma_s' \partial^2 V_s \sigma_s\right).$$

We further assume that $R_s = \lambda\ Id$, for some $\lambda < 0$, where Id stands for the $(r \times r)$-identity matrix. We also set

$$a_s(i,j)(x) = (\sigma_s(x) \sigma_s'(x))_{i,j} = \sum_{1 \leq k \leq r} \sigma_{s,k}^i(x) \sigma_{s,k}^j(x)$$

and we set

$$V_s(x) = -\lambda \log q_s(x)$$

with the terminal condition $V_t = -\lambda \log q_t = f_t$.

Observe that

$$\partial_s V_s = -\lambda q_s^{-1} \partial_s q_s \quad \text{and} \quad \partial V_s = -\lambda q_s^{-1} \partial q_s,$$

as well as

$$\partial^2 V_s = -\lambda q_s^{-1} \partial^2 q_s + \lambda\ q^{-2}\ \partial q_s\, (\partial q_s)' = -\lambda q_s^{-1} \partial^2 q_s + \lambda^{-1} \partial V_s\, (\partial V_s)'.$$

This yields

$$\begin{aligned}
\mathrm{tr}\left(\partial^2 V_s a_s\right) &= \sum_{1 \leq i,j \leq r} a_s(i,j)\ \partial_{x_i, x_j} V_s \\
&= -\lambda q_s^{-1} \sum_{1 \leq i,j \leq r} a_s(i,j) \partial_{x_i, x_j} q_s + \lambda\ q_s^{-2} \sum_{1 \leq i,j \leq r} a_s(i,j)\ \partial_{x_i} q_s \partial_{x_j} q_s.
\end{aligned}$$

Recalling that

$$\partial_{x_i} V_s = -\lambda q_s^{-1} \partial_{x_i} q_s$$

we prove that

$$\lambda\ q_s^{-2} \sum_{1 \leq i,j \leq r} a_s(i,j)\ \partial_{x_i} q_s \partial_{x_j} q_s = \lambda^{-1} \sum_{1 \leq i,j \leq r} a_s(i,j)\ \partial_{x_i} V_s \partial_{x_j} V_s = \lambda^{-1}\ (\partial V_s)' a_s (\partial V_s).$$

Therefore we conclude that

$$\mathrm{tr}\left(\partial^2 V_s a_s\right) = -\lambda q_s^{-1} \mathrm{tr}\left(\partial^2 q_s a_s\right) + \lambda^{-1}\ (\partial V_s)' a_s (\partial V_s).$$

In this situation (recalling that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$) we have

$$\begin{aligned} -\partial_s V_s &= \lambda q_s^{-1} \partial_s q_s \\ &= h_s - \lambda q_s^{-1} \left(\partial q_s\right)'\ b_s - \frac{1}{2}\ \lambda q_s^{-1} \mathrm{tr}\left(\partial^2 q_s a_s\right). \end{aligned}$$

This implies that

$$-\partial_s q_s = -\lambda^{-1} h_s q_s + \left(\partial q_s\right)'\ b_s + \frac{1}{2} \mathrm{tr}\left(\partial^2 q_s a_s\right).$$

Arguing as in the end of section 29.3.3, this equation can be rewritten as follows:

$$-\partial_s q_s = \mathcal{L}(q_s) + \overline{h}_s\ q_s$$

with the potential function $\overline{h}_s = -\lambda^{-1} h_s$ and the infinitesimal generator $\mathcal{L}$ of the diffusion process

$$dY_s = b_s(Y_s)\ ds + \sigma_s(Y_s)\ dW_s.$$

The solution of this equation is given by the Feynman-Kac formula

$$q_s(y) = Q_{s,t}\left(e^{\overline{f}_t}\right) := \mathbb{E}\left[\exp\left(\overline{f}_t(Y_t)\right)\ \exp\left(\int_s^t\ \overline{h}_r(Y_r)dr\right) \mid Y_s = y\right]$$

with $\overline{f}_t = -\lambda^{-1} f_t$.

This ends the proof of the exercise.

■

Solution to exercise 481:

We consider the optimal stopping problem defined in section 29.4.2 by replacing the maximization problem (29.33) my the minimization problem

$$U_k := \inf_{T \in \mathcal{T}_k} \mathbb{E}\left(f_T(X_T) \mid \mathcal{F}_k\right).$$

$$-U_k := \sup_{T \in \mathcal{T}_k} \mathbb{E}\left(g_T(X_T) \mid \mathcal{F}_k\right) \quad \text{with} \quad g_k(x) = -f_k(x).$$

The solution of the problem is given by the sequence of stopping times defined using the backward induction

$$T_k = k\ 1_{f_k(X_k)\ \leq\ \mathbb{E}\left(f_{T_{k+1}}(X_{T_{k+1}})\ \mid\ \mathcal{F}_k\right)} + T_{k+1}\ 1_{f_k(X_k)\ >\ \mathbb{E}\left(f_{T_{k+1}}(X_{T_{k+1}})\ \mid\ \mathcal{F}_k\right)}$$

with the terminal condition $T_n = n$. The optimal stopping times formula (29.31) takes the form

$$T_k := \inf\left\{l \in \{k, k+1, \ldots, n\}\ :\ U_l = f_l(X_l)\ \right\}.$$

Using (29.32) Snell envelope $(-U_k) = (-V_k(X_k))$ is solved using the functions V_k defined by the backward induction

$$V_k(x_k) = f_k(x_k) \wedge \mathbb{E}(V_{k+1}(X_{k+1}) \mid X_k = x_k), \tag{30.67}$$

with the terminal condition $V_n = f_n$.

This ends the proof of the exercise.

■

Solution to exercise 482:
We have

$$\begin{array}{rcl} V_n(x) & = & x = x \vee (-\infty) = x \vee m_0 \\ V_{n-1}(x) & = & x \vee \mathbb{E}(X_n) = X_{n-1} \vee m_1 \\ V_{n-2}(x) & = & x \vee \mathbb{E}\left(X_{n-1} \vee \mathbb{E}(X_n)\right) = x \vee m_2 \\ \vdots & = & \vdots \\ V_{n-k}(x) & = & x \vee m_k \\ \vdots & = & \vdots \\ V_0(x) & = & x \vee m_n. \end{array}$$

The optimal stopping strategy (29.31) is given by given by

$$\begin{array}{rcl} T_k & = & \inf\left\{l \in \{k, k+1, \ldots, n\} \ : \ X_l \vee m_{n-l} = X_l\right\} \\ & = & \inf\left\{l \in \{k, k+1, \ldots, n\} \ : \ X_l \geq m_{n-l}\right\}. \end{array}$$

For i.i.d. copies X_k of an uniform random variable X on $[0,1]$ we have

$$\begin{array}{rcl} m_1 & = & \mathbb{E}(X) = 1/2 \\ m_2 & = & \mathbb{E}(X \vee (1/2)) = 1/2 \ \mathbb{P}(X < 1/2) + \mathbb{E}(X \ 1_{X \geq 1/2}) \\ & = & 1/4 + \displaystyle\int_{1/2}^{1} x \ dx = 1/4 + (1/2 - 1/8) = 1/8 + 1/2 = 5/8 \\ \vdots & = & \vdots \\ m_{k+1} & = & \mathbb{E}(X \vee m_k) = m_k \ \mathbb{P}(X < m_k) + \mathbb{E}(X \ 1_{X \geq m_k}) \\ & = & m_k^2 + \displaystyle\int_{m_k}^{1} x \ dx = m_k^2 + \frac{1}{2}\left(1 - m_k^2\right) = \frac{1}{2}\left(m_k^2 + 1\right). \end{array}$$

For i.i.d. copies X_k of an exponential random variable X with parameter λ we have

$$\begin{array}{rcl} m_1 & = & \mathbb{E}(X) = \dfrac{1}{\lambda} \\ m_{k+1} & = & \mathbb{E}(X \vee m_k) = m_k \ \mathbb{P}(X < m_k) + \mathbb{E}(X \ 1_{X \geq m_k}) \\ & = & m_k \displaystyle\int_0^{m_k} \lambda \ e^{-\lambda x} dx + \int_{m_k}^{+\infty} \lambda \ x \ e^{-\lambda x} dx \\ & = & m_k \displaystyle\int_0^{m_k} \lambda \ e^{-\lambda x} dx + \left(\frac{1}{\lambda} - \int_0^{m_k} \lambda \ x \ e^{-\lambda x} dx\right) \\ & = & \dfrac{1}{\lambda} + e^{-m_k} \displaystyle\int_0^{m_k} \lambda \ (m_k - x) \ e^{\lambda(m_k - x)} \ dx \\ & = & \dfrac{1}{\lambda} + e^{-m_k} \displaystyle\int_0^{m_k} \lambda \ x \ e^{\lambda x} \ dx = \frac{1}{\lambda} + m_k - \frac{1}{\lambda} \ \left[1 - e^{-\lambda m_k}\right] = m_k + \frac{1}{\lambda} \ e^{-\lambda m_k}. \end{array}$$

The last assertion is checked using the integration by parts

$$\begin{array}{rcl} \displaystyle\int_0^{m_k} \lambda x \ e^{\lambda x} \ dx & = & \left[x \ e^{\lambda x}\right]_0^{m_k} - \displaystyle\int_0^{m_k} e^{\lambda x} \ dx \\ & = & m_k \ e^{\lambda m_k} - \dfrac{1}{\lambda} \ \left[e^{\lambda x}\right]_0^{m_k} = m_k \ e^{\lambda m_k} - \dfrac{1}{\lambda} \ \left[e^{\lambda m_k} - 1\right]. \end{array}$$

This ends the proof of the exercise.

■

Solution to exercise 483:
We have

$$f_n(1) \quad = \quad (1-p)\ \partial_p \left[\sum_{k\geq 0} p^k\right] = (1-p)/(1-p)^2 = 1/(1-p).$$

Following the arguments presented in exercise 481 and in exercise 482, the Snell envelope is given by

$$\begin{array}{rcl} V_{n-(k+1)}(x) & = & f_{n-(k+1)}(x) \wedge \mathbb{E}(V_{n-k}(X)) = f_{n-(k+1)}(x) \wedge m_{k+1} \leq f_{n-(k+1)}(x) \\ & = & ((k+1)\ 1_{x=0} + \infty\ 1_{x=1}) \wedge m_{k+1} \\ & = & 1_{x=0}\ [(k+1) \wedge m_{k+1}] + 1_{x=1}\ m_{k+1}, \end{array}$$

with the non increasing sequence of parameters $(m_k)_{1\leq k\leq n}$ given by the recursion

$$\begin{array}{rcl} m_{k+1} & := & \mathbb{E}(V_{n-k}(X)) = (1-p)\ V_{n-k}(0) + p\ V_{n-k}(1) \\ & = & (1-p)\ [k \wedge m_k] + p\ m_k\ \leq\ m_k \wedge k \leq m_k \end{array}$$

and the initial condition

$$m_1 = \mathbb{E}(V_n(X)) = \mathbb{E}(f_n(X)) = f_n(1)\ p = p/(1-p) \quad (\Longrightarrow\ f_n(0) = 0)\,.$$

The optimal stopping times at rank k are given by

$$T_k := \inf \left\{ l \in \{k, k+1, \ldots, n\}\ :\ \overbrace{f_l(X_l) \wedge m_{n-l}}^{=V_l(X_l)} = \underbrace{(n-l)\ 1_{X_l=0} + \infty\ 1_{X_l=1}}_{f_l(X_l)} \right\}.$$

Notice that

$$f_l(X_l) \wedge m_{n-l} = f_l(X_l) \quad \Longleftrightarrow \quad X_l = 0 \quad \text{and} \quad (n-l) \leq m_{n-l}.$$

This yields the stopping rules

$$T_k := \inf \{ l \in \{k, k+1, \ldots, n\}\ :\ X_l = 0 \quad \text{and} \quad m_{n-l} \geq n-l\ \}.$$

This ends the proof of the exercise.

■

Solution to exercise 484:
By (29.31), the optimal stopping rule at time $k = 0$ is given by

$$\begin{array}{rcl} T & := & \inf\{l \in \{0, \ldots, n\}\ :\ V_l(X_l) \leq f_l(X_l)\} \\ & = & \inf\{l \in \{0, \ldots, n\}\ :\ [f_l(X_l) \vee \mathbb{E}(V_{l+1}(X_{l+1}) \mid X_l)] \leq f_l(X_l)\} \\ & = & \inf\{l \in \{0, \ldots, n\}\ :\ f_l(X_l) \geq \mathbb{E}(V_{l+1}(X_{l+1}) \mid X_l)\}, \end{array}$$

where V_l stands for the Snell envelope defined by the backward recursion (29.32). We recall that $V_k(X_k) \geq f_k(X_k)$, for any $0 \leq k \leq n$. We have

$$\begin{array}{rcl} S > l & \Rightarrow & \forall 0 \leq k \leq l \quad f_k(X_k) < \mathbb{E}\left(f_{k+1}(X_{k+1}) \mid X_k\right) \ \ (\leq \mathbb{E}\left(V_{k+1}(X_{k+1}) \mid X_k\right)) \\ & \Rightarrow & \forall 0 \leq k \leq l \quad f_k(X_k) < \mathbb{E}\left(V_{k+1}(X_{k+1}) \mid X_k\right) \Rightarrow T > l. \end{array}$$

In the reverse angle, we have

$$\begin{aligned}
\{S=l\} &= \bigcap_{0\leq k<l}\{f_k(X_k)<\mathbb{E}\left(f_{k+1}(X_{k+1})\mid X_k\right)\}\cap\{f_l(X_l)\geq\mathbb{E}\left(f_{l+1}(X_{l+1})\mid X_l\right)\}\\
&\subset \bigcap_{0\leq k<l}\{f_k(X_k)<\mathbb{E}\left(V_{k+1}(X_{k+1})\mid X_k\right)\}\\
&\qquad\qquad\cap\bigcap_{l<m\leq n}\{f_{m-1}(X_{m-1})\geq\mathbb{E}\left(f_m(X_m)\mid X_{m-1}\right)\}.
\end{aligned}$$

This implies that

$$\{T\leq l\}\subset\{S\leq l\}.$$

Thus, on the event $\{S=l\}$ we have

$$\begin{array}{rclccc}
f_{n-1}(X_{n-1}) & \overset{(m=n)}{\geq} & \mathbb{E}\left(f_n(X_n)\mid X_{n-1}\right) & \Rightarrow & V_{n-1}(X_{n-1}) = & f_{n-1}(X_{n-1})\\
f_{n-2}(X_{n-2}) & \overset{(m=(n-1))}{\geq} & \mathbb{E}\left(f_{n-1}(X_{n-1})\mid X_{n-2}\right) & & & \\
 & & =\mathbb{E}\left(V_{n-1}(X_{n-1})\mid X_{n-2}\right) & \Rightarrow & V_{n-2}(X_{n-2}) = & f_{n-2}(X_{n-2})\\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
f_l(X_l) & \overset{(m=(l+1))}{\geq} & \mathbb{E}\left(f_{l+1}(X_{l+1})\mid X_l\right) & & & \\
 & & =\mathbb{E}\left(V_{l+1}(X_{l+1})\mid X_l\right) & \Rightarrow & V_l(X_l) = & f_l(X_l).
\end{array}$$

This yields the inclusion

$$\{S=l\}\subset\{T\geq l\}\cap\bigcap_{l\leq m\leq n}\{V_m(X_m)=f_m(X_m)\}=\{T=l\}.$$

Hence we have

$$\{T\leq l\}\subset\{S\leq l\}\subset\{T\leq l\}\implies\{S\leq l\}=\{T\leq l\}\implies\{S>l\}=\{T>l\}$$

from which we conclude that

$$\{S=l\}=\{S\leq l\}\cap\{S>(l-1)\}=\{T\leq l\}\cap\{T>(l-1)\}=\{T=l\}.$$

This ends the proof of the exercise.

■

Solution to exercise 485:
We have

$$\begin{aligned}
f_n(X_n) &= X_n^1X_n^2=\epsilon_n\ X_{n-1}^1\left(X_{n-1}^2+W_n\right)\\
&= \epsilon_n\ f_{n-1}(X_{n-1})+\epsilon_n\ X_{n-1}^1\ W_n\geq\epsilon_n\ f_{n-1}(X_{n-1}).
\end{aligned}$$

We also readily check that

$$\begin{aligned}
\mathbb{E}\left(f_n(X_n)\mid\mathcal{F}_{n-1}\right) &= \mathbb{E}\left(\epsilon_n\ f_{n-1}(X_{n-1})+\epsilon_n\ X_{n-1}^1\ W_n\mid\mathcal{F}_{n-1}\right)\\
&= \mathbb{E}\left(\epsilon\right)\ f_{n-1}(X_{n-1})+\mathbb{E}\left(\epsilon\ W\right)\ X_{n-1}^1\\
&= p\ f_{n-1}(X_{n-1})+p\ w\ X_{n-1}^1,
\end{aligned}$$

from which we prove that

$$f_{n-1}(X_{n-1})-\mathbb{E}\left(f_n(X_n)\mid\mathcal{F}_{n-1}\right)=(1-p)\ f_{n-1}(X_{n-1})-p\ w\ X_{n-1}^1.$$

This yields

$$
\begin{aligned}
& f_n(X_n) - \mathbb{E}\left(f_{n+1}(X_{n+1}) \mid \mathcal{F}_n\right) \\
& = (1-p)\ f_n(X_n) - p\ w\ X_n^1 \\
& = (1-p)\ \left(\epsilon_n\ f_{n-1}(X_{n-1}) + \epsilon_n\ X_{n-1}^1\ W_n\right) - p\ w\ \epsilon_n\ X_{n-1}^1 \\
& = \epsilon_n\ \left[f_{n-1}(X_{n-1}) - \mathbb{E}\left(f_n(X_n) \mid \mathcal{F}_{n-1}\right)\right] + (1-p)\ \epsilon_n\ X_{n-1}^1\ W_n \\
& \geq \epsilon_n\ \left[f_{n-1}(X_{n-1}) - \mathbb{E}\left(f_n(X_n) \mid \mathcal{F}_{n-1}\right)\right].
\end{aligned}
$$

This shows that the optimal stopping problem is monotone. Using exercise 484 the optimal stopping rule on any finite time horizon is defined by

$$
\begin{aligned}
S &= \inf\left\{n \geq 0\ :\ f_n(X_n) \geq \mathbb{E}\left(f_{n+1}(X_{n+1}) \mid X_n\right)\right\} \\
&= \inf\left\{n \geq 0\ :\ X_n^1\ X_n^2 \geq p\ X_n^1\ X_n^2 + p\ w\ X_n^1\right\} \\
&= \inf\left\{n \geq 0\ :\ X_n^1 = 0 \quad \text{or} \quad X_n^2\ \geq pw/(1-p)\right\}.
\end{aligned}
$$

This shows that the best strategy (before to be caught) is to stop as soon as the accumulated earnings are at least $pw/(1-p)$.

This ends the proof of the exercise.

■

Solution to exercise 486:

We have

$$
\begin{aligned}
\mathbb{E}\left(f_{n+1}(X_{n+1}) \mid \mathcal{F}_n\right) - f_n(X_n) &= \mathbb{E}\left(\left[(X_n \vee W_{n+1}) - X_n\right] \mid \mathcal{F}_n\right) - a \\
&= \mathbb{E}\left(\left[W_{n+1} - X_n\right]\ 1_{W_{n+1} \geq X_n} \mid X_n\right) - a \\
&= \mathbb{E}\left(\left[W_{n+1} - X_n\right]_+ \mid X_n\right) - a \\
&= \int_{X_n}^{\infty} (w - X_n)\ \mu(dw) - a.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
X_{n+1} - X_n \geq 0 \Rightarrow \int_{X_n}^{\infty} (w - X_n)\ \mu(dw) &\geq \int_{X_{n+1}}^{\infty} \left(\left[(w - X_{n+1}) + (X_{n+1} - X_n)\right]\ \mu(dw)\right. \\
&\geq \int_{X_{n+1}}^{\infty} (w - X_{n+1})\ \mu(dw).
\end{aligned}
$$

This implies that

$$
\mathbb{E}\left(f_{n+1}(X_{n+1}) \mid \mathcal{F}_n\right) - f_n(X_n) \geq \mathbb{E}\left(f_n(X_n) \mid \mathcal{F}_{n-1}\right) - f_{n-1}(X_{n-1}).
$$

We conclude that the optimal stopping problem is monotone. Using exercise 484 the optimal stopping rule on any finite time horizon is defined by

$$
\begin{aligned}
S &= \inf\left\{n \geq 0\ :\ f_n(X_n) \geq \mathbb{E}\left(f_{n+1}(X_{n+1}) \mid X_n\right)\right\} \\
&= \inf\left\{n \geq 0\ :\ \mathbb{E}\left(f_{n+1}(X_{n+1}) | X_n\right) - f_n(X_n) \leq 0\right\} \\
&= \inf\left\{n \geq 0\ :\ \int_{X_n}^{\infty} (w - X_n)\ \mu(dw) \leq a\right\}.
\end{aligned}
$$

When W is an exponential random variable with parameter $\lambda > 0$ we have

$$\begin{aligned}
\int_x^\infty (w-x)\ \mu(dw) &= -e^{-\lambda x}\int_x^\infty (w-x)\ \partial_w\left(e^{-\lambda(w-x)}\right)\ dw \\
&= -e^{-\lambda x}\underbrace{\left[(w-x)\ e^{-\lambda(w-x)}\right]_{w=x}^{w=\infty}}_{=0}+\frac{e^{-\lambda x}}{\lambda}\int_x^\infty \lambda\ e^{-\lambda(w-x)}\ dw \\
&= \frac{e^{-\lambda x}}{\lambda}.
\end{aligned}$$

In this situation, we have

$$S=\inf\left\{n\geq 0\ :\ X_n\geq -\frac{1}{\lambda}\log(\lambda a)\right\}.$$

When $(\lambda a)\geq 1$, the best strategy is to sell the asset immediately. When $(\lambda a)\leq 1$, the optimal stopping time is given by

$$S=\inf\left\{n\geq 0\ :\ X_n\geq \frac{1}{\lambda}\left|\log(\lambda a)\right|\right\}.$$

When W is an uniform random variable on $[w_1,w_2]$, for any $x,w\in[w_1,w_2]$ we have

$$\int_x^\infty (w-x)\ \mu(dw) = \frac{1}{2}\int_x^{w_2}\partial_w\left((w-x)^2\right)\ dw=\frac{1}{2}\ (w_2-x)^2.$$

In this situation, we have

$$\begin{aligned}
S &= \inf\left\{n\geq 0\ :\ (X_n-w_2)^2\leq 2a\right\} \\
&= \inf\left\{n\geq 0\ :\ \left(X_n-w_2-\sqrt{2a}\right)\left(X_n-w_2+\sqrt{2a}\right)\leq 0\right\} \\
&= \inf\left\{n\geq 0\ :\ X_n\in\left[w_2-\sqrt{2a},w_2+\sqrt{2a}\right]\right\}.
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 487:
For any $m\geq\sum_{1\leq k\leq n}w_k$ we have

$$\begin{aligned}
&\mathbb{P}\left(N_n=m-\textstyle\sum_{1\leq k\leq n}w_k,\ W_1=w_1,\ldots,W_n=w_n\right) \\
&= e^{-\lambda}\ \frac{\lambda^m}{m!}\ \frac{m!}{w_1!(m-w_1)!}\ p^{w_1}(1-p)^{m-w_1}\times\frac{(m-w_1)!}{w_2!(m-w_1-w_2)!}\ p^{w_2}(1-p)^{m-w_1-w_2} \\
&\qquad\times\ldots\times\frac{(m-w_1-\ldots-w_{n-1})!}{w_n!(m-w_1-\ldots-w_n)!}\ p^{w_n}(1-p)^{m-w_1-\ldots-w_n} \\
&= e^{-\lambda(1-(1-p)^n)}\left\{\prod_{1\leq k\leq n}\lambda^{w_k}\ \frac{p^{w_k}}{w_k!}\right\}\left\{\prod_{1<k\leq n}(1-p)^{w_k+\ldots+w_n}\right\} \\
&\qquad\times e^{-\lambda(1-p)^n}\ \frac{(\lambda(1-p)^n)^{m-w_1-\ldots-w_n}}{(m-w_1-\ldots-w_n)!}.
\end{aligned}$$

This yields

$$\begin{aligned}
&\mathbb{P}\left(W_1 = w_1, \ldots, W_n = w_n\right) \\
&= \sum_{m \geq \sum_{1\leq k\leq n} w_k} \mathbb{P}\left(N_n = m - \sum_{1\leq k\leq n} w_k,\ W_1 = w_1, \ldots, W_n = w_n\right) \\
&= \ e^{-\lambda(1-(1-p)^n)} \left\{\prod_{1\leq k\leq n} \lambda^{w_k}\ \frac{p^{w_k}}{w_k!}\right\} \left\{\prod_{1<k\leq n} (1-p)^{w_k+\ldots+w_n}\right\},
\end{aligned}$$

from which we check that

$$\mathbb{P}\left(N_n = m - \sum_{1\leq k\leq n} w_k \mid W_1 = w_1, \ldots, W_n = w_n\right) = e^{-\lambda(1-p)^n} \frac{(\lambda(1-p)^n)^{m-w_1-\ldots-w_n}}{(m-w_1-\ldots-w_n)!}.$$

We conclude that for any $m \geq 0$

$$\mathbb{P}\left(N_n = m \mid W_1 = w_1, \ldots, W_n = w_n\right) = e^{-\lambda(1-p)^n} \frac{(\lambda(1-p)^n)^m}{m!}.$$

This implies that

$$f_n(X_n) = \mathbb{E}\left(na_1 + (N - X_n)\ a_2 \mid X_n\right) = na_1 + \mathbb{E}\left(N_n \mid X_n\right)\ a_2 = na_1 + \lambda(1-p)^n a_2.$$

On the other hand

$$\begin{aligned}
&\mathbb{E}\left(f_{n+1}(X_{n+1}) \mid \mathcal{F}_n\right) - f_n(X_n) \\
&= a_1 + \lambda\left[(1-p)^{n+1} - (1-p)^n\right]\ a_2 = a_1 - \lambda\ p(1-p)^n\ a_2.
\end{aligned}$$

This shows that

$$n \mapsto \mathbb{E}\left(f_{n+1}(X_{n+1}) \mid \mathcal{F}_n\right) - f_n(X_n)$$

is an increasing function, from which we conclude that

$$\mathbb{E}\left(f_{n+1}(X_{n+1}) \mid \mathcal{F}_n\right) - f_n(X_n) \geq \mathbb{E}\left(f_n(X_n) \mid \mathcal{F}_{n-1}\right) - f_{n-1}(X_{n-1}).$$

This shows that the optimal stopping problem is monotone (recall that we are dealing with a minimization problem).

Using exercise 484 the optimal stopping rule on any finite time horizon is defined by

$$\begin{aligned}
S &= \inf\left\{n \geq 0\ :\ f_n(X_n) \leq \mathbb{E}\left(f_{n+1}(X_{n+1}) \mid X_n\right)\right\} \\
&= \inf\left\{n \geq 0\ :\ \mathbb{E}\left(f_{n+1}(X_{n+1})|X_n\right) - f_n(X_n) \geq 0\right\} \\
&= \inf\left\{n \geq 0\ :\ \lambda\ p(1-p)^n\ \leq a_1/a_2\right\}.
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 488:

We check this claim by using backward induction. At the final time horizon n the result is immediate:

$$V_n(x_n) := \sum_{0\leq l<n} g'_l(x'_l) + V'_n(x'_n),$$

with $V'_n = f'_n$. We further assume that the result has been checked at some rank $(k+1)$. The backward formula (29.32) implies that

$$\begin{aligned}
&V_k(x_k)\\
&= \left[\sum_{0\leq l<k} g'_l(X'_l) + f'_k(X'_k)\right] \vee \left[\sum_{0\leq l\leq k} g'_l(x'_l) + \mathbb{E}(V'_{k+1}(x') \mid X'_k = x'_k)\right]\\
&= \sum_{0\leq l<k} g'_l(X'_l) + \underbrace{\left[f'_k(X'_k) \vee \left[g'_k(x'_k) + \mathbb{E}(V'_{k+1}(x') \mid X'_k = x'_k)\right]\right]}_{:=V'_k(x'_k)}.
\end{aligned}$$

We conclude that the result is satisfied at rank k. This ends the proof of the claim.

This ends the proof of the exercise.

■

Solution to exercise 489:

As in exercise 488, we check this claim by using backward induction. At the final time horizon n the result is immediate since we have

$$V_n(x_n) := \left[\prod_{0\leq l<n} g'_l(X'_l)\right] \times V'_n(X'_k)$$

with $V'_n = f'_n$. We further assume that the result has been checked at some rank $(k+1)$. The backward formula (29.32) implies that

$$\begin{aligned}
&V_k(x_k)\\
&= \left[\left\{\prod_{0\leq l<k} g'_l(X'_l)\right\}\ f'_k(X'_k)\right] \vee \left[\left\{\prod_{0\leq l\leq k} g'_l(x'_l)\right\}\ \mathbb{E}(V'_{k+1}(x') \mid X'_k = x'_k)\right]\\
&= \left\{\prod_{0\leq l<k} g'_l(X'_l)\right\} \times \underbrace{\left[f'_k(X'_k) \vee \left[g'_k(x'_k)\ \mathbb{E}(V'_{k+1}(x') \mid X'_k = x'_k)\right]\right]}_{:=V'_k(x'_k)}.
\end{aligned}$$

We conclude that the result is satisfied at rank k. This ends the proof of the claim.

This ends the proof of the exercise.

■

Solution to exercise 490:

By (29.1) the fortune of the gambler is given by

$$M_{n+1} = M_n + H_n\, X_{n+1} = M_n + \alpha M_n\, X_{n+1} = (1+\alpha\ X_{n+1})\, M_n = M_0 \prod_{1\leq k\leq n+1} (1+\alpha\ X_k).$$

The growth rate is given by

$$L_n(\alpha) = \frac{1}{n}\log\left(M_n/M_0\right) = \frac{1}{n}\sum_{1\leq k\leq n} \log\left(1+\alpha\ X_k\right).$$

By the Law of Large Numbers, we have

$$\begin{aligned}
L_\infty(\alpha) &= \lim_{n\to\infty} L_n(\alpha) = \mathbb{E}\left(\log\left(1+\alpha\ X_1\right)\right)\\
&= (1-q)\ \log\left(1-\alpha\right) + q\ \log\left(1+\alpha\right).
\end{aligned}$$

We also have

$$\begin{aligned}\partial_\alpha L_\infty(\alpha) &= (q-1)\ \frac{1}{1-\alpha}+q\ \frac{1}{1+\alpha}\\ &= \frac{q(1-\alpha)+(q-1)(1+\alpha)}{1-\alpha^2}=\frac{(2q-1)-\alpha}{1-\alpha^2}.\end{aligned}$$

This shows that the optimal proportion is defined by $\alpha^\star = 2q-1=\mathbb{E}(X)=(1-q)\times(-1)+q\times(1)$. In addition:

$$\begin{aligned}L_\infty(\alpha^\star) &= (1-q)\ \log\left(1-(2q-1)\right)+q\ \log\left(1+(2q-1)\right)\\ &= (1-q)\ \log\left(2(1-q)\right)+q\ \log\left(2q\right)\\ &= \log\left(2\right)+(1-q)\ \log\left(1-q\right)+q\ \log\left(q\right).\end{aligned}$$

In the second part of the exercise, we have

$$\begin{aligned}M_{n+1} &= M_n+(1-\alpha)\ M_n\ r+\ \alpha M_n\ X_{n+1}\\ &= (1+\alpha\ X_{n+1}+(1-\alpha)\ r)\ \ M_n=M_0\prod_{1\leq k\leq n+1}(1+\alpha\ X_k+(1-\alpha)\ r)\,.\end{aligned}$$

The growth rate is now given by the formula

$$L_n(\alpha)=\frac{1}{n}\log\left(M_n/M_0\right)=\frac{1}{n}\sum_{1\leq k\leq n}\log\left(1+\alpha\ X_k+(1-\alpha)\ r\right).$$

By the Law of Large Numbers, we have

$$\begin{aligned}L_\infty(\alpha) &= \lim_{n\to\infty}L_n(\alpha)=\mathbb{E}\left(\log\left(1+\alpha\ X_1+(1-\alpha)\ r\right)\right)\\ &= \mathbb{E}\left(\log\left((1+r)+\alpha(X_1-r)\right)\right)=\mathbb{E}\left(\log\left((1+r)\left[1+\alpha\ \frac{(X_1-r)}{1+r}\right]\right)\right)\\ &= \log\left(1+r\right)+\mathbb{E}\left(\log\left[1+\alpha\ \frac{(X_1-r)}{1+r}\right]\right)\\ &= \underbrace{\log\left(1+r\right)}_{L_\infty(0)\geq 0}+q\ \log\left[1+\alpha\ \delta_r\right]+p\ \log\left[1-\alpha\right],\end{aligned}$$

with $\delta_r=(1-r)/(1+r)\in]0,1[$. In this case, we have

$$\partial_\alpha L_\infty(\alpha)=q\ \frac{\delta_r}{1+\alpha\ \delta_r}-p\ \frac{1}{1-\alpha}=\frac{q\delta_r(1-\alpha)-p(1+\alpha\ \delta_r)}{(1-\alpha)(1+\alpha\delta_r)}.$$

Observe that

$$\partial_\alpha L_\infty(\alpha)\geq 0$$

if and only if

$$q\delta_r(1-\alpha)\geq p(1+\alpha\ \delta_r)\Leftrightarrow (1\geq)\ q-p/\delta_r\geq\alpha\ .$$

Recalling that $p/q\in[0,1]$ and $\delta_r\in[0,1]$ need to consider the two cases: 1) $\delta_r\leq p/q$ or 2) $p/q\leq\delta_r$.

In the first case, we have $q-p/\delta_r\leq 0$. This shows that $L_\infty\ :\ \alpha\in[0,1]\mapsto L_\infty(\alpha)$ is decreasing from $L_\infty(0)=\log\left(1+r\right)$. In this situation, the optimal strategy is given by $\alpha=\alpha^\star=0$.

In the first case, we have $1\geq\alpha_r:=q-p/\delta_r\geq 0$. This shows that $L_\infty\ :\ \alpha\in[0,1]\mapsto L_\infty(\alpha)$ is increasing for $\alpha\in[0,\alpha_r]\subset[0,1]$ from $L_\infty(0)=\log\left(1+r\right)$ to $L_\infty(\alpha_r)$;

and decreasing for $\alpha \in [q - p/\delta_r, 1] \subset [0,1]$ from $L_\infty(\alpha_r)$ to $L_\infty(1)$. In this situation the optimal strategy is given by $\alpha = \alpha^\star = \alpha_r$.

This ends the proof of the exercise.

∎

Solution to exercise 491:

We simply use the Bayes' rule. Let D_1, D_2, D_3 be the three doors. We assume that the contestant first selects the door D_1 with nothing behind, and then Monty opens the second door D_2 with nothing behind. We denote by H the event *"Monty chooses the door D_2 and nothing is behind"*, and we let P_i be the event *"the price is hidden behind D_i"* with $i = 1, 2, 3$.

We have the three cases

$$\mathbb{P}(P_1 \mid H) = \frac{\overbrace{\mathbb{P}(H \mid P_1)}^{=1/2}}{\underbrace{\mathbb{P}(H)}_{1/2}} \overbrace{\mathbb{P}(P_1)}^{=1/3} = 1/3$$

$$\mathbb{P}(P_2 \mid H) = \frac{\overbrace{\mathbb{P}(H \mid P_2)}^{=0}}{\underbrace{\mathbb{P}(H)}_{1/2}} \overbrace{\mathbb{P}(P_2)}^{=1/3} = 0$$

and finally

$$\mathbb{P}(P_3 \mid H) = \frac{\overbrace{\mathbb{P}(H \mid P_3)}^{=1}}{\underbrace{\mathbb{P}(H)}_{1/2}} \overbrace{\mathbb{P}(P_3)}^{=1/3} = 2/3.$$

Since $\mathbb{P}(P_1 \mid H) < \mathbb{P}(P_3 \mid H)$ the best strategy is to switch the door.

If we have 1000 doors $D_1, D_2, \ldots, D_{1000}$, we let H be the event *"Monty chooses all the doors $D_2; \ldots, D_{999}$ and nothing is behind"*. In this case, when the host chooses the 998 doors $D_2; \ldots, D_{999}$ among the 999 with nothing behind. If the door chosen by the contestant is the correct one, Monty could have selected only one of the 999 sequences of 998 doors among $\{D_2, \ldots, D_{1000}\}$ excluding D_2, or D_3,..., or D_{1000}. This yields the Bayes' formula

$$\mathbb{P}(P_1 \mid H) = \frac{\overbrace{\mathbb{P}(H \mid P_1)}^{=1/999}}{\underbrace{\mathbb{P}(H)}_{1/999}} \overbrace{\mathbb{P}(P_1)}^{=1/1000} = 10^{-3}.$$

When the prize is behind one of the doors D_i selected by the host for some $i = 2, \ldots, 999$, we have

$$\mathbb{P}(P_i \mid H) = \frac{\overbrace{\mathbb{P}(H \mid P_i)}^{=0}}{\underbrace{\mathbb{P}(H)}_{1/999}} \overbrace{\mathbb{P}(P_i)}^{=1/1000} = 0.$$

Finally, we have

$$\mathbb{P}\left(P_{1000} \mid H\right) = \frac{\overbrace{\mathbb{P}(H \mid P_{1000})}^{=1}}{\underbrace{\mathbb{P}(H)}_{1/999}} \overbrace{\mathbb{P}(P_{1000})}^{=1/1000} = 0.999.$$

Clearly, switching is the best strategy. This ends the proof of the exercise.

∎

Solution to exercise 492:
We have

$$\begin{array}{rcl} \Delta Y_n & := & Y_n - Y_{n-1} \\ & = & X_n^2 - X_{n-1}^2 - c^2 = (X_{n-1} + \Delta X_n)^2 - X_{n-1}^2 - c^2 \quad \text{with} \quad \Delta X_n = X_n - X_{n-1} \\ & = & 2X_{n-1}\Delta X_n + (\Delta X_n)^2 - c^2. \end{array}$$

In addition, we have

$$\mathbb{E}((\Delta X_n)^2 \mid \mathcal{F}_{n-1}) = \frac{1}{2}\ c^2 + \frac{1}{2}\ (-c)^2 = c^2$$

and

$$\mathbb{E}(2X_{n-1}\Delta X_n \mid \mathcal{F}_{n-1}) = 2X_{n-1}\Delta X_n\ \underbrace{\mathbb{E}(\Delta X_n \mid \mathcal{F}_{n-1})}_{=\frac{1}{2}\ c + \frac{1}{2}\ (-c)=0}.$$

This shows that

$$\mathbb{E}(\Delta Y_n \mid \mathcal{F}_{n-1}) = 0 \implies Y_n \text{ is a martingale.}$$

By the optional stopping theorem, theorem 8.4.16 (after checking that T is finite and has a finite mean, cf. lemma 8.4.18), we have

$$0 = \mathbb{E}(Y_T) - \mathbb{E}(Y_0) = \mathbb{E}(X_T^2) - c^2\mathbb{E}(T) \leq (a^2 \vee b^2) - c^2\mathbb{E}(T) \Rightarrow \mathbb{E}(T) \leq \frac{a^2 \vee b^2}{c^2}.$$

This ends the proof of the exercise.

∎

Solution to exercise 493:
Recall that $-1 = 2 \bmod(3)$

$$\begin{array}{rcl} M & := & \left(\begin{array}{ccc} M(0,0) & M(0,1) & M(0,2) = M(0,-1) \\ M(1,0) & M(1,1) & M(1,2) \\ M(2,0) = M(2,3) & M(2,1) & M(2,2) \end{array}\right) \\ & = & \left(\begin{array}{ccc} 0 & p_m & q_m \\ q & 0 & p \\ p & q & 0 \end{array}\right) = \left(\begin{array}{ccc} 0 & \frac{1}{10} & \frac{9}{10} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{array}\right). \end{array}$$

When $\epsilon = 0$, the probability $P(n)$ to win game (G_2) at time n is given by

$$P(n) = \mathbb{P}(Y_n = 0)\ p_m + \mathbb{P}(Y_n \in \{1,2\})\ p = \mathbb{P}(Y_n = 0)\ \frac{1}{10} + (1 - \mathbb{P}(Y_n = 0))\ \frac{3}{4}.$$

Notice that

$$[5,2,6]\left(\begin{array}{ccc} 0 & \frac{1}{10} & \frac{9}{10} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{array}\right) = \left[\frac{2}{4}+\frac{18}{4}, \frac{1}{2}+\frac{6}{4}, \frac{9}{2}+\frac{3}{2}\right] = [5,2,6]\,.$$

This shows that $\pi = \frac{1}{13}[5,2,6]$ is the invariant measure of M. Thus, the stationary probability P to win game (G_2) after long runs is given by

$$P = \pi(0)\ p_m + (1-\pi(0))\ p = \frac{5}{13}\ \frac{1}{10} + \frac{8}{13}\ \frac{3}{4} = \frac{1}{26} + \frac{6}{13} = \frac{13}{26} = \frac{1}{2}.$$

When $\epsilon > 0$, the transition of the chain is given by

$$M_\epsilon := \left(\begin{array}{ccc} 0 & \frac{1}{10}-\epsilon & \frac{9}{10}+\epsilon \\ \frac{1}{4}+\epsilon & 0 & \frac{3}{4}-\epsilon \\ \frac{3}{4}-\epsilon & \frac{1}{4}+\epsilon & 0 \end{array}\right).$$

Using exercise 98, the invariant measure $\pi_\epsilon M_\epsilon = \pi_\epsilon$ is given by

$$\begin{array}{rcl} \pi_\epsilon(0) & \propto & M_\epsilon(1,0)M_\epsilon(2,0) + M_\epsilon(1,2)M_\epsilon(2,0) + M_\epsilon(2,1)M_\epsilon(1,0) \\ & = & \left(\frac{1}{4}+\epsilon\right)\left(\frac{3}{4}-\epsilon\right) + \left(\frac{3}{4}-\epsilon\right)^2 + \left(\frac{1}{4}+\epsilon\right)^2 \\ & = & 1-\left(\frac{1}{4}+\epsilon\right)\left(\frac{3}{4}-\epsilon\right) = \frac{13}{16} - \frac{1}{2}\ \epsilon\ + \epsilon^2, \end{array}$$

$$\begin{array}{rcl} \pi_\epsilon(1) & \propto & M_\epsilon(0,1)M_\epsilon(2,1) + M_\epsilon(0,2)M_\epsilon(2,1) + M_\epsilon(2,0)M_\epsilon(0,1) \\ & = & \left(\frac{1}{10}-\epsilon\right)\left(\frac{1}{4}+\epsilon\right) + \left(\frac{9}{10}+\epsilon\right)\left(\frac{1}{4}+\epsilon\right) + \left(\frac{3}{4}-\epsilon\right)\left(\frac{1}{10}-\epsilon\right) \\ & = & \left(\frac{1}{10}-\epsilon\right) + \left(\frac{9}{10}+\epsilon\right)\left(\frac{1}{4}+\epsilon\right) \\ & = & \frac{13}{40} + \frac{3}{20}\ \epsilon + \epsilon^2, \end{array}$$

and

$$\begin{array}{rcl} \pi_\epsilon(2) & \propto & M_\epsilon(0,2)M_\epsilon(1,2) + M_\epsilon(0,1)M_\epsilon(1,2) + M_\epsilon(1,0)M_\epsilon(0,2) \\ & = & \left(\frac{9}{10}+\epsilon\right)\left(\frac{3}{4}-\epsilon\right) + \left(\frac{1}{10}-\epsilon\right)\left(\frac{3}{4}-\epsilon\right) + \left(\frac{1}{4}+\epsilon\right)\left(\frac{9}{10}+\epsilon\right) \\ & = & \left(\frac{9}{10}+\epsilon\right) + \left(\frac{1}{10}-\epsilon\right)\left(\frac{3}{4}-\epsilon\right) \\ & = & \frac{39}{40} + \frac{3}{20}\ \epsilon + \epsilon^2. \end{array}$$

Thus, the normalizing constant is given by

$$\frac{13}{4}\ \left[\frac{1}{4}+\frac{1}{10}+\frac{3}{10}\right] - \frac{1}{2}\left(1-\frac{3}{5}\right)\ \epsilon\ + 3\epsilon^2 = \frac{13}{4}\ \frac{13}{20} - \frac{1}{5}\ \epsilon\ + 3\epsilon^2.$$

We conclude that

$$\begin{aligned}
\pi_\epsilon(0) &= \frac{\frac{13}{16} - \frac{1}{2}\ \epsilon\ + \epsilon^2}{\frac{13}{4}\ \frac{13}{20} - \frac{1}{5}\ \epsilon\ + 3\epsilon^2} \\
&= \frac{5}{13}\ \frac{1 - \frac{8}{13}\ \epsilon + \mathrm{O}(\epsilon^2)}{1 - \frac{16}{13^2}\ \epsilon + \mathrm{O}(\epsilon^2)} \\
&= \frac{5}{13}\ \left(1 - \epsilon\ \frac{1}{13}\left(8 - \frac{16}{13}\right)\right) + \mathrm{O}(\epsilon^2) = \pi(0) - 40\ \frac{11}{13^3}\ \epsilon + \mathrm{O}(\epsilon^2).
\end{aligned}$$

The stationary probability P_ϵ to win game (G_2) after long runs is given by

$$\begin{aligned}
P_\epsilon &= \pi_\epsilon(0)\ \left(\frac{1}{10} - \epsilon\right) + (1 - \pi_\epsilon(0))\ \left(\frac{3}{4} - \epsilon\right) \\
&= \pi_\epsilon(0)\ \frac{1}{10} + (1 - \pi_\epsilon(0))\ \frac{3}{4} - \epsilon \\
&= \frac{1}{2} - \left(1 - 2\ \frac{11}{13^2}\right)\ \epsilon + \mathrm{O}(\epsilon^2) = P - \frac{147}{13^2}\ \epsilon + \mathrm{O}(\epsilon^2).
\end{aligned}$$

In game (G_1) the transition matrix of the chain Y_n is given by

$$\begin{aligned}
\overline{M}_\epsilon &:= \begin{pmatrix} \overline{M}_\epsilon(0,0) & \overline{M}_\epsilon(0,1) & \overline{M}_\epsilon(0,-1) \\ \overline{M}_\epsilon(1,0) & \overline{M}_\epsilon(1,1) & \overline{M}_\epsilon(1,2) \\ \overline{M}_\epsilon(2,3) & \overline{M}_\epsilon(2,1) & \overline{M}_\epsilon(2,2) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{1}{2} - \epsilon & \frac{1}{2} + \epsilon \\ \frac{1}{2} + \epsilon & 0 & \frac{1}{2} - \epsilon \\ \frac{1}{2} - \epsilon & \frac{1}{2} + \epsilon & 0 \end{pmatrix}.
\end{aligned}$$

This implies that in game (G_3) the transition matrix of the chain Y_n is given by

$$\begin{aligned}
\widehat{M}_\epsilon &= \frac{1}{2}(M_\epsilon + \overline{M}_\epsilon) \\
&= \frac{1}{2}\left[\begin{pmatrix} 0 & \frac{1}{2} - \epsilon & \frac{1}{2} + \epsilon \\ \frac{1}{2} + \epsilon & 0 & \frac{1}{2} - \epsilon \\ \frac{1}{2} - \epsilon & \frac{1}{2} + \epsilon & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{10} - \epsilon & \frac{9}{10} + \epsilon \\ \frac{1}{4} + \epsilon & 0 & \frac{3}{4} - \epsilon \\ \frac{3}{4} - \epsilon & \frac{1}{4} + \epsilon & 0 \end{pmatrix}\right] \\
&= \begin{pmatrix} 0 & \frac{3}{10} - \epsilon & \frac{7}{10} + \epsilon \\ \frac{3}{8} + \epsilon & 0 & \frac{5}{8} - \epsilon \\ \frac{5}{8} - \epsilon & \frac{3}{8} + \epsilon & 0 \end{pmatrix}.
\end{aligned}$$

Using exercise 98, the invariant measure $\widehat{\pi}_\epsilon \widehat{M}_\epsilon = \widehat{M}_\epsilon$ is given by

$$\begin{aligned}
\widehat{\pi}_\epsilon(0) &\propto \widehat{M}_\epsilon(1,0)\widehat{M}_\epsilon(2,0) + \widehat{M}_\epsilon(1,2)\widehat{M}_\epsilon(2,0) + \widehat{M}_\epsilon(2,1)\widehat{M}_\epsilon(1,0) \\
&= \left(\frac{3}{8} + \epsilon\right)\left(\frac{5}{8} - \epsilon\right) + \left(\frac{5}{8} - \epsilon\right)^2 + \left(\frac{3}{8} + \epsilon\right)^2 \\
&= 1 - \left(\frac{3}{8} + \epsilon\right)\left(\frac{5}{8} - \epsilon\right) = \left(\frac{7}{8}\right)^2 - \frac{1}{4}\ \epsilon\ + \epsilon^2,
\end{aligned}$$

by

$$\begin{aligned}
\widehat{\pi}_\epsilon(1) &\propto \widehat{M}_\epsilon(0,1)\widehat{M}_\epsilon(2,1)+\widehat{M}_\epsilon(0,2)\widehat{M}_\epsilon(2,1)+\widehat{M}_\epsilon(2,0)\widehat{M}_\epsilon(0,1)\\
&= \left(\frac{3}{10}-\epsilon\right)\left(\frac{3}{8}+\epsilon\right)+\left(\frac{7}{10}+\epsilon\right)\left(\frac{3}{8}+\epsilon\right)+\left(\frac{5}{8}-\epsilon\right)\left(\frac{3}{10}-\epsilon\right)\\
&= \left(\frac{3}{10}-\epsilon\right)+\left(\frac{7}{10}+\epsilon\right)\left(\frac{3}{8}+\epsilon\right)\\
&= \frac{45}{80}+\frac{3}{40}\ \epsilon+\epsilon^2,
\end{aligned}$$

and finally by

$$\begin{aligned}
\widehat{\pi}_\epsilon(2) &\propto \widehat{M}_\epsilon(0,2)\widehat{M}_\epsilon(1,2)+\widehat{M}_\epsilon(0,1)\widehat{M}_\epsilon(1,2)+\widehat{M}_\epsilon(1,0)\widehat{M}_\epsilon(0,2)\\
&= \left(\frac{7}{10}+\epsilon\right)\left(\frac{5}{8}-\epsilon\right)+\left(\frac{3}{10}-\epsilon\right)\left(\frac{5}{8}-\epsilon\right)+\left(\frac{3}{8}+\epsilon\right)\left(\frac{7}{10}+\epsilon\right)\\
&= \left(\frac{7}{10}+\epsilon\right)+\left(\frac{3}{10}-\epsilon\right)\left(\frac{5}{8}-\epsilon\right)\\
&= \frac{71}{80}+\frac{3}{40}\ \epsilon+\epsilon^2.
\end{aligned}$$

Thus, the normalizing constant is now given by

$$\left(\frac{7}{8}\right)^2+\frac{116}{80}-\frac{1}{10}\ \epsilon+3\epsilon^2=\frac{1}{8^2}\frac{709}{5}-\frac{1}{10}\ \epsilon+3\epsilon^2.$$

This implies that

$$\begin{aligned}
\widehat{\pi}_\epsilon(0) &= \frac{\left(\frac{7}{8}\right)^2-\frac{1}{4}\ \epsilon\ +\epsilon^2}{\frac{1}{8^2}\frac{709}{5}-\frac{1}{10}\ \epsilon+3\epsilon^2}\\
&= \frac{245}{709}\ \frac{1-\left(\frac{4}{7}\right)^2\ \epsilon\ +\left(\frac{8}{7}\epsilon\right)^2}{1-\frac{32}{709}\ \epsilon+(8\epsilon)^2\frac{15}{709}}\simeq 0.35+\mathrm{O}(\epsilon).
\end{aligned}$$

The stationary probability P_ϵ to win game (G_3) after long runs is given by

$$\begin{aligned}
\widehat{P}_\epsilon &= \widehat{\pi}_\epsilon(0)\ \left(\frac{3}{10}-\epsilon\right)+(1-\widehat{\pi}_\epsilon(0))\ \left(\frac{5}{8}-\epsilon\right)\\
&= \frac{1}{2}\ \frac{727}{709}+\mathrm{O}(\epsilon)>\frac{1}{2}=P \quad \text{when } \epsilon \text{ is sufficiently small.}
\end{aligned}$$

The effect we just observed is the “Parrondo's paradox". As we have just realised, two losing games, when alternated in a periodic or random fashion, can produce a winning game.

This ends the proof of the exercise.

■

Solution to exercise 494:

At each point in time n, the gambler bets $a\times S(X_n)$ with the re-scaled gambling strategy defined by

$$S(X_n):=\left\{\begin{array}{ll} X_n & \text{if} \quad X_n\leq 1/2\\ 1-X_n & \text{if} \quad 1/2\leq 1-X_n\leq 1\end{array}\right.$$

When the relative fortune $x \leq 1/2$ (i.e. real fortune $ax := y \leq a/2$) he bets the relative amount x (i.e. the real amount $y := ax$) and wins $2x$ (i.e. the real amount $2y := 2ax$). (Of course, the win is zero when he loses.) Thus, when $x \leq 1/2 \Leftrightarrow y = ax \leq a/2$ we have

$$\begin{aligned} P(y) &= \mathbb{P}\,(\text{Reach the fortune } a \mid Y_0 = y) \\ &= \mathbb{P}\,(\text{Reach the fortune } a \mid Y_1 = 2y)\,\mathbb{P}(Y_1 = 2y \mid Y_0 = y) \\ &\quad + \mathbb{P}\,(\text{Reach the fortune } a \mid Y_1 = 0)\,\mathbb{P}(Y_1 = 0 \mid Y_0 = y) \\ &= p\ P(2y) + q\ 0. \end{aligned}$$

In other words

$$x := y/a \in [0,1] \quad \text{and} \quad Q(y/a) := P(y) \Longrightarrow \quad Q(x) = p\ Q(2x).$$

When the relative fortune $1/2 \leq x \leq 1$ (i.e. real fortune $a \geq ax := y \geq a/2$) to reach the rescaled fortune 1 (i.e. the real target fortune a,) one needs to bet the relative amount $1 - x$ (i.e. the real amount $a - ax = a - y$.) Arguing as above we find that

$$\begin{aligned} &P(y) = \mathbb{P}\,(\text{Reach the fortune } a \mid Y_0 = y) \\ &= \underbrace{\mathbb{P}\left(\text{Reach the fortune } a \mid \underbrace{Y_1 = 2y}_{\geq 2\ a/2 = a}\right)}_{=1} \mathbb{P}(Y_1 = 2y \mid Y_0 = y) \\ &\qquad + \mathbb{P}\,(\text{Reach the fortune } a \mid Y_1 = y - (a - y))\,\mathbb{P}(Y_1 = y - (a - y) \mid Y_0 = y) \\ &= p\ + q\ P(2y - a). \end{aligned}$$

In other words:

$$x := y/a \quad \text{and} \quad Q(y/a) := P(y) \Longrightarrow \quad Q(x) = p + q\ Q(2x - 1).$$

The probability to reach the fortune a starting with an initial fortune $y \in [0, a]$ is defined by the function $Q(y/a)$ with

$$Q(x) := \begin{cases} p\ Q(2x) & \text{if} \quad x \leq 1/2 \\ p + q\ Q(2x - 1) & \text{if} \quad 1/2 \leq x \leq 1, \end{cases}$$

with the boundary conditions $(Q(0), Q(1)) = (0, 1)$.

Now compute $Q(i/2^n)$ for $i < 2^n$ and $n = 1, 2, 3$. For $n = 1$ and $i = 1$ we have we have

$$Q(1/2) = p\ Q(1) = p.$$

For $n = 2$ and $i = 1, 2, 3$, we have

$$Q(1/4) = p\ Q(1/2) = p^2 \quad \text{and} \quad Q(3/4) = p + q\ Q\left(2\ \frac{3}{4} - 1\right) = p + q\ Q(1/2) = p(1 + q).$$

For $n = 3$ and $i \in \{1, \ldots, 7\}$ we have

$$\begin{aligned} Q(1/8) &= pQ(1/4) = p^2 Q(1/2) = p^3 \\ Q(3/8) &= pQ(3/4) = p^2(1 + q) \\ Q(4/8) &= Q(1/2) = p \end{aligned}$$

and

$$
\begin{aligned}
Q(5/8) &= p + q\,Q\left(2\,\frac{5}{8} - 1\right) = p + q\,Q\left(\frac{1}{4}\right) = p + p^2 q = p(1+pq)\\
Q(6/8) &= Q\,(3/4) = p(1+q)\\
Q(7/8) &= p + q\,Q\left(2\,\frac{7}{8} - 1\right) = p + q\,Q\left(\frac{3}{4}\right) = p + qp(1+q) = p\,(1 + q(1+q))\,.
\end{aligned}
$$

This ends the proof of the exercise.

■

Solution to exercise 495:

After $n = (a+b)$ votes are counted, the difference of A votes and B votes is given by $X_n = (a-b)$. For any k such that $0 \leq k \leq n$ we let X_k be the difference of A votes and B votes after counting k votes. After counting k votes, A has $\frac{k+X_k}{2}$ votes while B has $\frac{k-X_k}{2}$ votes. Given X_{k+1}, the difference of votes X_k can be $X_{k+1}+1$ if the $(k+1)$-th vote was for B; or $X_{k+1}-1$ if the $(k+1)$-th vote was for A. In addition, we have

$$
\begin{aligned}
\mathbb{P}(X_k = X_{k+1}+1 \mid X_{k+1}) &= \frac{[(k+1) - X_{k+1}]/2}{k+1}\\
\mathbb{P}(X_k = X_{k+1}-1 \mid X_{k+1}) &= \frac{[(k+1) + X_{k+1}]/2}{k+1}.
\end{aligned}
$$

If we set

$$
M_k := \frac{X_{n-k}}{n-k} \qquad (\Rightarrow\ M_0 = (a-b)/(a+b))
$$

then we have

$$
\begin{aligned}
&\mathbb{E}(M_k \mid M_0, M_1, \ldots, M_{k-1})\\
&= \mathbb{E}(\frac{X_{n-k}}{n-k} \mid X_n, X_{n-1}, \ldots, X_{(n-k)+1}) = \mathbb{E}(X_{n-k} \mid X_{(n-k)+1})\\
&= \frac{[((n-k)+1) - X_{(n-k)+1}]/2}{(n-k)+1}\ \left(\frac{X_{(n-k)+1}+1}{n-k}\right)\\
&\qquad\qquad + \frac{[((n-k)+1) + X_{(n-k)+1}]/2}{(n-k)+1}\ \left(\frac{X_{(n-k)+1}-1}{n-k}\right)\\
&= \frac{1}{2(n-k)+2}\left[\left(\left\{1 + \frac{1}{n-k}\right\} - \frac{X_{(n-k)+1}}{n-k}\right)\left(X_{(n-k)+1}+1\right)\right.\\
&\qquad\qquad \left. + \left(\left\{1 + \frac{1}{n-k}\right\} + \frac{X_{(n-k)+1}}{n-k}\right)\left(X_{(n-k)+1}-1\right)\right]\\
&= \frac{1}{2(n-k)+2}\left[2\left(1 + \frac{1}{n-k}\right)\ X_{(n-k)+1} - 2\,\frac{X_{(n-k)+1}}{n-k}\right] = \frac{X_{(n-k)+1}}{(n-k)+1} = M_{k-1}.
\end{aligned}
$$

In addition we have $|M_k| \leq n$. We let T be the first time $k = 0, \ldots, n$ we have $X_k = 0$. At that time 2 situations may occur:

- The candidate A is always ahead if and only if we have $T = (n-1)$ and $M_T = M_{n-1} = X_1 = 1$

- Otherwise, as some time X_k hits the null axis and $X_T = 0 = M_T$.

We conclude that

$$\mathbb{E}(M_T) = \mathbb{E}(M_0) = \frac{X_n}{n} = \frac{a-b}{a+b}$$

$$\Rightarrow \frac{a-b}{a+b} = \mathbb{P}(M_T = 1) \ \times 1 + \mathbb{P}(M_T = 0) \ \times 0 \Rightarrow \mathbb{P}(M_T = 1) = \frac{a-b}{a+b}.$$

This ends the proof of the first assertion.
Now we turn to the path counting problems.

- The number of permutations of the $(a+b)$ ballots is clearly given by the number of possible positions of A or B among $(a+b)$ ballots. One instance of a position is:

$$\underbrace{AA\ldots AAA}_{a\text{-times}}\underbrace{BBBB\ldots B}_{b\text{-times}}.$$

This number is clearly given by $\begin{pmatrix} a+b \\ a \end{pmatrix} = \begin{pmatrix} a+b \\ b \end{pmatrix}$.

- We need to find the number of paths with a up-steps and b down-steps where no step ends on or below the $(0,x)$-axis. We observe that for $k = n$, $\mathcal{P}_n$ is the set of all the $\begin{pmatrix} a+b-1 \\ a \end{pmatrix}$ bad paths starting with a down-step $(1,-n)$, and $\cup_{0\leq k\leq n}\mathcal{P}_k$ coincides with all the possible "bad" paths.

Let $P = P_1P_2$ be a path in $\mathcal{P}_k$, with $k < n$ and path P_1 ending with the first bad step ending k units below $(0,x)$. We denote by Q_1 the path obtained by rotating P_1 by π and exchanging the end-points. By construction $Q = Q_1P_1$ starts with a down-step so that $Q \in \mathcal{P}_n$ (and inversely). This shows that $\mathrm{Card}(\mathcal{P}_n) = \mathrm{Card}(\mathcal{P}_k)$ for any k.

We deduce that

$$\begin{aligned} \mathrm{Card}(\mathcal{P}) &= \begin{pmatrix} a+b \\ a \end{pmatrix} - \sum_{0\leq k\leq n} \mathrm{Card}(\mathcal{P}_k) \\ &= \begin{pmatrix} a+b \\ a \end{pmatrix} - (n+1)\begin{pmatrix} a+b-1 \\ a \end{pmatrix} = \frac{a-nb}{a+b}\begin{pmatrix} a+b \\ a \end{pmatrix} \end{aligned}$$

The last assertion follows trivially from:

$$\begin{pmatrix} a+b-1 \\ a \end{pmatrix} = \frac{(a+b-1)!}{a!(b-1)!} = \frac{b}{a+b}\begin{pmatrix} a+b \\ a \end{pmatrix}$$

$$\Longrightarrow 1 - \frac{(n+1)b}{a+b} = \frac{a-nb}{a+b}.$$

- Checking the formulae $N_n(a,0) = 1$ and $N_n(nb,b) = 0$ for any $a,b > 0$ is direct. The recurrence formula is

$$N_n(a,b) = N_n(a-1,b) + N_n(a,b-1).$$

To show it inductively using the previous result, we calculate the r.h.s. in detail:

$$N_n(a-1,b) + N_n(a,b-1) = \frac{(a-1)-nb}{(a-1)+b}\underbrace{\begin{pmatrix} (a-1)+b \\ a-1 \end{pmatrix}}_{=\frac{a}{a+b}\begin{pmatrix} a+b \\ a \end{pmatrix}} + \frac{a-n(b-1)}{a+(b-1)}\underbrace{\begin{pmatrix} a+(b-1) \\ a \end{pmatrix}}_{=\frac{b}{a+b}\begin{pmatrix} a+b \\ a \end{pmatrix}}$$

The last assertion provides a proof by induction. Indeed:

$$\begin{aligned}
& a\ \frac{(a-1)-nb}{(a-1)+b} + b\ \frac{a-n(b-1)}{a+(b-1)} \\
& = a\ \frac{((a-1)+b)-(n+1)b}{(a-1)+b} + b\ \frac{(a+(b-1))-(n+1)(b-1)}{a+(b-1)} \\
& = (a+b)-(n+1)b\ \frac{a+(b-1)}{(a-1)+b} = a-nb.
\end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 496:

- The random variables X_k are independent with distribution

$$\mathbb{P}(X_n = 1) = \ldots = \mathbb{P}(X_n = n) = 1/n.$$

- The chance for $Y_k = q_1$ coincides with the probability that the candidate with qualification q_1 appears among the first k interviewed candidates. This implies that $\mathbb{P}(Y_k = q_1 \mid X_1, \ldots, X_k) = 1_{X_k=1}\ \frac{k}{n}$.
- We have

$$\mathbb{P}(Y_T = q_1) \quad = \quad \sum_{1\leq k\leq n} \mathbb{E}\left(1_{T=n}\mathbb{E}\left(1_{Y_n=q_1} \mid (X_1,\ldots,X_n)\right)\right) = \mathbb{E}(f_T(X_T)).$$

- The Snell envelope associated with this optimal stopping problem is defined by the backward induction

$$V_k(x) = f_k(x) \vee \mathbb{E}(V_{k+1}(X_{k+1}) \mid X_k = x) = \mathrm{Max}\left(f_k(x), \frac{1}{k+1}\sum_{1\leq l\leq (k+1)} V_{k+1}(l)\right)$$

with the terminal condition $V_n = f_n$.
We let m_n be the first and unique value $k : 2 \leq k \leq n$ such that

$$\frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{n-1} \leq 1 < \frac{1}{k-1} + \frac{1}{k+1} + \ldots + \frac{1}{n-1}.$$

For any n we notice that

$$V_n(x) = 1_{x=1} \quad \Longrightarrow \quad \frac{1}{n}\sum_{1\leq l\leq n} V_n(l) = \frac{1}{n}.$$

This implies that

$$\begin{aligned}
V_{n-1}(x) \quad &= \quad \mathrm{Max}\left(1_{x=1}\ \frac{n-1}{n}, \frac{1}{n}\sum_{1\leq l\leq n} V_n(l)\right) \\
&= \quad \mathrm{Max}\left(1_{x=1}\ \frac{n-1}{n}, \frac{1}{n}\right) = \frac{n-1}{n}\ \mathrm{Max}\left(1_{x=1}, \frac{1}{n-1}\right),
\end{aligned}$$

so that

$$\frac{1}{n-1}\sum_{1\leq l\leq(n-1)} V_{n-1}(l)=\frac{n-2}{n}\left(\frac{1}{n-2}+\frac{1}{n-1}\right).$$

In the same way, when $(n-1)\geq m_n$ we have

$$\begin{aligned}V_{n-2}(x) &= \text{Max}\left(1_{x=1}\ \frac{n-2}{n},\frac{n-2}{n}\left(\frac{1}{n-2}+\frac{1}{n-1}\right)\right)\\ &= \frac{n-2}{n}\ \text{Max}\left(1_{x=1},\left(\frac{1}{n-2}+\frac{1}{n-1}\right)\right),\end{aligned}$$

so that

$$\begin{aligned}\frac{1}{n-2}\sum_{1\leq l\leq(n-2)} V_{n-2}(l) &= \frac{1}{n}\left(1+(n-3)\ \left(\frac{1}{n-2}+\frac{1}{n-1}\right)\right)\\ &= \frac{n-3}{n}\left(\frac{1}{n-3}+\frac{1}{n-2}+\frac{1}{n-1}\right).\end{aligned}$$

Iterating we find that

$$V_k(x)=\frac{k}{n}\ \text{Max}\left(1_{x=1},\left(\frac{1}{n-1}+\ldots+\frac{1}{k+1}+\frac{1}{k}\right)\right)$$

for any $k\geq m_n$ with

$$\begin{aligned}\frac{1}{k}\sum_{1\leq l\leq k} V_k(l) &= \frac{1}{n}\ \left(1+(k-1)\ \left(\frac{1}{n-1}+\ldots+\frac{1}{k+1}+\frac{1}{k}\right)\right)\\ &= \frac{k-1}{n}\ \left(\frac{1}{n-1}+\ldots+\frac{1}{k}+\frac{1}{k-1}\right).\end{aligned}$$

Notice that for $k=m_n$ we have

$$\frac{1}{m_n}\sum_{1\leq l\leq m_n} V_{m_n}(l)=\frac{m_n-1}{n}\ \underbrace{\left(\frac{1}{n-1}+\ldots+\frac{1}{m_n}+\frac{1}{m_n-1}\right)}_{>1}.$$

This implies that the function V_{m_n-1} is the constant function

$$V_{m_n-1}(x)\quad=\quad\frac{m_n-1}{n}\ \left(\frac{1}{n-1}+\ldots+\frac{1}{m_n}+\frac{1}{m_n-1}\right)$$

so that

$$\frac{1}{m_n-1}\sum_{1\leq l\leq m_n-1} V_{m_n-1}(l)=V_{m_n-1}(x).$$

This implies that

$$V_{m_n-2}(x)=\frac{m_n-1}{n}\ \text{Max}\left(\underbrace{\frac{m_n-2}{m_n-1}}_{<1},\underbrace{\left(\frac{1}{n-1}+\ldots+\frac{1}{m_n}+\frac{1}{m_n-1}\right)}_{>1}\right)$$

and therefore

$$V_{m_n-2}=V_{m_n-1}.$$

Iterating this procedure, we prove that

$$\forall k < m_n \qquad V_k = V_{m_n-1}.$$

This implies that

$$\sup_{T\in\mathcal{T}_n} \mathbb{P}(Y_T = q_1) = V_1(1) = \frac{m_n - 1}{n}\left(\frac{1}{n-1} + \ldots + \frac{1}{m_n} + \frac{1}{m_n - 1}\right).$$

- The optimal policy is defined by

$$\begin{aligned} T &= \inf\{1 \leq k \leq n \;:\; k\ 1_{X_k=1} = n\ V_k(X_k)\} \\ &= \inf\{m_n \leq k \leq n \;:\; k\ 1_{X_k=1} = n\ V_k(X_k)\} = \inf\{m_n \leq k \leq n \;:\; X_k = 1\}. \end{aligned}$$

  The last assertion follows from the fact that

$$\forall m_n \leq k \leq n \quad \forall x \in \{2,\ldots,n\} \qquad (0 < n\ V_k(x) <)\, n\ V_k(1) = k.$$

  We conclude that the optimal strategy is to reject the first $m_n - 1$ candidates, and to continue the interviewing until we find the best candidate among those examined so far.

This ends the proof of the exercise.

■

Chapter 30

Solution to exercise 497:

If we set

$$\begin{aligned}
\mathbb{P}\left(\overline{\mathcal{S}}_1 = s_{0,1} \mid \overline{\mathcal{S}}_0 = s_0\right) &= \frac{s_{0,2} - s_0}{s_{0,2} - s_{0,1}} \\
\mathbb{P}\left(\overline{\mathcal{S}}_1 = s_{0,2} \mid \overline{\mathcal{S}}_0 = s_0\right) &= \frac{s_0 - s_{0,1}}{s_{0,2} - s_{0,1}} = 1 - \frac{s_{0,2} - s_0}{s_{0,2} - s_{0,1}}
\end{aligned}$$

then we find that

$$\begin{aligned}
\mathbb{E}\left(\overline{\mathcal{S}}_1 \mid \overline{\mathcal{S}}_0 = s_0\right) &= s_{0,1}\ \frac{s_{0,2} - s_0}{s_{0,2} - s_{0,1}} + s_{0,2}\ \frac{s_0 - s_{0,1}}{s_{0,2} - s_{0,1}} \\
&= \frac{s_0 s_{0,2} - s_0 s_{0,1}}{s_{0,2} - s_{0,1}} = s_0.
\end{aligned}$$

In the same way, if we set

$$\begin{aligned}
\mathbb{P}\left(\overline{\mathcal{S}}_2 = s_{(0,1),1} \mid \overline{\mathcal{S}}_1 = s_{0,1}\right) &= \frac{s_{(0,1),2} - s_{(0,1)}}{s_{(0,1),2} - s_{(0,1),2}} \\
\mathbb{P}\left(\overline{\mathcal{S}}_2 = s_{(0,1),2} \mid \overline{\mathcal{S}}_1 = s_{0,1}\right) &= \frac{s_{(0,1)} - s_{(0,1),1}}{s_{(0,1),2} - s_{(0,1),1}} = 1 - \frac{s_{(0,1),2} - s_{(0,1)}}{s_{(0,1),2} - s_{(0,1),1}}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}\left(\overline{\mathcal{S}}_2 = s_{(0,2),1} \mid \overline{\mathcal{S}}_1 = s_{(0,2)}\right) &= \frac{s_{(0,2),2} - s_{(0,2)}}{s_{(0,2),2} - s_{(0,2),2}} \\
\mathbb{P}\left(\overline{\mathcal{S}}_2 = s_{(0,2),2} \mid \overline{\mathcal{S}}_1 = s_{(0,2)}\right) &= \frac{s_{(0,2)} - s_{(0,2),1}}{s_{(0,2),2} - s_{(0,2),1}} = 1 - \frac{s_{(0,2),2} - s_{(0,2)}}{s_{(0,2),2} - s_{(0,2),1}}
\end{aligned}$$

then we have

$$\mathbb{E}\left(\overline{\mathcal{S}}_2 \mid \overline{\mathcal{S}}_1 = s_{(0,1)}\right) = s_{(0,1)} \quad \text{and} \quad \mathbb{E}\left(\overline{\mathcal{S}}_2 \mid \overline{\mathcal{S}}_1 = s_{(0,2)}\right) = s_{(0,2)}.$$

This ends the proof of the exercise. ■

Solution to exercise 498:

Applying the Doeblin-Itō formula to the function

$$g(t,x) = \exp\left(\sigma x + t\left(r - \frac{\sigma^2}{2}\right)\right)$$

we find that

$$\begin{aligned}
dg(t,W_t) &= \partial_t g(t,W_t)\ dt + \partial_x f(t,W_t)\ dW_t + \frac{1}{2}\ \partial_x^2 f(t,W_t) dW_t dW_t \\
&= \left(r - \frac{\sigma^2}{2}\right)\ g(t,W_t)\ dt + \sigma\ g(t,W_t)\ dW_t + \frac{1}{2}\ \sigma^2\ g(t,W_t) dt \\
&= r\ g(t,W_t) dt + \sigma\ g(t,W_t)\ dW_t.
\end{aligned}$$

We conclude that

$$\begin{aligned}(S_t/S_0) = g(t, W_t) \quad &\Longrightarrow \quad d(S_t/S_0) = r\ (S_t/S_0)\ dt + \sigma\ (S_t/S_0)\ dW_t \\ &\Longleftrightarrow \quad dS_t = r\ S_t\ dt + \sigma\ S_t\ dW_t.\end{aligned}$$

This ends the proof of the exercise. ∎

Solution to exercise 499:
We have

$$S_{\alpha t} := \exp\left(\sigma W_{\alpha t} + \alpha t \left(r - \tfrac{\sigma^2}{2}\right)\right) \geq S_t^{\beta} := \exp\left(\beta\sigma W_t + \beta t\left(r - \tfrac{\sigma^2}{2}\right)\right)$$

$$\Leftrightarrow (\beta - \alpha)\left(\tfrac{r}{\sigma} - \tfrac{\sigma}{2}\right)\ t \leq (W_{\alpha t} - \beta W_t) = (W_{\alpha t} - W_t) + (1-\beta) W_t.$$

Observe that

$$(W_{\alpha t} - W_t) + (1-\beta) W_t$$

is a centered Gaussian random variable with variance

$$\begin{aligned}\mathbb{E}\left([(W_{\alpha t} - W_t) + (1-\beta)W_t]^2\right) &= \mathbb{E}\left((W_{\alpha t} - W_t)^2\right) + (1-\beta)^2\ \mathbb{E}\left(W_t^2\right) \\ &= \left[(\alpha - 1) + (1-\beta)^2\right]\ t.\end{aligned}$$

This implies that

$$\mathbb{P}\left(S_{\alpha t} \geq S_t^{\beta}\right) = \mathbb{P}\left(W_1 \leq \frac{(\beta-\alpha)\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)}{\sqrt{[(\alpha-1) + (1-\beta)^2]}}\ \sqrt{t}\right).$$

This ends the proof of the exercise. ∎

Solution to exercise 500:
By (30.12) the process $\overline{\mathcal{S}}_t$ is an $\mathcal{F}_t$-martingale. In addition by (30.8) we have

$$\overline{\mathcal{S}}_t := \overline{\mathcal{S}}_0\ \exp\left(\sigma W_t - \frac{\sigma^2 t}{2}\right) = e^{-rt}\ S_t.$$

This implies that

$$\mathbb{E}\left(e^{-rt}\ S_t \mid \mathcal{F}_s\right) = \mathbb{E}\left(\overline{\mathcal{S}}_t \mid \mathcal{F}_s\right) = \overline{\mathcal{S}}_s = e^{-rs}\ S_s$$

$$\Longrightarrow \mathbb{E}\left(S_t \mid \mathcal{F}_s\right) = e^{r(t-s)}\ S_s.$$

This ends the proof of the exercise. ∎

Solution to exercise 501:
Using the elementary formula $a = a^+ - a^-$ which is valid for any real number $a \in \mathbb{R}$, we check that

$$P_t^{\text{call}} - P_t^{\text{put}} = \mathbb{E}\left(\overline{\mathcal{S}}_T \mid \overline{\mathcal{S}}_0\right) - K_T = \overline{\mathcal{S}}_0 - K_T.$$

This ends the proof of the exercise.

∎

Solution to exercise 502:
We clearly have $\mathcal{V}_2 = f$. Using the neutral probability obtained in exercise 497, we have

$$\begin{aligned}
\mathcal{V}_1(s_{0,1}) &:= \mathbb{E}\left(f(\overline{\mathcal{S}}_2) \mid \overline{\mathcal{S}}_1 = s_{0,1}\right) \\
&= f(s_{(0,1),1})\ \frac{s_{(0,1),2} - s_{(0,1)}}{s_{(0,1),2} - s_{(0,1),2}} + f(s_{(0,1),2})\ \frac{s_{(0,1)} - s_{(0,1),1}}{s_{(0,1),2} - s_{(0,1),1}}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{V}_1(s_{0,2}) &:= \mathbb{E}\left(f(\overline{\mathcal{S}}_2) \mid \overline{\mathcal{S}}_1 = s_{0,2}\right) \\
&= f(s_{(0,2),1})\ \frac{s_{(0,2),2} - s_{(0,2)}}{s_{(0,2),2} - s_{(0,2),2}} + f(s_{(0,2),2})\ \frac{s_{(0,2)} - s_{(0,2),1}}{s_{(0,2),2} - s_{(0,2),1}}.
\end{aligned}$$

Finally, for $k = 0$ we have

$$\begin{aligned}
\mathcal{V}_0(s_0) &:= \mathbb{E}\left(f(\overline{\mathcal{S}}_2) \mid \overline{\mathcal{S}}_0 = s_0\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(f(\overline{\mathcal{S}}_2) \mid \overline{\mathcal{S}}_1\right)\ \mid \overline{\mathcal{S}}_0 = s_0\right) \\
&= \mathbb{E}\left(\mathcal{V}_1(\overline{\mathcal{S}}_1) \mid \overline{\mathcal{S}}_0 = s_0\right) \\
&= \mathcal{V}_1(s_{0,1})\ \frac{s_{0,2} - s_0}{s_{0,2} - s_{0,1}} + \mathcal{V}_1(s_{0,2})\ \frac{s_0 - s_{0,1}}{s_{0,2} - s_{0,1}}.
\end{aligned}$$

This ends the proof of the first assertion. Now we turn to the self-financing portfolio strategy. The idea is to have

$$\forall 0 \leq k \leq 2 \qquad \mathcal{P}_k(b) = \mathcal{V}_k(\overline{\mathcal{S}}_k).$$

This shows that $\mathcal{P}_0(b) = \mathcal{V}_0(s_0)$, and

$$\Delta\mathcal{P}_k(b) = b_{k-1}\ \Delta\overline{S}_k = \Delta\mathcal{V}_k(\overline{\mathcal{S}}_k) = \mathcal{V}_k(\overline{\mathcal{S}}_k) - \mathcal{V}_{k-1}(\overline{\mathcal{S}}_{k-1}).$$

Considering the two cases $\overline{S}_1 \in \{s_{0,1}, s_{0,2}\}$, this implies that

$$\left\{\begin{array}{rcl} b_0\ (s_{0,1} - s_0) &=& \mathcal{V}_1(s_{0,1}) - \mathcal{V}_0(s_0) \\ b_0\ (s_{0,2} - s_0) &=& \mathcal{V}_1(s_{0,2}) - \mathcal{V}_0(s_0) \end{array}\right\} \Rightarrow b_0 = \frac{\mathcal{V}_1(s_{0,2}) - \mathcal{V}_1(s_{0,1})}{(s_{0,2} - s_{0,1})}.$$

In much the same way, if $\overline{S}_1 = s_{0,1}$, then we have

$$\left\{\begin{array}{rcl} b_1\ (s_{(0,1),1} - s_{0,1}) &=& \mathcal{V}_2(s_{(0,1),1}) - \mathcal{V}_1(s_{0,1}) \\ b_1\ (s_{(0,1),2} - s_{0,1}) &=& \mathcal{V}_2(s_{(0,1),2}) - \mathcal{V}_1(s_{0,1}) \end{array}\right\} \Rightarrow b_1 = \frac{\mathcal{V}_2(s_{(0,1),2}) - \mathcal{V}_2(s_{(0,1),1})}{(s_{(0,1),2} - s_{(0,1),1})}$$

and when $\overline{S}_1 = s_{0,2}$ we have

$$\left\{\begin{array}{rcl} b_1\ (s_{(0,2),1} - s_{0,2}) &=& \mathcal{V}_2(s_{(0,2),1}) - \mathcal{V}_1(s_{0,2}) \\ b_1\ (s_{(0,2),2} - s_{0,2}) &=& \mathcal{V}_2(s_{(0,2),2}) - \mathcal{V}_1(s_{0,2}) \end{array}\right\} \Rightarrow b_1 = \frac{\mathcal{V}_2(s_{(0,2),2}) - \mathcal{V}_2(s_{(0,2),1})}{(s_{(0,2),2} - s_{(0,2),1})}.$$

In summary, the strategy is given by

$$b_1 = \frac{\mathcal{V}_2(s_{(0,1),2}) - \mathcal{V}_2(s_{(0,1),1})}{(s_{(0,1),2} - s_{(0,1),1})}\ 1_{\overline{S}_1 = s_{0,1}} + \frac{\mathcal{V}_2(s_{(0,2),2}) - \mathcal{V}_2(s_{(0,2),1})}{(s_{(0,2),2} - s_{(0,2),1})}\ 1_{\overline{S}_1 = s_{0,1}}.$$

The initial value of the portfolio corresponds to the price of the call option. This ends the proof of the exercise. ■

Solution to exercise 503:

- We have

$$\begin{aligned} I_n &= a + b\,(I_{n-1} - a) + \sigma\, W_n = \alpha + bI_{n-1} + \sigma W_n \quad \text{with} \quad \alpha = a(1-b) \\ &= \alpha + b\,(\alpha + bI_{n-2} + \sigma W_{n-1}) + \sigma W_n \\ &= \alpha(1+b) + b^2 I_{n-2} + b\sigma W_{n-1} + \sigma W_n \\ &= \alpha(1+b) + b^2\,(\alpha + bI_{n-3} + \sigma\, W_{n-2}) + \sigma\,(b\, W_{n-1} + W_n) \\ &= \alpha(1+b+b^2) + b^3 I_{n-3} + \sigma\left(b^2 W_{n-2} + b\, W_{n-1} + W_n\right). \end{aligned}$$

Iterating this procedure, we find that

$$I_n = b^n I_0 + a(1-b) \sum_{0 \le k < n} b^k + \sigma \sum_{0 \le k < n} b^k W_{n-k} = a + b^n (I_0 - a) + \sigma \sum_{0 \le k < n} b^k W_{n-k}.$$

Notice that

$$\text{Law}\left(\sum_{0 \le k < n} b^k W_{n-k}\right) = \text{Law}\left(\sum_{0 \le k < n} b^k W_k\right) = \mathcal{N}\left(0, \sum_{0 \le k < n} b^{2k}\right) = \mathcal{N}\left(0, \frac{1-b^{2n}}{1-b^2}\right).$$

This yields

$$\lim_{n \uparrow \infty} \text{Law}\,(I_n) = \mathcal{N}\left(a, \frac{\sigma^2}{1-b^2}\right).$$

On the other hand, we have

$$\begin{aligned} \text{Law}\,(I_0 - a) = \mathcal{N}\left(0, \frac{\sigma^2}{1-b^2}\right) \Rightarrow \ \text{Law}\,(I_1 - a) &= \text{Law}\,(b\,(I_0 - a) + \sigma\, W_n) \\ &= \mathcal{N}\left(0, b^2 \tfrac{\sigma^2}{1-b^2} + \sigma^2\right) = \mathcal{N}\left(0, \tfrac{\sigma^2}{1-b^2}\right). \end{aligned}$$

The first assertion is now easily completed.

- We have

$$\begin{aligned} M_n &= b^{-n}\,(I_n - a) = (I_0 - a) + \sigma \sum_{0 \le k < n} b^{-(n-k)} W_{n-k} \\ &= (I_0 - a) + \sigma \sum_{1 \le k \le n} b^{-k} W_k = M_{n-1} + b^{-n} W_n. \end{aligned}$$

This implies that M_n is a martingale. In addition, we have

$$\begin{aligned} M_n^2 - M_{n-1}^2 &= \left(M_{n-1} + b^{-n} W_n\right)^2 - M_{n-1}^2 \\ &= 2b^{-n}\, M_{n-1}\, W_n + b^{-2n} W_n^2. \end{aligned}$$

This yields

$$\begin{aligned} \overline{M}_n - \overline{M}_{n-1} &= (1-b^2)\left[\ \left(M_n^2 - M_{n-1}^2\right) - b^{-2n}\sigma^2\right] \\ &= (1-b^2)\ b^{-n}\ \left[2M_{n-1} W_n + b^{-n}\left(W_n^2 - \sigma^2\right)\right]. \end{aligned}$$

This clearly implies that $\overline{M}_n$ is a martingale.

This ends the proof of the exercise. ■

Solution to exercise 504:

By construction, we have

$$\begin{aligned}
&\mathbb{E}(M_n \mid L_0, \ldots, L_{n-1}) \\
&= \sum_{0 \leq k \leq L_{n-1}} \left(\begin{array}{c} L_{n-1} \\ k \end{array} \right) \left(\tfrac{l_n}{l_{n-1}}\right)^k \left(1 + \tfrac{\alpha}{l_n}\right)^k \left(1 - \tfrac{l_n}{l_{n-1}}\right)^{L_{n-1}-k} \\
&= \left(\tfrac{l_n}{l_{n-1}} \left(1 + \tfrac{\alpha}{l_n}\right) + \left(1 - \tfrac{l_n}{l_{n-1}}\right)\right)^{L_{n-1}} = \left(1 + \tfrac{\alpha}{l_{n-1}}\right)^{L_{n-1}} = M_{n-1}.
\end{aligned}$$

This ends the proof of the exercise. ■

Solution to exercise 505:

In the time homogeneous settings, formula (30.6) is clearly given by

$$\overline{\mathcal{S}}^h_{t_n+h} = \overline{\mathcal{S}}^h_{t_n} \ \exp\left(-rh + \epsilon_n \sigma \sqrt{h}\right)$$

with a collection of independent $\{-1,+1\}$-valued Bernoulli random variables with common law

$$\begin{aligned}
p_h &= \mathbb{P}\left(\epsilon_n = -1\right) = \frac{e^{\sigma\sqrt{h}} - e^{rh}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} \\
q_h &= \mathbb{P}\left(\epsilon_n = +1\right) = \frac{e^{rh} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{\sigma-\sqrt{h}}}.
\end{aligned}$$

This shows that the one step transitions T_h of the Markov chain $M^h_{t_n}$ on a time step h are given for any bounded function f by the formula

$$T_h(f)(x) = f\left(x\ y_h\right)\ p_h + f\left(x\ z_h\right)\ q_h$$

with

$$y_h = e^{-rh-\sigma\sqrt{h}} \quad \text{and} \quad z_n = x\ e^{-rh+\sigma\sqrt{h}}.$$

A simple Taylor expansion of the second order gives

$$\begin{aligned}
f\left(x\ y_h\right) - f(x) &= f'(x)\ x(y_h - 1) + \frac{1}{2}\ f''(x)\ x^2\ (y_h - 1)^2 + \mathrm{O}(h\sqrt{h}) \\
f\left(x\ z_h\right) - f(x) &= f'(x)\ x(z_h - 1) + \frac{1}{2}\ f''(x)\ x^2\ (z_h - 1)^2 + \mathrm{O}(h\sqrt{h}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(z_h - 1)\ q_h &= \left[e^{-rh+\sigma\sqrt{h}} - 1\right]\ \frac{e^{rh} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{\sigma-\sqrt{h}}} \\
&= \frac{1}{2\sinh\left(\sigma\sqrt{h}\right)}\ \left\{\left[e^{\sigma\sqrt{h}} + e^{-\sigma\sqrt{h}}\right] - \left[e^{rh} + e^{-rh}\right]\right\} \\
&= \frac{1}{\sinh\left(\sigma\sqrt{h}\right)}\ \left[\cosh\left(\sigma\sqrt{h}\right) - \cosh\left(rh\right)\right],
\end{aligned}$$

and

$$\begin{aligned}
(y_h - 1)\ p_h &= \left[e^{-rh-\sigma\sqrt{h}} - 1\right]\ \frac{e^{\sigma\sqrt{h}} - e^{rh}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} \\
&= -\left[e^{-\sigma\sqrt{h}} - e^{rh}\right]\ e^{-rh}\ \frac{e^{rh} - e^{\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = -(z_h - 1)\ q_h.
\end{aligned}$$

This implies that

$$\begin{aligned}
&[T_h(f)(x) - f(x)] \\
&= [f(x\ y_h) - f(x)]\ p_h + [f(x\ z_h) - f(x)]\ q_h \\
&= \tfrac{1}{2}\ f''(x)\ x^2\ \left[(y_h-1)^2 p_h + (z_h-1)^2 q_h\right] + \mathrm{O}(h\sqrt{h}).
\end{aligned}$$

To take the final step, we observe that

$$\begin{array}{rcl}
(y_h-1)^2 p_h + (z_h-1)^2 q_h & = & [z_h - y_h]\,[z_h - 1]\ q_h \\
& = & e^{-rh}\ \left[^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}\right]\ [z_h - 1]\ q_h \\
& = & 2e^{-rh}\sinh(\sigma\sqrt{h})\ [z_h-1]\ q_h \\
& = & 2\ e^{-rh}\ \left[\cosh(\sigma\sqrt{h}) - \cosh(rh)\right] \\
& = & \sigma^2 h + \mathrm{O}(h^2).
\end{array}$$

This ends the proof of the exercise.

■

Solution to exercise 506:
By construction, we have

$$q_s(x) = P^V_{s,t}(1)(x)$$

with the Feynman-Kac semigroup

$$P^V_{s,t}(f)(x) := \mathbb{E}\left(f(X_t)\ \exp\left[-\int_s^t\ V(X_r)\ dr\right] \mid X_s = x\right).$$

By (15.31) we have

$$\partial_s P^V_{s,t}(1) = -L^V_s(P^V_{s,t}(1)) \Longleftrightarrow \partial_s q_s = -L^V_s(q_s).$$

In the first situation, we have

$$X_t = X_s + b\ (t-s) + \sigma\ (W_t - W_s).$$

This implies that

$$\begin{array}{rcl}
q_s(x) & = & \mathbb{E}\left(\exp\left[-\int_s^t\ [x + b\ (r-s) + \sigma\ (W_r - W_s)]\ dr\right] \mid X_s = x\right) \\
& = & e^{-x(t-s)-b\ (t-s)^2/2}\ \mathbb{E}\left[\exp\left[-\sigma\overline{W}_{t-s}\right]\right]
\end{array}$$

with

$$\overline{W}_{t-s} := \int_0^{t-s}\ W_r\ dr.$$

We also notice that $\overline{W}_s$ is a centered and Gaussian random variable with variance

$$\begin{array}{rcl}
\mathbb{E}\left[\left(\int_0^{t-s}\ W_r\ dr\right)^2\right] & = & \int_0^s\int_0^s\ \mathbb{E}\left[W_{r_1}W_{r_2}\right]\ dr_1 dr_2 \\
& = & 2\ \int_0^s\left[\int_0^{r_2}\ r_1\ dr_1\right]\ dr_2 = s^3/3.
\end{array}$$

The last assertion comes from symmetry arguments and the fact that

$$r_1 \leq r_2 \Rightarrow \mathbb{E}\left(W_{r_1} W_{r_2}\right) = \mathbb{E}\left(W_{r_1} W_{r_1}\right) + \overbrace{\mathbb{E}\left(W_{r_1}(W_{r_2} - W_{r_1})\right)}^{=0} = r_1.$$

This implies that

$$\mathbb{E}\left[\exp\left[-\sigma \overline{W}_{t-s}\right]\right] = \exp\left[\sigma^2\ \mathbb{E}\left(\overline{W}^2_{t-s}\right)/2\right]$$

from which we conclude that

$$\log\left[q_s(x)\right] = -x(t-s) - b\ (t-s)^2/2 + \sigma^2 (t-s)^3/6.$$

In the second case, using the exercise 255, we have

$$\begin{aligned} X_t - X_0 &= \int_0^t a\ (b - X_s)\ ds + \sigma\ W_t \\ &= abt - a \int_0^t X_s\ ds + \sigma W_t. \end{aligned}$$

This implies that

$$-\int_0^t X_s\ ds = \frac{X_t - X_0}{a} - bt - \frac{\sigma}{a} W_t.$$

On the other hand, by exercise 255 we have

$$\frac{X_t - X_0}{a} = \frac{(b - X_0)}{a}\ \left(1 - e^{-at}\right) + \frac{\sigma}{a} \int_0^t\ e^{-a(t-s)}\ dW_s.$$

This implies that

$$\begin{aligned} -\int_0^t X_s ds &= \frac{(b-X_0)}{a}\ \left(1 - e^{-at}\right) - bt\ + \frac{\sigma}{a}\left[\int_0^t\ e^{-a(t-s)}\ dW_s - W_t\right] \\ &= \frac{(b-X_0)}{a}\ \left(1 - e^{-at}\right) - bt\ - \frac{\sigma}{a} \int_0^t\ \left(1 - e^{-a(t-s)}\right)\ dW_s. \end{aligned}$$

To take the final step, observe that $\overline{W}_{s,t} := \int_0^t\ \left(1 - e^{-a(t-s)}\right)\ dW_s$ is a centered and Gaussian random variable with variance

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^t\ \left(1 - e^{-a(t-s)}\right)\ dW_s\right)^2\right] &= \int_0^t\ \left(1 - e^{-a(t-s)}\right)^2\ ds = \int_0^t\ \left(1 - e^{-as}\right)^2\ ds \\ &= t - \frac{2}{a}\left(1 - e^{-at}\right) + \frac{1}{2a}\ \left(1 - e^{-2at}\right) =: \alpha(t). \end{aligned}$$

This implies that

$$\begin{aligned} q_s(x) &= \mathbb{E}\left(\exp\left[-\int_s^t X_r dr\right] \mid X_s = x\right) = \mathbb{E}\left(\exp\left[-\int_0^{t-s} X_r dr\right] \mid X_0 = x\right) \\ &= \exp\left[\frac{(b-x)}{a}\ \left(1 - e^{-a(t-s)}\right) - b(t-s)\ \right]\ \mathbb{E}\left[\exp\left[-\frac{\sigma}{a}\ \overline{W}_{s,t}\right]\right] \\ &= \exp\left[\frac{(b-x)}{a}\ \left(1 - e^{-a(t-s)}\right) - b(t-s)\ + \frac{1}{2}\ \left(\frac{\sigma}{a}\right)^2\ \alpha(t-s)\right]. \end{aligned}$$

This ends the proof of the exercise.

■

Solution to exercise 507:
By (15.31) we have

$$\partial_s u_s(x) = -\left[L_s(u_s) - r_s u_s\right](x) = -x\ r_s(x)\partial_x u_s(x) - \frac{1}{2}\ \sigma_s(x) x^2 \partial_x^2 u_s(x) + r_s(x) u_s(x)$$

with the terminal condition $u_t = f_t$ for $s = t$. In the above display L_s stands for the generator of the diffusion X_s defined by

$$L_s(f)(x) = r_s(x)\ x\ \partial_x(f)(x) + \frac{1}{2}\ \sigma_s(x)\ x^2\ \partial_x^2(f)(x).$$

This ends the proof of the first assertion.
To check the second one, by (15.31) we have the forward equation

$$\forall t \in [s,\infty[\qquad \partial_t Q_{s,t}(f) = Q_{s,t}(L_t(f)) - r_t f$$

with the initial condition $Q_{s,s}(f) = f$. Using the fact that

$$Q_{s,t}(f)(x) = \int\ q_{s,t}(x,y)\ f(y)\ dy$$

a simple integration by parts yields

$$\begin{aligned}
\partial_t Q_{s,t}(f)(x) &= = \int\ \left[\partial_t q_{s,t}(x,y)\right]\ f(y)\ dy \\
&= \int\ q_{s,t}(x,y)\ y\ r_t(y)\ \partial_y f(y)\ dy \\
&\qquad + \frac{1}{2} \int\ q_{s,t}(x,y)\ \sigma_t(y) y^2\ \partial_y^2 f(y) - \int\ q_{s,t}(x,y)\ f(y)\ r_t(y)\ dy \\
&= -\int \partial_y\left(y\ r_t(y)\ q_{s,t}(x,y)\right)\ f(y)\ dy \\
&\qquad + \int\ \left[\frac{1}{2}\partial_y^2\left(\sigma_t(y) y^2 q_{s,t}(x,.)\right) + r_t(y) q_{s,t}(x,y)\right]\ f(y)\ dy
\end{aligned}$$

for any smooth function f with compact support. We conclude that the density function $(t,y) \mapsto q_{s,t}(x,y)$ is a weak solution of the forward equation

$$\partial_t q_{s,t}(x,y) = -\partial_y\left(y\ r_t(y)\ q_{s,t}(x,y)\right) + \frac{1}{2}\ \partial_y^2\left(\sigma_t(y) y^2 q_{s,t}(x,y)\right) - r_t(y) q_{s,t}(x,y)$$

for any $t \in [s,\infty[$, with the initial condition $q_{s,s}(x,y)dy = \delta_x(dy)$. This yields

$$\partial_z v_{s,t}(x,z) = -\int_z^\infty\ q_{s,t}(x,y)\ dz \quad \Rightarrow \quad \partial_z^2 v_{s,t}(x,z) = q_{s,t}(x,z).$$

This ends the proof of the exercise.

■