Particle Methods in Filtering & Applications in Finance

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IPAM / Financial Math
January 8-12, 2001
Partially Observed Stochastic Systems

- **State** of the system $X_n$
  \[ X_{n+1} = F(X_n, V_n) \]
- $F(x, v)$ is a known function of $x$ and $v$
- \{V_n\}_n white noise (so $X_n$ is Markov)
- **Observation** $Y_n$
  \[ Y_n = H(X_n, W_n) \]
- \{W_n\}_n white noise
- $H(x, v)$ is a known function of $x$ and $w$
  Often additive (for math. proofs) $H(x, w) = h(x) + w$
- \{V_n\}_n and \{W_n\}_n assumed independent (for math. proofs)
The Optimal Filter

Goal

Estimate, at each time $n$, the state $X_n$ from all the observations $Y_n = \{Y_n, Y_{n-1}, \cdots, Y_0\}$ up to that time.

Optimal solution (in the least squares sense)

conditional distribution of $X_n$ given all the observations $Y_n$

$$\pi_n(\cdot | Y_n) = \pi_n(\cdot | \{Y_n, Y_{n-1}, \cdots, Y_0\})$$
Stettner’s Equation

\[ \pi_{n+1} = \phi_n(\pi_n, Y_{n+1}) \]

Dynamical system in the (infinite dimensional) space of probability measures

**Conditionally Gaussian Case** Kalman-Bucy
The Particle Approximation

\[ \pi_n(dx) = \mathbb{P}\{X_n \in dx | Y_n\} \approx \frac{1}{m} \sum_{j=1}^{m} \delta_{f^j_n}(dx) \]

Need two kinds of particles:

- those used to simulate \( \mathbb{P}\{X_n | Y_n\} \)
  \[ f^1_n, \ldots, f^m_n \]

- those used to simulate \( \mathbb{P}\{X_{n+1} | Y_n\} \)
  \[ p^1_{n+1}, \ldots, p^m_{n+1} \]
One Step Ahead Prediction

- Assume

$$f_n^1, \cdots, f_n^m$$

form a random sample from the distribution $$\pi_n = \mathbb{P}\{X_n|Y_n\}$$

- Assume $$v_n^1, \ldots, v_n^m$$ are $$m$$ independent realizations of the noise $$V_n$$

Then

$$p_{n+1}^j = F(f_n^j, v_n^j)$$

gives a random sample

$$p_{n+1}^1, \cdots, p_{n+1}^m$$

from the conditional distribution $$\mathbb{P}\{X_{n+1}|Y_n\}.$$
Filtering, or *Updating*

- Assume

\[ p_{n+1}^1, \ldots, p_{n+1}^m \]

sample from the conditional distribution \( \mathbb{P}\{X_{n+1} | Y_n\} \)

- Given a new observation \( y_{n+1} \)

\[
\alpha_{n+1}^j = \mathbb{P}\{(Y_{n+1} = y_{n+1} | p_{n+1}^j) \text{ likelihood of each particle } p_{n+1}^j \}
\]

\[
\alpha_{n+1}^j = r(G(y_{n+1}, p_{n+1}^j) | \frac{\partial G}{\partial Y_n}(G(y_{n+1}, p_{n+1}^j))}
\]
\[
\mathbb{P}\{X_{n+1} = p^i_{n+1}|Y_{n+1}\} = \frac{\mathbb{P}\{X_{n+1} = p^i_{n+1}, Y_{n+1}|Y_n\}}{\mathbb{P}\{Y_{n+1}|Y_n\}} \frac{\mathbb{P}\{Y_{n+1}|p^i_{n+1}\}\mathbb{P}\{X_{n+1} = p^i_{n+1}|Y_n\}}{\sum_{j=1}^{m} \mathbb{P}\{Y_{n+1}|p^j_{n+1}\}\mathbb{P}\{X_{n+1} = p^j_{n+1}|Y_n\}} \\
= \mathbb{P}\{X_{n+1} = p^i_{n+1}|Y_n\} \frac{\alpha^i_{n+1} \cdot \frac{1}{m}}{\frac{1}{m} \sum_{j=1}^{m} \alpha^j_{n+1}} \\
= \frac{\alpha^i_{n+1}}{\sum_{j=1}^{m} \alpha^j_{n+1}}
\]

\(r\) (observation) white (additive) noise density
Given this computation we define:

\[ f_{n+1}^j = \begin{cases} 
  p_{n+1}^1 & \text{with probability } \frac{\alpha_{n+1}^1}{\alpha_{n+1}^1 + \ldots + \alpha_{n+1}^m} \\
  \vdots & \text{with probability } \frac{\alpha_{n+1}^m}{\alpha_{n+1}^1 + \ldots + \alpha_{n+1}^m} \\
  p_{n+1}^m & \text{with probability } \frac{\alpha_{n+1}^m}{\alpha_{n+1}^1 + \ldots + \alpha_{n+1}^m} 
\end{cases} \]

These particles form a random sample of the conditional distribution \( \pi_{n+1} = \mathbb{P}\{X_{n+1} \mid Y_{n+1}\} \).
Algorithm Summary

Kitagawa (1994)

1. Initialization: generate an initial random sample of $m$ particles $f_1^0, \ldots, f_m^0$.

2. For each time step $n$, we repeat the following process:
   - Generate independent particles $v_n^j$ from the distribution of the system noise
   - Generate the particles $p_{n+1}^j$ using the formula $p_{n+1}^j = F(f_n^j, v_n^j)$
   - Given a new observation, compute the likelihood $\alpha_{n+1}^j$
   - Resample the $p_{n+1}^j$ to produce the $f_{n+1}^j$'s

Proved Results For fixed $n$ particle approximation converges toward $\pi_n$. (Del Moral, Guillonnet, Lyons, Crisan, ⋅⋅⋅)
In Car Navigation Systems

The data

• Static data: representation of the network of roads and streets on which the vehicles travel

• Dynamic data: sequences of time stamped estimates of the position of the GPS receiver GPS tracks
The Street Network

A Schematic of the Streets of Princeton
An Example of a Track

Beginning Part of a Track Illustrating some of the Pitfalls
Animation Example

Z. Peng (C)
A. Bibas (matlab)
Stochastic Volatility Estimation

\[ dS_t = S_t(\mu dt + \sigma_t dW_t), \]

where \( \sigma_t \) satisfies

\[ d\sigma_t = -\lambda(\sigma_t - \sigma_0)dt + rd\tilde{W}_t. \]

Descretization:

\[ X_{t+\Delta t} = \frac{S_{t+\Delta t}}{S_t} = (1 + \mu \Delta t) + \sigma_t \sqrt{\Delta t} \epsilon_{t+\Delta t} \]

and

\[ \sigma_{t+\Delta t} \sim \mathcal{N} \left( \sigma_0 + e^{-\lambda \Delta t}(\sigma_t - \sigma_0), \sqrt{\frac{r^2}{2\lambda}(1 - e^{-2\lambda \Delta t})} \right). \]
Perfect Observation

State Equation

\[
\begin{pmatrix}
X_{t+\Delta t} \\
\sigma_{t+\Delta t}
\end{pmatrix} = \begin{pmatrix}
1 + \mu dt \\
\sigma_0 + e^{-\lambda \Delta t} (\sigma_t - \sigma_0)
\end{pmatrix} + \begin{pmatrix}
\sigma_t \sqrt{\Delta t} \epsilon_{t+\Delta t} \\
\sqrt{\frac{r^2}{2\lambda} (1 - e^{\lambda \Delta t})} \tilde{\epsilon}_{t+\Delta t}
\end{pmatrix}
\]

Observation Equation

\[
Y_t = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} X_t \\ \sigma_t \end{pmatrix}
\]
Including the Parameters in the State Space Model

State Equation

$$\begin{pmatrix} X_{t+\Delta t} \\ \sigma_{t+\Delta t} \\ \lambda_{t+\Delta t} \\ c_{t+\Delta t} \end{pmatrix} = \begin{pmatrix} 1 + \mu dt \\ \sigma_0 + e^{-\lambda t \Delta t} (\sigma_t - \sigma_0) \\ \lambda_t \\ c_t \end{pmatrix} + \begin{pmatrix} \sigma_t \sqrt{\Delta t} \epsilon_{t+\Delta t} \\ \sqrt{c_t (1 - e^{\lambda t \Delta t})} \tilde{\epsilon}_{t+\Delta t} \\ 0 \\ 0 \end{pmatrix}$$

Observation Equation

$$Y_t = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_t \\ \sigma_t \\ \lambda_{t+\Delta t} \\ c_{t+\Delta t} \end{bmatrix}$$
Noisy Observations

- $\sigma_t$ unobserved state
- $X_t$ observation

State Evolution

$$
\begin{pmatrix}
\sigma_{t+\Delta t} \\
\lambda_{t+\Delta t} \\
c_{t+\Delta t}
\end{pmatrix} =
\begin{pmatrix}
\sigma_0 + e^{-\lambda t\Delta t}(\sigma_t - \sigma_0) \\
\lambda_t \\
c_t
\end{pmatrix} +
\begin{pmatrix}
\sqrt{c_t(1 - e^{\lambda t\Delta t})}\tilde{\epsilon}_{t+\Delta t} \\
0 \\
0
\end{pmatrix}
$$

Observation Equation

$$X_{t+\Delta t} = \frac{S_{t+\Delta t}}{S_t} = (1 + \mu \Delta t) + \sigma_t \sqrt{\Delta t} \epsilon_{t+\Delta t}$$
Regime Switching Stochastic Volatility

A. Papavasiliou \( \Delta t = 0.004, \mu = 0.006, \lambda = 2, c = 0.5, \sigma_0 = 0.1 \) & \( \sigma_0 = 0.1 \)
Volatility

Simulated volatility
estimated volatility
historical volatility
Another Example
Still Another Example