A mean field theory of sequential Monte Carlo methods

P. Del Moral

INRIA, Centre Bordeaux-Sud Ouest

Department of Statistics, Oxford Univ., december 2008.

$\sim$ **Series of joint works :** *F. Cérou, D. Crisan, D. Dawson, A. Doucet, A. Guyader, J. Jacod, A. Jasra, A. Guionnet, M. Ledoux, L. Miclo, F. Patras, T. Lyons, S. Rubenthaler,...*

A pair of http-references :

$\hookrightarrow$ `Feynman-Kac formulae. Genealogical and interacting particle systems, Springer (2004),` `+ References`

$\hookrightarrow$ `DM, Doucet, Jasra. SMC Samplers.` *JRSS B (2006).*

Outline

Summary

Particle Interpretation models

- **Mathematical physics and molecular chemistry** ($\geq 1950's$) : Particle/microscopic interpretation models, particle absorption, macro-molecular chains, quantum and diffusion Monte Carlo.
- **Environmental studies and biology** ($\geq 1950's$): Population, gene evolutions, species genealogies, branching/birth and death models.
- **Evolutionary mathematics and engineering sciences** ($\geq 1970's$): Adaptive stochastic search method, evolutionary learning models, interacting stochastic grids approximations, genetic algorithms.
- **Applied Probability and Bayesian Statistics** ($\geq 1990's$): Approximating simulation technique (recursive acceptance-rejection model), Sequential Monte Carlo, `new interacting MCMC technology (Andrieu, Bercu, DM, Doucet, Jasra).`
- **Pure mathematics ($\geq 1960's$ for fluid models, $\geq 1990's$ for discrete time and interacting jump models)**: Stochastic linearization tech., mean field particle interpretations of nonlinear PDE and measure valued equations.

- **Central idea of particle/SMC/evolutionary sampling :**

$$\left\{\begin{array}{l}\textbf{Physical and Biological intuitions}\\ \textit{[learning, adaptation, optimization,...]}\end{array}\right\} \in \textbf{Engineering problems}$$

Sequential Monte Carlo	Sampling	Resampling
Particle Filters	Prediction	Updating
Genetic Algorithms	Mutation	Selection
Evolutionary Population	Exploration	Branching
Diffusion Monte Carlo	Free evolutions	Absorption
Quantum Monte Carlo	Walkers motions	Reconfiguration
Sampling Algorithms	Transition proposals	Acceptance-rejection

More botanical names : spawning, cloning, pruning, enrichment, go with the winner, and many others.

- **Pure mathematical point of view :**
  = Mean field particle interpretation of Feynman-Kac measures

Some application areas of Feynman-Kac formulae

- **Physics** :
  - Feynman-Kac-Schroedinger semigroups $\in$ nonlinear integro-differential equations ($\sim$ generalized Boltzmann models).
  - Spectral analysis of Schrödinger operators and large matrices with nonnegative entries.
  - Particle evolutions in disordered/absorbing media.
  - Multiplicative Dirichlet problems with boundary conditions.
  - Microscopic and macroscopic interacting particle interpretations.
- **Chemistry and Biology**:
  - Self-avoiding walks, macromolecular simulation, directed polymers.
  - Spatial branching and evolutionary population models.
  - Coalescent and Genealogical tree based evolutions.

Some application areas of Feynman-Kac formulae

- **Rare events analysis**:
  - Multisplitting and branching particle models (Restart type methods).
  - Importance sampling and twisted probability measures.
  - Genealogical tree based simulations (default tree sampling models).
- **Advanced Signal processing**:
  - Optimal filtering, prediction, smoothing.
  - Open loop optimal control, optimal regulation.
  - Interacting Kalman-Bucy filters.
  - Stochastic and adaptative grid approximation-models
- **Statistics/Probability**:
  - Restricted Markov chains (w.r.t terminal values, visiting regions, constraints simulation problems,...)
  - Analysis of Boltzmann-Gibbs type distributions (simulation, partition functions, localization models...).
  - Random search evolutionary algorithms, interacting Metropolis/simulated annealing algo, combinatorial counting.

Summary

Standard notation

E measurable space, $\mathcal{P}(E)$ proba. on E, $\mathcal{B}(E)$ bounded meas. functions.

- $(\mu, f) \in \mathcal{P}(E) \times \mathcal{B}(E) \quad \longrightarrow \quad \mu(f) = \int \mu(dx)\ f(x)$
- $M(x, dy)$ **integral operator on E**

$$\begin{aligned} M(f)(x) &= \int M(x, dy) f(y) \\ [\mu M](dy) &= \int \mu(dx) M(x, dy) \qquad (\Longrightarrow [\mu M](f) = \mu[M(f)]\ ) \end{aligned}$$

- **Bayes-Boltzmann-Gibbs transformation :** $G : E \to [0, \infty[$ with $\mu(G) > 0$

$$\Psi_G(\mu)(dx) = \frac{1}{\mu(G)}\ G(x)\ \mu(dx)$$

Only 3 Ingredients

- **A state space :**

  E_n with $n =$ time/level index **[transitions, paths, excursions,...].**

$$X_n := (X'_{n-1}, X'_n), \quad X'_{[0,n]}, \quad X'_{[t_{n-1},t_n]}, \quad X'_{[T_{n-1},T_n]}, \cdots$$

- **A Markov Proposal/Exploration/Mutation transition :**

$$M_n(x_{n-1}, dx_n) := \mathbb{P}\left(X_n \in dx_n \mid X_{n-1} = x_{n-1}\right)$$

- **A potential/likelihood/fitness/weight function on E_n :**

$$G_n \ : \ x_n \in E_n \ \longrightarrow \ G_n(x_n) \in [0, \infty[$$

Running Examples :

- [Confinement] X_n=Simple random walk (SRW) on $E_n = \mathbb{Z}$ and $G_n = 1_A$.
- [Filtering] M_n=signal transitions, G_n=Likelihood weight function.

SMC/Genetic type branching particle model :

$$\begin{bmatrix} \xi_n^1 \\ \vdots \\ \xi_n^i \\ \vdots \\ \xi_n^N \end{bmatrix} \xrightarrow{\text{Updating/Branching}} \begin{bmatrix} \widehat{\xi}_n^1 \\ \vdots \\ \widehat{\xi}_n^i \\ \vdots \\ \widehat{\xi}_n^N \end{bmatrix} \begin{array}{c} \xrightarrow{\quad M_{n+1} \quad} \\ \vdots \\ \xrightarrow{\text{Prediction/Exploration}} \\ \vdots \\ \longrightarrow \end{array} \begin{bmatrix} \xi_{n+1}^1 \\ \vdots \\ \xi_{n+1}^i \\ \vdots \\ \xi_{n+1}^N \end{bmatrix}$$

Selection/Branching : $(\forall \epsilon_n \geq 0 \text{ s.t. } \epsilon_n(x^1, \ldots, x^N) \times G_n(x^i) \in [0,1])$

- **Acceptance probability:**

$$\widehat{\xi}_n^i = \xi_n^i \quad \text{with probability} \quad \epsilon_n(\xi_n^1, \ldots, \xi_n^N) \; G_n(\xi_n^i)$$

- **Otherwise :**

$$\widehat{\xi}_n^i = \xi_n^j \quad \text{with probability} \quad \frac{G_n(\xi_n^j)}{\sum_{k=1}^N G_n(\xi_n^k)}$$

Running examples: [Confinement & Filtering] = $[(G_n = 1_A)$ & $(G_n = \text{Likelihood})]$.

Some remarks :

- $\epsilon_n = 0 \implies$ *Simple Mutation-Selection Genetic model.*
- $G_n = \exp\{-V_t \Delta t\}$ & $\epsilon_n = 1 \implies$ *V_t-expo-clocks* $\oplus$ uniform selection
- $G_n \in [0,1]$ & $\epsilon_n = 1 \Rightarrow$ *Interacting Acceptance-Rejection Sampling.*
- Better fitted individuals acceptance :

$$\text{For} \qquad \epsilon_n(x^1, \ldots, x^N) G_n(x^i) = G_n(x^i) / \sup_{1 \leq j \leq N} G_n(x^j)$$

- Related branching rules:

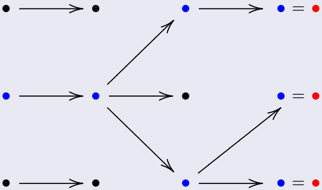
  [DM-Crisan-Lyons MPRF 99, DM 04] (Given $\xi_n = (\xi_n^i)_i$)

$$P_n^i := \text{Proportion of offsprings of the individual } \xi_n^i$$

  - Unbiasedness property : $\mathbb{E}\left(P_n^i\right) = G_n(\xi_n^i) / \sum_{k=1}^N G_n(\xi_n^k)$
  - Local mean error : $\mathbb{E}\left(\left[\sum_{i=1}^N \left[P_n^i - \mathbb{E}\left(P_n^i\right)\right] f(\xi_n^i)\right]^2\right) \leq \frac{Cte}{N}$

Interacting-Branching proc. $\hookrightarrow$ 3 Particle/SMC occupation measures:

$(N = 3)$



- Current population $\hookrightarrow \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_n^i} \leftarrow$ *i*-th individual at time *n*
- Historical/genealogical tree $\hookrightarrow \frac{1}{N}\sum_{i=1}^{N} \delta_{\left(\xi_{0,n}^i,\xi_{1,n}^i,\ldots,\xi_{n,n}^i\right)} \leftarrow$ *i*-th ancestral line
- **Complete genealogical tree** $\hookrightarrow \frac{1}{N}\sum_{i=1}^{N} \delta_{\left(\xi_0^i,\xi_1^i,\ldots,\xi_n^i\right)}$
- $\oplus$ *Mean potential values [Success proportions $(G_n = 1_A)$]* $\hookrightarrow \frac{1}{N}\sum_{i=1}^{N} G_n(\xi_n^i)$

Limiting measures ("Test" functions $f : E_n \to \mathbb{R}$)

- **Occupation measures of the Current population**

$$\eta_n^N(f) := \frac{1}{N}\sum_{i=1}^{N} f(\xi_n^i) \longrightarrow_{N\uparrow\infty} \eta_n(f) := \frac{\gamma_n(f)}{\gamma_n(1)}$$

with the Feynman-Kac measures (X_n Markov with transitions M_n):

$$\gamma_n(f) := \mathbb{E}\left(f_n(X_n) \prod_{0\leq p<n} G_p(X_p)\right)$$

- *Running examples :*
  - *Confinement $G_n = 1_A$:*

$$\gamma_n(1) = \mathbb{P}\left(\forall 0 \leq p < n \quad X_p \in A\right) \quad \& \quad \eta_n = \mathrm{Law}\left(X_n \mid \forall 0 \leq p < n \quad X_p \in A\right)$$

  - *Filtering: G_n=Likelihood function :*

$$\gamma_n(1) = p_n(y_0, \ldots, y_{n-1}) \quad \& \quad \eta_n = \mathrm{Law}\left(X_n \mid Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1}\right)$$

Limiting mean potential/success proportions ($G_n = 1_A$)

$$\eta_n^N(G_n) := \frac{1}{N}\sum_{i=1}^{N} G_n(\xi_n^i) \longrightarrow_{N\uparrow\infty} \eta_n(G_n) \stackrel{def.}{=} \frac{\gamma_n(G_n)}{\gamma_n(1)} = \frac{\gamma_{n+1}(1)}{\gamma_n(1)} \tag{1}$$

$\Rightarrow$**Unbiased estimate of the normalizing cts/partition functions :**

$$\gamma_n^N(1) := \prod_{0\leq p<n} \eta_p^N(G_p) \longrightarrow_{N\uparrow\infty} \gamma_n(1) = \prod_{0\leq p<n} \eta_p(G_p)$$

with the key product formula :

$$(1) \quad \Longrightarrow \quad \gamma_n(1) := \mathbb{E}\left(\prod_{0\leq p<n} G_p(X_p)\right) = \prod_{0\leq p<n} \eta_p(G_p)$$

Running ex. : $[X_n$ SRW & $G_n = 1_A]$

$$\prod_{0\leq p<n} \textbf{(Success proportion time p)} \simeq \mathbb{P}\left(\forall 0 \leq p < n \quad X_p \in A\right)$$

- **Occupation measures of the historical/genealogical tree**

$$\eta_n^N(f_n) := \frac{1}{N}\sum_{i=1}^{N} f_n\left(\xi_{0,n}^i, \xi_{1,n}^i, \ldots, \xi_{n,n}^i\right) \longrightarrow_{N\uparrow\infty} \eta_n(f_n) := \frac{\gamma_n(f_n)}{\gamma_n(1)}$$

with the Feynman-Kac measures on path space :

$$\gamma_n(f_n) := \mathbb{E}\left(f_n(X'_0, \ldots, X'_n) \prod_{0\leq p<n} G_p(X'_0, \ldots, X'_p)\right)$$

- *Running examples :* $X_n = (X'_0, \ldots, X'_n)$ SRW & $G_n(X_n) = 1_A(X'_n)$

$$\begin{aligned} \gamma_n(1) &= \mathbb{P}\left(\forall 0 \leq p < n \quad X'_p \in A\right) \\ \eta_n &= \mathrm{Law}\left((X'_0, \ldots, X'_n) \mid \forall 0 \leq p < n \quad X'_p \in A\right) \end{aligned}$$

Filtering $\rightsquigarrow \eta_n = \mathrm{Law}\left((X'_0, \ldots, X'_n) \mid Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1}\right)$

Limiting measures
("Test" function on path space $F_n : (E_0 \times \ldots \times E_n) \rightarrow \mathbb{R}$)

- **Occupation measures of the complete genealogical tree ($\epsilon_n = 0$)**

$$\frac{1}{N}\sum_{i=1}^{N} F_n\left(\xi_0^i, \xi_1^i, \ldots, \xi_n^i\right) \longrightarrow_{N\uparrow\infty} (\eta_0 \otimes \ldots \otimes \eta_n)(F_n)$$

with the Feynman-Kac tensor product measures :

$$(\eta_0 \otimes \ldots \otimes \eta_n)(F_n) = \int_{E_0} \ldots \int_{E_n} \eta_0(dx_0) \ldots \eta_n(dx_n)\ F_n(x_0, \ldots, x_n)$$

- Acceptance parameter $\epsilon_n \neq 0 \rightsquigarrow \neq$ Limiting McKean measures.

$$\eta_n = \mathrm{Law}(\overline{X}_n) \quad \text{with Markov transition} \quad \overline{X}_n \overset{\eta_n}{\rightsquigarrow} \overline{X}_{n+1}$$

Interacting-Branching model $=$ Mean-field interpretation of $\overline{X}_n$

Summary-Conclusions

SMC/Genetic type branching/particle model

[M_n-**free exploration** $\oplus$ G_n-**weighted branchings/adaptation**]

$\Downarrow$ & $\Uparrow$

Feynman-Kac measures

[M_n-**free motion** $\oplus$ G_n-**potential functions**]

Summary

Some "wrong" approximation ideas

- "Pure" weighted Monte Carlo methods : X^i iid copies of X

$$\frac{1}{N}\sum_{i=1}^{N} f_n(X_n^i)\left\{\prod_{0\leq p<n} G_p(X_p^i)\right\} \simeq \mathbb{E}\left(f_n(X_n)\prod_{0\leq p<n} G_p(X_p)\right)$$

$\rightsquigarrow$ bad grids X^i $\oplus$ degenerate weights (running ex $G_n = 1_A$).

- Uncorrelated MCMC for each target measure η_n ($\uparrow$ complex.).
- "Pure" branching $\rightsquigarrow$ critical random population sizes

$$G_n(x) = \mathbb{E}(g_n(x)) \quad \text{with} \quad g_n(x) \text{ r.v. } \in \mathbb{N}$$

- Harmonic/(Gaussian+linearisation) approximations.
- $G.M(H) \propto H \rightsquigarrow G \propto H/M(H) \rightsquigarrow H$-process X^H (unknown).

Nonlinear distribution flows

Evolution equation: [$\eta_n \in \mathcal{P}(E_n)$ probability measures $\uparrow$ complexity].

$$\eta_{n+1} = \Phi_{n+1}(\eta_n) = \Psi_{G_n}(\eta_n) M_{n+1}$$

With only 2 transformations:

- X-Free Markov transport eq. : [$M_n(x_{n-1}, dx_n)$ from E_{n-1} into E_n]

$$(\eta_{n-1} M_n)(dx_n) := \int_{E_{n-1}} \eta_{n-1}(dx_{n-1})\ M_n(x_{n-1}, dx_n)$$

- Bayes-Boltzmann-Gibbs transformation :

$$\Psi_{G_n}(\eta_n)(dx_n) := \frac{1}{\eta_n(G_n)}\ G_n(x_n)\ \eta_n(dx_n)$$

Nonlinear distribution flows

- **Nonlinear Markov models :** always $\exists K_{n,\eta}(x, dy)$ Markov s.t.

$$\eta_n = \Phi_n(\eta_{n-1}) = \eta_{n-1} K_{n,\eta_{n-1}} = \mathrm{Law}\left(\overline{X}_n\right)$$

i.e. :

$$\mathbb{P}(\overline{X}_n \in dx_n \mid \overline{X}_{n-1}) = K_{n,\eta_{n-1}}(\overline{X}_{n-1}, dx_n)$$

Mean field particle interpretation

- Markov chain $\xi_n = (\xi_n^1, \ldots, \xi_n^N) \in E_n^N$ s.t.

$$\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq_{N \uparrow \infty} \eta_n$$

- Particle approximation transitions ($\forall 1 \leq i \leq N$)

$$\xi_{n-1}^i \rightsquigarrow \xi_n^i \sim K_{n,\eta_{n-1}^N}(\xi_{n-1}^i, dx_n)$$

Discrete generation mean field particle model

Schematic picture : $\xi_n \in E_n^N \rightsquigarrow \xi_{n+1} \in E_{n+1}^N$

$$
\begin{array}{ccc}
\xi_n^1 & \xrightarrow{\qquad K_{n+1,\eta_n^N} \qquad} & \xi_{n+1}^1 \\
\vdots & & \vdots \\
\xi_n^i & \xrightarrow{\qquad\qquad\qquad} & \xi_{n+1}^i \\
\vdots & & \vdots \\
\xi_n^N & \xrightarrow{\qquad\qquad\qquad} & \xi_{n+1}^N
\end{array}
$$

Rationale :

$$
\begin{array}{rcl}
\eta_n^N \simeq_{N\uparrow\infty} \eta_n & \Longrightarrow & K_{n+1,\eta_n^N} \simeq_{N\uparrow\infty} K_{n+1,\eta_n} \\
& \Longrightarrow & \xi_n^i \text{ almost iid copies of } \overline{X}_n
\end{array}
$$

Advantages

- Mean field model=Stoch. linearization/perturbation tech. :

$$\eta_n^N = \Phi_n(\eta_{n-1}^N) + \frac{1}{\sqrt{N}}\ W_n^N$$

  with $W_n^N \simeq W_n$ independent and centered Gauss field.

- $\eta_n = \Phi_n(\eta_{n-1})$ stable $\Rightarrow$ local errors do not propagate

$$\Longrightarrow \text{uniform control of errors w.r.t. the time parameter}$$

- "No need" to study the cv of equilibrium of MCMC models.
- Adaptive stochastic grid approximations
- Take advantage of the nonlinearity of the system to define beneficial interactions. Non intrusive methods.
- Natural and easy to implement, etc.

Summary

1. Foundations & application areas
2. A simple mathematical model
3. Mean field particle models
4. Some theoretical aspects
   - Intuitive picture
   - Non asymptotic results (bias, $\mathbb{L}_p$ and exponential estimates)
   - A stochastic perturbation model $\Leftrightarrow$ Uniform estimates w.r.t. time
   - Asymptotic results (+ sketched proof of a functional CLT)
   - Unnormalized particle models

Feynman-Kac (nonlinear) semigroup $\eta_p \longrightarrow \Phi_{p,n}(\eta_p) := \eta_n$

LOCAL FLUCTUATION THEOREM : $W_n^N := \sqrt{N}\,\left[\eta_n^N - \Phi_n\left(\eta_{n-1}^N\right)\right] \simeq W_n$ **Centered and Independent Gaussian field**

Local transport formulation :

$$
\begin{array}{ccccccccc}
\eta_0 & \rightarrow & \eta_1 = \Phi_1(\eta_0) & \rightarrow & \eta_2 = \Phi_{0,2}(\eta_0) & \rightarrow & \cdots & \rightarrow & \Phi_{0,n}(\eta_0) \\
\Downarrow & & & & & & & & \\
\eta_0^N & \rightarrow & \Phi_1(\eta_0^N) & \rightarrow & \Phi_{0,2}(\eta_0^N) & \rightarrow & \cdots & \rightarrow & \Phi_{0,n}(\eta_0^N) \\
 & & \Downarrow & & & & & & \\
 & & \eta_1^N & \rightarrow & \Phi_2(\eta_1^N) & \rightarrow & \cdots & \rightarrow & \Phi_{1,n}(\eta_1^N) \\
 & & & & \Downarrow & & & & \\
 & & & & \eta_2^N & \rightarrow & \cdots & \rightarrow & \Phi_{2,n}(\eta_2^N) \\
 & & & & & & & & \vdots \\
 & & & & & & \Downarrow & & \\
 & & & & & & \eta_{n-1}^N & \rightarrow & \Phi_n(\eta_{n-1}^N) \\
 & & & & & & & & \Downarrow \\
 & & & & & & & & \eta_n^N
\end{array}
$$

$\rightsquigarrow$ **Key decomposition formula entering the stability of the limiting system:**

$$
\begin{aligned}
\eta_n^N - \eta_n &= \sum_{q=0}^{n} \left[\Phi_{q,n}(\eta_q^N) - \Phi_{q,n}(\Phi_q(\eta_{q-1}^N))\right] \text{ "} \simeq \text{" } \sum_{q=0}^{n} \frac{1}{\sqrt{N}}\, e^{-\lambda\,(n-q)} \\
&\simeq \frac{1}{\sqrt{N}} \sum_{q=0}^{n} W_q^N D_{q,n} \hookleftarrow \text{First order decomp. } \Phi_{p,n}(\eta) - \Phi_{p,n}(\mu) \simeq (\eta-\mu)D_{p,n} + (\eta-\mu)^{\otimes 2} \ldots
\end{aligned}
$$

$\Rightarrow$ **Two lines proof of a Functional CLT :** $\sqrt{N}\left[\eta_n^N - \eta_n\right] \simeq \sum_{q=0}^{n} W_q D_{q,n}$

Non asymptotic weak/bias estimates

Weak estimates $\leftrightarrow$ Bias estimates ($\leftrightarrow$ Propagations of chaos)

Law(q particles among N at time n) $\simeq_{N\uparrow\infty}$ Law(q iid r.v. w.r.t. η_n)

1. Total variation $= \frac{q^2}{N} c(n)$, Boltzmann entropy $= \frac{q}{N} c(n)$.
2. Stable models: uniform estimates w.r.t. time $\sup_n c(n) < \infty$.
3. Path space and genealogical tree models $c(n) = c \times n$.
4. Explicit weak decompositions at any order $\frac{1}{N^k}$.

   $\hookrightarrow$`http-ref : DM-Patras-Rubenthaler, Coalescent tree based functional representations for some Feynman-Kac particle models, Hal-INRIA (2006).`

Some crude uniform estimates w.r.t. time

Hypothesis : (Time homogeneous models) $\exists(m,r)$ s.t. for any (x,y)

$$M^m(x,.) \ \geq\ \epsilon\, M^m(y,.) \quad \text{and} \quad G_n(x)\ \leq\ r\ G_n(y)$$

- **Limiting system stability properties :**

$$\|\Phi_{p,p+nm}(\eta)-\Phi_{p,p+nm}(\mu)\|_{tv}\ \leq\ (1-\epsilon^2/r^{m-1})^n$$

  and w.r.t. Csiszár's *H*-entropy criteria

$$H(\Phi_{p,p+nm}(\mu),\Phi_{p,p+nm}(\eta))\ \leq\ \alpha_H(r^m/\epsilon)\ (1-\epsilon^2/r^{m-1})^n\ H(\mu,\eta)$$

- **Examples :**

  $\alpha_H(t)=t$ (tv norm & Boltzmann entropy), $\alpha_H(t)=t^{1+p}$ (Havrda-Charvat & Kakutani-Hellinger p-integrals, $\alpha_H(t)=t^3$ ($\mathbb{L}_2$-norm),...

Some crude uniform estimates w.r.t. time

Hypothesis : (Time homogeneous models) $\exists (m,r)$ s.t. for any (x,y)

$$M^m(x,.) \;\geq\; \epsilon\; M^m(y,.) \quad \text{and} \quad G_n(x) \;\leq\; r\; G_n(y)$$

- $\mathbb{L}_p$**-mean error bounds**

$$\sup_{n\geq 0}\; \sup_{N\geq 1} \sqrt{N}\; \mathbb{E}\left(\left|\left[\eta_n^N - \eta_n\right](f)\right|^p\right)^{\frac{1}{p}} \leq 2\; b(p)\; m\; r^{2m-1}/\epsilon^3$$

- **Uniform concentration estimates :**

$$\sup_{n\geq 0} \mathbb{P}\left(\left|\left[\eta_n^N - \eta_n\right](f)\right| \geq\; \delta\right) \leq 6\; \exp\left(-N\; \delta^2\; \epsilon^5/(32 m r^{4m-1})\right)$$

- **Extensions to Zolotarev's seminorms** $\|\eta_n^N - \eta_n\|_{\mathcal{F}} = \sup_{f\in\mathcal{F}} |[\eta_n^N - \eta_n](f)|$

Fluctuations and large deviations

- **Central Limit Theorems** [Sharp $\mathbb{L}_p$ estimates]

`{http-ref : 1999`$\rightsquigarrow$`2004 : DM, Guionnet, Jacod, Ledoux, Tindel}`

$$V_n^N(f) := \sqrt{N}\ \left[\eta_n^N(f) - \eta_n(f)\right] \Longrightarrow V_n(f) = \text{Centered Gaussian r.v.}$$

1. **Functional Central Limit Theorems.** $[\forall d,\ \forall (f^i)_{1\leq i\leq d}]$

$$(V_n^N(f^1), \ldots, V_n^N(f^d)) \Longrightarrow \left(V_n(f^1), \ldots, V_n(f^d)\right)$$

2. Empirical processes $\rightsquigarrow$ Donsker type theorems.
3. Convergence rates $\rightsquigarrow$ Berry Esseen type theorems.
4. Path space models (Complete tree and genealogical tree).

- **Large deviations principles** [Sharp asymptotic expo estimates]

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(\eta_n^N \not\in \mathcal{V}(\eta_n)\right)$$

Example : $\mathcal{V}(\eta_n) = \{\mu\ :\ |\eta_n^N(f) - \eta_n(f)| \leq \epsilon\}$ (weak and strong τ-topo).

`{http-ref 1998`$\rightsquigarrow$`2004 : DM, Dawson, Guionnet, Zajic}`

Unnormalized particle models $\sim$ the key product formula

$$\gamma_n(f) := \mathbb{E}\left(f(X_n) \prod_{0\leq p<n} G_p(X_p)\right) = \eta_n(f) \prod_{0\leq p<n} \eta_p(G_p)$$

$$\simeq \eta_n^N(f) \prod_{0\leq p<n} \eta_p^N(G_p) := \gamma_n^N(f)$$

- CLT & LDP [+ *A. Guionnet (*`AAP 99`, `SPA 98`*) & L. Miclo* (`SP 2000`)]
- Polynomial expansion *(+ F. Patras & S. Rubenthaler* (`AAP 09`) :

$$\mathbb{E}\left((\gamma_n^N)^{\otimes q}(F)\right) =: \mathbb{Q}_{n,q}^N(F) = \gamma_n^{\otimes q}(F) + \sum_{1\leq k\leq (q-1)(n+1)} \frac{1}{N^k}\, \partial^k \mathbb{Q}_{n,q}(F)$$

- Variance estimates *(+ F. Cerou & A. Guyader* `Hal-INRIA nov.08`*)* :

$$\mathbb{E}\left(\left[\gamma_n^N(f_n) - \gamma_n(f_n)\right]^2\right) \leq c\, \frac{n}{N} \times \gamma_n(1)^2$$

Adaptive resampling times $\sim$ Continuous time particle models

- Continuous time models = "uniform" resampling at stochastic exponential times (Self adaptive resampling strategy)
- Discrete time models :

$$\text{Random resampling times } T_n^N \sim \text{empirical adaptive criteria } C_n^N$$

$$\longrightarrow_{N\uparrow\infty}$$

$$\text{Deterministic resampling times } T_n \sim \text{limiting adaptive criteria } C_n$$

Theorem [+ A. Doucet & A. Jasra, `HAL-INRIA 08`] :

$$(T_1^N,\ T_2^N,\ \ldots, T_n^N) = (T_1,\ T_2,\ \ldots, T_n) \quad \text{up to an expo small event}$$

$$\Downarrow$$

Direct consequences : Exponential estimates, functional CLT, $\mathbb{L}_p$-rates,....