LIMIT THEOREMS FOR SOME BRANCHING MEASURE-VALUED PROCESSES

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ABSTRACT. We consider a particles system, where, the particles move independently according to a Markov process and branching event occurs at an inhomogeneous time. The offspring locations and their number may depend on the position of the mother. Our setting capture, for instance, the processes indexed by Galton-Watson tree. We first determine the asymptotic behaviour of the empirical measure. The proof is based on an expression of the empirical measure using an auxiliary process. This latter is not distributed as a one cell lineage, there is a biased phenomenon. Our model is a microscopic description of a random (discrete) population of individuals. We then obtain a large population approximation as weak solution of a growth-fragmentation equation. We illustrate our result with two examples. The first one is a size-structured population model which describes the mitosis and the second one can model a parasite infection.

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1. INTRODUCTION AND STATEMENT OF RESULT

This work is devoted to a continuous time model for dividing cells already studied in [2, 4, 5, 7, 29]. This model comes from biology and physic, we can interpret it as the size of cells or polymers. In [5], it is proved that X can represent the growth of some biological content of the cell (nutriments, parasites...). With biological reference, it is also explained why the division time must depend of the motion. A long time behaviour for a similar discrete model is developed in [18]. The proof is based on a many-to-one formula and an auxiliary process. In [24], we get a law of large number for long time for a model with a continuum population. The proof is based on a spectral analysis and an auxiliary process.

Let us begin by describe our model. Let E be a Polish space. We start with one cell that have a weight $x_0 \in E$. For each cell u, its weight X^u evolves as a càdlàg strong Markov process $(X_t)_{t\geq 0}$, until it dies , an event such that

$$\int_{\alpha(u)}^{\beta(u)} r(X^u_s)\, ds \sim \text{ Exp}(1)$$

where $\alpha(u), \beta(u)$ are respectively the birth date and the death date of the cell u. r is a non-negative, measurable and locally bounded function. The cell u is then replaced by a random number K of offsprings, that follows a law $(p_k(X^u_{\beta(u)-}))_{k\in\{1,\dots,\bar{k}\}}$, on $\{1, \dots, \bar{k}\}$, which depends of the mother's weight. The states of the offspring are given

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by $(F_j^{(K)}(X^u_{\beta(u)-},\Theta))_{1\le j\le K}$, where Θ is a uniform variable on $[0,1]$, and, $(F_j^{(k)})_{j\le k,k\in\mathbb{N}}$ a family of measurable functions. The new born branches evolve then independently from each other. Let $m=\sum_{k\ge1}kp_k$ be the mean of news offspring, we always assume $m>1$ (supercrical case).

Before giving the main and general result, let us give an example. This models a size-structured population which represents the cell mitosis. It is described as follows: X is a deterministic and linear function and, when a cell dies, it divides in two equal parts. Formally,

$$E=[0,+\infty),\ \forall f\in C^1,\ Af=f' \text{ and } p_2=1. \tag{1}$$

$$\forall x\in E,\ \forall\theta\in[0,1],\ F_1^{(2)}(x,\theta)=F_2^{(2)}(x,\theta)=\frac{x}{2}. \tag{2}$$

In this case, one cell lineage is generated by:

$$\forall f\in C^1,\forall x\ge0,\ Gf=f'(x)+r(x)\left[f\left(\frac{x}{2}\right)-f(x)\right].$$

This process have some application in computer science, it is sometimes called the TCP (Transmission Control Protocol) process. The emergence of TCP has spurred an enormous amount of research, we refer to [11, 23, 28, 37, 44] for some result about approximation, long time behaviour or moments estimates. Our main result about this model is :

Theorem 1.1 (Convergence of the empirical measure for a mitosis model)**.** *Assume (1-2). If there exists $\underline{r},\bar{r}$, such that $0<\underline{r}\le r\le\bar{r}$ and $r(x)$ is constant equal to $\bar{r}$ for a large enough x, then there exists a probability measure π such that*

$$\lim_{t\to+\infty}\frac{1}{N_t}\sum_{u\in V_t}g(X_t^u)=\int g\,d\pi$$

where the convergence holds in probability and for any continuous and bounded function g. In particular for a constant rate r, π has Lebesgue density:

$$x\mapsto\frac{2r}{\prod_{n=1}^{+\infty}(1-2^{-n})}\sum_{n=0}^{+\infty}\left(\prod_{k=1}^{n}\frac{2}{1-2^k}\right)e^{-2^{n+1}rx}. \tag{3}$$

The explicit formula (3) is not new [45, 46], but here, we have a convergence, in probability, of the empirical measure instead a convergence for the mean measure . We give an analogue result for r affine (see proposition 4.4).

When the rate r is constant, the process is simpler to study. For instance, we can calculate the moment (see proposition 4.6). We also obtain a speed of convergence for $Z_t=\sum_{u\in V_t}\delta_{X_t^u}$, the measure which describes the population. Let us explain how we estimate the distance between two random measure M_1,M_2. We embed the space of random measure with the Wasserstein distance [49, 54], defined by

$$W_d^{(p)}(\mathcal{L}(M_1),\mathcal{L}(M_2))=(\inf\mathbb{E}[d(M_1,M_2)^p])^{1/p}$$

where the infimum runs over all couples (M_1,M_2) such that $M_1\sim\mathcal{L}(M_1)$ and $M_2\sim\mathcal{L}(M_2)$. d is a distance on the measure and $\mathcal{L}(\cdot)$ stands for the law of the random variable. We take $d=W_{|\cdot|}^{(1)}=W_{|\cdot|}$ is the Wasserstein distance on $(E,|\cdot|)$. And we have:

Theorem 1.2 (Quantitative bounds)**.** *Under the same assumptions of theorem 1.1 and if r is constant, we get, for every $t\ge0$,*

$$W^{(1)}_{W_{|\cdot|}}\left(\mathcal{L}\left(\frac{Z_t^x}{N_t}\right),\mathcal{L}\left(\frac{Z_t^y}{N_t}\right)\right)\le|x-y|e^{-rt}$$
$$W^{(1)}_{W_{|\cdot|}}\left(\mathcal{L}\left(\frac{Z_t^x}{\mathbb{E}[N_t]}\right),\mathcal{L}\left(\frac{Z_t^y}{\mathbb{E}[N_t]}\right)\right)\le|x-y|e^{-rt}$$

where Z^x (resp. Z^y) is the empirical measure starting with one cell that have the weight x (resp. y) in $[0,+\infty)$.

The proof is based on coupling and matching arguments. This result does not give a bound of $W^{(1)}_{W_{|\cdot|}}\left(\mathcal{L}\left(Z_t^x/\mathbb{E}[N_t]\right),\mathcal{L}(\pi)\right)$ or $W^{(1)}_{W_{|\cdot|}}\left(\mathcal{L}\left(Z_t^x/N_t\right),\mathcal{L}(\pi)\right)$, where π is the limit measure of the theorem 1.1 (see remark 4.7).

To obtain a limit theorem, we follow the approach of [4]. In this paper, the cell's death rate r is constant and the law of offspring $(p_k)_{k\ge1}$ do not depend to the mother. A many-to-one formula, which looks like the Wald formula, is proved:

$$\frac{1}{\mathbb{E}[N_t]}\mathbb{E}\left[\sum_{u\in V_t}f(X_t^u)\right]=\mathbb{E}[f(Y_t)]. \tag{4}$$

Where V_t denote the set of the cell alive at time t, $N_t = \text{card}(V_t)$ and Y is an auxiliary process with infinitesimal generator

$$\forall f \in \mathcal{D}(A),\ \forall x \in E,\ Gf(x) = Af(x) + rm \sum_{k\geq 1} \frac{kp_k}{m} \int_0^1 \left(\frac{1}{k} \sum_{j=1}^{k} f(F_j^{(k)}(x,\theta)) - f(x) \right) d\theta \tag{5}$$

where $(A, \mathcal{D}(A))$ is the generator of X. This process evolves as X, until it jumps, at an exponential time with mean $1/rm$. We observe that r is not the jump rate of the auxiliary process. There is a biased phenomenon, already described in [4, 29] and their references. We can interpret it by the fact that the faster the cells divide, the more descendants they have. That is why a uniformly chosen individual has an accelerated rate of division. It is like the bus paradox already observed for the Poisson process. A possible generalisation of (4) is a Feynman-Kac interpretation as in [17, 29]:

$$\mathbb{E}\left[\sum_{u\in V_t} f(X_t^u)\right] = \mathbb{E}\left[f(Y_t) e^{\int_0^t r(Y_s)(m(Y_s)-1)ds}\right]$$

where Y is an auxiliary process starting at x_0 and generated by (5). An other formula with Poisson measure is given in [5] to prove criterion for extinction. However, it is difficult to exploit these formulas. In this paper, we follow an alternative approach, which is inspired by [36, 45, 46]. In the expression (4), Y can be understood as a uniformly chosen individual. The problem is, if r is not constant, a uniformly chosen individual is not a Markov process. Our solution is to choose this individual, with an appropriate weight which gives a Markov process. This weight is the eigenvector of the following operator which is not a Markovian generator,

$$\tilde{A}f(x) = Af(x) + r(x) \left[\left(\sum_{k\geq 0} \sum_{j=1}^{k} \int_0^1 f(F_j^{(k)}(x,\theta))\, d\theta\, p_k(x) \right) - f(x) \right].$$

Under some assumptions, which are given thereafter, we have the following many-to-one formula:

$$\frac{1}{\mathbb{E}[\sum_{u\in V_t} V(X_t^u)]} \mathbb{E}\left[\sum_{u\in V_t} f(X_t^u) V(X_t^u)\right] = \mathbb{E}[f(Y_t)] \tag{6}$$

where Y is an auxiliary Markov process, starting at x_0, generated by

$$\mathcal{G}f(x) = Bf(x) + \Lambda(x) \left[\frac{\sum_{k\in\mathbb{N}} \sum_{j=1}^{k} \int_0^1 V(F_j^{(k)}(x,\theta)) f(F_j^{(k)}(x,\theta))\, d\theta\, p_k(x)}{\sum_{k\in\mathbb{N}} \sum_{j=1}^{k} \int_0^1 V(F_j^{(k)}(x,\theta))\, d\theta\, p_k(x)} - f(x) \right] \tag{7}$$

where

$$Bf(x) = \frac{A(f\times V)(x) - f(x)AV(x)}{V(x)} = \frac{2\Gamma_A(f,V)(x)}{V(x)} + Af(x)$$

and Γ_A is the "carré du champs"operator associated to A (see (12)) and

$$\Lambda(x) = \left[\sum_{k\in\mathbb{N}} \sum_{j=1}^{k} \int_0^1 V(F_j^{(k)}(x,\theta))\, d\theta\, p_k(x) \right] \times \frac{r(x)}{V(x)}.$$

Let us further agree to call $\mathcal{E}$ a determining class if two probability measures P, Q are identical whenever they agree on $\mathcal{E}$.

Theorem 1.3 (Weighted many-to-one formula)**.** *If*

- *for all* $t \geq 0,\ N_t < +\infty$ *a.s.*
- $\tilde{A}$ *have eigenelements* (V, λ_0) *with a positive* V
- $\mathcal{G}$ *generate a non explosive strong Markov process*
- $\mathcal{D}_b(\mathcal{G}) = \{f \in \mathcal{D}(A) \mid \forall x \in E,\ |\mathcal{G}f(x)| \leq 1\}$ *is a determining class.*

then (6) holds for any non negative and measurable function f*.*

This formula seems to be complicated, but for the mitosis model it reduces to:

$$\forall f \in C^1, \forall x \geq 0,\ \mathcal{G}f = f'(x) + r(x)\frac{2V(x/2)}{V(x)} \left[f\left(\frac{x}{2}\right) - f(x) \right].$$

We also observe a biased phenomenon. But contrary to [4, 29], in general, the bias is present in the motion and the branching mechanism. it is, to our knowledge, a novelty. We can interpret the bias in the division part as follow: When a cell dies, we have more chance to choose the daughter that is more appropriate for r (the bigger or the smaller for example). For the bias in the motion, we can observe that if A is a vector field, $Af(x) = \alpha(x).\nabla f(x)$, (i.e. X is deterministic) then $B = A$. But if A is the generator of a diffusion, B is also the generator of a diffusion but with biased drift. One interpretation is that we have more chance to choose the cell with smaller or bigger noise. Notice also that we do not assume that λ_0 is the first eigenvalue. So, it is possible to have some auxiliary processes. We can find some result about existence of eigenelements in [19, 43] and theirs references. A first application of this formula is that if Y is ergodic, with invariant measure π, we obtain

$$\lim_{t\to+\infty} \frac{1}{\mathbb{E}[\sum_{u\in V_t} V(X_t^u)]}\mathbb{E}\left[\sum_{u\in V_t} f(X_t^u)V(X_t^u)\right] = \int f\, d\pi$$

for all bounded function f. We improve this result:

Theorem 1.4 (Convergence of the empirical measure for the long time)**.** *Assume the hypothesis of theorem 1.3 and Y is ergodic with invariant measure π. Consider a real function g and assume that:*

- *There exists $C > 0$, such that for all $x \in E$, $g(x) \le CV(x)$.*
- *There exists $\alpha < \lambda_0$, such that $\mathbb{E}[V^2(Y_t)] \le Ce^{\alpha t}$ and*

$$\mathbb{E}\left[\frac{r(Y_s)}{V(Y_s)}\int_0^1 \sum_{a,b\in\mathbb{N}^*,a\neq b}\ \sum_{k\ge\max(a,b)} p_k(x)V(F_a^{(k)}(x,\theta))V(F_b^{(k)}(x,\theta))\right] \le Ce^{\alpha t}.$$

Then we get,

$$\lim_{t\to+\infty} e^{-\lambda_0 t}\sum_{u\in V_t} g(X_t^u) = W\int \frac{g}{V}\, d\pi$$

where $W = \lim_{t\to+\infty} e^{-\lambda_0 t}\mathbb{E}[\sum_{u\in V_t} V(X_t^u)]$ and the convergence holds in probability. If furthermore, $\mathbb{E}[V(Y_t)] \le Ce^{\alpha t}$ and there exists $c > 0$ such that $\forall x \in E, V(x) \ge c$, then,

$$\lim_{t\to+\infty} \frac{1}{N_t}\sum_{u\in V_t} g(X_t^u) = \int \frac{g}{V}\, d\pi \Big/ \int \frac{1}{V}\, d\pi \text{ in probability}$$

For r constant, we have $V \equiv 1$ is an eigenvector and this theorem generalises [4, theorem 1.1].

In the other hand, our model is a microscopic interpretation, the population is discrete. And, we are also interested by the behaviour of our process in a large population. More precisely, we take a sequence $Z^{(n)}$ distributed as Z, the empirical measure, such that the starting distribution $Z_0^{(n)}$ grows to infinity with n. Consider the following renormalised process $X^{(n)} = Z^{(n)}/n$, and we get:

Theorem 1.5 (Law of large number for the large population)**.** *Let $T > 0$, assume r is bounded and one of the following hypothesis:*

(i) E is compact
(ii) $E \subset \mathbb{R}$, $|F_j^{(k)}(x,\theta)| \le |x|$, and for all $k \in \mathbb{N}^*$, there exists $\psi_k : E \to \mathbb{R}$ such that:

$$\forall x \in E,\ \mathbf{1}_{[k;+\infty[}(x) \le \psi_k(x) \le \mathbf{1}_{[k-1;+\infty[}(x) \text{ and } \exists C,\ A\psi_k \le C\psi_{k-1}$$

So, If $X_0^{(n)}$ converges in distribution to a deterministic measure X_0 in $\mathcal{M}(E)$ (embedded with the weak topology), then $X^{(n)}$ converges in distribution in $\mathbb{D}([0,T],\mathcal{M}(E))$ to a deterministic measure X, such that, for all $f \in \mathcal{D}(A)$,

$$\int_E f(x)\, X_t(dx) = \int_E f(x)\, X_0(dx) + \int_0^t\int_E \tilde{A}f(x)\, X_s(dx)ds \tag{8}$$

where $\mathbb{D}([0,T],\mathcal{M}(E))$ is the space of càd-làg functions embedded with the Skorohod topology [8, 33]

The second assumption is verified by any operator upper bounded by a differential operator [34, 40]. We can observe that the equation (8) is the Fokker-Planck (or Kolmogorov) equation. Thus X is equal to the mean measure of Z (e.g. $f \mapsto \mathbb{E}[\int_E f(x)\, Z_t(dx)]$). This average phenomenon is predicable for two same reasons. The first is that after a branching event, each cell evolves independently from each other, there is not interaction or mutation. The second is the linearity of the operator $\tilde{A}$. From theorem 1.3, one can see that, in large population, the empiral measure (not the mean measure!) behaves as the auxiliary process. The proof is based on the Aldous-Rebolledo criterion [33, 51] and it is inspired by [25, 40, 52]. In these papers, there are other models of structured populations.

In the mitosis case, the equation, (8) can be written by:

$$\partial_t n(t,x) + \partial_x n(t,x) + r(x)n(t,x) = 4r(2x)n(t,2x) \tag{9}$$

This equation was studied in [36, 45, 46]. In these papers, the constant case and the non constant case are separated. For a constant r, the authors prove the following exponential decay

$$\|n(t,.)e^{-rt} - N\|_{L^1} \leq e^{-rt}C,$$

where N is the density of the stationary distribution. There implies a convergence in total variation. In contrast, we also obtain the convergence to an equilibrium state for the size-structured population, and we have an exponential decay in Wasserstein distance (see theorem 1.2). It is showed that this rate of convergence is optimal in [36]. For the non constant case, we can also find an exponential decay,

$$\|(n(t,.)e^{-\lambda_0 t} - N)V\|_{L^1} \leq e^{-\alpha t}C,$$

proved by a perturbation method (α is explicit). This expression can be understood as a total variation decay for one cell lineage. It is not easy to find a total variation bound by coupling method. When r is affine, we can find Wasserstein bound in [11], for one cell lineage. In contrast, without speed of convergence, we find a convergence in the case where r is affine (which means non bounded). Furthermore, for this model, we estimate the fluctuation between the empirical measure and its approximation. It is defined by,

$$\forall t \geq 0,\ \eta_t^{(n)} = \sqrt{n}(X_t^{(n)} - X_t)$$

Theorem 1.6 (Central limit Theorem for size-structured population)**.** *Let $T > 0$. Assume (1-2), r is bounded and $\eta_0^{(n)}$ converges and*

$$\mathbb{E}\left[\sup_{n\geq 1} \int_0^{+\infty} 1 + x\ X_0^{(n)}(dx)\right] < +\infty.$$

Then the sequence $(\eta^{(n)})_{n\geq 1}$ converges in $\mathbb{D}([0,T], C^{-2,0})$ to the unique solution of the following evolution equation: For all $f \in C^{2,0}$,

$$\int_0^{+\infty} f(x)\ \eta_t(dx) = \int_0^{+\infty} f(x)\ \eta_0(dx) + \int_0^t \int_0^{+\infty} f'(x) + r(x)\left(2f\left(\frac{x}{2}\right) - f(x)\right)\ \eta_s(dx)\ ds + \tilde{M}(f) \tag{10}$$

where $\tilde{M}(f)$ is a martingale and a Gaussian process with bracket:

$$\langle \tilde{M}(f)\rangle_t = \int_0^t \int_0^{+\infty} 2r(x)\left(f\left(\frac{x}{2}\right) - f(x)\right)^2\ X_s(dx)\ ds.$$

And $C^{2,0}$ is the set of function C^2, such that f, f', f'' vanish to zero when x vanishes to infinity. $C^{-2,0}$ is its dual space.

Strucure of the paper: In the next section, we introduce some notations and give the generator of the measure-valued process. In section 3, we focus our interest in the long time. We prove the theorem 1.3, others many-to-one formulas and we deduce a general limit theorem. Theorem 1.4 is a consequence of Theorem 3.7 which gives similar result. Then we give two instructive examples in section 4. The first one describes the cell mitosis, the proofs of theorem 1.1 and theorem 1.2 are in this section. The second example can describe cell division with parasite infection. In this example, we give different eigenelements. Finally the section 5 is devoted to the study of the large population. We prove the theorem 1.5, and a central limit theorem for asymmetric cell division which implies the theorem 1.6. The last section is devoted to several open problem around our model.

2. Notation and preliminaries results

When we start with one individual with a weight $x_0 \in E$, we use the Ulam-Harris-Neveu notation [4, 16] to describe the population. We denote by $\emptyset$ the first cell. $X^{\emptyset}$ is its weight. Then every cell is indexed by a label $u = (u_1, ..., u_m)$ in the set:

$$\mathcal{U} = \bigcup_{m=0}^{\infty} (\mathbb{N}^*)^m$$

with the convention $(\mathbb{N}^*)^0 = \emptyset$. The cell indexed by u is the daughter of the cell indexed by $(u_1, ..., u_{m-1})$ and the mother of the cell indexed by $uv = (u_1, ..., u_m, v)$. v is between 1 and the number of offspring. We introduce the following measure to represent the population at time t:

$$Z_t^{x_0} = \sum_{u\in V_t} \delta_{X_t^u}.$$

We get that the process $Z^{x_0} = (Z_t^{x_0})_{t\geq 0}$ is a càd-làg measure-valued Markov process of $\mathbb{D}(\mathbb{R}_+, \mathcal{M}(E))$, the space of càd-làg functions with values in $\mathcal{M}(E)$, the set of finite measures on E. And, if there will be no ambiguity we shall note Z.

Example 2.1 (Branching diffusion)**.** *If X is a real diffusion, its generator is defined, for all smooth enough function f, by*

$$Af(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2} f''(x) \tag{11}$$

where we assume that b and σ are such that there exists a unique process with this generator (see for instance the [30, theorem 3.2 p.168]*) and $E = \mathbb{R}$ or $\mathbb{R}_+^*$. In this case, we can describe the population with a Poisson point measure* [25, 50]. *This S.D.E. is defined, for all $f : (t,x) \mapsto f_t(x)$ in $C_b^{1,2}$, by*

$$\begin{aligned} Z_t(f_t) =& Z_0(f_0) + \int_0^t \int_E (Af_s(x) + \partial_s f_s(x) Z_s(dx) ds + \int_0^t \sum_{u \in V_s} \sqrt{2}\sigma(X_s^u)\partial_x f_s(X_s^u) dB_s^u \\ &+ \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+\times\mathbb{N}^*\times[0,1]} \mathbf{1}_{\{u \in V_{s-}, l \le r(X_{s-}^u)\}} \left(\sum_{j=1}^k f_s(F_j^{(k)}(X_{s-}^u, \theta)) - f_s(X_{s-}^u) \right) \rho(ds, du, dl, dk, d\theta) \end{aligned}$$

where $(B^u)_{u\in\mathcal{U}}$ is a family of independent standard Brownian motions and $\rho(ds, du, dl, dk, d\theta)$ a Poisson point measure on $\mathbb{R}_+ \times \mathcal{U} \times \mathbb{R}_+ \times \mathbb{N}^* \times [0,1]$ of intensity $\bar{\rho}(ds, du, dl, dk, d\theta) = ds\ n(du)\ dl\ dp_k\ d\theta$ independent from the Brownian Motion. We have denoted by $n(du)$ the counting measure on $\mathcal{U}$ and $ds\ dl\ d\theta$ are Lebesgue measures.

A necessary and sufficient condition for the existence of our process is there is no explosion, indeed $N_t < +\infty$ a.s.. This hypothesis is always assumed. For instance, we can assume that r is bounded by $\bar{r}$. In this case, a coupling argument implies $\mathbb{E}[N_t] \le \mathbb{E}[N_0]\ \mathrm{e}^{(\bar{k}-1)\ \bar{r}\ T}$.

In the next sections, the notation C_x means a constant which only depend to x, and the notation $\mu(1+x^p)$ means for $\int 1 + x^p \mu(dx)$.

2.1. **Infinitesimal generator and martingale properties.** Denoted by $(A, \mathcal{D}(A))$ the generator of X and L the generator of Z. For ϕ, ψ be two bounded functions belong to the domain of a generator $\mathcal{A}$ such that $\phi \times \psi$ belong it too, we recall that the associated "carré du champ" operator is defined by:

$$\Gamma_{\mathcal{A}}(\phi, \psi) = \frac{1}{2}(\mathcal{A}(\phi \times \psi) - \psi\mathcal{A}\phi - \phi\mathcal{A}\psi). \tag{12}$$

Lemma 2.2 (Semi-martingale Decomposition)**.** *Let ϕ be a bounded function belong to the domain of L. Then there is a square-integrable and cádlág martingale M such that:*

$$\forall t \ge 0,\ M_t = \phi(Z_t) - \phi(Z_0) - \int_0^t L\phi(Z_s)\, ds\ a.s.$$

and if furthermore ϕ^2 be belong to the domain of L too, we get:

$$\langle M \rangle_t = \int_0^t 2\Gamma(\phi,\phi)(Z_s) ds.$$

So, for all $\varphi \in \mathcal{D}(A)$ and $t \ge 0$,

$$Z_t(\varphi) = Z_0(\varphi) + M_t(\varphi) + V_t(\varphi)$$

where

$$\begin{aligned} V_t(\varphi) &= \int_0^t A\varphi(x) + \int_E r(x) \int_0^1 \sum_{k\in\mathbb{N}^*} \left(\sum_{j=1}^k \varphi\left(F_j^{(k)}(x,\theta)\right)\right) - \varphi(x)\ p_k(x)\ d\theta\ Z_s(dx)\ ds \\ &= \int_0^t Z_s(\tilde{A}\varphi)\ ds \end{aligned}$$

and if $\varphi^2 \in \mathcal{D}(A)$, the bracket of $M_t(\varphi)$ equal to

$$\int_0^t 2Z_s\left(2\Gamma_A(\varphi,\varphi)\right) + \int_E r(x) \int_0^1 \sum_{k\in\mathbb{N}^*} \left(\sum_{j=1}^k \varphi(F_j^{(k)}(x,\theta)) - \varphi(x) \right)^2 p_k(x)\ d\theta\ Z_s(dx) ds$$

Proof. For the first part, it is an application of Dynkin and Itô formulas, see [32, lemma 3.68] for instance. For the second part a computation gives the generator of Z that is applied in i_φ and i_φ^2 where:

$$i_\varphi : \mu \mapsto \mu(\varphi) = \int \varphi\, d\mu \text{ and } i_\varphi^2 : \mu \mapsto (\mu(\varphi))^2.$$

So

$$Li_\varphi(\mu) = \int_E A\varphi(x) + r(x) \int_0^1 \sum_{k\in\mathbb{N}^*} \sum_{j=1}^{k} \varphi(F_j^{(k)}(x,\theta)) - \varphi(x) p_k(x)\, d\theta\, \mu(dx)$$

$$\begin{aligned} Li^2_\varphi(\mu) &= \mu(A\varphi^2) + 2\mu(\varphi)\mu(A\varphi) - 2\mu(\varphi \times A\varphi) \\ &+ \int_E r(x) \int_0^1 \sum_{k\in\mathbb{N}^*} 2\mu(\varphi) \times \left(\sum_{j=1}^{k} \varphi(F_j^{(k)}(x,\theta)) - \varphi(x) \right) + \left(\sum_{j=1}^{k} \varphi(F_j^{(k)}(x,\theta)) - \varphi(x) \right)^2 p_k(x)\, d\theta\, \mu(dx) \end{aligned}$$

□

We define the mean measure z, for all smooth enough function φ, by $z(\varphi) = \mathbb{E}(Z(\varphi)) = \mathbb{E}[\sum_{u\in V_t} \varphi(X_t^u)]$.

Corollary 2.3 (Evolution equation for the mean measure)**.** *If* $\mathcal{D}_b(\tilde{A}) = \{f \in \mathcal{D}(A) \mid \forall x \in E,\ |\tilde{A}f(x)| \leq 1\}$ *is a determining class, for* $\varphi \in \mathcal{D}(A)$*, we get*

$$z_t(\varphi) = z_0(\varphi) + \int_0^t z_s(A\varphi) + \int_E r(x) \sum_{k\geq 1} \sum_{j=1}^{k} \int_0^1 \varphi\left(F_j^{(k)}(x,\theta)\right) d\theta\, p_k(x) - \varphi(x)\, z_s(dx)\, ds$$

and it is the unique solution of this integro-differential equation for a fixed initial condition.

Proof. We have just to prove the uniqueness. Consider two probability measures $(\mu_t)_t$ and $(\nu_t)_t$ solution of this P.D.E. with same starting distribution $\mu_0 = \nu_0$. We consider the following norm defined by

$$\|m_1 - m_2\| = \sup_{\varphi\in\mathcal{D}_b(\tilde{A})} |m_1(\varphi) - m_2(\varphi)|$$

Then we consider one function φ in $\mathcal{D}(A)$ such that $|\tilde{A}\varphi| < 1$, we have,

$$\begin{aligned} |\mu_t(\varphi) - \nu_t(\varphi)| &= \left| \int_0^t \int_E A\varphi(x) + r(x) \left[\mathbb{E}\left[\sum_{k\geq 1} p_k(x) \sum_{j=1}^{k} \varphi(F_j^{(k)}(x,\Theta)) \right] - \varphi(x) \right] (\mu_s - \nu_s)(dx) \right| \\ &\leq C_{\bar{r},\bar{k}} \int_0^t \|\mu_s - \nu_s\| ds \end{aligned}$$

Taking the supremum and using the Gronwall lemma we fill deduce that :

$$\forall t \geq 0,\ \|\mu_t - \nu_t\| = 0$$

and, as $\mathcal{D}_b(\tilde{A})$ is a determining class, uniqueness holds. □

Example 2.4 (Branching diffusion)**.** *We return at the example 2.1, in this case the generator is more explicit. We give it for the function defined by* $F_\varphi : \mu \to F(\int \varphi\, d\mu) = F(\mu(\varphi))$*, with* $F \in C_b^2(\mathbb{R},\mathbb{R})$ *and* $\varphi \in C_b^2(E,\mathbb{R})$ *(which is known to be convergence determining* [16]*).*

$$\begin{aligned} LF_\varphi(\mu) =& \mu(A\varphi)F'(\mu(\varphi)) + \mu(\sigma\varphi'^2)F''(\mu(\varphi)) \\ &+ \int_E r(x) \int_0^1 \sum_{k\in\mathbb{N}} F\left(\mu(\varphi) + \sum_{j=1}^{k} \varphi\left(F_j^{(k)}(x,\theta)\right) - \varphi(x) \right) - F(\mu(\varphi))\, p_k(x)\, d\theta\, \mu(dx). \end{aligned}$$

3. Long time's behaviour

We recall that

$$\tilde{A}\varphi(x) = A\varphi(x) + r(x) \left[\sum_{k\geq 0} \sum_{j=1}^{k} \int_0^1 \varphi(F_j^{(k)}(x,\theta))\, d\theta\, p_k(x) - \varphi(x) \right],$$

and in all this section, we assume $\tilde{A}$ have as eigenelements (V, λ_0) such that $\tilde{A}V = \lambda_0 V$ and V positive.

3.1. **Eigenelements and auxiliary process (Proof of theorem 1.3).** Before the proof of theorem 1.3, we show that $Z_t(V) = \sum_{u\in V_t} V(X_t^u)$ have the same part that $N_t = \sum_{u\in V_t} 1$ for constant r.

Proposition 3.1 (Martingale properties)**.** *Under the assumptions of theorem 1.3, the process $(Z_t(V)e^{-\lambda_0 t})_{t\geq 0}$ is a martingale thus it converges to a random variable W almost surely.*

Proof. First, by corollary 2.3 we have:

$$\begin{aligned} z_t(V) &= z_0(V) + \int_0^t z_s(\tilde{A}V)ds \\ &= z_0(V) + \lambda_0 \int_0^t z_s(V)ds \end{aligned}$$

and then $z_t(V) = z_0(V)e^{\lambda_0 t}$. Then, denote $\mathcal{F}_t = \sigma\{Z_s \mid s \leq t\}$. The Markov properties, applies on Z, gives

$$\mathbb{E}[Z_{t+s}(V)|\mathcal{F}_s] = \mathbb{E}[\tilde{Z}_t(V)|\tilde{Z}_0 = Z_s]$$

where $\tilde{Z}$ is distributed as Z. Then $\mathbb{E}[Z_{t+s}(V)|\mathcal{F}_s] = Z_s(V)e^{\lambda_0 t}$ and thus

$$\mathbb{E}[Z_{t+s}(V)e^{-\lambda_0(t+s)}|\mathcal{F}_s] = Z_s(V)e^{\lambda_0 s}$$

□

proof of theorem 1.3. Let $\gamma_t : f \mapsto z_t(f \times V)e^{-\lambda_0 t}V(x_0)^{-1}$. We get, for all $t \geq 0$,

$$\partial_t\gamma_t(f) = z_t(\tilde{A}(f\,V))e^{-\lambda_0 t}V(x_0)^{-1} - z_t(f\,V)\lambda_0 e^{-\lambda_0 t}V(x_0)^{-1} = e^{-\lambda_0 t}V(x)^{-1}\left[z_t(\tilde{A}(f\,V)) - z_t(f \times \tilde{A}V)\right]$$

and thus,

$$e^{\lambda_0 t}\partial_t\gamma_t(f) = \int_E \frac{V(x)}{V(x_0)}Bf(x) + \frac{V(x)}{V(x_0)}\Lambda(x)\left[\frac{\sum_{k\in\mathbb{N}}\sum_{j=1}^k \int_0^1 V(F_j^{(k)}(x,\theta)f(F_j^{(k)}(x,\theta))\,d\theta\,p_k(x)}{\sum_{k\in\mathbb{N}}\sum_{j=1}^k \int_0^1 V(F_j^{(k)}(x,\theta))\,d\theta\,p_k(x)} - f(x)\right]z_t(dx).$$

Finally, $\partial_t\gamma_t(f) = \gamma_t(\mathcal{G}f)$. Now, by Dynkin formula, the law of the auxiliary process $(f \mapsto \mathbb{E}[f(Y_t)])$ verifies the same equation. The uniqueness, proved at corollary 2.3, gives the result. □

Remark 3.2 (Schrödinger operator and h-transform)**.** *In introduction, we said that $\tilde{A}$ is not a Markov generator. We can rewrite, for all φ smooth enough,*

$$\tilde{A}\varphi = G\varphi + r(m-1)\varphi$$

where G is the Markov generator defined at (5) and $r(m-1)$ is a potential. $\tilde{A}$ is called a Schrödinger operator, and its study is connected to the Feynman-Kac formula [17]. *Thus, the key point of our weighted many-to-one formula is a h-transform (Girsanov type transformation) of the Feynman-Kac semigroup as in* [26, 48] *(here, $Ve^{-\lambda_0 t}$ is a space-time harmonic function).*

Remark 3.3 (Malthus parameter)**.** *Since, Thomas Malthus (1766-1834) were introduced the simpler model to describe the population:*

$$\partial_t N_t = \textit{ birth} - \textit{death} = bN_t - dN_t = \lambda_0 N_t \implies N_t = e^{\lambda_0 t},$$

in biology and genetic population study, λ_0 is sometimes called the Malthus parameter.

Example 3.4 (Galton-Watson tree)**.** *If r and p are constant, $V \equiv 1$ is an eigenvector for the eigenvalue $\lambda_0 = r(m-1)$. So, $Z_t(V) = N_t$, and the population grows exponentially. This result is already know for N_t. It is a continuous branching process* [3, 4].

3.2. **Many-to-one formulas.** In order to compute our limit theorem, we need to control the second moment. As in [4], we begin by describe the population over whole the tree. Then we give a many-to-one formula for forks. Let $\mathcal{T}$ be the random set according to represent cells having lived at a certain moment. It is defined by

$$\mathcal{T} = \{u \in \mathcal{U} \mid \exists t > 0, X_t^u \in V_t\}.$$

In the following, the propositions 3.5 and 3.6 are respectively the generalisation of [4, proposition 3.5] and [4, proposition 3.9].

Proposition 3.5 (Many-to-one formula over the whole tree). *Under the assumptions of theorem 1.3, for any non-negative measurable function $f : E \times [0, +\infty) \to \mathbb{R}$ we get,*

$$\mathbb{E}\left[\sum_{u\in\mathcal{T}} f\left(X^u_{\beta(u)-}, \beta(u)\right)\right] = V(x_0)\int_0^{+\infty} \mathbb{E}\left[f(Y_s, s)\,\frac{r(Y_s)}{V(Y_s)}\right] e^{\lambda_0 s} ds$$

Proof. First we have, for all $u \in \mathcal{U}$,

$$\mathbb{E}\left[\mathbf{1}_{\{u\in\mathcal{T}\}}\int_{\alpha(u)}^{\beta(u)} f(X^u_s, s) r(X^u_s) ds\right] = \mathbb{E}\left[\mathbf{1}_{\{u\in\mathcal{T}\}} f\left(X^u_{\beta(u)-}, \beta(u)\right)\right]$$

because

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{u\in\mathcal{T}\}}\int_{\alpha(u)}^{\beta(u)} f(X^u_s, s) r(X^u_s) ds\right] &= \mathbb{E}\left[\mathbf{1}_{\{u\in\mathcal{T}\}}\int_0^{+\infty}\int_{\alpha(u)}^{\tau} f(X^u_s, s) r(X^u_s) ds\, r(X^u_\tau) e^{-\int_{\alpha(u)}^{\tau} r(X^u_t)dt}\, d\tau\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{u\in\mathcal{T}\}}\int_{\alpha(u)}^{+\infty}\int_s^{+\infty} r(X^u_\tau) e^{-\int_{\alpha(u)}^{\tau} r(X^u_t)dt}\, d\tau\, f(X^u_s, s) r(X^u_s) ds\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{u\in\mathcal{T}\}}\int_{\alpha(u)}^{+\infty} e^{-\int_{\alpha(u)}^{s} r(X^u_t)dt} f(X^u_s, s) r(X^u_s) ds\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{u\in\mathcal{T}\}} f\left(X^u_{\beta(u)-}, \beta(u)\right)\right]
\end{aligned}$$

thus,

$$\mathbb{E}\left[\mathbf{1}_{\{u\in\mathcal{T}\}} f\left(X^u_{\beta(u)-}, \beta(u)\right)\right] = \mathbb{E}\left[\int_0^{+\infty} \mathbf{1}_{\{u\in V_s\}} f(X^u_s) r(X^u_s) ds\right]$$

and then,

$$\begin{aligned}
\mathbb{E}\left[\sum_{u\in\mathcal{T}} f\left(X^u_{\beta(u)-}, \beta(u)\right)\right] &= \int_0^{+\infty} \mathbb{E}\left[\sum_{u\in V_s} f(X^u_s, s) r(X^u_s)\right] ds \\
&= \int_0^{+\infty} V(x_0)\mathbb{E}\left[f(Y_s, s)\frac{r(Y_s)}{V(Y_s)}\right] e^{\lambda_0 s} ds.
\end{aligned}$$

□

If f has the form $f(x, s) = g(x, s)V(x)$, then we have:

$$\mathbb{E}\left[\sum_{u\in\mathcal{T}} g\left(X^u_{\beta(u)-}, \beta(u)\right) V\left(X^u_{\beta(u)-}\right)\right] = \int_0^{+\infty} \mathbb{E}\left[g(Y_s, s) r(Y_s)\right] \times \mathbb{E}\left[Z_s(V)\right] ds.$$

This equality means that adding the contributions over all the individuals corresponds to integrating the contribution of the auxiliary process over the average number of living individuals at time s. Let $(P_t)_{t\geq 0}$ be the semigroup of the auxiliary process,

$$P_t f(x) = \mathbb{E}[f(Y_t) \mid Y_0 = x]$$

Proposition 3.6 (Many-to-one formula for forks). *Under the assumptions of theorem 1.3, for all non-negative and measurable function f, g we get*

$$\begin{aligned}
\mathbb{E}\left[\sum_{u,v\in V_t, u\neq v} f(X^u_t)V(X^u_t)g(X^v_t)V(X^v_t)\right] &= e^{2\lambda_0 t}V(x_0)\int_0^t \mathbb{E}\left[J_2(VP_{t-s}f, VP_{t-s}g)(Y_s)\,\frac{r(Y_s)}{V(Y_s)}\right] e^{-\lambda_0 s} ds \\
&= \mathbb{E}[Z_t(V)]^2\int_0^t \frac{1}{\mathbb{E}[Z_s(V)]}\mathbb{E}\left[J_2(VP_{t-s}f, VP_{t-s}g)(Y_s)\,\frac{r(Y_s)}{V(Y_s)}\right] ds
\end{aligned}$$

where J_2 is defined by

$$J_2(\varphi, \psi)(x) = \int_0^1 \sum_{a\neq b}\sum_{k\geq \max(a,b)} p_k(x)\,\varphi\left(F^{(k)}_a(x, \theta)\right)\psi\left(F^{(k)}_a(x, \theta)\right)\, d\theta$$

J_2 represent the starting distributions of the offspring picked at random.

Proof. Let $u, v \in V_t$ such that $u \neq v$, there exist $(w, \tilde{u}, \tilde{v}) \in \mathcal{U}^3$ and $a, b \in \mathbb{N}^*$, $a \neq b$ such that $u = wa\tilde{u}$ and $v = wb\tilde{v}$. w is the most recent common ancestor. Thus,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{u,v\in V_t, u\neq v} f(X_t^u)V(X_t^u)g(X_t^v)V(X_t^v)\right] \\
&= \sum_{w\in\mathcal{U}}\sum_{a\neq b}\sum_{\tilde{u},\tilde{v}\in\mathcal{U}} \mathbb{E}\left[\mathbf{1}_{\{wa\tilde{u}\in V_t\}} f(X_t^{wa\tilde{u}})V(X_t^{wa\tilde{u}})\mathbf{1}_{\{wa\tilde{v}\in V_t\}} g(X_t^{wa\tilde{v}})V(X_t^{wa\tilde{v}})\right]
\end{aligned}$$

We recall that $\mathcal{F}_t = \sigma\{Z_s \mid s \leq t\}$ and, by the conditional independence between descendants, we get,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{u,v\in V_t, u\neq v} f(X_t^u)V(X_t^u)g(X_t^v)V(X_t^v)\right] \\
&= \sum_{w\in\mathcal{U}}\sum_{a\neq b}\mathbb{E}\left[\mathbb{E}\left[\sum_{\tilde{u}\in\mathcal{U}} \mathbf{1}_{\{wa\tilde{u}\in V_t\}} f(X_t^u)V(X_t^u)|\mathcal{F}_{\beta(w)}\right]\mathbb{E}\left[\sum_{\tilde{v}\in\mathcal{U}} \mathbf{1}_{\{wa\tilde{v}\in V_t\}} g(X_t^v)V(X_t^v)|\mathcal{F}_{\beta(w)}\right]\right]
\end{aligned}$$

Therefore, as $\beta(w)$ is a stopping time, using the strong Markov property and theorem 1.3, we get,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{u,v\in V_t, u\neq v} f(X_t^u)V(X_t^u)g(X_t^v)V(X_t^v)\right] \\
&= \sum_{w\in\mathcal{U}}\sum_{a\neq b}\mathbb{E}\left[\mathbf{1}_{\{wa,wb\in\mathcal{T},\ t\geq\beta(w)\}} P_{t-\beta(w)}f(X^{wa}_{\beta(w)})V(X^{wa}_{\beta(w)})\ P_{t-\beta(w)}g(X^{wb}_{\beta(w)})V(X^{wb}_{\beta(w)})e^{2\lambda_0(t-\beta(w))}\right] \\
&= \mathbb{E}\left[\sum_{w\in\mathcal{T}} \mathbf{1}_{\{t\geq\beta(w)\}} J_2(VP_{t-\beta(w)}f, VP_{t-\beta(w)}g)(X^w_{\beta(w)-})\ e^{2\lambda_0(t-\beta(w))}\right] \\
&= e^{2\lambda_0 t}V(x_0)\int_0^t \mathbb{E}\left[J_2(VP_{t-s}f, VP_{t-s}g)(Y_s)\ \frac{r(Y_s)}{V(Y_s)}\right] e^{-\lambda_0 s}ds.
\end{aligned}$$

□

3.3. **Limit theorem (proof of theorem 1.4).** Here we give the main limit theorem which implies the theorem 1.4.

Theorem 3.7 (General Condition for the convergence of the empirical measure)**.** *We assume that the hypothesis of theorem 1.3 are verified. Let f be a real measurable function defined on E and μ a probability measure such that there exists a probability measure π, and two constants $\alpha < \lambda_0$ and $C > 0$ such that*

$$\pi(|f|) < +\infty \ \text{ and } \ \forall x \in E \lim_{t\to+\infty} P_t f(x) = \pi(f) \tag{13}$$

$$\mu(V) < +\infty,\ \mu P_t(f^2 \times V) \leq Ce^{\alpha t} \ \text{ and } \ \mu P_s\left(J_2(VP_{t-s}f, VP_{t-s}f)\ \frac{r}{V}\right) \leq Ce^{\alpha t}. \tag{14}$$

If $x_0 = X_0^{\emptyset} \sim \mu$, then we have

$$\lim_{t\to+\infty} \frac{1}{\mathbb{E}[Z_t(V)]}\sum_{u\in V_t} f(X_t^u)V(X_t^u) = W \times \pi(f)$$

where the convergence holds in probability. If furthermore $Z_t(V)$ is bounded into L^2 then the convergence holds in L^2.

Notice that the constants and π may be depend on f and μ! Notice also that λ_0 is not supposed to be the first eigenvalue.

Proof. As in [4, theorem 4.2], we first prove the convergence for f such that $\pi(f) = 0$. We have $\mathbb{E}[Z_t(V)] = \mu(V)e^{\lambda t}$, then,

$$\mathbb{E}\left[\left(\frac{1}{\mathbb{E}[Z_t(V)]}\sum_{u\in V_t} f(X_t^u)V(X_t^u)\right)^2\right] = \mathbb{E}\left[Z_t(f\times V)^2 e^{-2\lambda_0 t}\mu(V)^{-2}\right] = A_t + B_t$$

where

$$A_t = e^{-2\lambda_0 t}\mu(V)^{-2}\mathbb{E}\left[\sum_{u\in V_t} f^2(X_t^u)V^2(X_t^u)\right] = e^{-\lambda_0 t}\mu(V)^{-1}\mathbb{E}\left[f^2(Y_t)V(Y_t)\right]$$

and

$$\begin{aligned}B_t &= e^{-2\lambda_0 t}\mu(V)^{-2}\mathbb{E}\left[\sum_{u,v\in V_t,\ u\neq v} f(X_t^u)V(X_t^u)f(X_t^v)V(X_t^v)\right]\\ &= \mu(V)^{-1}\int_0^t \mathbb{E}\left[J_2(VP_{t-s}f, VP_{t-s}f)(Y_s)\,\frac{r(Y_s)}{V(Y_s)}\right]e^{-\lambda_0 s}ds\end{aligned}$$

From (14), we have $\lim_{t\to+\infty} A_t = 0$ and, since

$$J_2(\varphi,\psi)(x) = \int_0^1 \sum_{a\neq b}\sum_{k\geq \max(a,b)} p_k(x)\varphi\left(F_a^{(k)}(x,\theta)\right)\psi\left(F_a^{(k)}(x,\theta)\right)d\theta,$$

from (13) and as $\pi(f) = 0$, we get, for all $s\geq 0$ and $x\in E$,

$$\lim_{t\to+\infty} J_2(VP_{t-s}f, VP_{t-s}f)(x) = 0.$$

And thus, by (14) and dominated convergence, we obtain $\lim_{t\to+\infty} B_t = 0$. Now for a general f, we have

$$Z_t(fV)e^{-\lambda_0 t} - W\pi(f) = Z_t\left((f-\pi(f))V\right)e^{-\lambda_0 t} + \pi(f)\left(Z_t(V)e^{-\lambda_0} - W\right)$$

Then, thanks to the first part of the proof, the first term of the sum, in the right hand side, converges to 0 in L^2. The second term converges to 0 in probability thanks proposition 3.1. □

It is enough to consider $g = f\times V$ to deduce theorem 1.4.

4. Examples

Here, we give two examples. The first one describes the cell mitosis for a very smooth r and an affine r. In the second one, we illustrate the fact that we can use different eigenelement. This example can model a parasite infection.

4.1. **Size-structured population (equal mitosis) : Inhomogeneous rate of division (proof of theorem 1.1).** As say in introduction, the cell size grows linearly and divides into two parts. Formally, with the notation of the example 2.1,

$$E = \mathbb{R}_+^*,\ \sigma = 0,\ b = 1,\ p_2 = 1\ \text{ and } F_1^{(2)}(x,\theta) = F_2^{(2)}(x,\theta) = x/2.$$

First prove that our process is well defined:

Lemma 4.1 (Non explosion)**.** *Let $p\geq 1$. If for all $x\in\mathbb{R}_+^*$, $r(x)\leq C_0(1+x^p)$, and $z_0(1+x^p) < +\infty$, then our process is well defined for all $t\geq 0$. Moreover*

$$\mathbb{E}\left[\sup_{s\in[0,T]} Z_s(1+x^p)\right] \leq z_0(1+x^p)e^{C_pT}$$

Proof. As in the example 2.1, we can write

$$\begin{aligned}Z_t(f) &= Z_0(f) + \int_0^t\int_E f'(x)\,Z_s(dx)\,ds\\ &+ \int_0^t\int_{\mathcal{U}\times\mathbb{R}_+\times[0,1]} \mathbf{1}_{\{u\in V_{s-}, l\leq r(X_{s-}^u)\}} f(\theta X_{s-}^u) + f((1-\theta)X_{s-}^u) - f(X_{s-}^u)\ \rho(ds,du,dl,d\theta)\end{aligned}$$

Using the same argument to [25, theorem 3.1], we introduce $\tau_n = \inf\{\ t\geq 0 \mid Z_t(1+x^p) > n\ \}$ and,

$$\begin{aligned}\sup_{u\in[0,t\wedge\tau_n]} Z_u(1+x^p) &\leq Z_0(1+x^p) + \int_0^{t\wedge\tau_n} Z_s(px^{p-1})ds\\ &+ \int_0^{t\wedge\tau_n}\int_{\mathcal{U}\times\mathbb{R}_+\times[0,1]} \mathbf{1}_{u\in V_{s-}, l\leq r(X_{s-}^u)}(1 + (\theta^p + (1-\theta)^p - 1)(X_{s-}^u)^p)\ \rho(ds,du,dl,d\theta)\\ &\leq Z_0(1+x^p) + \int_0^{t\wedge\tau_n} p\times \sup_{u\in[0,s\wedge\tau_n]} Z_u(1+x^p)ds.\\ &+ \int_0^t\int_{\mathcal{U}\times\mathbb{R}_+\times[0,1]} \mathbf{1}_{\{u\in V_{s-}, l\leq r(X_{s-}^u)\}}\ \rho(ds,du,dl,d\theta)\end{aligned}$$

Then,

$$\mathbb{E}\left[\sup_{u\in[0,t\wedge\tau_n]} Z_u(1+x^p)\right] \leq z_0(1+x^p) + \int_0^t C_{p,C_0}\,\mathbb{E}\left[\sup_{u\in[0,s\wedge\tau_n]} Z_u(1+x^p)\right] ds.$$

So, by the Gronwall lemma,

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_n]} Z_s(1+x^p)\right] \leq z_0(1+x^p)e^{C_p t} \leq z_0(1+x^p)e^{C_p T}.$$

We deduce that τ_n tends a.s. to infinity, and our process is well defined. □

In order to have the many-to-one formula, we give a condition for the existence of eigenelement extracted to [46] (see [45], for an asymmetric division cell, and [19], for a non linear motion between the division).

Theorem 4.2 (Sufficient condition for the existence of eigenelements)**.** *Assume* $\exists \underline{r}, \bar{r}$ *such that:*

$$\forall x \geq 0,\ 0 < \underline{r} \leq r(x) \leq \bar{r}$$

Then there is a unique eigenelement (λ_0, V) *and we have:*

$$\underline{r} \leq \lambda_0 \leq \bar{r}$$
$$\frac{c}{1+x^k} \leq V(x) \leq C(1+x^k)$$

where C, c *are two positive constants and* $2^k\lambda_{min} > \lambda_{max}$

So, we get a many-to-one formula with an auxiliary process generated by

$$\mathcal{G}f(x) = f'(x) + r(x)\frac{2V(x/2)}{V(x)}\left(f(x/2) - f(x)\right). \tag{15}$$

But , even if this theorem gives us a many-to-one formula, we need a smoother r to have a convergence:

Theorem 4.3 (Sufficient condition for the existence of smooth eigenelements)**.** *Under the same assumption and if furthermore* $r(x)$ *is constant equal at* r_∞ *for a* x *large enough then*

$$c(1+x^k) \leq V(x) \leq C(1+x^k)$$

where C, c *are two constant and* $2^k = \frac{2r_\infty}{\lambda_0 + r_\infty}$.

Proof of theorem 1.1. Under the assumptions of theorem 1.1 and theorem 4.3, $V(x/2)/V(x)$ is bounded. Thus, the auxiliary process is ergodic and admits a unique invariant law, as can be checked using a suitable Foster-Lyapunov function [13, 42] (for instance, $V(x) = 1 + x$). Finally, we use theorem 1.4 to conclude. The explicit formula is an application of the theorem of [44]. □

We can see that the assumptions of theorem 4.3 are strong, and not necessary. Because if $r(x) = ax + b$ (with $a, b \geq 0$ and a or b positive) then $V(x) = x\frac{\sqrt{b^2+4a}-b}{2} + 1$ is an eigenvector and $\frac{2a}{\sqrt{b^2+4a}-b}$ the eigenvalue. Thus we deduce,

Proposition 4.4 (Convergence of the empirical measure when $r(x) = ax + b$)**.** *For* $r(x) = x$ *there exists a measure* π *such that*

$$\lim_{t\to+\infty} \frac{1}{N_t}\sum_{u\in V_t} g(X_t^u) = \int g\, d\pi$$

where the convergence holds in probability and for any continuous function g *on* E *such that* $\forall x \in E,\ |g(x)| \leq C(1+x)$.

It is a pity not to manage to obtain $\Lambda(x) = x$, because in this case the invariant measure of the auxiliary process possesses an explicit form [28]. So, we also obtain

$$\lim_{t\to+\infty} N_t\, e^{-\lambda_0 t} = W\int_E \frac{1}{V}\, d\pi$$

and $\lambda_0 = \frac{2a}{\sqrt{b^2+4a}-b}$ is the Malthus parameter (see remark 3.3).

Remark 4.5 (Value of r for the Escherichia coli cell)**.** *We can find some estimate of the division rate in the literature (for the macroscopic model). An inverse problem is developed in* [22, 47]. *In* [21]*, this method is applied with experimental data extracted to* [35]. *It is also explain why our model is realistic for the Escherichia coli cell. More recently,* [20] *gives a nonparametric estimation of the division rate.*

4.2. **Size-structured population (equal mitosis) : Homogeneous rate of division (proof of theorem 1.2).** When r is constant, the process is easier to be studied and we can find some result about the auxiliary process in [11, 37, 44]. It is the most homogeneous possible case. r and p constant and X is linear. Furthermore, the generator conserves the polynomial function. So, we can calculate the moments (proposition 4.6). This knowledge gives us the Laplace transformation of the equilibrium, and by inversion, the formula (3). Now, we give the moments, the proof of theorem 1.2 and some remarks about this result. Let $\mu = \sum_{i=1}^{n} x_i$ be a deterministic measure, we denote by Z^{μ} the process, distributed as Z starting at μ, indeed:

$$Z^{\mu} \overset{d}{=} \sum_{i=1}^{n} Z^{x_i}$$

where Z^{x_i} are i.i.d. and distributed as Z starting with one point with size x_i.

Proposition 4.6 (Moments of the empirical measure)**.** *For all $m \in \mathbb{N}$, and for all $t \geq 0$, we have,*

$$\mathbb{E}[Z_t^{\mu}(x^m)] = \mathbb{E}\left[\sum_{u \in V_t^{\mu}} (X_t^u)^m\right] = \int_0^{+\infty} e^{rt}\left[\frac{m!}{\prod_{i=1}^{m}\theta_i} + m!\sum_{i=1}^{m}\left(\sum_{k=0}^{i}\frac{x^k}{k!}\prod_{j=k,j\neq i}^{m}\frac{1}{\theta_j - \theta_i}\right)e^{-\theta_i t}\right]\mu(dx)$$

where $\theta_i = 2r\left(1 - 2^{-i}\right)$. In particular,

$$\begin{aligned}\mathbb{E}[Z_t^{\mu}(x)] = e^{rt}\mathbb{E}\left[\sum_{u\in V_t^{\mu}} X_t^u\right] &= \int_0^{+\infty}\frac{1}{r} - \left(\frac{1}{r} - x\right)e^{-rt}\,\mu(dx)\\ &= \frac{n}{r}(e^{rt}-1) + \sum_{i=1}^{n} x_i\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[Z_t^{\mu}(x^2)] &= e^{rt}\mathbb{E}\left[\sum_{u\in V_t^{\mu}}(X_t^u)^2\right]\\ &= e^{rt}\int_0^{+\infty}\frac{4}{3r^2} + 2\left[e^{-rt}\left(\frac{-2}{r^2}+\frac{2x}{r}\right) + e^{-3rt/2}\left(\frac{4}{3r^2} - \frac{2x}{3r} + \frac{x^2}{2}\right)\right]\mu(dx).\\ &= \frac{4n}{3r^2}\left(e^{rt} - 3 + 2e^{-rt/2}\right) + \left(\sum_{i=1}^{n} x_i\right)\left(\frac{4}{r} - \frac{4}{3r}e^{-rt/2}\right) + e^{-rt/2}\sum_{i=1}^{n} x_i^2.\end{aligned}$$

Proof. It is an application of the moment estimate of the homogeneous TCP windows size process [37, Theorem 8] and theorem 1.3. □

proof of theorem 1.2. We have to prove

$$\forall t \geq 0,\ W^{(1)}_{W_{|\cdot|}}(\mathcal{L}(Z_t^x), \mathcal{L}(Z_t^y)) \leq |x - y|.$$

We recall again, the Wasserstein distance between two laws, m_1 and m_2, with finite mean on a metric space (F, d_F), is defined by

$$W^{(p)}_{d_F}(m_1, m_2) = (\inf \mathbb{E}[d_F(X,Y)^p])^{1/p}$$

where the infimum runs over all coupling of $X \sim m_1$ and $Y \sim m_2$ (see for instance [49, 54]). Let us explain how we build our coupling. Since this process is homogeneous, we can see it as a process indexed by a tree [4]. For our coupling, we take two process indexed by the same tree. In other word, like the time of branching do not depend of the position, we can take the same for our two processes. Let $\mathcal{T} = \bigcup_{n\in\mathbb{N}}\{1,2\}^n$ be the set according to represent cells having lived at a certain moment. Let $(d_u)_{u\in\mathcal{U}}$ a family of i.i.d. exponential with mean $1/r$, which will model the lifetimes. We build Z^x and Z^y by recurrence. $\forall t \in [0, d_{\emptyset}),\ X_t^{\emptyset} = x + t$ (resp. $Y_t^{\emptyset} = y + t$), $\alpha(u) = 0$. Then for all $u \in \mathcal{T}$, for all $k \in \{1, 2\}$, $\alpha(uk) = \alpha(u) + d_u$ and

$$\forall u \in \mathcal{T}, \forall k \in \{1,\ldots,\nu_u\},\ \forall t \in [\alpha(uk), \alpha(uk) + d_{uk}),\ X_t^{uk} = \frac{1}{2}X^u_{\alpha(uk)-} + t - \alpha(uk)$$

(resp. $Y_t^{uk} = Y^u_{\alpha(uk)-}/2 + t - \alpha(uk)$). Finally, $V_t = \{u \in \mathcal{T} \mid \alpha(u) \leq t < \alpha(u) + d_u\}$ and

$$Z_t^x = \sum_{u\in V_t}\delta_{X_t^u} \text{ and } Z_t^y = \sum_{u\in V_t}\delta_{Y_t^u}.$$

Then, we see that the trajectories are parallels between the branching events. At this time, $\sum_{u\in V_t} |X_t^u - Y_t^u|$ is constant. Hence, we easily prove

$$\sum_{u\in V_t} |X_t^u - Y_t^u| = |x-y|.$$

But, If $m_1 = \frac{1}{n}\sum_{k=0}^n \delta_{x_i}$ and $m_2 = \frac{1}{n}\sum_{k=0}^n \delta_{y_i}$ are two discrete measures, where $n \in \mathbb{N}^*$ and $x_i,\ y_i \in F$, we have the following matching representation [54]:

$$W_{d_F}^{(p)}(m_1,m_2)^p = \inf_{\tau\in S_n} \frac{1}{n}\sum_{i=1}^n d_F(x_i, y_{\tau(i)})^p$$

where S_n denote the symmetric group. Thus,

$$W_{|\cdot|}^{(1)}(Z_t^x, Z_t^y) \le |x-y|$$

and the others inequalities follow. □

Remark 4.7 (Convergence to equilibrium)**.** *Usually, for the real Markov processes, if we have a bound of*

$$W(\mathcal{L}(X_t|X_0\sim\mu), \mathcal{L}(X_t|X_0\sim\nu)),$$

it is enough to take the invariant probability measure for μ to obtain a speed of convergence toward the equilibrium. But here, it is not possible because the equilibrium is not a Dirac mass. But, we can try to estimate the distance between Z^x and Z^{π_n}, such that

$$\frac{1}{n}\pi_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i} \to \pi.$$

By the branching properties, we get, ,

$$Z_t^{\pi_n} \overset{d}{=} \sum_{i=1}^n Z_t^{x_i}$$

where $Z_t^{x_i}$ are independent and distributed as Z starting at δ_{x_i}. Thus,

$$W_{|\cdot|}\left(\frac{Z_t^x}{\mathbb{E}[N_t^x]}, \frac{Z_t^{\pi_n}}{\mathbb{E}[N_t^{\pi_n}]}\right) \le e^{-r(m-1)t}\ \frac{1}{n}\sum_{i=1}^n W_{|\cdot|}(Z_t^x, Z_t^{x_i})$$

Now, we want to take the infimum and obtain a result such that,

$$\begin{aligned} W_{W_{|\cdot|}}^{(1)}\left(\mathcal{L}\left(\frac{Z_t^x}{\mathbb{E}[N_t^x]}\right), \mathcal{L}\left(\frac{Z_t^{\pi_n}}{\mathbb{E}[N_t^{\pi_n}]}\right)\right) &\le e^{-r(m-1)t}\frac{1}{n}\sum_{i=1}^n W_{W_{|\cdot|}}^{(1)}(\mathcal{L}(Z_t^x), \mathcal{L}(Z_t^{x_i})) \\ &\le e^{-r(m-1)t}\frac{1}{n}\sum_{i=1}^n |x-x_i| \\ &\le e^{-r(m-1)t} W_{|\cdot|}(\delta_x, \pi_n) \end{aligned}$$

But, these inequalities are false. It seems to be impossible to use the inequalities, of theorem 1.2, to obtain a bound to the equilibrium. One explication is that this problem is similar to the following: Let X, Y, Z three random variables such that X and Y are independent. Is there a constant C such that,

$$W\left(\mathcal{L}\left(\frac{X+Y}{2}\right), \mathcal{L}(Z)\right) \le C \times \frac{W(\mathcal{L}(X),\mathcal{L}(Z)) + W(\mathcal{L}(Y),\mathcal{L}(Z))}{2}.$$

But it is enough to consider X, Y, Z are three Bernoulli variables with same parameter to see that it is not possible. We can only find

$$W_{W_{|\cdot|}}^{(1)}\left(\mathcal{L}\left(\frac{Z_t^x}{\mathbb{E}[N_t^x]}\right), \mathcal{L}\left(\frac{Z_t^{\pi_n}}{\mathbb{E}[N_t^{\pi_n}]}\right)\right) \le e^{-r(m-1)t}\frac{1}{n}\sum_{i=1}^n \mathbb{E}[W_{|\cdot|}(\mathcal{L}(Z_t^x), \mathcal{L}(Z_t^{x_i}))]$$

where $Z_t^{x_i}$ and $Z_t^{x_j}$ are independent for all $i \neq j$. This inequalities suggests that we must consider the independent coupling, but it is not satisfactory too (see proposition 4.9 latter).

Remark 4.8 (Generalisation of theorem 1.2)**.** *In the proof of theorem 1.2, we only need that, for all n, θ, x and y,*

$$\sum_{j=1}^n |F_j^{(k)}(X_T,\theta) - F_j^{(k)}(Y_T,\theta)| \le |x-y|$$

where X, Y are generated by A and start respectively at x, y and T is exponentially distributed. For instance we can consider X is a continuous lévy process and a sub-critical fragmentation:

$$\forall x \in E,\ \forall k \in \mathbb{N}^*, \forall j \leq k,\ F_j^{(k)}(x,\Theta) = \Theta_j^k x,\ \sum_{j=1}^{k} \Theta_j^k \leq 1 \text{ and } \forall j \in \{1,\dots,k\},\ \Theta_j^k \in [0,1].$$

Proposition 4.9 (Independent coupling)**.** *Let $\mu = \sum_{i=1}^n x_i$ $\nu = \sum_{i=1}^m y_i$ be two discrete measures and Z^μ and Z^ν be two independent processes starting at μ and ν. We get,*

$$\forall t > 0,\ \mathbb{E}\left[W_{|\cdot|}^{(2)}\left(\frac{Z_t^\mu}{N_t^\mu}, \frac{Z_t^\nu}{N_t^\nu}\right)\right]^2 \leq \frac{2t^2}{(1-e^{-t})^2} + \mathcal{O}(t^2 e^{-rt}). \tag{16}$$

Proof. By matching and Cauchy-Schwarz formulas, we get,

$$\begin{aligned}
\mathbb{E}\left[W_{|\cdot|}^{(2)}\left(\frac{Z_t^\mu}{N_t^\mu}, \frac{Z_t^\nu}{N_t^\nu}\right)\right]^2 &= \mathbb{E}\left[W_{|\cdot|}^{(2)}\left(\frac{N_t^\nu \times Z_t^\mu}{N_t^\nu N_t^\mu}, \frac{N_t^\mu \times Z_t^\nu}{N_t^\nu N_t^\mu}\right)\right]^2 \\
&\leq \mathbb{E}\left[\sqrt{\frac{1}{N_t^\nu N_t^\mu}\sum_{u\in V_t^\nu}\sum_{v\in V_t^\mu}|X_t^u - Y_t^v|^2}\right]^2 \qquad (17)\\
&\leq \mathbb{E}\left[\frac{1}{N_t^\nu N_t^\mu}\right] \times \mathbb{E}\left[\sum_{u\in V_t^\nu}\sum_{v\in V_t^\mu}|X_t^u - Y_t^v|^2\right]
\end{aligned}$$

where,

$$Z_t^\mu = \sum_{u \in V_t^\mu} \delta_{X_t^u} \text{ and } Z_t^\nu = \sum_{v\in V_t^\nu} \delta_{Y_t^v}.$$

Then,

$$\mathbb{E}\left[\frac{1}{N_t^\nu N_t^\mu}\right] = \mathbb{E}\left[\frac{1}{N_t^\mu}\right]\mathbb{E}\left[\frac{1}{N_t^\nu}\right] = \frac{1}{n\times m}\mathbb{E}\left[\frac{1}{N_t}\right]^2$$

where N_t is the classical Yule process starting at $N_0 = 1$. Then, since N_t is is geometric with parameter e^{-rt} [4], we get

$$\forall t > 0,\ \mathbb{E}\left[\frac{1}{N_t^\nu N_t^\mu}\right] = \frac{r^2 t^2}{nm}\frac{e^{-2rt}}{(1-e^{-rt})^2}.$$

In the other hand, we have, by proposition 4.6,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{u\in V_t^\nu}\sum_{v\in V_t^\mu}|X_t^u - Y_t^v|^2\right] \\
=&\mathbb{E}[N_t^y]\mathbb{E}\left[\sum_{u\in V_t^x}(X_t^u)^2\right] + \mathbb{E}[N_t^x]\mathbb{E}\left[\sum_{v\in V_t^y}(Y_t^v)^2\right] - 2\mathbb{E}\left[\sum_{v\in V_t^y} Y_t^v\right]\mathbb{E}\left[\sum_{u\in V_t^x} X_t^u\right] \\
=&\frac{8mn}{3r^2}\left(e^{2rt} - 3e^{rt} + 2e^{rt/2}\right) + (m\mu(x) + n\nu(x))\left(\frac{4}{r}e^{rt} - \frac{4}{3r}e^{rt/2}\right) + \left(m\mu\left(x^2\right) + n\nu\left(x^2\right)\right)e^{rt/2} \\
&-2\left(\frac{nm}{r^2}(e^{rt}-1)^2 + \frac{1}{r}(e^{rt}-1)(m\mu(x) + n\nu(x)) + \nu(x)\mu(x)\right) \\
=&\frac{2mn}{r^2}e^{2rt} - \frac{4mn}{r^2}e^{rt} + \frac{16mn}{3r^2}e^{rt/2} - \frac{2nm}{r^2} \\
&+\frac{1}{r}(m\mu(x) + n\nu(x))\left(2e^{rt} - \frac{4}{3}e^{-rt/2} + 2\right) + \left(m\mu\left(x^2\right) + n\nu\left(x^2\right)\right)e^{-rt/2} - 2\mu(x)\nu(x).
\end{aligned}$$

Thus, we deduce (16). □

The coupling choice does not seem to be responsible of the non-optimality (the limit is deterministic). The error results maybe from the matching choice (17). But it is the only one such that we can estimate the distance. In spite of everything, we have

Proposition 4.10 (Wasserstein convergence)**.** *Under the assumptions of theorem 1.2, we have*

$$\lim_{t\to+\infty} W_{|\cdot|}^{(1)}\left(\frac{Z_t}{N_t}, \pi\right) = 0 \text{ in probability.}$$

Proof. As $x \mapsto 1 + x$ is a Lyapounov function for the auxiliary process, we have

$$\lim_{t\to+\infty} \frac{Z_t}{N_t}(f) = \pi(f) \text{ in probability}$$

for all function f such that $f(x) \leq C(1+x)$. The convergence also holds in distribution. By the Prokhorov theorem, in an other probability space, we have,

$$\lim_{t\to+\infty} \frac{Z_t}{N_t}(f) = \pi(f) \text{ a.s.}$$

for all bounded function and for $f(x) = x$. This convergence is equivalently to a Wasserstein convergence. Thus, by a classical argument of discreteness (Varadarajan theorem type), we get,

$$\lim_{t\to+\infty} W^{(1)}_{|\cdot|}\left(\frac{Z_t}{N_t}, \pi\right) = 0 \text{ a.s..}$$

Hence, in our probability space we get $\lim_{t\to+\infty} W^{(1)}_{|\cdot|}(Z_t/N_t, \pi) = 0$ in distribution. And like the convergence is deterministic, we get the result. □

4.3. **Explicit eigenelements for a parasite infection model.** In theorem 1.3, we did not required that λ_0 was the first eigenvalue. So, it is possible to have different eigenelements and auxiliary processes. Consider the following example, where some eigenelements are explicit. :

$$\forall x > 0,\ Af(x) = axf'(x) + b(x)f''(x) \tag{18}$$

with b smooth enough. We also consider that for $j \leq k$ and for all measurable and non-negative f,

$$\mathbb{E}[f(F^{(k)}_j(x,\Theta))] = \mathbb{E}[f(\Theta^k_j x)] \tag{19}$$

where

$$\sum_{j=1}^{k} \Theta^k_j = 1 \text{ and } \Theta^k_j \in [0,1] \text{ a.s..} \tag{20}$$

This process can model physical or biological polymers. It can also models cell division with parasite infection [5]. We easily find a is an eigenvalue and $V(x) = x$ is its eigenvector. So, for all measurable and non-negative function f,

$$\mathbb{E}\left[\sum_{u\in V_t} X^u_t f(X^u_t)\right] = \mathbb{E}[f(Y_t)]e^{at}x_0$$

where Y is a Markov process, generated by,

$$G_Y f(x) = \left(ax + 2\frac{b(x)}{x}\right) f'(x) + b(x)f''(x) + r(x)\left(\left(\sum_{k\in\mathbb{N}} p_k(x)\sum_{j=1}^{k}\mathbb{E}[\Theta^k_j f(\Theta^k_j x)]\right) - f(x)\right).$$

When r is affine, we obtain a second formula. Assume m is constant and $r(x) = cx + d$, with $c \geq 0$ and $d(m-1) > a$ (or $d > 0$ and $c = 0$). So, $V_1(x) = \frac{c(m-1)}{d(m-1)-a}x + 1$ is an eigenvector associated to the eigenvalue $\lambda_1 = d(m-1)$ ($\Rightarrow \lambda_1 > \lambda_0 = a$). Thus, for all measurable and positive function,

$$\mathbb{E}\left[\sum_{u\in V_t} f(X^u_t)\right] e^{-dt} = \mathbb{E}\left[\frac{f(U_t)}{\tau U_t + 1}\right](\tau x_0 + 1)$$

where $\tau = \frac{c(m-1)}{d(m-1)-a}$ and U is generated, for all $f \in \mathcal{D}(A)$ and for all $x > 0$, by

$$G_U f(x) = \left(ax + \frac{2b(x)\tau}{\tau x + 1}\right) f'(x) + b(x)f''(x) + \frac{r(x)(\tau x + m)}{\tau x + 1}\left(\frac{\mathbb{E}[\sum_{k\geq 1} p_k(x)(\tau\Theta^k_j x + 1)f(\Theta^k_j x)]}{\tau x + m} - f(x)\right).$$

So, if we start with one cell infected by x_0 parasite then $d(m-1)$ is the Malthus parameter (see remark 3.3):

Proposition 4.11 (Properties of the number of individual alive)**.** *Under (18-20) and if $r(x) = cx + d$, with $c \geq 0$ and $d(m-1) > a$ (or $d > 0$ and $c = 0$). N_t verifies,*

$$\mathbb{E}[N_t] = e^{d(m-1)t} + \tau x_0\,(e^{d(m-1)t} - e^{at}).$$

And $(N_t e^{-d(m-1)t})_{t\geq 0}$ and $(N_t/\mathbb{E}[N_t])_{t\geq 0}$ converge a.s..

Proof. First, a same computation of lemma 4.1 gives that the process is well defined and that the weighted many-to-one formula holds. So, as we get:

$$N_t = \left(\sum_{u\in V_t} 1 + \tau X_t^u\right) - \tau \left(\sum_{u\in V_t} X_t^u\right)$$

the proposition follows. □

Consider the same parameter of [5], that is $b(x) = \sigma^2 x$ and $p_2 = 1$. $X_t = Z_t(V)$ is the total number of parasite. It is a martingale, so we easily obtain $E[X_t] = e^{at}\mathbb{E}[X_0]$ and $X_t e^{-at}$ converge a.s.. But since his bracket is $2\sigma^2(1-e^{-at})$, we have a convergence a.s and in L^2. This result is already know, because in this case, $(X_t)_{t\geq 0}$ is a Feller diffusion.

5. Macroscopic interpretation

To prove theorem 1.5, we need to use different topology on $\mathcal{M}(E)$. We note $(\mathcal{M}(E), d_v)$ (resp. $(\mathcal{M}(E), d_w)$) when it is embedded with the vague (resp. weak) topology. These topologies will be understood in the following sense:

$$\lim_{n\to+\infty} d_v(X_n, X_\infty) = 0 \iff \forall f \in C_0,\ \lim_{n\to+\infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X_\infty)]$$

$$\lim_{n\to+\infty} d_w(X_n, X_\infty) = 0 \iff \forall f \in C_b,\ \lim_{n\to+\infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X_\infty)]$$

where C_0 is the set of continuous function that vanishes to zero at the infinity and C_b the set of bounded continuous function. We also will use $\mathbb{D}([0,T],E)$ and $C([0,T],E)$ be respectively the set of càd-làg function embedded with the Skorohod topology and the set of continuous function embedded with the uniform topology [8].

5.1. Law of large number (proof of theorem 1.5).

In this section, we consider a sequence $Z^{(n)}$ distributed as Z, starting at some measure of probability $Z_0^{(n)}$, and the following scaling: $X^{(n)} = \frac{1}{n}Z^{(n)}$. We describe the behavior of this renormalized process when n go to infinity.

Heuristically, to understand the behaviour of our process when we start with a large population distributed by a deterministic measure X_0, we can approximate X_0 by the interesting sequence defined by $X_0^{(n)} = \frac{1}{n}\sum_{k=0}^n \delta_{Y_k}$,where $(Y_k)_{k\geq 1}$ is a sequence i.i.d. distributed by X_0. Thus, we get,

$$X^{(n)} = \frac{1}{n}Z^{(n)} \overset{d}{=} \frac{1}{n}\sum_{k=0}^n Z^{Y_k}$$

where $Z_t^{Y_k}$ are i.i.d. distributed as Z, with $Z_0 = \delta_{Y_k}$. So, let φ a bounded function, the law of large number gives:

$$\forall t \geq 0,\ \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^n Z_t^{X_k}(\varphi) = \mathbb{E}\left[Z_t^{Y_1}(\varphi)\right]$$

So by corollary 2.3 , it implies that $X^{(n)}$ converges to the solution of the following integro-differential equation:

$$\mu_t(\varphi) = \mu_0(\varphi) + \int_0^t \mu_s(A\varphi) + \int_E r(x)\sum_{k\geq 0} p_k(x)\int_0^1 \sum_{j=1}^k \varphi(F_j^{(k)}(x,\theta))d\theta - \varphi(x)\ \mu_s(dx)\ ds \tag{21}$$

In fact, this convergence is better. It is a processes convergence. There is that the theorem 1.5 said.

Lemma 5.1 (Semi-martingale decomposition). *for all $\varphi \in \mathcal{D}(A^2)$ and $t \geq 0$,*

$$X_t^{(n)}(\varphi) = X_0^n(\varphi) + M_t^{(n)}(\varphi) + V_t^{(n)}(\varphi)$$

with

$$V_t^{(n)}(\varphi) = \int_0^t \int_E A\varphi(x) + r(x)\int_0^1 \sum_{k\in\mathbb{N}}\sum_{j=1}^k \varphi(F_j^{(k)}(x,\theta)) - \varphi(x)p_k\ d\theta\ X_s^{(n)}(dx)\ ds$$

and $M_t^{(n)}(\varphi)$ is a square-integrable and càdlàg martingale with bracket

$$\frac{1}{n}\left(\int_0^t 2X_s^{(n)}(A\varphi^2) - 2X_s^{(n)}(\varphi\times A\varphi) + \int_E r(x)\int_0^1 \sum_{k\in\mathbb{N}^*}\left(\sum_{j=1}^k \varphi(F_j^{(k)}(x,\theta)) - \varphi(x)\right)^2 p_k(x)\ d\theta\ X_s^{(n)}(dx)\ ds\right)$$

Proof. It is an application of the lemma 2.2 because the generator of $X^{(n)}$, denoted by $L^{(n)}$, verifies:

$$L^{(n)}F_\varphi(\mu) = \partial_t \mathbb{E}[F_\varphi(X^{(n)})|X_0^{(n)} = \mu]\big|_{t=0} = \partial_t \mathbb{E}[F_{\varphi/n}(Z^{(n)})|Z_0^{(n)} = n\mu]\big|_{t=0} = LF_{\varphi/n}(n\mu)$$

where $F_\varphi(\mu) = F(\mu(\varphi))$. □

Lemma 5.2. *Under the assumptions of theorem 1.5, $X^{(n)}$ is tight for the vague topology.*

Proof. For this proof, we are inspired by [25]. According to [51], it is enough to show that, for any continuous bounded function f, the sequence of laws of $X^{(n)}(f)$ is tight in $\mathbb{D}([0,T],\mathbb{R})$. To prove this, we use the Aldous-Rebolledo criterion. Let S be a dense subset of C_0 that contained the function $x \mapsto 1$. We have the following two points to be verified: For all function $f \in S$,

(1) for all $t \geq 0$, $\left(X_t^{(n)}(f)\right)_{n\geq 0}$ is tight.
(2) for all $n \in \mathbb{N}$, and $\varepsilon, \eta > 0$, there exists δ such that for each stopping times S_n bounded by T,

$$\limsup_{n\to+\infty} \sup_{0\leq u\leq\delta} \mathbb{P}(|V_{S_n+u}^{(n)}(f) - V_{S_n}^{(n)}(f)| \geq \eta) \leq \varepsilon.$$

$$\limsup_{n\to+\infty} \sup_{0\leq u\leq\delta} \mathbb{P}(|\langle M^{(n)}(f)\rangle_{S_n+u} - \langle M^{(n)}(f)\rangle_{S_n}| \geq \eta) \leq \varepsilon.$$

The first point explain a pointwise tightness and the second point, called the Aldous condition, gives a "stochastic continuity". It look like the Arzelà-Ascoli theorem. For our problem we can take $S = \mathcal{D}(A^2)$. The first point gives,

$$\begin{aligned}\mathbb{P}(|X_t^{(n)}(f)| > k) &\leq \frac{\|f\|_\infty\ \mathbb{E}[X_t^{(n)}(1)]}{k} \\ &\leq \frac{\|f\|_\infty\ \mathbb{E}[N_0^{(n)}]\ C_{\bar{r},\bar{k}}}{n\ k}.\end{aligned}$$

Since $\mathbb{E}[N_0^{(n)}]/n$ converges , it is bounded, and for a large k, we have the tightness. Let $\delta > 0$ and $S_n \leq T_n \leq (S_n + \delta) \leq T$, we get

$$\begin{aligned}\mathbb{E}[|V_{T_n}^{(n)}(f) - V_{S_n}^{(n)}(f)|] &= \mathbb{E}\left[\left|\int_{S_n}^{T_n} X_s^{(n)}(Af) + \int_E r(x)\int_0^1 \sum_{k\in\mathbb{N}}\sum_{j=1}^k f(F_j^{(k)}(x,\theta)) - f(x)p_k\ d\theta\ X_s^{(n)}(dx)\ ds\right|\right] \\ &\leq C_{\bar{r},\bar{k},T}[\|Af\|_\infty + \|f\|_\infty] \times (T_n - S_n) \\ &\leq C_{\bar{r},\bar{k},T,f}\ \delta.\end{aligned}$$

In the other hand,

$$\begin{aligned}&\mathbb{E}[|\langle M^{(n)}(f)\rangle_{T_n} - \langle M^{(n)}(f)\rangle_{S_n}|] \\ =&\frac{1}{n}\mathbb{E}\left[\left|\int_{S_n}^{T_n} 2X_s^{(n)}(Af^2) - 2X_s^{(n)}(f\times Af) + \int_E r(x)\int_0^1 \sum_{k\in\mathbb{N}}\sum_{j=1}^k (f(F_j^{(k)}(x,\theta)) - f(x))^2 p_k\ d\theta\ X_s^{(n)}(dx)\ ds\right|\right] \\ \leq&\frac{1}{n} \times C_{\bar{r},\bar{k},T,f} \times (T_n - S_n) \\ \leq&\frac{C_{\bar{r},\bar{k},T,f}\,\delta}{n}.\end{aligned}$$

Then, for a sufficiently small δ the second point is verified and we conclude that $\left(X^{(n)}\right)_{n\geq 1}$ is uniformly tight in $\mathbb{D}([0,T],\mathcal{M}(E))$ for the vague topology. □

Proof of theorem 1.5. First, by the Doob's inequality, we get,

$$\sup_\varphi \mathbb{E}\left[\sup_{t\leq T}\left|M^{(n)}(\varphi)_t\right|\right] \leq 2\sup_\varphi \mathbb{E}[\langle M^{(n)}(\varphi)\rangle_T] \leq \frac{C_{\bar{r},\bar{k}}}{n}$$

where the supremum is taken over all the function $\varphi \in \mathcal{D}(A^2)$ such that $\|\varphi\|_\infty \leq 1$. Hence,

$$\lim_{n\to+\infty} \sup_\varphi \mathbb{E}\left[\sup_{t\leq T}\left|M^{(n)}(\varphi)_t\right|\right] = 0. \tag{22}$$

But,

$$M_t^{(n)}(\varphi) = X_t^{(n)}(\varphi) - X_0^{(n)}(\varphi) - \int_0^t \int_E A\varphi(x) + r(x)\int_0^1 \sum_{k\in\mathbb{N}}\sum_{j=1}^k \varphi(F_j^{(k)}(x,\theta)) - \varphi(x)p_k\ d\theta\ X_s^{(n)}(dx)\ ds.$$

So, we have to prove that the limit of $(M_t^{(n)}(\varphi))$ is also

$$X_t(\varphi) - X_0(\varphi) - \int_0^t X_s(A\varphi) + \int_E \lambda(x) \left(\sum_{j=1}^{k} \varphi(F_j^{(K)}(x,\theta))\, p_k(x)\, d\theta - \varphi(x). \right) X_s(dx)\, ds$$

Since this equation has a unique solution, it is enough to prove that the convergence of $X^{(u_n)}$ is in $\mathbb{D}([0,T],(\mathcal{M}(E),d_w))$, for each convergent subsequence $(u_n)_{n\in\mathbb{N}^*}$. If E is compact, the vague topology and the weak topology coincide, and we have the result. For the case (ii) of the assumptions, we can use the Méléard-Roelly criterion [39]. Let $(u_n)_n$ a subsequence such that $(X^{(u_n)})_n$ converges in distribution to X in $\mathbb{D}([0,T],(\mathcal{M}(E),d_v))$. We have to prove that X is in $C([0,T],(\mathcal{M}(E),w))$ and $X^{(n)}(1)$ converges to $X(1)$. To prove it, as in [34, 40], we can use the following lemma:

Lemma 5.3 (Analogous of the lemma 3.3 of [40])**.** *Under the same assumptions of theorem 1.5,*

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \mathbb{E}\left[\sup_{t\le T} X_t^{(n)}(\psi_k)\right] = 0$$

where $(\psi_k)_{k\ge0}$ are defined at theorem 1.5.

This lemma explain that we can commute the limit, The proof is postponed after. Hence, a same computation to [40] give us the convergence in $\mathbb{D}([0,T],(\mathcal{M}(E),w))$ to our process. Thus, each subsequence converges to the equation (21). There is a unique solution, and our sequence converges in $\mathbb{D}([0,T],(\mathcal{M}(E),w))$ to z (defined at the corollary 2.3) the unique solution about the equation (21).

But the lemma 5.3 is so strong, we can give another argument, without to use the Méléard-Roelly criterion [39]. As in [40], we can prove that X is continuous, from $[0,T]$ to $(\mathcal{M}(E),d_w)$, because

$$\sup_{t\ge0} \sup_{f,\|f\|_\infty\le1} |X_{t-}^{(n)}(f) - X_t^{(n)}(f)| \le \frac{\bar{k}}{n}.$$

Then, let G be a Lipschitz function on $C([0,T],(\mathcal{M}(E),d_w))$, we get,

$$\begin{aligned}
|\mathbb{E}[G(X^{u_n})] - G(X)| &\le \mathbb{E}\left[\sup_{t\in[0,T]} d_w\left(X_t^{(u_n)}, X_t\right)\right] \\
&\le \mathbb{E}\left[\sup_{t\in[0,T]} d_w\left(X_t^{(u_n)}, X_t^{(u_n)}(.\times(1-\psi_k))\right)\right] \\
&+ \mathbb{E}\left[\sup_{t\in[0,T]} d_w\left(X_t^{(u_n)}(.\times(1-\psi_k)), X_t(.\times(1-\psi_k))\right)\right] \\
&+ \sup_{t\in[0,T]} d_w\left(X_t(.\times(1-\psi_k)), X_t\right).
\end{aligned}$$

According the lemma 5.3, we obtain that

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \mathbb{E}\left[\sup_{t\in[0,T]} d_w\left(X_t^{(u_n)}, X_t^{(u_n)}(.\times(1-\psi_k))\right)\right] = 0$$

and

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \sup_{t\in[0,T]} d_w(X_t(.\times(1-\psi_k)), X_t) = 0.$$

Then, we have $d_w(X_t^{(u_n)}(.\times(1-\psi_k)), X_t(.\times(1-\psi_k))) = d_v(X_t^{(u_n)}(.\times(1-\psi_k)), X_t(.\times(1-\psi_k)))$. Thus,

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \mathbb{E}\left[\sup_{t\in[0,T]} d_w(X_t^{(u_n)}(.\times(1-\psi_k)), X_t(.\times(1-\psi_k)))\right] = 0$$

by contuinity of $\nu \mapsto \nu(1-\psi_k)$ in $\mathbb{D}(\mathcal{M}(E),d_v))$. □

proof of lemma 5.3. we denote by $\mu_t^{n,k} = \mathbb{E}(X_t^{(n)}(\psi_k))$, and we get:

$$\begin{aligned}
\mu_t^{n,k} &= \mathbb{E}[X_0^{(n)}(\psi_k)] + \int_0^t \mathbb{E}\left[\int_E A\psi_k(x) + r(x)\left(\sum_{k\geq 1}\sum_{j=1}^{k} p_k(x)\int_0^1 \psi_k(F_j^{(k)}(x,\theta)) - \psi_k(x)\right) X_s^{(n)}(dx)\right] ds \\
&\leq \mu_0^{n,k} + C\int_0^t \mu_s^{n,k-1} + \mu_s^{n,k} ds
\end{aligned}$$

and by Gronwall's lemma, iteration, monotonicity and the boundedness of $\frac{1}{n}\mathbb{E}[\sup_{t\leq T} N_t^n]$:

$$\begin{aligned}
\mu_t^{n,k} &\leq C_1(\mu_0^{n,k} + \int_0^t \mu_s^{n,k-1} ds) \\
&\leq C_1\mu_0^{n,k} + C_1^2 T\mu_0^{n,k-1} + \int_0^t\int_0^s \mu_u^{n,k-2} duds \\
&\leq \sum_{l=0}^{k-1} \mu_0^{n,k-l} C_1 \frac{(C_1T)^l}{l!} + C_2 \times \frac{(C_1T)^k}{k!} \\
&\leq \mu_0^{n,\lfloor k/2\rfloor} C_1 e^{C_1T} + C_3 \sum_{l>\lfloor k/2\rfloor} \frac{(C_1T)^l}{l!} + C_2 \times \frac{(C_1T)^k}{k!}
\end{aligned}$$

where C_1, C_2 and C_3 are three constants. Thus,

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \mu_t^{n,k} = 0.$$

Then, the following expression concludes the proof,

$$\mathbb{E}\left[\sup_{t\leq T} |X_t^n(\psi_k)|\right] \leq \mu_0^{n,k} + C\int_0^t \mu_s^{n,k-1} + \mu_s^{n,k} ds + \mathbb{E}\left[\sup_{t\leq T} |M_t^{(n)}(\psi_k)|\right].$$

□

Example 5.4 (Asymmetric mitosis)**.** *Let $F_1^{(2)}(x,\theta) = G^{-1}(\theta)x$ and $F_2^{(2)}(x,\theta) = (1 - G^{-1}(\theta))x$. Where G is the cumulative distribution function of the random fraction in $[0,1]$ associated with the branching event. It verifies $G(x) = 1 - G(1-x)$. If $n(t,.)$ is the density of z_t, then it is a weak solution solution of the following P.D.E. :*

$$\partial_t n(t,x) + \partial_x n(t,x) + r(x)\, n(t,x) = 2\mathbb{E}[\frac{1}{\Theta} r(x/\Theta) n(t,x/\Theta)].$$

Especially, we deduce that the following P.D.E. gets a weak solution:

$$\partial_t n(t,x) + \partial_x n(t,x) + r(x)\, n(t,x) = \int_x^{+\infty} b(x,y) n(t,y) dy$$

where b verify the following properties:

$$b(x,y) \geq 0, b(x,y) = 0 \ \text{for}\ y < x \tag{23}$$

$$\int_0^{+\infty} b(x,y)dx = 2r(y) \tag{24}$$

$$\int_0^{+\infty} xb(x,y)dx = yr(y) \tag{25}$$

$$b(x,y) = b(y-x,y). \tag{26}$$

This equation was studied in [45]. *$b(x,y) = \frac{2}{y} r(y) g(\frac{x}{y})$, where g is the density of G. b has this form is equivalently at verify the following points (23 - 26).*

5.2. **Central Limit Theorem for size-structured population (proof of theorem 1.6).** Our aim in this section is to describe the limit of the fluctuation process defined by:

$$\forall t \in [0,T], \forall n \in \mathbb{N}^*,\ \eta_t^{(n)} = \sqrt{n}(X_t^{(n)} - X_t).$$

For a better understanding, we only give the convergence on the example of the size-structured population (asymmetric mitosis). The result of this section are easily generalisable for splitted diffusion, but we do not want to weigh down the hypotheses and the notations.

Theorem 5.5 (Central limit theorem for asymmetric size-structured population)**.** *Let $T > 0$. Assume $\eta_0^{(n)}$ converges and*

$$\mathbb{E}\left[\sup_{n\geq 1} \int_E 1 + x\, X_0^{(n)}(dx)\right] < +\infty.$$

Then the sequence $(\eta^{(n)})_{n\geq 1}$ converges in $\mathbb{D}([0,T], C^{-2,0})$ to the unique solution of the evolution equation: for all $f \in C^{2,0}$,

$$\eta_t(f) = \eta_0(f) + \int_0^t \int_0^{+\infty} f'(x) + r(x)\left(\int_0^1 f(qx) + f((1-q)x)G(dq) - f(x)\right) \eta_s(dx)\, ds + \tilde{M}(f) \tag{27}$$

where $\tilde{M}(f)$ is a martingale and a Gaussian process with bracket:

$$\langle \tilde{M}_t(f)\rangle = \int_0^t \int_0^{+\infty} 2f'(x)f(x) + 2r(x)\int_0^1 (f(qx) - f(x))^2 G(dq)\, X_s(dx)\, ds.$$

And $C^{2,0}$ is the set of function C^2, such that f, f', f'' vanish to zero when x vanishes to infinity. $C^{-2,0}$ is its dual space.

By lemma 5.1, we have the following representation:

$$\forall\, t \geq 0,\ \eta_t^{(n)} = \eta_0^{(n)} + \tilde{V}_t^{(n)} + \tilde{M}_t^{(n)}$$

where

$$\forall \varphi \in C_b \cap C^1,\ \tilde{V}_t^{(n)}(\varphi) = \int_0^t \int_0^{+\infty} \varphi'(x) + r(x)\left(\int_0^1 \varphi(qx) + \varphi((1-q)x)G(dq) - \varphi(x)\right) \eta_s^{(n)}(dx)\, ds$$

and $\tilde{M}_t^{(n)}$ is a martingale with bracket:

$$\langle \tilde{M}_t^{(n)}(\varphi)\rangle = \int_0^t \int_0^{+\infty} 2r(x)\int_0^1 (\varphi(qx) - \varphi(x))^2 G(dq)\, X_s^{(n)}(dx)\, ds. \tag{28}$$

The set of signed measure is not metrizable, so we can not adapt the proof of theorem 1.5. As in [38, 52], we consider $\eta^{(n)}$ like an operator in a Sobolev space, and use the Hilbertian properties of this space to have tightness (see for instance [41] for tightness condition on Hilbert spaces). Let us explain the Sobolev space that we will use. Let $p > 0$, $j \in N$, and $W^{j,p}$ be the closure of the set of function C^∞ to $[0,+\infty)$ into $\mathbb{R}$ with compact support with the following norm:

$$\forall f \in W^{j,p},\ \|f\|^2_{W^{j,p}} = \sum_{k=0}^{j} \int_0^\infty \left(\frac{f^{(k)}(x)}{1+x^p}\right)^2 dx.$$

$W^{j,p}$ is an Hilbert space and we consider $W^{-j,p}$ the dual space. Let $C^{j,p}$, the space of function f, C^j, such that:

$$\forall k \leq j,\ \lim_{x\to+\infty} \frac{f^{(k)}(x)}{1+x^p} = 0$$

and we embed it by the following norm:

$$\forall f \in C^{j,p},\ \|f\|_{C^{j,p}} = \sum_{k=0}^{j} \sup_{x\geq 0} \frac{f^{(k)}(x)}{1+x^p}.$$

Thus, $C^{j,p}$ is a Banach space and we denote by $C^{-j,p}$ its dual space. These spaces verify the following continuous injection [38, 1]:

$$C^{j,p} \subset W^{j,p+1} \text{ and } W^{1+j,p} \subset C^{j,p}. \tag{29}$$

Or equivalently, if f is smooth enough,

$$\|f\|_{W^{j,p+1}} \leq C\|f\|_{C^{j,p}} \text{ and } \|f\|_{C^{j,p}} \leq C\|f\|_{W^{j+1,p}}.$$

The first embedding/inequality prove that the tightness in $W^{j,p+1}$ implies the tightness in $C^{j,p}$. The second is useful for some upper bound:

Lemma 5.6. *If $(e_k)_{k\geq 1}$ is a basis of $W^{2,1}$, we get:*

$$\sum_{k\geq 1} e_k(x)^2 \leq C(1+x).$$

Proof. Let $D^0_x : f \mapsto f(x)$ and $D^1_x : f \mapsto f'(x)$ be an operator on $W^{2,1}$. We have, for all $f \in W^{2,1}$,

$$|D^0_x f| \leq (1+x)\|f\|_{C^{0,1}} \leq C(1+x)\|f\|_{W^{1,1}} \leq C(1+x)\|f\|_{W^{2,1}}$$

But, by Parseval identity we get,

$$\|D^0_x\|^2_{W^{-2,1}} = \sum_{k\geq 1} e_k(x)^2.$$

It ends the proof. □

We introduce the trace $\left(\langle\langle \tilde{M}^{(n)}\rangle\rangle_t\right)_{t\geq 0}$ of $\left(\tilde{M}^{(n)}_t\right)_{t\geq 0}$ defined such that $\left(\|\tilde{M}^{(n)}_t\|^2_{W^{-2,1}} - \langle\langle \tilde{M}^{(n)}\rangle\rangle_t\right)_t$ is a local martingale. Then since

$$\|\tilde{M}^{(n)}_t\|^2_{W^{-2,1}} = \sum_{k\geq 1} \tilde{M}^{(n)}_t(e_k)$$

where $(e_k)_{k\geq 1}$ is a basis of $W^{2,1}$, and by (28), we get,

$$\langle\langle \tilde{M}^{(n)}\rangle\rangle_t = \sum_{k\geq 1} \int_0^t \int_0^{+\infty} 2r(x) \int_0^1 (e_k(qx) - e_k(x))^2 G(dq)\; X^{(n)}_s(dx)\; ds.$$

Now, we first prove the tightness of $(\eta^{(n)})_{n\geq 1}$ then we prove theorem 5.5.

Lemma 5.7. *$(\eta^n)_{n\geq 1}$ is tight in $\mathbb{D}([0,T], W^{-2,1})$.*

Proof. By [33, theorem 2.2.2] and [33, theorem 2.3.2] (see also [38, lemma C]), it is enough to prove

(1) $\mathbb{E}\left[\sup_{s\leq t}\|\eta^n_s\|^2_{W^{-2,1}}\right] < +\infty$.
(2) $\forall n \in \mathbb{N},\ \forall \varepsilon, \rho > 0,\ \exists \delta$ such that for each stopping times S_n bounded by T,

$$\limsup_{n\to+\infty} \sup_{0\leq u\leq \delta} \mathbb{P}\left(\|V^{(n)}_{S_n+u} - V^{(n)}_{S_n}\|_{W^{-2,1}} \geq \eta\right) \leq \varepsilon$$

$$\limsup_{n\to+\infty} \sup_{0\leq u\leq \delta} \mathbb{P}\left(\left|\left\langle\left\langle \tilde{M}^{(n)}\right\rangle\right\rangle_{S_n+u} - \left\langle\left\langle \tilde{M}^{(n)}\right\rangle\right\rangle_{S_n}\right| \geq \eta\right) \leq \varepsilon.$$

These two points are the Aldous-Rebolledo criterion. For the first point, we get,

$$\begin{aligned}\sum_{k\geq 1}\langle \tilde{M}^{(n)}_t(e_k)\rangle &\leq \int_0^t 2\bar{r}\int_0^1 2\sum_{k\geq 1} e_k^2(qx) + 2\sum_{k\geq 1} e_k^2(x)\; G(dq)\; X^{(n)}_s(dx)\; ds \\ &\leq C_T\; X^{(n)}_0(1+x)\end{aligned}$$

then, by the assumptions of theorem 5.5, we have the boundedness. Thus since,

$$\|\tilde{M}^{(n)}_t\|^2_{W^{-2,1}} = \sum_{k\geq 1} (\tilde{M}^{(n)}_t(e_k))^2$$

we have by Doob inequality,

$$\mathbb{E}\left[\sup_{t\in[0,t]} \|\tilde{M}^{(n)}_t\|^2_{W^{-2,1}}\right] \leq C.$$

Then

$$\|\eta^{(n)}_t\|^2_{W^{-2,1}} \leq \|\eta^{(n)}_0\|^2_{W^{-2,1}} + \|\tilde{V}^{(n)}_t\|^2_{W^{-2,1}} + \|\tilde{M}^{(n)}_t\|^2_{W^{-2,1}} \leq C + \|\tilde{V}^{(n)}_t\|^2_{W^{-2,1}}.$$

And

$$\|\tilde{V}^{(n)}_t\|^2_{W^{-2,1}} \leq C\int_0^t \sup_{w\leq s} \|\eta^{(n)}_s\|^2_{W^{-2,1}} ds.$$

So by Gronwall lemma we obtain

$$\mathbb{E}\left[\sup_{s\leq t} \|\eta^{(n)}_s\|^2_{W^{-2,1}}\right] \leq C$$

Then for the second point, we have

$$\mathbb{E}[\|V^{(n)}_{S_n+u} - V^{(n)}_{S_n}\|_{W^{-2,1}}] \leq \mathbb{E}\left[C\int_{S_n}^{S_n+u} \sup_{s\leq T} \|\eta^{(n)}_s\|^2_{W^{-2,1}}\right] \leq Cu.$$

So, by Markov-Chebyshev inequality, we get the Aldous condition. A same proof gives $\langle\langle \tilde{M}^{(n)}\rangle\rangle$ also verify the Aldous condition. Thus, $(\eta^{(n)})_{n\geq 1}$ is tight. □

Proof of theorem 5.5. Let $\tilde{M}$ a continuous Gaussian process with quadratic variation, given for every $f \in C^{2,0}$ ($\subset W^{2,1}$) and $t \in [0,T]$ by:

$$\sum_{k\geq 1}\int_0^t \int_0^{+\infty} 2r(x)\int_0^1 (f(qx)-f(x))^2 G(dq) X_s(dx).$$

Since we have,

$$\forall f \in C^{2,0},\ \sup_{t\in[0,T]} |\tilde{M}^{(n)}(f)| \leq \frac{C_f}{\sqrt{n}}$$

and $\langle \tilde{M}^{(n)}_t\rangle$ converge in law to $\langle \tilde{M}_t\rangle$, we obtain, by [32, theorem 3.11 p.473], the convergence of $\tilde{M}^{(n)}(f)$ to $\tilde{M}(f)$.

By lemma 5.7 and (29) , the sequence $(\eta^{(n)})_{n\geq 1}$ is also tight in $C^{-2,0}$. Let η be an accumulation point. Since the martingale part $\tilde{M}$, η is almost surely continuous. Hence, η solves (27). Using Gronwall's inequality, we obtain that this equation admits in $C([0,T], C^{-2,0})$ a unique solution for a given Gaussian white noise $\tilde{M}$. We deduce the announced result. □

6. Open problems

In the literature, the auxiliary process is sometimes called an hybrid process [6]. When the motion between the branching times is deterministic, indeed A is a vector fields, the auxiliary process is a piecewise deterministic Markov process (PDMP). These processes were introduced in the literature by Davis [14] as a general class of non diffusion stochastic models. Some properties of the PDMPs are given in [15, 31]. But, there is a lot of question about this process.

Speed of convergence for piecewise deterministic Markov processes:
In [11], we see that it is sometimes easier to have a speed of convergence for the embedded chain than for the continuous process (the embedded chain is the continuous process indexed at the jump times). We have some link about the invariant measure of the process and its embedded chain in [12, 15], but it would be interesting to find a link between their long time behaviour. We can also research a criterion, like the Bakry-Emery criterion, to have a quantitative rate of decay for the entropy. We can find a first approach in [9, 10]. It is also interesting to improve theorem 1.2 or proposition 4.9.

Regularity of the stationary distribution:
In [13], we can find some criterion for ergodicity. A natural question is the regularity of the invariant distribution (support, density,...). For instance, is there Hörmander's condition? At the moment, there is some properties of PDMP semi-group in [27, 53].

Other functional of the empirical measure:
this paper gives some result about the convergence of the empirical measure $\sum_{u\in V_t}\delta_{X^u_t}$, but it do not capture other symmetric functional of the population, like the bigger cell or the more infected cell:

$$\max_{u\in V_t} f(X^u_t),$$

or the following functional:

$$\int_0^t \sum_{u\in V_s} f(X^u_s)\, ds = \sum_{u\in\mathcal{T}}\int_{\alpha(u)\wedge t}^{\beta(u)\wedge t} f(X^u_s)\, ds$$

Interesting result for the maximum for branching Brownian motion are developed in [2].

Statistic:
A natural application of our limit theorem is the parameter estimation. Working in the Kolmogorov equation and the macroscopic process, [20] gives a non parametric estimation of r.

Eigenproblem:
The existence of eigenelement is fundamental to have our many-to-one formula. As say in introduction, [19, 43] give some condition to have it. The problem is that, in these papers, the eigenvector are not lower bounded. Hence, it will be interesting to find a theorem like the theorem 4.3.

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