Stability of positive semigroups & their mean field particle interpretations

P. Del Moral

On Future Trends and Opportunities for Monte Carlo methods, Warsaw Dec. 8th 2022

To warm up
- An exo for probabilists (intro stochastic perturbation)
- Stability Markov sg. without tears
- Motivating example = Branching processes

Positive semigroups

A series of good news & mean field particle systems

Stochastic perturbation analysis

Stability positive semigroups

Stochastic interpolation formulae

To warm up

- An exo for probabilists (intro stochastic perturbation)
- Stability Markov sg. without tears
- Motivating example = Branching processes

Positive semigroups

A series of good news & mean field particle systems

Stochastic perturbation analysis

Stability positive semigroups

Stochastic interpolation formulae

Continuous time stochastic interpolation

Continuous time stochastic interpolation

Exo: $(\overline{X}_{s,t} - X_{s,t})$? **for** 2 **diffusions** $(s \leq t)$ **starting at** x **at time** s:

$$dX_{s,t}(x) = b_t\left(X_{s,t}(x)\right)\ dt + \sigma_t\left(X_{s,t}(x)\right)\ dW_t$$

$$d\overline{X}_{s,t}(x) = \overline{b}_t\left(\overline{X}_{s,t}(x)\right)\ dt + \overline{\sigma}_t\left(\overline{X}_{s,t}(x)\right)\ dW_t$$

Continuous time stochastic interpolation

Exo: $(\overline{X}_{s,t} - X_{s,t})$? **for** 2 **diffusions** $(s \leq t)$ **starting at** x **at time** s**:**

$$
\begin{aligned}
dX_{s,t}(x) &= b_t\left(X_{s,t}(x)\right)\ dt + \sigma_t\left(X_{s,t}(x)\right)\ dW_t \\
d\overline{X}_{s,t}(x) &= \overline{b}_t\left(\overline{X}_{s,t}(x)\right)\ dt + \overline{\sigma}_t\left(\overline{X}_{s,t}(x)\right)\ dW_t
\end{aligned}
$$

Semigroup formula

$$
\forall s \leq u \leq t \qquad X_{s,t} = X_{u,t} \circ X_{s,u} \quad \text{and} \quad X_{s,s} = Id
$$

Continuous time stochastic interpolation

Exo: $(\overline{X}_{s,t} - X_{s,t})$? for 2 diffusions $(s \leq t)$ starting at x at time s:

$$
\begin{aligned}
dX_{s,t}(x) &= b_t\left(X_{s,t}(x)\right)\, dt + \sigma_t\left(X_{s,t}(x)\right)\, dW_t \\
d\overline{X}_{s,t}(x) &= \overline{b}_t\left(\overline{X}_{s,t}(x)\right)\, dt + \overline{\sigma}_t\left(\overline{X}_{s,t}(x)\right)\, dW_t
\end{aligned}
$$

Semigroup formula

$$
\forall s \leq u \leq t \qquad X_{s,t} = X_{u,t} \circ X_{s,u} \quad \text{and} \quad X_{s,s} = Id
$$

key idea $=$ Interpolating flow

$$
\overline{X}_{s,t} - X_{s,t} = \int_s^t d_u(X_{u,t} \circ \overline{X}_{s,u})
$$

Forward-Backward interpolation $(\overline{a}, a) := (\overline{\sigma}\ \overline{\sigma}', \sigma\sigma')$

$$d_u \left(X_{u,t} \circ \overline{X}_{s,u}\right)(x) = (d_u X_{u,t})(\overline{X}_{s,u}(x))$$

$$+ \left(\nabla X_{u,t}\right)(\overline{X}_{s,u}(x))'\ d_u \overline{X}_{s,u}(x) + \frac{1}{2}\ \left(\nabla^2 X_{u,t}\right)(\overline{X}_{s,u}(x))'\ \overline{a}_u(\overline{X}_{s,u}(x))\ du$$

Forward-Backward interpolation $(\overline{a}, a) := (\overline{\sigma}\,\overline{\sigma}', \sigma\sigma')$

$$d_u \left(X_{u,t} \circ \overline{X}_{s,u}\right)(x) = (d_u X_{u,t})(\overline{X}_{s,u}(x))$$

$$+ (\nabla X_{u,t})\,(\overline{X}_{s,u}(x))'\ d_u\overline{X}_{s,u}(x) + \frac{1}{2}\ (\nabla^2 X_{u,t})\,(\overline{X}_{s,u}(x))'\ \overline{a}_u(\overline{X}_{s,u}(x))\ du$$

with the backward term

$$-(d_u X_{u,t})(\overline{X}_{s,u}(x))$$

$$= \nabla X_{u,t}(\overline{X}_{s,u}(x))'\ \left(b_u(\overline{X}_{s,u}(x))du + \sigma_u(\overline{X}_{s,u}(x))\ dW_u\right)$$

$$+\frac{1}{2}\ \nabla^2 X_{u,t}(\overline{X}_{s,u}(x))'\ a_u(\overline{X}_{s,u}(x))\ du$$

$\Longrightarrow$ Forward-Backward interpolation formula

$$d_u\left(X_{u,t}\circ\overline{X}_{s,u}\right)(x)$$

$$= \nabla X_{u,t}(\overline{X}_{s,u}(x))' \ \left((\overline{b}_u - b_u)(\overline{X}_{s,u}(x))\ du + (\overline{\sigma}_u - \sigma_u)(\overline{X}_{s,u}(x))\ dW_u\right)$$

$$+\frac{1}{2}\ \nabla^2 X_{u,t}(\overline{X}_{s,u}(x))'\ (\overline{a}_u - a_u)(\overline{X}_{s,u}(x))\ du$$

$\Longrightarrow$ Forward-Backward interpolation formula

$$d_u\left(X_{u,t}\circ\overline{X}_{s,u}\right)(x)$$

$$= \nabla X_{u,t}(\overline{X}_{s,u}(x))' \;\left((\overline{b}_u - b_u)(\overline{X}_{s,u}(x))\;du + (\overline{\sigma}_u - \sigma_u)(\overline{X}_{s,u}(x))\;dW_u\right)$$

$$+\frac{1}{2}\;\nabla^2 X_{u,t}(\overline{X}_{s,u}(x))'\;(\overline{a}_u - a_u)(\overline{X}_{s,u}(x))\;du$$

Extended version when:

$$d\overline{X}_{s,t}(x) = \underbrace{\overline{b}_t\left(\overline{X}_{s,t}(x)\right)}_{=\overline{B}_{s,t}(x)}\;dt + \underbrace{\overline{\sigma}_t\left(\overline{X}_{s,t}(x)\right)}_{=\overline{\Sigma}_{s,t}(x)}\;dW_t$$

Forward-Backward interpolation formula

Ex. 1) $\sigma = \overline{\sigma} \rightsquigarrow$ *drift change formula*

$$\overline{X}_{s,t}(x) - X_{s,t}(x) = \int_s^t \nabla X_{u,t}(\overline{X}_{s,u}(x))' \, (\overline{b}_u - b_u)(\overline{X}_{s,u}(x)) \, du$$

Forward-Backward interpolation formula

Ex. 1) $\sigma = \overline{\sigma} \rightsquigarrow$ *drift change formula*

$$\overline{X}_{s,t}(x) - X_{s,t}(x) = \int_s^t \nabla X_{u,t}(\overline{X}_{s,u}(x))' \, (\overline{b}_u - b_u)(\overline{X}_{s,u}(x)) \, du$$

Ex. 2) $b = \overline{b}$ *&* $\sigma = 0 \rightsquigarrow X$ *deterministic &* $\overline{\sigma} = \epsilon \, \widehat{\sigma}$ *small perturbation*

$$\overline{X}_{s,t}(x) - X_{s,t}(x)$$

$$= \epsilon \int_s^t \nabla X_{u,t}(\overline{X}_{s,u}(x))' \, \widehat{\sigma}_u(\overline{X}_{s,u}(x) \, dW_u + \underbrace{\frac{\epsilon^2}{2} \int_s^t \nabla^2 X_{u,t}(\overline{X}_{s,u}(x))' \, \widehat{a}_u(\overline{X}_{s,u}(x)) \, du}_{\text{bias}}$$

Forward-Backward interpolation formula

Ex. 1) $\sigma = \overline{\sigma} \rightsquigarrow$ *drift change formula*

$$\overline{X}_{s,t}(x) - X_{s,t}(x) = \int_s^t \nabla X_{u,t}(\overline{X}_{s,u}(x))' \, (\overline{b}_u - b_u)(\overline{X}_{s,u}(x)) \, du$$

Ex. 2) $b = \overline{b}$ & $\sigma = 0 \rightsquigarrow X$ *deterministic* & $\overline{\sigma} = \epsilon \, \widehat{\sigma}$ *small perturbation*

$$\overline{X}_{s,t}(x) - X_{s,t}(x)$$

$$= \epsilon \int_s^t \nabla X_{u,t}(\overline{X}_{s,u}(x))' \, \widehat{\sigma}_u(\overline{X}_{s,u}(x) \, dW_u + \underbrace{\frac{\epsilon^2}{2} \int_s^t \nabla^2 X_{u,t}(\overline{X}_{s,u}(x))' \, \widehat{a}_u(\overline{X}_{s,u}(x)) \, du}_{\text{bias}}$$

Stability terms $\subset$ Gradient and Hessian of the stochastic flow $X_{s,t}(x)$

Tangent process, first variational equations, spectral techniques,...

Stab. Markov sg. without tears - V-norms ($V \geq 1/2$)

$$\|f\|_V = \|f/V\| \quad \rightsquigarrow \mathcal{B}_V(E) \quad \text{and} \quad \|\!|\mu - \eta|\!\|_V = |\mu - \eta|(V) \quad \rightsquigarrow \mathcal{P}_V(E)$$

Stab. Markov sg. without tears - V-norms ($V \geq 1/2$)

$$\|f\|_V = \|f/V\| \quad \rightsquigarrow \mathcal{B}_V(E) \quad \text{and} \quad |||\mu - \eta|||_V = |\mu - \eta|(V) \quad \rightsquigarrow \mathcal{P}_V(E)$$

V-Dobrushin contraction coef.:

$$\beta_V(P_{s,s+\tau}) := \sup_{(\mu,\eta)\in\mathcal{P}_V(E)} \frac{|||(\mu-\eta)P_{s,s+\tau}|||_V}{|||\mu-\eta|||_V} \leq 1 - \delta_\tau$$

Stab. Markov sg. without tears - V-norms ($V \geq 1/2$)

$$\|f\|_V = \|f/V\| \quad \rightsquigarrow \mathcal{B}_V(E) \quad \text{and} \quad |||\mu - \eta|||_V = |\mu - \eta|(V) \quad \rightsquigarrow \mathcal{P}_V(E)$$

V-Dobrushin contraction coef.:

$$\beta_V(P_{s,s+\tau}) := \sup_{(\mu,\eta)\in\mathcal{P}_V(E)} \frac{|||(\mu-\eta)P_{s,s+\tau}|||_V}{|||\mu-\eta|||_V} \leq 1 - \delta_\tau$$

$$\Downarrow$$

V-contraction :

$$|||(\mu-\eta)P_{s,s+n\tau}|||_V \leq (1-\delta_\tau)^n \; |||\mu-\eta|||_V$$

Lyapunov cond. (+Penev [CRC-16], +Horton,Jasra Arxiv (21), AAP (22))

$$(1) \qquad P_{s,s+\tau}(V) \leq \epsilon_\tau V + c_\tau$$

$$(2) \qquad \sup_{V(x)\vee V(y)\leq r} \|(\delta_x - \delta_y)P_{s,s+\tau}\|_{tv} \leq 1 - \epsilon_\tau(r)$$

Lyapunov cond. (+Penev [CRC-16], +Horton,Jasra `Arxiv (21), AAP (22)`)

(1) $$P_{s,s+\tau}(V) \leq \epsilon_\tau V + c_\tau$$

(2) $$\sup_{V(x)\vee V(y)\leq r} \|(\delta_x - \delta_y)P_{s,s+\tau}\|_{tv} \leq 1 - \epsilon_\tau(r)$$

$$\Downarrow$$

$$\beta_V(P_{s,s+\tau}) \leq 1 - \delta_\tau$$

Lyapunov cond. (+Penev [CRC-16], +Horton,Jasra `Arxiv (21), AAP (22)`)

$$(1) \qquad P_{s,s+\tau}(V) \leq \epsilon_\tau V + c_\tau$$

$$(2) \qquad \sup_{V(x)\vee V(y)\leq r} \|(\delta_x - \delta_y)P_{s,s+\tau}\|_{tv} \leq 1 - \epsilon_\tau(r)$$

$$\Downarrow$$

$$\beta_V(P_{s,s+\tau}) \leq 1 - \delta_\tau$$

But also

$$(2)' = \left\{ \begin{array}{c} P_{s,s+\tau}(x,dy) \geq q_{s,s+\tau}(x,y)\ \nu_\tau(dy) \\ \\ \inf_{V(x)\vee V(y)\leq r} q_{s,s+\tau}(x,y) \geq \iota_r^-(\tau) > 0 \end{array} \right\} \Longrightarrow (2)$$

Note:
V bounded $\Longrightarrow$ Dobrushin coef. $:= \beta(P_{s,t})$ and $(2)'$=Doeblin min cond.

Branching processes - discrete time

- $M_n(x_{n-1}, dx_n)$ Markov $X_{n-1} \in E_{n-1} \rightsquigarrow X_n \in E_n$ (ex. historical)
- $g_n^i(x_n) \in \mathbb{N}$ = i.i.d. branching r.v., indexed by $x_n \in E_n$ and $i \geq 1$

Branching processes - discrete time

- $M_n(x_{n-1}, dx_n)$ Markov $X_{n-1} \in E_{n-1} \rightsquigarrow X_n \in E_n$ (ex. historical)
- $g_n^i(x_n) \in \mathbb{N}$ = i.i.d. branching r.v., indexed by $x_n \in E_n$ and $i \geq 1$

Branching process

$$\xi_n = (\xi_n^i)_{1 \leq i \leq N_n} \xrightarrow{branching} \widehat{\xi}_n = (\widehat{\xi}_n^i)_{1 \leq i \leq \widehat{N}_n = N_{n+1}} \xrightarrow{exploration} \xi_{n+1}$$

Branching processes - discrete time

- $M_n(x_{n-1}, dx_n)$ Markov $X_{n-1} \in E_{n-1} \rightsquigarrow X_n \in E_n$ (ex. historical)
- $g_n^i(x_n) \in \mathbb{N} =$ i.i.d. branching r.v., indexed by $x_n \in E_n$ and $i \geq 1$

Branching process

$$\xi_n = (\xi_n^i)_{1 \leq i \leq N_n} \xrightarrow{\textit{branching}} \widehat{\xi}_n = (\widehat{\xi}_n^i)_{1 \leq i \leq \widehat{N}_n = N_{n+1}} \xrightarrow{\textit{exploration}} \xi_{n+1}$$

Occupation measure

$$\mathcal{X}_n := \sum_{1 \leq i \leq N_n} \delta_{\xi_n^i} \quad \& \quad G_n(x_n) := \mathbb{E}(g_n^i(x_n))$$

Branching processes - discrete time

- $M_n(x_{n-1}, dx_n)$ Markov $X_{n-1} \in E_{n-1} \rightsquigarrow X_n \in E_n$ (ex. historical)
- $g_n^i(x_n) \in \mathbb{N}$ = i.i.d. branching r.v., indexed by $x_n \in E_n$ and $i \geq 1$

Branching process

$$\xi_n = (\xi_n^i)_{1 \leq i \leq N_n} \xrightarrow{\textit{branching}} \widehat{\xi}_n = (\widehat{\xi}_n^i)_{1 \leq i \leq \widehat{N}_n = N_{n+1}} \xrightarrow{\textit{exploration}} \xi_{n+1}$$

Occupation measure

$$\mathcal{X}_n := \sum_{1 \leq i \leq N_n} \delta_{\xi_n^i} \quad \& \quad G_n(x_n) := \mathbb{E}(g_n^i(x_n))$$

$$\Downarrow$$

First moment = Feynman-Kac measure

$$\mathbb{E}(\mathcal{X}_n(f))/N_0 = \gamma_n(f) := \mathbb{E}\left(f(X_n) \prod_{0 \leq p < n} G_p(X_p)\right)$$

Historical/Path-space models

$$X_n := (X'_0, \ldots, X'_n) \quad \& \quad G_n(X_n) = G'_n(X'_n)$$

$$\Downarrow$$

$$\gamma_n(f) := \mathbb{E}\left(f(X'_0, \ldots, X'_n) \prod_{0 \le p < n} G'_p(X'_p) \right)$$

Historical/Path-space models

$$X_n := (X'_0, \ldots, X'_n) \quad \& \quad G_n(X_n) = G'_n(X'_n)$$

$$\Downarrow$$

$$\gamma_n(f) := \mathbb{E}\left(f(X'_0, \ldots, X'_n) \prod_{0 \leq p < n} G'_p(X'_p) \right)$$

Back to *n*-th time marginals

$$f(X'_0, \ldots, X'_n) = f'(X'_n) \quad \rightharpoonup \quad \gamma_n(f) = \gamma'_n(f')$$

Branching processes - continuous time (easier)

- L_t generator $X_t \in E_t$ (ex. historical $X_t = (X'_s)_{0 \leq s \leq t}$)
- $(V^d, V^b_t) =$ (death,duplication-birth)-rates

Occupation measure

$$\mathcal{X}_t := \sum_{1 \leq i \leq N_t} \delta_{\xi^i_t} \quad \& \quad U_t := V^b_t - V^d_t$$

Branching processes - continuous time (easier)

- L_t generator $X_t \in E_t$ (ex. historical $X_t = (X'_s)_{0 \leq s \leq t}$)
- $(V^d, V^b_t) =$ (death,duplication-birth)-rates

Occupation measure

$$\mathcal{X}_t := \sum_{1 \leq i \leq N_t} \delta_{\xi^i_t} \quad \& \quad U_t := V^b_t - V^d_t$$

$$\Downarrow$$

First moment = Feynman-Kac measure

$$\mathbb{E}(\mathcal{X}_t(f))/N_0 = \gamma_t(f) := \mathbb{E}\left(f(X_t)\ \exp \int_0^t U_s(X_s)\ ds\right)$$

Some application domains

Signal processing/Bayesian inference/machine learning/Physics/Risk-rare events/Maths-bio/...
$\supset$ *Filtering/smoothing, tube conditioning, self-avoiding RW, "quasi"-invariant, rare events, level splitting, interacting MCMC, ground states, leading eigen-triple Schrodinger semigroups, ...*

Real world examples (discrete time):

$$G'_n(x'_n) = p(y_n|x'_n) \qquad G'_n = 1_{A_n} \qquad G_n(x_0, \ldots, x_n) = 1_{x_n \not\in \{x_0, \ldots, x_{n-1}\}}$$

$$G_n = e^{-(\beta_{n+1}-\beta_n)V} \quad \& \quad M_n = MCMC \quad \text{target} \quad \propto \quad e^{-\beta_n V} \qquad \ldots$$

Calculations/Approximations of the measure γ_t ?

To warm up

Positive semigroups

A series of good news & mean field particle systems

Stochastic perturbation analysis

Stability positive semigroups

Stochastic interpolation formulae

Positive semigroups ($X_{s,t}(x)$=flow starting at $X_{s,s}(x) = x$)

Good news: linear evolution

$$\gamma_t = \gamma_s Q_{s,t} \quad \leftharpoondown \quad (\gamma Q)(dy) := \int \gamma(dx) Q(x, dy)$$

with the positive (positivity preserving) semigroup

$$Q_{s,t}(f)(x) := \mathbb{E}\left(f(X_{s,t}(x))\ \exp\left(\int_s^t U_u(X_{s,u}(x))\ du\right)\right)$$

Positive semigroups ($X_{s,t}(x)$=flow starting at $X_{s,s}(x) = x$)

Good news: linear evolution

$$\gamma_t = \gamma_s Q_{s,t} \quad \leftharpoondown \quad (\gamma Q)(dy) := \int \gamma(dx) Q(x, dy)$$

with the positive (positivity preserving) semigroup

$$Q_{s,t}(f)(x) := \mathbb{E}\left(f(X_{s,t}(x)) \; \exp\left(\int_s^t U_u(X_{s,u}(x)) \; du \right) \right)$$

Bad news (practical viewpoint): degenerate total mass

$$\gamma_t(1) \longrightarrow_{t \to \infty} \{0, 1, \infty\}$$

Positive semigroups ($X_{s,t}(x)$=flow starting at $X_{s,s}(x) = x$)

Good news: linear evolution

$$\gamma_t = \gamma_s Q_{s,t} \quad \leftharpoondown \quad (\gamma Q)(dy) := \int \gamma(dx) Q(x, dy)$$

with the positive (positivity preserving) semigroup

$$Q_{s,t}(f)(x) := \mathbb{E}\left(f(X_{s,t}(x)) \ \exp\left(\int_s^t U_u(X_{s,u}(x)) \ du\right)\right)$$

Bad news (practical viewpoint): degenerate total mass

$$\gamma_t(1) \longrightarrow_{t\to\infty} \{0, 1, \infty\}$$

i.i.d. copies of branching process $\rightsquigarrow$ *very bad news for the computer*

To warm up

Positive semigroups

A series of good news & mean field particle systems

Stochastic perturbation analysis

Stability positive semigroups

Stochastic interpolation formulae

Key idea = Normalization to deal with proba. $= \mathcal{P}(E_t)$

1-st Good news $\rightharpoonup \exists$ perfect sampler !

$$\eta_t(f) := \gamma_t(f)/\gamma_t(1) \quad \rightharpoonup \quad \eta_t = \mathrm{Law}(\overline{X}_t) \in \mathcal{P}(E_t)$$

Key idea = Normalization to deal with proba. $= \mathcal{P}(E_t)$

1-st Good news $\rightharpoonup \exists$ **perfect sampler !**

$$\eta_t(f) := \gamma_t(f)/\gamma_t(1) \quad \rightharpoonup \quad \eta_t = \mathrm{Law}(\overline{X}_t) \in \mathcal{P}(E_t)$$

with (non unique) the generator $\overline{L}_t = \mathcal{L}_{t,\eta_t}$ of $\overline{X}_t$ given (example) by

$$\mathcal{L}_{t,\eta_t}(f)(x) := L_t(f)(x)$$

$$+V_t^d(x) \int (f(y) - f(x))\, \eta_t(dy) + \int (f(y) - f(x))\, V_t^b(y)\, \eta_t(dy)$$

More good news

2-nd Good news $\rightharpoonup$ we can come back to γ_t

$$\gamma_t(f) = \mathbb{E}\left(f(X_t)\exp\left(\int_0^t U(X_s)ds\right)\right) = \eta_t(f)\ \exp\left(\int_0^t \ \eta_s(U_s)\ ds\right)$$

More good news

2-nd Good news $\rightharpoonup$ we can come back to γ_t

$$\gamma_t(f) = \mathbb{E}\left(f(X_t)\exp\left(\int_0^t U(X_s)ds\right)\right) = \eta_t(f)\ \exp\left(\int_0^t\ \eta_s(U_s)\ ds\right)$$

$$\Downarrow$$

Time average formula

$$\frac{1}{t}\log\gamma_t(1) = \frac{1}{t}\int_0^t\ \eta_s(U_s)\ ds$$

More good news

For any choice $\mathcal{L}_{t,\eta}$ satisfying the compatibility condition

$$\partial_t \eta_t(f) = \Lambda_t(\eta_t)(f) := \eta_t(L_t f) + \eta_t(f(U_t - \eta_t(U_t))) = \eta_t(\mathcal{L}_{t,\eta_t}(f))$$

More good news

For any choice $\mathcal{L}_{t,\eta}$ satisfying the compatibility condition

$$\partial_t \eta_t(f) = \Lambda_t(\eta_t)(f) := \eta_t(L_t f) + \eta_t(f(U_t - \eta_t(U_t))) = \eta_t(\mathcal{L}_{t,\eta_t}(f))$$

$$\Downarrow$$

3-rd Good news
$\rightharpoonup$ **mean field "continuous-time sampler"** $\xi_t = (\xi_t^i)_{1\leq i\leq N} \in E_t^N$ **!**

$$\mathcal{G}_t(F)(x_1,\ldots,x_N) := \sum_{1\leq i\leq N} \mathcal{L}_{t,m(x)}\left(F_{x_{-i}}\right)(x_i)$$

with $x_{-i} = (x_j)_{j\in\{1,\ldots,N\}-\{i\}}$ and

$$m(x) := \frac{1}{N}\sum_{1\leq i\leq N}\delta_{x_i} \quad \& \quad F_{x_{-i}}(x_i) = F(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_N)$$

More good news

4-th Good news $\rightharpoonup$ path-space/historical $=$ Genealogical trees!

$$\xi^i_t = \left(\xi^i_{s,t}\right)_{0\leq s\leq t} \quad \text{with} \quad \xi^i_{s,t} = \quad \text{ancestor of } \xi^i_{t,t} \text{ at level } s$$

$$\Downarrow$$

Occupation measure of the ancestral tree:

$$m(\xi_t) := \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi^i_t} = \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\left(\xi^i_{s,t}\right)_{0\leq s\leq t}}$$

More good news

Mimicking the formula

$$\gamma_t(f) = \eta_t(f)\ \exp\left(\int_0^t \eta_s(U_s)\ ds\right)$$

$$\Downarrow$$

5-th Good news $\rightharpoonup$ unbiased estimates available !

$$\gamma_t^N(f) = \eta_t^N(f)\ \exp\left(\int_0^t \eta_s^N(U_s)ds\right) \quad \text{with} \quad \eta_t^N := m(\xi_t)$$

More good news

Mimicking the formula

$$\gamma_t(f) = \eta_t(f)\ \exp\left(\int_0^t\ \eta_s(U_s)\ ds\right)$$

$$\Downarrow$$

5-th Good news $\rightharpoonup$ unbiased estimates available !

$$\gamma_t^N(f) = \eta_t^N(f)\ \exp\left(\int_0^t\ \eta_s^N(U_s)ds\right) \quad \text{with} \quad \eta_t^N := m(\xi_t)$$

$$\Downarrow$$

$$\mathbb{E}(\gamma_t^N(f)) = \gamma_t(f) \quad \textbf{EVEN IF} \quad \mathbb{E}(\eta_t^N(f)) = \eta_t(f) + \mathrm{O}(1/N)$$

To warm up

Positive semigroups

A series of good news & mean field particle systems

Stochastic perturbation analysis

Stability positive semigroups

Stochastic interpolation formulae

Stochastic perturbation analysis

Itô formula

$$dF(\xi_t) = \mathcal{G}_t(F)(\xi_t) + d\mathcal{M}_t(F) \quad \text{with} \quad \partial_t \langle \mathcal{M}(F) \rangle_t = \Gamma_{\mathcal{G}_t}(F, F)(\xi_t)$$

$$\Downarrow$$

For <u>ANY</u> mean field particle system/nonlinear evolution:

$$\left\{ \begin{array}{rcl} d\eta_t & = & \Lambda_t(\eta_t)\, dt \\ \\ d\eta_t^N & = & \Lambda_t(\eta_t^N)\, dt + \frac{1}{\sqrt{N}}\, dM_t \quad \text{with} \quad \partial_t \langle M(f) \rangle_t = \eta_t^N \Gamma_{\mathcal{L}_{t,\eta_t^N}}(f, f) \end{array} \right.$$

Stochastic perturbation analysis

Itô formula

$$dF(\xi_t) = \mathcal{G}_t(F)(\xi_t) + d\mathcal{M}_t(F) \quad \text{with} \quad \partial_t \langle \mathcal{M}(F) \rangle_t = \Gamma_{\mathcal{G}_t}(F,F)(\xi_t)$$

$$\Downarrow$$

For ANY mean field particle system/nonlinear evolution:

$$\left\{ \begin{array}{rcl} d\eta_t & = & \Lambda_t(\eta_t)\, dt \\ \\ d\eta_t^N & = & \Lambda_t(\eta_t^N)\, dt + \frac{1}{\sqrt{N}}\, dM_t \quad \text{with} \quad \partial_t \langle M(f) \rangle_t = \eta_t^N \Gamma_{\mathcal{L}_{t,\eta_t^N}}(f,f) \end{array} \right.$$

$$\Downarrow$$

Back to perturbation analysis $\rightsquigarrow$ Need to study the stability of $\eta_t := \Phi_{s,t}(\eta_s)$

To warm up

Positive semigroups

A series of good news & mean field particle systems

Stochastic perturbation analysis

Stability positive semigroups

Stochastic interpolation formulae

Stability of the normalized sg $\Phi_{s,t}$

Key observation

$$\Phi_{s,t}(\eta)(f) = \frac{\eta Q_{s,t}(f)}{\eta Q_{s,t}(1)} = \frac{\eta\left(Q_{s,t}(1) R^{(t)}_{s,t}(f)\right)}{\eta\left(Q_{s,t}(1)\right)}$$

with the triangular array of Markov transitions

$$R^{(t)}_{u,v}(f) := \frac{Q_{u,v}(f\ Q_{v,t}(1))}{Q_{u,v}(Q_{v,t}(1))} \Longrightarrow R^{(t)}_{s_0,s_n} = R^{(t)}_{s_0,s_1} \ldots R^{(t)}_{s_{n-1},s_n}$$

The Markov case:

$$Q_{u,v}(1) = 1 \Longrightarrow R^{(t)}_{u,v} = Q_{u,v} = P_{u,v} \quad \text{Markov semigroup}$$

Stability of the Markov triangular array

$$R^{(t)}_{u,v}(f) := \frac{Q_{u,v}(f\ Q_{v,t}(1))}{Q_{u,v}(Q_{v,t}(1))} \Longrightarrow R^{(t)}_{s_0,s_n} = R^{(t)}_{s_0,s_1} \ldots R^{(t)}_{s_{n-1},s_n}$$

$\Downarrow$ *(strong) Dobrushin contraction* $\beta(R^{(t)}_{s,s+\tau}) \leq (1-\epsilon_\tau)$ $\Downarrow$

Time-varying models

$$\sup_{(\mu,\eta)\in\mathcal{P}(E)^2} \|\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)\|_{tv} = \sup_{(x,y)\in E^2} \|\delta_x R^{(t)}_{s,t} - \delta_y R^{(t)}_{s,t}\|_{tv} \leq (1-\epsilon_\tau)^{(t-s)/\tau}$$

2-nd key observation (ex. $U_t(x)$ unif. bounded)

$$\frac{Q_{s,t}(1)(x)}{\eta Q_{s,t}(1)} = \exp\left(\int_s^t \left[\Phi_{s,u}(\delta_x) - \Phi_{s,u}(\eta)\right](U_u)\ du\right) < q(\eta) \quad \perp (s,t,x)$$

2-nd key observation (ex. $U_t(x)$ unif. bounded)

$$\frac{Q_{s,t}(1)(x)}{\eta Q_{s,t}(1)} = \exp\left(\int_s^t \left[\Phi_{s,u}(\delta_x) - \Phi_{s,u}(\eta)\right](U_u)\, du\right) < q(\eta) \quad \perp (s,t,x)$$

$$\Downarrow$$

Time homogeneous models

$$\|\Phi_t(\mu) - \Phi_t(\eta)\|_{tv} \leq (q(\mu) \wedge q(\eta))\ (1-\epsilon_\tau)^{t/\tau}\ \|\mu - \eta\|_{tv}.$$

2-nd key observation (ex. $U_t(x)$ unif. bounded)

$$\frac{Q_{s,t}(1)(x)}{\eta Q_{s,t}(1)} = \exp\left(\int_s^t \left[\Phi_{s,u}(\delta_x) - \Phi_{s,u}(\eta)\right](U_u)\, du\right) < q(\eta) \quad \perp (s,t,x)$$

$$\Downarrow$$

Time homogeneous models

$$\|\Phi_t(\mu) - \Phi_t(\eta)\|_{tv} \leq (q(\mu) \wedge q(\eta))\; (1-\epsilon_\tau)^{t/\tau}\; \|\mu - \eta\|_{tv}.$$

$\Rightarrow$ **Existence leading pairs (ρ, h)/"QSD" η_∞/expo decays/...**

$$Q_t(h) = e^{\rho t}\, h \quad \text{and} \quad \eta_\infty = \Phi_t(\eta_\infty)$$

Ex.: absolutely continuous sg density $q_{s,t}(x, y)$ on $E \subset \mathbb{R}^n$

(H_0) **Uniform positive sg:**

$$0 < \iota^-(\tau) := \inf q_{t,t+\tau}(x, y) \leq \iota^-(\tau) := \sup q_{t,t+\tau}(x, y) < \infty$$

Examples: finite space, diffusions on compact manifolds (Aronson/Nash/Varopolous/...), reflected diffusions on boundaries/compact connected subsets, discrete time bi-Laplace, ...

Ex.: absolutely continuous sg density $q_{s,t}(x,y)$ on $E \subset \mathbb{R}^n$

(H_0) **Uniform positive sg:**

$$0 < \iota^-(\tau) := \inf q_{t,t+\tau}(x,y) \leq \iota^-(\tau) := \sup q_{t,t+\tau}(x,y) < \infty$$

Examples: finite space, diffusions on compact manifolds (Aronson/Nash/Varopolous/...), reflected diffusions on boundaries/compact connected subsets, discrete time bi-Laplace, ...

$$(H_0) \implies (H_1): \quad \operatorname{osc}\left(R^{(t)}_{s,t}(f)\right) \leq (1-\epsilon_\tau)^{(t-s)/\tau} \operatorname{osc}(f)$$

$$\implies (H_2): \quad Q_{s,t}(1)(x)/Q_{s,t}(1)(y) \leq q$$

Ex.: absolutely continuous sg density $q_{s,t}(x,y)$ on $E \subset \mathbb{R}^n$

(H_0) **Uniform positive sg:**

$$0 < \iota^-(\tau) := \inf q_{t,t+\tau}(x,y) \leq \iota^-(\tau) := \sup q_{t,t+\tau}(x,y) < \infty$$

Examples: finite space, diffusions on compact manifolds (Aronson/Nash/Varopolous/...), reflected diffusions on boundaries/compact connected subsets, discrete time bi-Laplace, ...

$$\begin{aligned} (H_0) \quad \Longrightarrow \quad (H_1): & \quad \operatorname{osc}\left(R^{(t)}_{s,t}(f)\right) \leq (1-\epsilon_\tau)^{(t-s)/\tau} \operatorname{osc}(f) \\ \Longrightarrow \quad (H_2): & \quad Q_{s,t}(1)(x)/Q_{s,t}(1)(y) \leq q \end{aligned}$$

Key observation:
(H_2) **is met for path space** when (H_1) is met the marginal model.

Back to particles $\mathrm{osc}(f), \mathrm{osc}(g) \leq 1$

Uniform estimates w.r.t. time:

$$
\begin{aligned}
(H_1) \quad &\Longrightarrow \quad |\mathbb{E}(m(\xi_t)(f)) - \eta_t(f)| \leq c/N \\
(H_2) \quad &\Longrightarrow \quad |\mathbb{E}(m(\xi_t)(f)) - \eta_t(f)| \leq c\ t/N
\end{aligned}
$$

Back to particles $\mathrm{osc}(f), \mathrm{osc}(g) \leq 1$

Uniform estimates w.r.t. time:

$$
\begin{array}{ccl}
(H_1) & \Longrightarrow & |\mathbb{E}(m(\xi_t)(f)) - \eta_t(f)| \leq c/N \\
(H_2) & \Longrightarrow & |\mathbb{E}(m(\xi_t)(f)) - \eta_t(f)| \leq c\ t/N
\end{array}
$$

As well as for any $i \neq j$

$$
\begin{array}{ccl}
(H_1) & \Longrightarrow & \left|\mathbb{E}\left(f(\xi_t^i)\ g(\xi_t^j)\right) - \eta_t(f)\ \eta_t(g)\right| \leq c/N \\
(H_2) & \Longrightarrow & \left|\mathbb{E}\left(f(\xi_t^i)\ g(\xi_t^j)\right) - \eta_t(f)\ \eta_t(g)\right| \leq c\ t/N
\end{array}
$$

Particle Lyapunov estimate ($U_t = U$)

what we know

$$\frac{1}{t}\log\gamma_t(1) = \frac{1}{t}\int_0^t \eta_s(U)ds = \eta_\infty(U) + \mathrm{O}(1/t)$$

recalling that

$$\gamma_t^N(f) = \eta_t^N(f)\ \exp\left(\int_0^t \eta_s^N(U)ds\right) \quad \text{with} \quad \eta_t^N := m(\xi_t)$$

we have

$$\frac{1}{t}\log\left(\gamma_t^N(1)/\gamma_t(1)\right) = \frac{1}{t}\int_0^t \left(\eta_s^N(U) - \eta_s(V)\right)\,ds$$

Particle Lyapunov estimate ($U_t = U$)

what we know

$$\frac{1}{t}\log\gamma_t(1) = \frac{1}{t}\int_0^t \eta_s(U)ds = \eta_\infty(U) + \mathrm{O}(1/t)$$

recalling that

$$\gamma_t^N(f) = \eta_t^N(f)\ \exp\left(\int_0^t \eta_s^N(U)ds\right) \quad \text{with} \quad \eta_t^N := m(\xi_t)$$

we have

$$\frac{1}{t}\log\left(\gamma_t^N(1)/\gamma_t(1)\right) = \frac{1}{t}\int_0^t \left(\eta_s^N(U) - \eta_s(V)\right)\, ds$$

Finally, using the uniform estimates

$$(H_1) \Longrightarrow |\mathbb{E}\left(((\eta_t^N(f)) - \eta_t(f))^{1 \text{ or } 2}\right)| \leq c/N \Longrightarrow \ldots$$

To warm up

Positive semigroups

A series of good news & mean field particle systems

Stochastic perturbation analysis

Stability positive semigroups

Stochastic interpolation formulae

1st. step proof - Taylor exp. $\mu_0 \simeq \mu_1$

$$\Phi_{s,t}(\mu_1) \simeq \Phi_{s,t}(\mu_0) + (\mu_1 - \mu_0)\partial_{\mu_0}\Phi_{s,t} + 2^{-1}\ (\mu_1 - \mu_0)^{\otimes 2}\partial^2_{\mu_0}\Phi_{s,t}$$

in terms of

$$Q^{\eta}_{s,t}(f) := \frac{Q_{s,t}(f)}{\eta Q_{s,t}(1)} = \underbrace{Q^{\eta}_{s,t}(1)}_{\text{bounded}} \times \underbrace{R^{(t)}_{s,t}(f)}_{\text{stable}}$$

$\Downarrow$ *(ratio form $\Rightarrow$ 2 lines proof)*

$$\partial_{\eta}\Phi_{s,t}(f) \quad = \quad Q^{\eta}_{s,t}[f - \Phi_{s,t}(\eta)f]$$

1st. step proof - Taylor exp. $\mu_0 \simeq \mu_1$

$$\Phi_{s,t}(\mu_1) \simeq \Phi_{s,t}(\mu_0) + (\mu_1 - \mu_0)\partial_{\mu_0}\Phi_{s,t} + 2^{-1}\ (\mu_1 - \mu_0)^{\otimes 2}\partial^2_{\mu_0}\Phi_{s,t}$$

in terms of

$$Q^{\eta}_{s,t}(f) := \frac{Q_{s,t}(f)}{\eta Q_{s,t}(1)} = \underbrace{Q^{\eta}_{s,t}(1)}_{\text{bounded}} \times \underbrace{R^{(t)}_{s,t}(f)}_{\text{stable}}$$

$\Downarrow$ *(ratio form $\Rightarrow$ 2 lines proof)*

$$\partial_\eta \Phi_{s,t}(f) = Q^{\eta}_{s,t}[f - \Phi_{s,t}(\eta)f]$$

$$\partial^2_\eta \Phi_{s,t}(f) = Q^{\eta}_{s,t}(1) \otimes \partial_\eta \Phi_{s,t}(f) + \partial_\eta \Phi_{s,t}(f) \otimes Q^{\eta}_{s,t}(1)$$

1st. step proof - Taylor exp. $\mu_0 \simeq \mu_1$

$$\Phi_{s,t}(\mu_1) \simeq \Phi_{s,t}(\mu_0) + (\mu_1 - \mu_0)\partial_{\mu_0}\Phi_{s,t} + 2^{-1}\ (\mu_1 - \mu_0)^{\otimes 2}\partial^2_{\mu_0}\Phi_{s,t}$$

in terms of

$$Q^{\eta}_{s,t}(f) := \frac{Q_{s,t}(f)}{\eta Q_{s,t}(1)} = \underbrace{Q^{\eta}_{s,t}(1)}_{\text{bounded}} \times \underbrace{R^{(t)}_{s,t}(f)}_{\text{stable}}$$

$\Downarrow$ *(ratio form $\Rightarrow$ 2 lines proof)*

$$\partial_{\eta}\Phi_{s,t}(f) \quad = \quad Q^{\eta}_{s,t}[f - \Phi_{s,t}(\eta)f]$$

$$\partial^2_{\eta}\Phi_{s,t}(f) \quad = \quad Q^{\eta}_{s,t}(1) \otimes \partial_{\eta}\Phi_{s,t}(f) + \partial_{\eta}\Phi_{s,t}(f) \otimes Q^{\eta}_{s,t}(1)$$

$$\partial^2_{\mu_1,\mu_0}\Phi_{s,t} \quad := \quad \mu_0 Q^{\mu_1}_{s,t}(1)\ \times \partial^2_{\mu_0}\Phi_{s,t}$$ $\rightsquigarrow$ remainder/equality in Taylor

Taylor exp. $\mu_0 \simeq \mu_1$ (also works for McKean-Vlasov diff.)

$$\Phi_{s,t}(\mu_1) \simeq \Phi_{s,t}(\mu_0) + (\mu_1 - \mu_0)\partial_{\mu_0}\Phi_{s,t} + 2^{-1}\ (\mu_1 - \mu_0)^{\otimes 2}\partial^2_{\mu_0}\Phi_{s,t}$$

$$\Downarrow$$

Operator norm/exponential decays

$$(H_1) \Longrightarrow \vee_{k=1,2} \|\!|\partial^k_{\mu_0}\Phi_{s,t}|\!\| \vee \|\!|\partial^2_{\mu_1,\mu_0}\Phi_{s,t}|\!\| \simeq e^{-\lambda(t-s)}$$

$$\Downarrow$$

AND almost ready for stochastic interpolation (cf. exo)

Reminder/Intermediate proof step

Mutation gen. $L_t = L_t^d + L_t^c$ **- perturbation martingales**

$$M_t(f) = M_t^c(f) + M_t^d(f) \quad \text{and} \quad \mathcal{M}_t(F) = \mathcal{M}_t^c(F) + \mathcal{M}_t^d(F)$$

Reminder/Intermediate proof step

Mutation gen. $L_t = L_t^d + L_t^c$ **- perturbation martingales**

$$M_t(f) = M_t^c(f) + M_t^d(f) \quad \text{and} \quad \mathcal{M}_t(F) = \mathcal{M}_t^c(F) + \mathcal{M}_t^d(F)$$

Jumps of order c/N $\rightsquigarrow$ Example

$$N\ \partial_s \mathbb{E}\left[(\Delta m(\xi_s))^{\otimes 2}(f \otimes f) \mid \mathcal{F}_{s-} \right] = m(\xi_{s-}) \Gamma_{L^d_{m(\xi_{s-})}}(f, f)$$

Reminder/Intermediate proof step

Mutation gen. $L_t = L_t^d + L_t^c$ **- perturbation martingales**

$$M_t(f) = M_t^c(f) + M_t^d(f) \quad \text{and} \quad \mathcal{M}_t(F) = \mathcal{M}_t^c(F) + \mathcal{M}_t^d(F)$$

Jumps of order $c/N \rightsquigarrow$ *Example*

$$N\ \partial_s \mathbb{E}\left[(\Delta m(\xi_s))^{\otimes 2}(f \otimes f) \mid \mathcal{F}_{s-}\right] = m(\xi_{s-})\Gamma_{L^d_{m(\xi_{s-})}}(f,f)$$

Variational/jump measures

$$\Upsilon^2_{m(\xi_{s-})}\Phi_{s,t}(f) := N\ \partial_s \mathbb{E}\left[(\Delta m(\xi_s))^{\otimes 2}\ \overline{\partial}^2_{m(\xi_s),m(\xi_{s-})}\Phi_{s,t}(f) \mid \mathcal{F}_{s-}\right]$$

Final step: interpolation formula

$$m(\xi_t) - \Phi_{0,t}(m(\xi_0)) = \int_0^t d_s\Phi_{s,t}(m(\xi_s))$$

with

$$d_s\Phi_{s,t}(m(\xi_s))(f) = \frac{1}{\sqrt{N}}\, dM_s^c\left(\partial_{m(\xi_\centerdot)}\Phi_{\centerdot,t}(f)\right) + d\mathcal{M}_s^d\left(\Phi_{\centerdot,t}(m(\centerdot))(f)\right)$$

Final step: interpolation formula

$$m(\xi_t) - \Phi_{0,t}(m(\xi_0)) = \int_0^t d_s\Phi_{s,t}(m(\xi_s))$$

with

$$d_s\Phi_{s,t}(m(\xi_s))(f) = \frac{1}{\sqrt{N}}\, dM_s^c\left(\partial_{m(\xi_{\bullet})}\Phi_{\bullet,t}(f)\right) + d\mathcal{M}_s^d\left(\Phi_{\bullet,t}(m(\bullet))(f)\right)$$

$$+\frac{1}{N}\underbrace{\left(\Upsilon^2_{m(\xi_{s-})}\Phi_{s,t}(f) - m(\xi_s)\Gamma_{L_s^c}\left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)}\Phi_{s,t}(f)\right)\right) ds}_{\text{bias}}$$

Extension to V-norms - V positive sg

(1) $\exists V \in \mathcal{B}_\infty(E) =$ unif.> 0, loc. bound, compact sub-level sets in E

$\rightsquigarrow 1/V \in \mathcal{B}_0(E) :=$ null a infinity.

Extension to V-norms - V positive sg

(1) $\exists V \in \mathcal{B}_\infty(E) =$ unif.> 0, loc. bound, compact sub-level sets in E

$\rightsquigarrow 1/V \in \mathcal{B}_0(E) :=$ null a infinity.

$$Q_{t,t+\tau}(V)/V \leq \Theta_\tau \in \mathcal{B}_0(E) \quad \text{or} \quad Q_\tau(V)/V \in \mathcal{B}_0(E)$$

Extension to V-norms - V positive sg

(1) $\exists V \in \mathcal{B}_\infty(E)$ = unif.>0, loc. bound, compact sub-level sets in E

$\rightsquigarrow 1/V \in \mathcal{B}_0(E) :=$ null a infinity.

$$Q_{t,t+\tau}(V)/V \leq \Theta_\tau \in \mathcal{B}_0(E) \quad \text{or} \quad Q_\tau(V)/V \in \mathcal{B}_0(E)$$

(2) On compact sub-level sets:

$$0 < \iota_r^-(\tau) \leq \inf_{V(x)\vee V(y)\leq r} q_{s,s+\tau}(x,y) \leq \sup_{V(x)\vee V(y)\leq r} q_{t,t+\tau}(x,y) \leq \iota_r(\tau) < \infty$$

Extension to V-norms - V positive sg

(1) $\exists V \in \mathcal{B}_\infty(E) =$ unif.>0, loc. bound, compact sub-level sets in E

$\rightsquigarrow 1/V \in \mathcal{B}_0(E) :=$ null a infinity.

$$Q_{t,t+\tau}(V)/V \leq \Theta_\tau \in \mathcal{B}_0(E) \quad \text{or} \quad Q_\tau(V)/V \in \mathcal{B}_0(E)$$

(2) On compact sub-level sets:

$$0 < \iota_r^-(\tau) \leq \inf_{V(x)\vee V(y)\leq r} q_{s,s+\tau}(x,y) \leq \sup_{V(x)\vee V(y)\leq r} q_{t,t+\tau}(x,y) \leq \iota_r(\tau) < \infty$$

$$\downarrow$$

Examples:

$$E = \mathbb{R} \implies V(x) = 1 + x^2, e^{|x|}, \ldots \in \mathcal{B}_\infty(E)$$

Extension to V-norms - V positive sg

(1) $\exists V \in \mathcal{B}_\infty(E) =$ unif.> 0, loc. bound, compact sub-level sets in E

$\rightsquigarrow 1/V \in \mathcal{B}_0(E) :=$ null a infinity.

$$Q_{t,t+\tau}(V)/V \leq \Theta_\tau \in \mathcal{B}_0(E) \quad \text{or} \quad Q_\tau(V)/V \in \mathcal{B}_0(E)$$

(2) On compact sub-level sets:

$$0 < \iota_r^-(\tau) \leq \inf_{V(x)\vee V(y)\leq r} q_{s,s+\tau}(x,y) \leq \sup_{V(x)\vee V(y)\leq r} q_{t,t+\tau}(x,y) \leq \iota_r(\tau) < \infty$$

$$\downarrow$$

Examples:

$$E = \mathbb{R} \implies V(x) = 1 + x^2, e^{|x|}, \ldots \in \mathcal{B}_\infty(E)$$
$$E =]0,1] \implies V(x) = 1/x \in \mathcal{B}_\infty(E)$$

Extension to V-norms - V positive sg

(1) $\exists V \in \mathcal{B}_\infty(E) =$ unif.> 0, loc. bound, compact sub-level sets in E

$\rightsquigarrow 1/V \in \mathcal{B}_0(E) :=$ null a infinity.

$$Q_{t,t+\tau}(V)/V \leq \Theta_\tau \in \mathcal{B}_0(E) \quad \text{or} \quad Q_\tau(V)/V \in \mathcal{B}_0(E)$$

(2) On compact sub-level sets:

$$0 < \iota_r^-(\tau) \leq \inf_{V(x)\vee V(y)\leq r} q_{s,s+\tau}(x,y) \leq \sup_{V(x)\vee V(y)\leq r} q_{t,t+\tau}(x,y) \leq \iota_r(\tau) < \infty$$

$$\downarrow$$

Examples:

$$E = \mathbb{R} \implies V(x) = 1 + x^2, e^{|x|}, \ldots \in \mathcal{B}_\infty(E)$$

$$E =]0,1] \implies V(x) = 1/x \in \mathcal{B}_\infty(E)$$

$$E =]0,1[\implies 1/x \notin \mathcal{B}_\infty(E)$$

Extension to V-norms - V positive sg

(1) $\exists V \in \mathcal{B}_\infty(E) =$ unif.> 0, loc. bound, compact sub-level sets in E

$\rightsquigarrow 1/V \in \mathcal{B}_0(E) :=$ null a infinity.

$$Q_{t,t+\tau}(V)/V \leq \Theta_\tau \in \mathcal{B}_0(E) \quad \text{or} \quad Q_\tau(V)/V \in \mathcal{B}_0(E)$$

(2) On compact sub-level sets:

$$0 < \iota_r^-(\tau) \leq \inf_{V(x)\vee V(y)\leq r} q_{s,s+\tau}(x,y) \leq \sup_{V(x)\vee V(y)\leq r} q_{t,t+\tau}(x,y) \leq \iota_r(\tau) < \infty$$

$$\downarrow$$

Examples:

$$E = \mathbb{R} \implies V(x) = 1 + x^2, e^{|x|}, \ldots \in \mathcal{B}_\infty(E)$$

$$E =]0,1] \implies V(x) = 1/x \in \mathcal{B}_\infty(E)$$

$$E =]0,1[\implies 1/x \notin \mathcal{B}_\infty(E) \quad \text{but} \quad V(x) = 1/(1-x) + 1/x \in \mathcal{B}_\infty(E)$$

Time varying V-positive sg

Theo [+Horton-Jasra Arxiv (21), AAP (22/23)]

$$(1) \;\; \& \;\; (2) \quad \Longrightarrow \quad \|\Phi_{s,t}(\mu)-\Phi_{s,t}(\eta)\|_V \leq \kappa(\eta,\mu)\; e^{-\lambda(t-s)} \;\; \|\mu-\eta\|_V$$

Time varying V-positive sg

Theo [+Horton-Jasra Arxiv (21), AAP (22/23) **]**

$$(1) \;\; \& \;\; (2) \quad \Longrightarrow \quad \|\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)\|_V \leq \kappa(\eta,\mu)\; e^{-\lambda(t-s)} \;\; \|\mu - \eta\|_V$$

$\rightsquigarrow$ *Extended version of* :

Whiteley AAP-13,

Time varying V-positive sg

Theo [+Horton-Jasra Arxiv (21), AAP (22/23)]

$$(1)\ \&\ (2) \quad \Longrightarrow \quad \|\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)\|_V \leq \kappa(\eta,\mu)\ e^{-\lambda(t-s)}\ \ \|\mu - \eta\|_V$$

⇝ *Extended version of* :

Whiteley AAP-13,itself based on Douc-Moulines-Ritov EJP-09,

Time varying V-positive sg

Theo [`+Horton-Jasra Arxiv (21), AAP (22/23)`]

$$(1) \;\&\; (2) \quad \Longrightarrow \quad \|\Phi_{s,t}(\mu)-\Phi_{s,t}(\eta)\|_V \leq \kappa(\eta,\mu)\ e^{-\lambda(t-s)}\ \ \|\mu-\eta\|_V$$

$\rightsquigarrow$ *Extended version of* :

Whiteley AAP-13,itself based on Douc-Moulines-Ritov EJP-09,itself based on Kleptsyna-Veretennikov PTRF-08,...

Time varying V-positive sg

Theo [+Horton-Jasra Arxiv (21), AAP (22/23)]

$$(1) \;\&\; (2) \quad \Longrightarrow \quad \|\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)\|_V \leq \kappa(\eta,\mu)\ e^{-\lambda(t-s)}\ \ \|\mu - \eta\|_V$$

$\rightsquigarrow$ *Extended version of* :

Whiteley AAP-13,itself based on Douc-Moulines-Ritov EJP-09,itself based on Kleptsyna-Veretennikov PTRF-08,...

to Neumann/Dirichlet boundaries/continuous time models, no use of coupling techniques, but not valid for random sg (filtering)

Time varying V-positive sg

Theo [+Horton-Jasra Arxiv (21), AAP (22/23)]

$$(1) \;\&\; (2) \quad \Longrightarrow \quad \|\Phi_{s,t}(\mu) - \Phi_{s,t}(\eta)\|_V \leq \kappa(\eta,\mu)\ e^{-\lambda(t-s)}\ \ \|\mu - \eta\|_V$$

$\rightsquigarrow$ *Extended version of* :

Whiteley AAP-13,itself based on Douc-Moulines-Ritov EJP-09,itself based on Kleptsyna-Veretennikov PTRF-08,. . .

to Neumann/Dirichlet boundaries/continuous time models, no use of coupling techniques, but not valid for random sg (filtering)

Particle estimates for this class of models?

Time varying Krein-Rutman theorem

$(0 < H/V \in \mathcal{B}_0(E)$ & $\mu_0 = \mu \in \mathcal{P}_V(E)) \rightsquigarrow$ **Rank one operators**

$$T^{\mu,H}_{s,t}(f) := \frac{Q_{s,t}(H)}{\mu_s(Q_{s,t}(1))}\ \mu_t(f)$$

Time varying Krein-Rutman theorem

$(0 < H/V \in \mathcal{B}_0(E)$ & $\mu_0 = \mu \in \mathcal{P}_V(E))$ $\rightsquigarrow$ **Rank one operators**

$$T^{\mu,H}_{s,t}(f) := \frac{Q_{s,t}(H)}{\mu_s(Q_{s,t}(1))}\ \mu_t(f)$$

$$\Longrightarrow \sup_{\|f\|_V \leq 1} \left\| \frac{Q_{s,t}(f)}{\mu_s Q_{s,t}(1)} - T^{\mu,H}_{s,t}(f) \right\|_V \leq\ c_H(\mu)\ e^{-b(t-s)}$$

Time varying Krein-Rutman theorem

$(0 < H/V \in \mathcal{B}_0(E)$ & $\mu_0 = \mu \in \mathcal{P}_V(E))$ $\rightsquigarrow$ **Rank one operators**

$$T^{\mu,H}_{s,t}(f) := \frac{Q_{s,t}(H)}{\mu_s(Q_{s,t}(1))}\ \mu_t(f)$$

$$\Longrightarrow \sup_{\|f\|_V \le 1} \left\| \frac{Q_{s,t}(f)}{\mu_s Q_{s,t}(1)} - T^{\mu,H}_{s,t}(f) \right\|_V \le\ c_H(\mu)\ e^{-b(t-s)}$$

$H = 1$ and $E = \mathbb{R}^n$ (no Dirichlet but random sg) $\subset$ N. Witheley [AAP-13]

Time varying Krein-Rutman theorem

$(0 < H/V \in \mathcal{B}_0(E)$ & $\mu_0 = \mu \in \mathcal{P}_V(E)) \rightsquigarrow$ **Rank one operators**

$$T^{\mu,H}_{s,t}(f) := \frac{Q_{s,t}(H)}{\mu_s(Q_{s,t}(1))}\ \mu_t(f)$$

$$\Longrightarrow \sup_{\|f\|_V \leq 1} \left\| \frac{Q_{s,t}(f)}{\mu_s Q_{s,t}(1)} - T^{\mu,H}_{s,t}(f) \right\|_V \leq\ c_H(\mu)\ e^{-b(t-s)}$$

$H = 1$ and $E = \mathbb{R}^n$ (no Dirichlet but random sg) $\subset$ N. Witheley [AAP-13]

Time homogenous: $\exists$ leading-triple (ρ, h, η_∞)/"QSD"/...

$$\eta_\infty = \Phi_t(\eta_\infty) \quad \text{and} \quad Q_t(h) = e^{\rho t} h$$

Time varying Krein-Rutman theorem

$(0 < H/V \in \mathcal{B}_0(E)\ \&\ \mu_0 = \mu \in \mathcal{P}_V(E)) \rightsquigarrow$ **Rank one operators**

$$T^{\mu,H}_{s,t}(f) := \frac{Q_{s,t}(H)}{\mu_s(Q_{s,t}(1))}\ \mu_t(f)$$

$$\Longrightarrow \sup_{\|f\|_V \leq 1} \left\| \frac{Q_{s,t}(f)}{\mu_s Q_{s,t}(1)} - T^{\mu,H}_{s,t}(f) \right\|_V \leq\ c_H(\mu)\ e^{-b(t-s)}$$

$H = 1$ and $E = \mathbb{R}^n$ (no Dirichlet but random sg) $\subset$ N. Witheley [AAP-13]

Time homogenous: $\exists$ leading-triple (ρ, h, η_∞)/"QSD"/...

$$\eta_\infty = \Phi_t(\eta_\infty) \quad \text{and} \quad Q_t(h) = e^{\rho t} h$$

and for $H = h$ and $\mu = \eta_\infty$ the above yields

$$\sup_{\|f\|_V \leq 1} \left\| e^{-\rho t}\ Q_t(f) - \frac{h}{\eta_\infty(h)}\ \eta_\infty(f) \right\|_V \leq\ c\ e^{-\lambda t}$$

Some refs/links

Stability positive semigroups/finite block propagation chaos/expo. concentration/empirical processes/large deviations, sharp bias $1/N^k$-expansions ($\rightsquigarrow$ Romberg-Richardson interpolation), discrete time stochastic interpolation, McKean-Vlasov/interacting diffusions, stochastic/Kushner-Strato nonlinnear PDE,...

Discrete time (+continuous)

- First old studies: (1996), + A. Guionnet (1998), + Crisan/Lyons (1998), + Ledoux (2000), + L. Miclo (2000), + Jacod/Protter (2001), + Dawson (2005)...
- Books +refs: Feynman-Kac (2004), Mean field simulation (2013), Stoch. Proc. from applications to theory (+Penev 2016).

Some refs/links

Continuous time (+discrete)

- +Miclo (Sem. prob. 2000, Fac. Toulouse-02 + ESAIM-03)
- +Arnaudon, Particle Gibbs (18) + EJP (20)
- +Arnaudon, Perturb. $\rightsquigarrow$ Arxiv (19), AAP (20)
- +Arnaudon, Variational tech. $\rightsquigarrow$ Arxiv (18) & SAA (19)
- +Horton-Jasra: stab. sg, Arxiv (21), AAP (22), +refs

More on stochastic interpolations

- Backward Ito-Ventzell and stochastic interpolation formulae - Arxiv-19; SPA-22. (CRAS-20).