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# On a Class of Genetic Genealogical Tree Models

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↪ `F-K Formulae, Genealogical and IPS, Springer` (2004)+**References therein**
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**Introduction**

– Mean field particle models $\simeq$ stochastic linearization Methods.
– Feynman-Kac formulae, Genetic and genealogical tree interpretation models
  *Discrete time models* $\rightsquigarrow$ *Continuous time version = Moran type genetic models*
  *($\sim$ joint works with L. Miclo, see also [PhD $\oplus$ articles] M. Rousset)*
– **Asympt. analysis $\oplus$ propagations of chaos expansions**
  ($\sim$ F. Patras, S. Rubenthaler)
– **Application model areas : particle physics (absorbing medium, ground states), biology (polymers, macromolecules), statistics (particle simulation, restricted Markov, target distributions), rare event analysis (importance sampling, multilevel branching)**.

– **Signal processing, Filtering : Particle filters**

1. `Filtering velocities in turbulent fluids` (`C. Baehr, Météo France Toulouse`)
   $\rightsquigarrow$ `Workshop on Particle methods in fluid mecanics` (March 16th 2007)
2. Vortex and source particles for fluid motion estimation (`A. Cuzol`, E. Mémin, IRISA projet VISTA `http://www.irisa.fr/vista`)

## Mean field particle models=Non Linear Monte Carlo Methods

$\eta_t$ Non linear PDE $\sim (L_{t,\eta})_{(t,\eta)\in(\mathbb{R}_+\times\mathcal{P}(E))}$ infinitesimal generators

$$\frac{d}{dt}\eta_t(f) = \eta_t L_{t,\eta_t}(f) =_{\text{def.}} \int_E \eta_t(dx)\ L_{t,\eta_t}(f)(x)$$

$$\downarrow$$

*Interacting Particle Approximation Model* $\xi_t = (\xi_t^i)_{1\leq i\leq N}$ infinitesimal generator $\mathcal{L}$

$$\mathcal{L}(F)(x^1,\ldots,x^N) =_{\text{def.}} \sum_{i=1}^N L^{(i)}_{t,m(x)} F(x^1,\ldots,x^i,\ldots,x^N) \quad \text{with} \quad m(x) = \frac{1}{N}\sum_{i=1}^N \delta_{x^i}$$

$$\downarrow \quad \left(\eta_t^N := m(\xi_t)\right)$$

$$d\eta_t^N(f) = \eta_t^N L_{t,\eta_t^N}(f)dt + dM_t^N(f) \quad \text{with} \quad \langle M^N(f)\rangle_t = \frac{1}{N}\int_0^t\ \eta_s^N \Gamma_{L_{s,\eta_s^N}}(f,f)\ ds$$

**Example : (**`L. Miclo & P. DM SPA 2000`**)**

$(X_t \simeq L-$ motion on $E), (V : E \rightarrow \mathbb{R}_+) \rightsquigarrow$ **F.K. norm. model** $\eta_t(f) := \gamma_t(f)/\gamma_t(1)$

$$\text{with} \quad \gamma_t(f) = \mathbb{E}\left(f(X_t)\,\text{exp}\left\{-\int_0^t V(X_s)ds\right\}\right) \left(= \eta_t(f)\ \ \text{exp}\left\{-\int_0^t \eta_s(V)ds\right\}\right)$$

$$\downarrow$$

$$\begin{aligned}
\frac{d}{dt}\gamma_t(f) &= \gamma_t(L^V(f)) \quad \text{with the Schrödinger operator} \quad L^V := L - V \\
\frac{d}{dt}\eta_t(f) &= \eta_t L_{\eta_t}(f) := \int \eta_t(dx)\ \left\{L(f)(x) + V(x) \int (f(y) - f(x))\ \eta_t(dy)\right\}
\end{aligned}$$

**Moran's type IPS Model** $(\xi_t^i)_{1\leq i\leq N}$ = L-motions $\oplus$ interacting jumps ($V$-intensity)

$$\begin{aligned}
\mathcal{L}_t(F)(x^1,\ldots,x^N) &= \sum_{i=1}^N L^{(i)} F(x^1,\ldots,x^i,\ldots,x^N) + \sum_{i=1}^N V(x^i) \\
&\quad \times \int \left(F(x^1,\ldots,y^i,\ldots,x^N) - F(x^1,\ldots,x^i,\ldots,x^N)\right)\ m(x)(dy^i)
\end{aligned}$$

**Discrete generation models**

$$\eta_n = \eta_{n-1} K_{n,\eta_{n-1}} =_{\text{def.}} \int_{E_{n-1}} \eta_{n-1}(dx) \quad \underbrace{K_{n,\eta_{n-1}}(x, \,.\,)}_{Markov\ \ transitions}$$

$$\downarrow$$

*Interacting Particle Approximation Model*

$$\forall 1 \leq i \leq N \qquad \xi^i_{n-1} \longrightarrow \xi^i_n \sim K_{n,\eta^N_{n-1}}(\xi^i_{n-1}, \,.\,) \quad \text{with} \quad \eta^N_{n-1} = \frac{1}{N}\sum_{i=1}^N \delta_{\xi^i_{n-1}}$$

$$\downarrow$$

$$\eta^N_n = \eta^N_{n-1} K_{n,\eta^N_{n-1}} + \frac{1}{\sqrt{N}}\, W^N_n \quad \text{with} \quad W^N_n \simeq \text{centered Gauss field}$$

$\rightsquigarrow$ adaptive dynamics, genetic and branching population model, etc.

## Evolutionary type models

| | | |
|---|---|---|
| **Simple Genetic Branching Algo.** | *Mutation* | *Selection/Branching* |
| **Metropolis-Hastings Algo.** | *Proposal* | *Acceptance/Rejection* |
| **Sequential Monte Carlo methods** | *Sampling* | *Resampling (SIR)* |
| **Filtering/Smoothing** | *Prediction* | *Updating/Correction* |
| **Particle $\in$ Absorbing Medium** | *Evolution* | *Killing/Creation/Anhiling* |

*Other Botanical Names :* multi-level splitting (Khan-Harris 51), prune enrichment (Rosenbluth 1955), switching algo. (Magill 65), matrix reconfiguration (Hetherington 84), restart (Villen-Altamirano 91), particle filters (Rigal-Salut-DM 92), SIR filters (Gordon-Salmon-Smith 93, Kitagawa 96), go-with-the-winner (Vazirani-Aldous 94), ensemble Kalman-filters (Evensen 1994), quantum Monte Carlo methods (Melik-Nightingale 1999), sequential Monte Carlo Methods (Arnaud Doucet 2001), spawning filters (Fisher-Maybeck 2002), SIR Pilot Exploration Resampling (Liu-Zhang 2002),...

## $\Longleftrightarrow$ Particle Interpretations of Feynman-Kac models

*Since R. Feynman's phD. on path integrals 1942*

Physics $\longleftrightarrow$ Biology $\longleftrightarrow$ Engineering Sciences $\longleftrightarrow$ Probability/Statistics

- **Physics** :
  - $FKS \in$ nonlinear integro-diff. éq. ($\sim$ generalized Boltzmann models).
  - Spectral analysis of Schrödinger operators and large matrices with nonnegative entries. (particle evolutions in disordered/absorbing media)
  - Multiplicative Dirichlet problems with boundary conditions.
  - Microscopic and macroscopic interacting particle interpretations.

- **Biology** :
  - Self-avoiding walks, macromolecular polymerizations.
  - Branching and genetic population models.
  - Coalescent and Genealogical evolutions.

- **Rare events analysis** :

  - Multisplitting and branching particle models (Restart).
  - Importance sampling and twisted probability measures.
  - Genealogical tree based simulation methods.

- **Advanced Signal processing** :

  - Optimal filtering/smoothing/regulation, open loop optimal control.
  - Interacting Kalman-Bucy filters.
  - Stochastic and adaptative grid approximation-models

- **Statistics/Probability** :
  - Restricted Markov chains (w.r.t terminal values, visiting regions,...)
  - Analysis of Boltzmann-Gibbs type distributions (simulation, partition functions,...).
  - Random search evolutionary algorithms, interacting Metropolis/simulated annealing algo.
  - Bayesian methodology.

**Simple Genetic evolution/simulation models** $\longrightarrow$ <u>only 2 ingredients</u>!!

(Discrete time parameter $n \in \mathbb{N} = \{0, 1, 2, ...\}$, state spaces $E_n$ ($\in \{\mathbb{Z}^d, \mathbb{R}^d, \underbrace{\mathbb{R}^d \times \ldots \times \mathbb{R}^d}_{(n+1)-times}\}$)

– *Mutation/exploration/prediction/proposal* :

$\rightarrow$ Markov transitions $M_n(x_{n-1}, dx_n)$ from $E_{n-1}$ into $E_n$.

– *Selection/absorption/updating/acceptance* :

$\rightarrow$ Potential functions $G_n$ from $E_n$ into $[0, 1]$.

**A Genetic Evolution Model** $\Rightarrow$ Markov chain $\xi_n = (\xi_n^1, \ldots, \xi_n^N) \in E_n^N = E_n \times \ldots \times E_n$

$$\xi_n \in E_n^N \xrightarrow{\text{selection}} \widehat{\xi}_n \in E_n^N \xrightarrow{\text{mutation}} \xi_{n+1} \in E_{n+1}^N$$

– **Selection transition** ($\exists \neq$ types $\rightarrow$ Ex. : accept/reject)

$$\xi_n^i \rightsquigarrow \widehat{\xi_n^i} = \xi_n^i \quad \text{with proba. } \quad G_n(\xi_n^i) \quad \textbf{[Acceptance]}$$

Otherwise we select a better fitted individual in the current configuration

$$\widehat{\xi_n^i} = \xi_n^j \quad \text{with proba. } \quad G_n(\xi_n^j) / \sum_{k=1}^N G_n(\xi_n^k) \quad \textbf{[Rejection + Selection]}$$

– **Mutation transition**

$$\widehat{\xi_n^i} \rightsquigarrow \xi_{n+1}^i \sim M_{n+1}(\widehat{\xi_n^i}, \,.\,)$$

*Continuous time models :*

$(M,G) = (Id + \Delta L, e^{-V\Delta}) \rightsquigarrow$ *$L$-motions $\oplus$ expo. random clocks rate $V$+Uniform selection*

**A Genealogical tree model**

*Important observation* [Historical process]

$$X'_n \in E'_n \quad \text{Markov chain}$$

$$\Downarrow$$

$$X_n = (X'_0, \ldots, X'_n) \in E_n = (E'_0 \times \ldots \times E'_n) \quad \text{Markov chain} \in \text{path spaces}$$

$\rightarrow$ *Markov transitions* $M_n(x_{n-1}, dx_n)$ [elementary extensions]

$$X_{n+1} = ((X'_0, \ldots, X'_n), X'_{n+1}) = (X_n, X'_{n+1})$$

**Genetic Evolution Model <u>on Path Spaces</u>=Genealogical tree model**

$$X_n = (X'_0, \ldots, X'_n) \quad \text{Markov transitions} \quad M_n \quad \text{and} \quad G_n(X_n) = G'_n(X'_n)$$

$$\downarrow$$

Genetic path-valued particle Model

$$\begin{cases} \xi^i_n & = & (\xi^i_{0,n}, \xi^i_{1,n}, \ldots, \xi^i_{n,n}) \\ \widehat{\xi}^i_n & = & (\widehat{\xi}^i_{0,n}, \widehat{\xi}^i_{1,n}, \ldots, \widehat{\xi}^i_{n,n}) \in E_n = (E'_0 \times \ldots \times E'_n) \end{cases}$$

– Path acceptance/(rejection+selection).
– Path mutation = path elementary extensions.

**Occupation/Empirical measures** ($\forall f_n$ test function on $E_n$)

$$\eta_n^N(f_n) = \frac{1}{N}\sum_{i=1}^N f_n(\xi_n^i) = \frac{1}{N}\sum_{i=1}^N f_n \underbrace{(\xi_{0,n}^i, \xi_{1,n}^i, \ldots, \xi_{n,n}^i)}_{i\text{-th ancestral lines}}$$

$$\downarrow$$

*Unbias-particle measures & Unnormalized Feynman-Kac measures :*

$$\gamma_n^N(f_n) \quad = \quad \eta_n^N(f_n) \; \times \prod_{0\leq p<n} \eta_p^N(G_p) \longrightarrow_{N\to\infty} \gamma_n(f_n) = \mathbb{E}(f_n(X_n) \prod_{0\leq p<n} G_p(X_p))$$

**Notes :**

- $f_n = 1 \Rightarrow \gamma_n^N(1) = \prod_{0\leq p<n} \eta_p^N(G_p) \longrightarrow_{N\to\infty} \gamma_n(1) = \mathbb{E}(\prod_{0\leq p<n} G_p(X_p))$
- *Path-space models*

$$[\; X_n = (X'_0, \ldots, X'_n) \text{ and } G_n(X_n) = G'_n(X'_n) \;] \Rightarrow \gamma_n(f_n) = \mathbb{E}(f_n(X'_0, \ldots, X'_n) \prod_{0\leq p<n} G'_p(X'_p))$$

$\Longrightarrow$ *Occupation measure & Normalized Feynman-Kac measures :*

$$\eta_n^N(f_n) \quad = \quad \frac{1}{N}\sum_{i=1}^N f_n(\xi_n^i) = \gamma_n^N(f_n)/\gamma_n^N(1) \longrightarrow_{N\to\infty} \eta_n(f_n) = \gamma_n(f_n)/\gamma_n(1)$$

*Path-space models*

$$[\ X_n = (X'_0, \ldots, X'_n) \text{ and } G_n(X_n) = G'_n(X'_n)\ ]$$

$$\Downarrow$$

$$\eta_n(f_n) = \frac{\mathbb{E}(f_n(X'_0, \ldots, X'_n) \prod_{0\leq p<n} G'_p(X'_p))}{\mathbb{E}(\prod_{0\leq p<n} G'_p(X'_p))}$$

*Note :*

$$\gamma_n(f_n) = \eta_n(f_n) \times \prod_{0\leq p<n} \eta_p(G_p) \quad (\longleftarrow \gamma_n^N(f_n) = \eta_n^N(f_n) \times \prod_{0\leq p<n} \eta_p^N(G_p))$$

**Asymptotic theory** $(n, N) \to \infty$ (usual LLN, CLT, LDP,...)

<u>**Some examples :**</u>

– *Weak convergence* $[p \geq 1$ + $\mathcal{F}_n$ not too large + regular mutations]
(JTP 2000, joint work with M. Ledoux+ FK Formulae Springer 2004)

$$\sup_{n\geq 0} \mathbb{E}(\sup_{f_n \in \mathcal{F}_n} |\eta_n^N(f_n) - \eta_n(f_n)|^p)^{1/p} \leq c(p)/\sqrt{N}$$

$$\sup_{n\geq 0} \mathbb{P}(|\eta_n^N(f_n) - \eta_n(f_n)| > \epsilon) \leq (1 + \epsilon\sqrt{N/2}) \ \exp -\frac{N\epsilon^2}{\sigma^2}$$

– *Propagation-of-chaos estimates* $[q \leq N$ finite block size]
(TVP+SIAM PTA 2006, joint work with A. Doucet)

$$\mathbb{P}_{n,q}^N := \mathrm{Law}(\xi_n^1, \ldots, \xi_n^q) \simeq \eta_n^{\otimes q} + \frac{1}{N} \ \partial^1 \mathbb{P}_{n,q} \quad \text{with} \quad \sup_{n\geq 0} \|\partial^1 \mathbb{P}_{n,q}\|_{\mathrm{tv}} \leq c \ q^2$$

**Functional representation at any order :** $(\mathbb{Q}^N_{n,q}(F) := \mathbb{E}((\gamma^N_n)^{\otimes q}(F)))$

$\hookrightarrow$ `Coalescent tree based functional representations for some Feynman-Kac particle models.`

(joint work with F. Patras and S. Rubenthaler)

**thm :**

$$\mathbb{Q}^N_{n,q} = \gamma_n^{\otimes q} + \sum_{1\leq k\leq (q-1)(n+1)} \frac{1}{N^k}\ \partial^k \mathbb{Q}_{n,q}$$

and

$$\mathbb{P}^N_{n,q} \simeq \eta_n^{\otimes q} + \sum_{1l\leq l\leq k} \frac{1}{N^l}\ \partial^l \mathbb{P}_{n,q} + \frac{1}{N^{k+1}}\ \partial^{k+1}\mathbb{P}^N_{n,q} \quad \text{with} \quad \sup_{N\geq 1} \|\partial^{k+1}\mathbb{P}^N_{n,q}\|_{\text{tv}} < \infty$$

**Consequences :** Sharp + strong propagations of chaos estimates at any order, Wick product formulae on forests, sharp $\mathbb{L}_p$-mean error bounds, law of large numbers for $U$-statistics for interacting processes,. . . .

**Applications :**

– Particle physics (absorbing medium, ground states)
– Biology (polymers, macromolecules)
– Statistics (particle simulation, restricted Markov, target distributions)
– Rare event analysis (importance sampling, multilevel branching)
– Signal processing, filtering

$\hookrightarrow$ *[Stochastic Engineering + Scilab Prog. (Master 2) (⇒ in french)]*

`http://math1.unice.fr/~delmoral/eng.html`

$\hookrightarrow$ *[IRISA ASPI project* ***A****ppl.* ***S****tat. Syst. de* ***P****articules en* ***I****nteraction]*

`http://www.irisa.fr/activites/equipes/aspi`

**Particle physics : Markov $X_n \in$ Absorbing medium** $G(x) = e^{-V(x)} \in [0,1]$

$$X_n^c \in E^c = E \cup \{c\} \xrightarrow{absorption} \widehat{X}_n^c \xrightarrow{exploration} X_{n+1}^c$$

*Absorption/killing :* $\longrightarrow \widehat{X}_n^c = X_n^c$, with proba $G(X_n^c)$ ; otherwise the particle is killed and $\widehat{X}_n^c = c$.

$$\Downarrow$$

$$\begin{array}{rcl} A & = & \{x \ : \ G(x) = 0\} \longrightarrow \text{Hard obstacles} \\ T & = & \inf\{n \geq 0 \ ; \ \widehat{X}_n^c = c\} \longrightarrow \text{Absorption time} \quad X_{T+n}^c = \widehat{X}_{T+n}^c = c \end{array}$$

$\Longrightarrow$ **<u>Feynman-Kac models $(G, X_n)$ :</u>** $\gamma_n = \text{Law}(X_n^c \ ; \ T \geq n)$ and $\gamma_n(1) = \text{Proba}(T \geq n)$

$$\Downarrow$$

$$\eta_n = \text{Law}(X_n^c \mid T \geq n) = \text{Law}((X_0'^c, \ldots, X_n'^c) \mid T \geq n)$$

**Lyapunov exponents and grounds states** ($X_n \sim M$, $G \in [0,1]$, $T$ abs. time)

($\oplus$ Schrodinger op. L+V $\rightsquigarrow$ joint work with `L. Miclo ESAIM 2003`, <u>**see also** `M. Rousset PhD.+articles`</u>)

$$\text{Proba}(T \geq n) = \mathbb{E}(\prod_{0 \leq p < n} G(X_p)) = \prod_{0 \leq p < n} \eta_p(G) \simeq e^{-\lambda n} \quad \text{with} \quad \lambda = -\log \eta_\infty(G)$$

$$M \quad \mu - \text{reversible} \Rightarrow \eta_\infty(f) = \frac{\mu(H\ M(f))}{\mu(H)} \quad \text{with} \quad GM(H) = \lambda H$$

and

$$\Psi(\eta_\infty)(f) := \frac{\eta_\infty(Gf)}{\eta_\infty(G)} = \frac{\mu(Hf)}{\mu(H)}$$

N-particle approx. :

$$\gamma_n^N(1) \quad = \quad \prod_{0 \leq p < n} \eta_p^N(G) \simeq_N \prod_{0 \leq p < n} \eta_p(G) = \gamma_n(1) \simeq_n e^{-\lambda n}$$

$$\Psi(\eta_n^N) \quad \simeq_N \quad \Psi(\eta_n) \simeq_n \Psi(\eta_\infty) = \frac{\mu(Hf)}{\mu(H)}$$

**Biology : Macromolecules and Directed Polymers**

– *Self avoiding walks* $X'_n \in \mathbb{Z}^d$

$$X_n = (X'_0, \ldots, X'_n) \quad \text{and} \quad G_n(X_n) = 1_{\not\in \{X'_0, \ldots, X'_{n-1}\}}(X'_n)$$

$$\gamma_n(1) = \text{Proba}(\forall 0 \leq p \neq q \leq n, \;\; X'_p \neq X'_q) \quad \text{and} \quad \eta_n = \text{Law}(X'_0, \ldots, X'_n \mid \forall 0 \leq p \neq q \leq n, \;\; X'_p \neq X'_q)$$

Ex. : (*Connectivity constants*)

$$\gamma_n(1) = \frac{1}{(2d)^n} \times \#\{\text{non intersecting walks with length n}\} \simeq \frac{\exp(\mathbf{c}\; n)}{(2d)^n}$$

– *Edwards' model*

$$X_n = (X'_0, \ldots, X'_n) \quad \text{and} \quad G_n(X_n) = \exp\{-\beta \sum_{0 \leq p < n} 1_{X'_p}(X'_n)\}$$

– Random medium $(G_n(x))_{n,x}$ i.i.d random variables ($\in \{0,1\} \Rightarrow$ Law($X_n \mid X \in$ percolation clusters)

## Statistics : Sequential MCMC and Feynman-Kac-Metropolis models

(Series joint works with A. Doucet and A. Jasra : Sem. Proba 2003, JRSS B 2006)

*Metropolis potential* [$\pi$ target measure]+[$(K, L)$ pair Markov transitions]

$$G(y_1, y_2) = \frac{\pi(dy_2)L(y_2, dy_1)}{\pi(dy_1)K(y_1, dy_2)}$$

**Ex. $\pi$ Gibbs measure :**

$$\pi(dy) \propto e^{-V(y)}\ \lambda(dy) \Rightarrow G(y_1, y_2) = e^{(V(y_1)-V(y_2))}\ \frac{\lambda(dy_2)L(y_2, dy_1)}{\lambda(dy_1)K(y_1, dy_2)}$$

*Note :* $(K = L \quad \lambda - \text{reversible})$ or $(\lambda K = \lambda$ and $L(y_2, dy_1) = \lambda(dy_1)\frac{dK(y_1, \centerdot)}{d\lambda}(y_2))$

$$\Downarrow$$

$$G(y_1, y_2) = \exp\left(V(y_1) - V(y_2)\right)$$

*Notation* $\mathbb{E}_{\nu}^{M}(\,\centerdot\,)$=Expectation w.r.t. Markov [transition $M$, initial condition $\nu$]

<u>**Theorem :**</u> (**Time reversal formula**), [*A. Doucet, P.DM ; (Séminaire Probab. 2003)*]

$$\mathbb{E}_{\pi}^{L}(f_n(Y_n, Y_{n-1} \ldots, Y_0)|Y_n = y) = \frac{\mathbb{E}_{y}^{K}(f_n(Y_0, Y_1, \ldots, Y_n)\ \{\prod_{0\leq p<n} G(Y_p, Y_{p+1})\})}{\mathbb{E}_{y}^{K}(\{\prod_{0\leq p<n} G(Y_p, Y_{p+1})\})}$$

**In addition :** $\oplus$ *FK-Metropolis $n$-marginal :* $\lim_{n\to\infty} \eta_n = \pi$ (cv. decays $\perp \pi$)

$\oplus$ *Nonhomogeneous models :* $(\pi_n, L_n, K_n)$ (joint work with A. Doucet and A. Jasra, JRSS B 2006)

$\pi_n(dy) \propto e^{-\beta_n V(y)}\ \lambda(dy)$, cooling schedule $\beta_n \uparrow \infty$, mutation s.t. $\pi_n = \pi_n K_n$, and $\text{Law}(X_0) = \pi_0$

$$\Downarrow$$

$G_n(y_1, y_2) = \exp\left[-(\beta_{n+1} - \beta_n)V(y_1)\right] \Longrightarrow \eta_n = \pi_n \hookrightarrow$ **stationary Metropolis-Hasting model**

**Rare events analysis** (joint work with `J. Garnier AAP 2005` $\oplus$ `Optics Comm. 2006` )

– **Importance sampling and Twisted Feynman-Kac measures**

$$\mathbb{P}(V_n(X_n) \geq a) \quad = \quad \mathbb{E}(1_{V_n(X_n)\geq a}\ e^{-\beta_n V_n(X_n)}\ e^{+\beta_n V_n(X_n)})$$

$$\Downarrow$$

*Importance potentials/measures :*

$$G_n(X_n, X_{n+1}) = e^{\beta_n(V_{n+1}(X_{n+1})-V_n(X_n))} \Longrightarrow \mathbb{P}(V_n(X_n) \geq a) = \gamma_n(1_{V_n\geq a}e^{-\beta_n V_n})$$

**In addition :**

$$\mathbb{E}(f_n(X_n) \mid V_n(X_n) \geq a) = \eta_n(f_n\ 1_{V_n\geq a}e^{-\beta_n V_n})/\eta_n(1_{V_n\geq a}e^{-\beta_n V_n})$$

$\oplus$ *Path-space models $\Rightarrow$ weighted genealogies*

$$X_n = (X'_0, \ldots, X'_n) \text{ and } V_n(X_n) = V'_n(X'^n)$$

$$\Downarrow$$

$$\mathbb{E}(f_n(X'_0, \ldots, X'_n) \mid V'_n(X'_n) \geq a) = \eta_n(f_n\ 1_{V_n\geq a}e^{-\beta_n V_n})/\eta_n(1_{V_n\geq a}e^{-\beta_n V_n})$$

– **Multi-splitting Feynman-Kac models** ($\neq$ importance sampling)
(*Series joint works with F. Cérou, A. Guyader, F. Le Gland, P. Lézaud,*
*[* `Alea 2006, Stochastic Hybrid Systems Springer 2006,...` *])*

$(E = A \cup A^c)$, $Y_n$ Markov, $Y_0 \in A_0 (\subset A) \rightsquigarrow A^c = (B \cup C)$, $C =$ absorbing set/hard obstacle

*Multi-level decomposition* $B = B_m \subset \ldots \subset B_1 \subset B_0$ $\quad (A_0 = B_1 - B_0,\ B_0 \cap C = \emptyset)$

$$\Downarrow$$

$$\mathbb{P}(Y_n \text{ hits } B \text{ before } C) = \mathbb{E}\Big(\prod_{1 \le p \le m} G_p(X_p)\Big)$$

*Inter-level excursions :* $T_n = \inf \{p \geq T_{n-1} \ : \ Y_p \in B_n \cup C\}$

$$X_n = (Y_p \ ; T_{n-1} \leq p \leq T_n) \in \text{Excursion space} \qquad G_n(X_n) = 1_{B_n}(Y_{T_n})$$

$$\Downarrow$$

**FK interpretation**

$$\mathbb{E}(f(Y_0, \ldots, Y_{T_m})\ 1_{B_m}(X_{T_m})) = \mathbb{E}(f(X_0, \ldots, X_m) \prod_{1 \le p \le m} G_p(X_p))$$

**Advanced signal processing** $\longrightarrow$ filtering/hidden Markov chains/Bayesian methodology

Signal process $\quad X_n$ = Markov chain $\in E_n$

Observation/Sensor eq. $\quad Y_n = H_n(X_n, V_n) \in F_n$ with $\mathbb{P}(H_n(x_n, V_n) \in dy_n) = g_n(x_n, y_n)\ \lambda_n(dy_n)$

*Example :* $Y_n = h_n(X_n) + V_n \in F_n = \mathbb{R}$, with Gaussian noise $V_n = \mathcal{N}(0,1)$

$$\Downarrow$$

$$\mathbb{P}(h_n(x_n) + V_n \in dy_n) = (2\pi)^{-1/2} e^{-\frac{1}{2}(y_n - h_n(x_n))^2}\ dy_n = \underbrace{\exp\left[h_n(x_n) y_n - h_n^2(x_n)/2\right]}_{g_n(x_n, y_n)}\ \underbrace{\mathcal{N}(0,1)(dy_n)}_{\lambda_n(dy_n)}$$

**Prediction/filtering/smoothing** $\rightarrow$ **Feynman-Kac representation** $G_n(x_n) = g_n(x_n, y_n)$

$$\eta_n = \text{Law}(X_n \mid Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1}) = \text{Law}(X'_0, \ldots, X'_n \mid Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1})$$

**Filtering velocities in turbulent fluids (`C. Baehr, Météo France Toulouse`)**

Simple Lagrangian Fluid Velocity Model (S.B. Pope 85)

$$\begin{aligned} dX_t &= V_t\ dt \\ dV_t &= \underbrace{-\frac{1}{\rho}\nabla\langle p\rangle\ dt}_{\text{average tendancy}}\ -\left(\frac{1}{2}+\frac{3}{4}C_0\right)\ \frac{\epsilon_t}{k_t}\ (V_t-\langle V\rangle)\ dt + \underbrace{\sqrt{C_0\epsilon_t}}_{dissipation\ \ rate}\ dB_t \end{aligned}$$

with ($\delta$ =localization parameter)

$$\langle V\rangle \simeq \mathbb{E}^\delta(V_t \mid X_t) \quad \text{and} \quad 2k_t := \langle (V-\langle V\rangle)^2\rangle \simeq \mathbb{E}^\delta((V_t-\mathbb{E}^\delta(V_t\mid X_t))^2 \mid X_t)$$

$$\downarrow$$

$$\text{"}\,\mathbb{E}^\delta(dV_t \mid X_t) \simeq -\frac{1}{\rho}\nabla\langle p\rangle\ dt\ , \quad \mathbb{E}^\delta(dV_t dV_t \mid X_t) \simeq C_0\epsilon_t\ dt\text{"}$$

Observations $dY_t = H_t(V_t)dt + dB'_t \Rightarrow$ Conditional mean field $\mathbb{E}^\delta(\centerdot \mid X_t) \rightsquigarrow \mathbb{E}^\delta(\centerdot \mid X_t, Y_{[0,t]})$