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On a Class of Genetic Genealogical Tree Models

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↪ *F-K Formulae, Genealogical and IPS, Springer (2004)*+References therein

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Introduction

- Evolutionary models and Feynman-Kac formulae
- Genetic genealogical models and Feynman-Kac limiting measures
- **Application model areas : particle physics (absorbing medium, ground states), biology (polymers, macromolecules), statistics (particle simulation, restricted Markov, target distributions), rare event analysis (importance sampling, multilevel branching), signal processing, filtering.**
- **Asymptotic analysis :**
 - ↔ *Functional representations \simeq precise propagations of chaos expansions.*
(joint work F. Patras, S. Rubenthaler)
 - *Combinatorial differential calculus*
 - *Permutation group analysis of (colored) forests*
(wreath product of permutation groups, Hilbert series techniques, . . .)

Discrete time models \rightsquigarrow Continuous time version = Moran type genetic models

(\sim joint works with L. Miclo, see also [PhD \oplus articles] M. Rousset)

Evolutionary type models

Simple Genetic Branching Algo.	<i>Mutation</i> <i>Selection/Branching</i>
Metropolis-Hastings Algo.	<i>Proposal</i> <i>Acceptance/Rejection</i>
Sequential Monte Carlo methods	<i>Sampling</i> <i>Resampling (SIR)</i>
Filtering/Smoothing	<i>Prediction</i> <i>Updating/Correction</i>
Particle \in Absorbing Medium	<i>Evolution</i> <i>Killing/Creation/Anhiling</i>

Other Botanical Names: multi-level splitting (Khan-Harris 51), prune enrichment (Rosenbluth 1955), switching algo. (Magill 65), matrix reconfiguration (Hetherington 84), restart (Villen-Altamirano 91), particle filters (Rigal-Salut-DM 92), SIR filters (Gordon-Salmon-Smith 93, Kitagawa 96), go-with-the-winner (Vazirani-Aldous 94), ensemble Kalman-filters (Evensen 1994), quantum Monte Carlo methods (Melik-Nightingale 1999), sequential Monte Carlo Methods (Arnaud Doucet 2001), spawning filters (Fisher-Maybeck 2002), SIR Pilot Exploration Resampling (Liu-Zhang 2002),...

\iff Particle Interpretations of Feynman-Kac models

Since R. Feynman's *phD. on path integrals 1942*

Physics \iff Biology \iff Engineering Sciences \iff Probability/Statistics

- **Physics :**

- $FKS \in$ nonlinear integro-diff. éq. (\sim generalized Boltzmann models).
- Spectral analysis of Schrödinger operators and large matrices with nonnegative entries. (particle evolutions in disordered/absorbing media)
- Multiplicative Dirichlet problems with boundary conditions.
- Microscopic and macroscopic interacting particle interpretations.

- **Biology:**

- Self-avoiding walks, macromolecular polymerizations.
- Branching and genetic population models.
- Coalescent and Genealogical evolutions.

- **Rare events analysis:**

- Multisplitting and branching particle models (Restart).
- Importance sampling and twisted probability measures.
- Genealogical tree based simulation methods.

- **Advanced Signal processing:**

- Optimal filtering/smoothing/regulation, open loop optimal control.
- Interacting Kalman-Bucy filters.
- Stochastic and adaptative grid approximation-models

- **Statistics/Probability:**

- Restricted Markov chains (w.r.t terminal values, visiting regions,...)
- Analysis of Boltzmann-Gibbs type distributions (simulation, partition functions,...).
- Random search evolutionary algorithms, interacting Metropolis/simulated annealing algo.

Simple Genetic evolution/simulation models \longrightarrow only 2 ingredients!!

(Discrete time parameter $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, state spaces E_n ($\in \{\mathbb{Z}^d, \mathbb{R}^d, \underbrace{\mathbb{R}^d \times \dots \times \mathbb{R}^d}_{(n+1)\text{-times}}, \dots\}$)

- *Mutation/exploration/prediction/proposal* :

\rightarrow Markov transitions $M_n(x_{n-1}, dx_n)$ from E_{n-1} into E_n .

- *Selection/absorption/updating/acceptance* :

\rightarrow Potential functions G_n from E_n into $[0, 1]$.

A Genetic Evolution Model \Rightarrow Markov chain $\xi_n = (\xi_n^1, \dots, \xi_n^N) \in E_n^N = \underbrace{E_n \times \dots \times E_n}_{N\text{-times}}$

$$\xi_n \in E_n^N \xrightarrow{\text{selection}} \widehat{\xi}_n \in E_n^N \xrightarrow{\text{mutation}} \xi_{n+1} \in E_{n+1}^N$$

- **Selection transition** ($\exists \neq$ types \rightarrow Ex.: accept/reject)

$$\xi_n^i \rightsquigarrow \widehat{\xi}_n^i = \xi_n^i \quad \text{with proba. } G_n(\xi_n^i) \quad \textbf{[Acceptance]}$$

Otherwise we select a better fitted individual in the current configuration

$$\widehat{\xi}_n^i = \xi_n^j \quad \text{with proba. } G_n(\xi_n^j) / \sum_{k=1}^N G_n(\xi_n^k) \quad \textbf{[Rejection + Selection]}$$

- **Mutation transition**

$$\widehat{\xi}_n^i \rightsquigarrow \xi_{n+1}^i \sim M_{n+1}(\widehat{\xi}_n^i, \cdot)$$

Continuous time models :

$(M, G) = (Id + \Delta L, e^{-V\Delta}) \rightsquigarrow L\text{-motions} \oplus \text{expo. random clocks rate } V + \text{Uniform selection}$

A Genealogical tree model

Important observation [Historical process]

$$X'_n \in E'_n \quad \text{Markov chain}$$

↓

$$X_n = (X'_0, \dots, X'_n) \in E_n = (E'_0 \times \dots \times E'_n) \quad \text{Markov chain} \in \text{path spaces}$$

→ *Markov transitions* $M_n(x_{n-1}, dx_n)$ [elementary extensions]

$$X_{n+1} = ((X'_0, \dots, X'_n), X'_{n+1}) = (X_n, X'_{n+1})$$

Genetic Evolution Model on Path Spaces = Genealogical tree model

$$X_n = (X'_0, \dots, X'_n) \quad \text{Markov transitions } M_n \quad \text{and} \quad G_n(X_n) = G'_n(X'_n)$$

↓

Genetic path-valued particle Model

$$\begin{cases} \xi_n^i &= (\xi_{0,n}^i, \xi_{1,n}^i, \dots, \xi_{n,n}^i) \\ \widehat{\xi}_n^i &= (\widehat{\xi}_{0,n}^i, \widehat{\xi}_{1,n}^i, \dots, \widehat{\xi}_{n,n}^i) \in E_n = (E'_0 \times \dots \times E'_n) \end{cases}$$

- Path acceptance/(rejection+selection).
- Path mutation = path elementary extensions.

Occupation/Empirical measures ($\forall f_n$ test function on E_n)

$$\eta_n^N(f_n) = \frac{1}{N} \sum_{i=1}^N f_n(\xi_n^i) = \frac{1}{N} \sum_{i=1}^N f_n \underbrace{(\xi_{0,n}^i, \xi_{1,n}^i, \dots, \xi_{n,n}^i)}_{i\text{-th ancestral lines}}$$

↓

Unbias-particle measures & Unnormalized Feynman-Kac measures :

$$\gamma_n^N(f_n) = \eta_n^N(f_n) \times \prod_{0 \leq p < n} \eta_p^N(G_p) \xrightarrow{N \rightarrow \infty} \gamma_n(f_n) = \mathbb{E}(f_n(X_n) \prod_{0 \leq p < n} G_p(X_p))$$

Notes:

- $f_n = 1 \Rightarrow \gamma_n^N(1) = \prod_{0 \leq p < n} \eta_p^N(G_p) \xrightarrow{N \rightarrow \infty} \gamma_n(1) = \mathbb{E}(\prod_{0 \leq p < n} G_p(X_p))$
- *Path-space models*

$$[X_n = (X'_0, \dots, X'_n) \text{ and } G_n(X_n) = G'_n(X'_n)] \Rightarrow \gamma_n(f_n) = \mathbb{E}(f_n(X'_0, \dots, X'_n) \prod_{0 \leq p < n} G'_p(X'_p))$$

\implies Occupation measure & Normalized Feynman-Kac measures:

$$\eta_n^N(f_n) = \frac{1}{N} \sum_{i=1}^N f_n(\xi_n^i) = \gamma_n^N(f_n) / \gamma_n^N(\mathbf{1}) \xrightarrow{N \rightarrow \infty} \eta_n(f_n) = \gamma_n(f_n) / \gamma_n(\mathbf{1})$$

Path-space models

$$[X_n = (X'_0, \dots, X'_n) \text{ and } G_n(X_n) = G'_n(X'_n)]$$

\Downarrow

$$\eta_n(f_n) = \frac{\mathbb{E}(f_n(X'_0, \dots, X'_n) \prod_{0 \leq p < n} G'_p(X'_p))}{\mathbb{E}(\prod_{0 \leq p < n} G'_p(X'_p))}$$

Note:

$$\gamma_n(f_n) = \eta_n(f_n) \times \prod_{0 \leq p < n} \eta_p(G_p) \quad (\leftarrow \gamma_n^N(f_n) = \eta_n^N(f_n) \times \prod_{0 \leq p < n} \eta_p^N(G_p))$$

Applications :

- Particle physics (absorbing medium, ground states)
- Biology (polymers, macromolecules)
- Statistics (particle simulation, restricted Markov, target distributions)
- Rare event analysis (importance sampling, multilevel branching)
- Signal processing, filtering

↪ *[Stochastic Engineering + Scilab Prog. (Master 2) (⇒ in french)]*

<http://math1.unice.fr/delmoral/eng.html>

↪ *[IRISA ASPI **A**pl. **S**tat. Syst. de **P**articules en **I**nteraction]*

<http://www.irisa.fr/activites/equipes/aspi>

Particle physics: Markov $X_n \in$ Absorbing medium $G(x) = e^{-V(x)} \in [0, 1]$

$$X_n^c \in E^c = E \cup \{c\} \xrightarrow{\text{absorption}} \widehat{X}_n^c \xrightarrow{\text{exploration}} X_{n+1}^c$$

Absorption/killing: $\rightarrow \widehat{X}_n^c = X_n^c$, with proba $G(X_n^c)$; otherwise the particle is killed and $\widehat{X}_n^c = c$.

\Downarrow

$A = \{x : G(x) = 0\} \rightarrow$ Hard obstacles

$T = \inf \{n \geq 0 ; \widehat{X}_n^c = c\} \rightarrow$ Absorption time $X_{T+n}^c = \widehat{X}_{T+n}^c = c$

\implies Feynman-Kac models $(G, X_n) : \gamma_n = \text{Law}(X_n^c ; T \geq n)$ and $\gamma_n(1) = \text{Proba}(T \geq n)$

\Downarrow

$$\eta_n = \text{Law}(X_n^c | T \geq n) = \text{Law}((X_0^{c'}, \dots, X_n^{c'}) | T \geq n)$$

Lyapunov exponents and grounds states ($X_n \sim M$, $G \in [0, 1]$, T abs. time)

(\oplus Schrodinger op. $L+V \rightsquigarrow$ joint work with L. Miclo ESAIM 2003, see also M. Rousset PhD.+articles)

$$\text{Proba}(T \geq n) = \mathbb{E}\left(\prod_{0 \leq p < n} G(X_p)\right) = \prod_{0 \leq p < n} \eta_p(G) \simeq e^{-\lambda n} \quad \text{with} \quad \lambda = -\log \eta_\infty(G)$$

$$M \quad \mu - \text{reversible} \Rightarrow \eta_\infty(f) = \frac{\mu(H M(f))}{\mu(H)} \quad \text{with} \quad GM(H) = \lambda H$$

and

$$\Psi(\eta_\infty)(f) := \frac{\eta_\infty(Gf)}{\eta_\infty(G)} = \frac{\mu(Hf)}{\mu(H)}$$

N-particle approx. :

$$\begin{aligned} \gamma_n^N(\mathbf{1}) &= \prod_{0 \leq p < n} \eta_p^N(G) \simeq_N \prod_{0 \leq p < n} \eta_p(G) = \gamma_n(\mathbf{1}) \simeq_n e^{-\lambda n} \\ \Psi(\eta_n^N) &\simeq_N \Psi(\eta_n) \simeq_n \Psi(\eta_\infty) = \frac{\mu(Hf)}{\mu(H)} \end{aligned}$$

Biology: Macromolecules and Directed Polymers

- Self avoiding walks $X'_n \in \mathbb{Z}^d$

$$X_n = (X'_0, \dots, X'_n) \quad \text{and} \quad G_n(X_n) = 1_{\notin \{X'_0, \dots, X'_{n-1}\}}(X'_n)$$

$$\gamma_n(1) = \text{Proba}(\forall 0 \leq p \neq q \leq n, X'_p \neq X'_q) \quad \text{and} \quad \eta_n = \text{Law}(X'_0, \dots, X'_n \mid \forall 0 \leq p \neq q \leq n, X'_p \neq X'_q)$$

Ex. : (Connectivity constants)

$$\gamma_n(1) = \frac{1}{(2d)^n} \times \#\{\text{non intersecting walks with length } n\} \simeq \frac{\exp(\mathbf{c} \, n)}{(2d)^n}$$

- Edwards' model

$$X_n = (X'_0, \dots, X'_n) \quad \text{and} \quad G_n(X_n) = \exp \left\{ -\beta \sum_{0 \leq p < n} 1_{X'_p}(X'_n) \right\}$$

Statistics: Sequential MCMC and Feynman-Kac-Metropolis models

(Series joint works with A. Doucet and A. Jasra : Sem. Proba 2003, JRSS B 2006)

Metropolis potential [π target measure]+[(K, L) pair Markov transitions]

$$G(y_1, y_2) = \frac{\pi(dy_2)L(y_2, dy_1)}{\pi(dy_1)K(y_1, dy_2)}$$

Ex. π Gibbs measure:

$$\pi(dy) \propto e^{-V(y)} \lambda(dy) \Rightarrow G(y_1, y_2) = e^{(V(y_1) - V(y_2))} \frac{\lambda(dy_2)L(y_2, dy_1)}{\lambda(dy_1)K(y_1, dy_2)}$$

Note: ($K = L$ λ - reversible) or ($\lambda K = \lambda$ and $L(y_2, dy_1) = \lambda(dy_1) \frac{dK(y_1, \cdot)}{d\lambda}(y_2)$)

↓

$$G(y_1, y_2) = \exp(V(y_1) - V(y_2))$$

Notation $\mathbb{E}_\nu^M(\cdot)$ = Expectation w.r.t. Markov [transition M , initial condition ν]

Theorem: (Time reversal formula), [A. Doucet, P.DM; (Séminaire Probab. 2003)]

$$\mathbb{E}_\pi^L(f_n(Y_n, Y_{n-1}, \dots, Y_0) | Y_n = y) = \frac{\mathbb{E}_y^K(f_n(Y_0, Y_1, \dots, Y_n) \{\prod_{0 \leq p < n} G(Y_p, Y_{p+1})\})}{\mathbb{E}_y^K(\{\prod_{0 \leq p < n} G(Y_p, Y_{p+1})\})}$$

In addition : \oplus FK-Metropolis n -marginal: $\lim_{n \rightarrow \infty} \eta_n = \pi$ (cv. decays $\perp \pi$)

\oplus Nonhomogeneous models: (π_n, L_n, K_n) (joint work with A. Doucet and A. Jasra, JRSS B 2006)

$\pi_n(dy) \propto e^{-\beta_n V(y)} \lambda(dy)$, cooling schedule $\beta_n \uparrow \infty$, mutation s.t. $\pi_n = \pi_n K_n$, and $\text{Law}(X_0) = \pi_0$

\Downarrow

$G_n(y_1, y_2) = \exp[-(\beta_{n+1} - \beta_n)V(y_1)] \implies \eta_n = \pi_n \leftrightarrow$ stationary Metropolis-Hasting model

Rare events analysis (joint work with J. Garnier AAP 2005 \oplus Optics Comm. 2006)

- **Importance sampling and Twisted Feynman-Kac measures**

$$\mathbb{P}(V_n(X_n) \geq a) = \mathbb{E}(\mathbf{1}_{V_n(X_n) \geq a} e^{-\beta_n V_n(X_n)} e^{+\beta_n V_n(X_n)})$$

\Downarrow

Importance potentials/measures:

$$G_n(X_n, X_{n+1}) = e^{\beta_n(V_{n+1}(X_{n+1}) - V_n(X_n))} \implies \mathbb{P}(V_n(X_n) \geq a) = \gamma_n(\mathbf{1}_{V_n \geq a} e^{-\beta_n V_n})$$

In addition:

$$\mathbb{E}(f_n(X_n) \mid V_n(X_n) \geq a) = \eta_n(f_n \mathbf{1}_{V_n \geq a} e^{-\beta_n V_n}) / \eta_n(\mathbf{1}_{V_n \geq a} e^{-\beta_n V_n})$$

\oplus Path-space models \Rightarrow weighted genealogies

$$X_n = (X'_0, \dots, X'_n) \text{ and } V_n(X_n) = V'_n(X'_n)$$

\Downarrow

$$\mathbb{E}(f_n(X'_0, \dots, X'_n) \mid V'_n(X'_n) \geq a) = \eta_n(f_n \mathbf{1}_{V_n \geq a} e^{-\beta_n V_n}) / \eta_n(\mathbf{1}_{V_n \geq a} e^{-\beta_n V_n})$$

- **Multi-splitting Feynman-Kac models** (\neq importance sampling)

(Series joint works with F. Cérou, A. Guyader, F. Le Gland, P. Lézaud,
[Alea 2006, Stochastic Hybrid Systems Springer 2006, . . .])

($E = A \cup A^c$), Y_n Markov, $Y_0 \in A_0 (\subset A) \rightsquigarrow A^c = (B \cup C)$, $C =$ absorbing set/hard obstacle

Multi-level decomposition $B = B_m \subset \dots \subset B_1 \subset B_0$ ($A_0 = B_1 - B_0$, $B_0 \cap C = \emptyset$)

\Downarrow

$$\mathbb{P}(Y_n \text{ hits } B \text{ before } C) = \mathbb{E}\left(\prod_{1 \leq p \leq m} G_p(X_p)\right)$$

Inter-level excursions : $T_n = \inf \{p \geq T_{n-1} : Y_p \in B_n \cup C\}$

$$X_n = (Y_p ; T_{n-1} \leq p \leq T_n) \in \text{Excursion space} \quad G_n(X_n) = 1_{B_n}(Y_{T_n})$$

\Downarrow

FK interpretation

$$\mathbb{E}(f(Y_0, \dots, Y_{T_m}) 1_{B_m}(X_{T_m})) = \mathbb{E}(f(X_0, \dots, X_m) \prod_{1 \leq p \leq m} G_p(X_p))$$

Advanced signal processing → filtering/hidden Markov chains/Bayesian methodology

Signal process $X_n = \text{Markov chain} \in E_n$

Observation/Sensor eq. $Y_n = H_n(X_n, V_n) \in F_n$ with $\mathbb{P}(H_n(x_n, V_n) \in dy_n) = g_n(x_n, y_n) \lambda_n(dy_n)$

Example: $Y_n = h_n(X_n) + V_n \in F_n = \mathbb{R}$, with Gaussian noise $V_n = \mathcal{N}(0, 1)$

↓

$$\mathbb{P}(h_n(x_n) + V_n \in dy_n) = (2\pi)^{-1/2} e^{-\frac{1}{2}(y_n - h_n(x_n))^2} dy_n = \underbrace{\exp [h_n(x_n)y_n - h_n^2(x_n)/2]}_{g_n(x_n, y_n)} \underbrace{\mathcal{N}(0, 1)(dy_n)}_{\lambda_n(dy_n)}$$

Prediction/filtering/smoothing → Feynman-Kac representation $G_n(x_n) = g_n(x_n, y_n)$

$$\eta_n = \text{Law}(X_n \mid Y_0 = y_0, \dots, Y_{n-1} = y_{n-1}) = \text{Law}(X'_0, \dots, X'_n \mid Y_0 = y_0, \dots, Y_{n-1} = y_{n-1})$$

Partially linear/Gaussian models

$$X_n^1 = \text{Markov} \in E_n \quad + \quad \begin{cases} X_n^2 = A_n(X_n^1) X_{n-1}^2 + a_n(X_n^1) + B_n(X_n^1) W_n \in \mathbb{R}^d \\ Y_n = C_n(X_n^1) X_n^2 + c_n(X_n^1) + D_n(X_n^1) V_n \in \mathbb{R}^{d'} \end{cases}$$

Given a realization $X^1 = x \rightarrow$ *Kalman-Bucy optimal one step predictor*

$$\hat{X}_{x,n+1}^{2-} = \mathbb{E}(X_{n+1}^2 \mid Y_0, \dots, Y_n, X^1 = x) \quad \text{and} \quad P_{x,n+1}^- = \mathbb{E}([X_{n+1}^2 - \hat{X}_{x,n+1}^{2-}][X_{n+1}^2 - \hat{X}_{x,n+1}^{2-}]')$$

↓

Quenched Kalman-Bucy recursion: $(\hat{X}_{x,n+1}^{2-}, P_{x,n+1}^-) = \mathcal{B}_{n+1}[(x_n, x_{n+1}), (\hat{X}_{x,n}^{2-}, P_{x,n}^-)]$

Feynman-Kac representation: $\eta_n \sim (\mathbf{X}_n, \mathbf{G}_n)$ s.t.

$$\begin{aligned} \mathbf{X}_n &= (X_n^1, (\widehat{X}_{X^1, n+1}^{2-}, P_{X^1, n+1}^-)) \text{ Markov chain} \in \mathbf{E}_n = (E_n \times \mathbb{R}^d \times \mathbb{R}^{d \times d}) \\ \mathbf{G}_n(x, m, P) &= \frac{d\mathcal{N}(C_n(x) m + c_n(x), C_n(x) P C_n(x)' + D_n(x) R_n^v D_n(x)')}{d\mathcal{N}(0, D_n(x) R_n^v D_n(x)')} (y_n) \end{aligned}$$

$$\Downarrow \text{ [virtual sensor : } Y_n = \{C_n(X_n^1) \widehat{X}_{X^1, n}^{2-} + c_n(X_n^1)\} + \widehat{V}_{X^1, n} \text{]}$$

$$\begin{aligned} F_n(x, m, P) = f_n(x) &\implies \eta_n(F_n) = \mathbb{E}(f_n(X_n^1) \mid Y_0, \dots, Y_{n-1}) \\ F_n(x, m, P) = \mathcal{N}(m, P)(f_n) &\implies \eta_n(F_n) = \mathbb{E}(f_n(X_n^2) \mid Y_0, \dots, Y_{n-1}) \end{aligned}$$

Note: \rightsquigarrow Interacting Kalman-Bucy filters and for path-space models we have

$$X_n^1 = (X_0^1, \dots, X_n^1) \rightsquigarrow \text{Law}((X_0^1, \dots, X_n^1) \mid Y_0, \dots, Y_{n-1})$$

Asymptotic theory $(n, N) \rightarrow \infty$ (usual LLN, CLT, LDP,...)

Some examples:

- *Weak convergence* [$p \geq 1$ + \mathcal{F}_n not too large + regular mutations]
(JTP 2000, joint work with M. Ledoux+ FK Formulae Springer 2004)

$$\sup_{n \geq 0} \mathbb{E} \left(\sup_{f_n \in \mathcal{F}_n} |\eta_n^N(f_n) - \eta_n(f_n)|^p \right)^{1/p} \leq c(p) / \sqrt{N}$$

$$\sup_{n \geq 0} \mathbb{P}(|\eta_n^N(f_n) - \eta_n(f_n)| > \epsilon) \leq (1 + \epsilon \sqrt{N/2}) \exp -\frac{N\epsilon^2}{\sigma^2}$$

- *Propagation-of-chaos estimates* [$q \leq N$ finite block size]
(TVP+SIAM PTA 2006, joint work with A. Doucet)

$$\mathbb{P}_{n,q}^N := \text{Law}(\xi_n^1, \dots, \xi_n^q) \simeq \eta_n^{\otimes q} + \frac{1}{N} \partial^1 \mathbb{P}_{n,q} \quad \text{with} \quad \partial^1 \mathbb{P}_{n,q} \quad \text{signed meas. s.t.} \quad \sup_{n \geq 0} \|\partial^1 \mathbb{P}_{n,q}\|_{\text{tv}} \leq c q^2$$

Functional representation at any order : $(\mathbb{Q}_{n,q}^N(F) := \mathbb{E}((\gamma_n^N)^{\otimes q}(F)))$

\hookrightarrow Coalescent tree based functional representations for some Feynman-Kac particle models.

<https://hal.ccsd.cnrs.fr/ccsd-00086532>

(joint work with F. Patras and S. Rubenthaler)

thm :

$$\mathbb{Q}_{n,q}^N = \gamma_n^{\otimes q} + \sum_{1 \leq k \leq (q-1)(n+1)} \frac{1}{N^k} \partial^k \mathbb{Q}_{n,q}$$

and

$$\mathbb{P}_{n,q}^N \simeq \eta_n^{\otimes q} + \sum_{1 \leq l \leq k} \frac{1}{N^l} \partial^l \mathbb{P}_{n,q} + \frac{1}{N^{k+1}} \partial^{k+1} \mathbb{P}_{n,q}^N \quad \text{with} \quad \sup_{N \geq 1} \|\partial^{k+1} \mathbb{P}_{n,q}^N\|_{\text{tv}} < \infty$$

Consequences : Sharp + strong propagations of chaos estimates at any order, Wick product formulae on forests, sharp \mathbb{L}_p -mean error bounds, law of large numbers for U -statistics for interacting processes,

k-order derivatives :

$$\partial^k Q_{n,q} = \sum_{\mathbf{r} < \mathbf{q}; |\mathbf{r}|=k} \sum_{\mathbf{f} \in \mathcal{F}_{n,q}(\mathbf{r})} \frac{1}{(\mathbf{q})_{|\mathbf{f}|}} s(|\mathbf{f}|, \mathbf{q} - \mathbf{r}) \#(\mathbf{f}) \Delta_{n,q}^{\mathbf{f}}$$

with

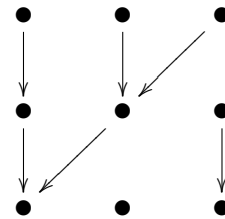
- $\mathcal{F}_{n,q}(\mathbf{r}) :=$ Forests with less than $\mathbf{r} = (r_0, \dots, r_n) (< \mathbf{q} = (q_0, \dots, q_n))$ coalescences.

- $\#(\mathbf{f}) :=$ nb of equivalent "forests" up to a change of leaves labeling (\exists closed formulae)

- $|\mathbf{f}| := (v_0, \dots, v_n)$ multi index $\#$ vertices at each level.

- $(\mathbf{q})_{|\mathbf{f}|} = (q)_{v_0} \dots (q)_{v_n}$ and $s(\mathbf{q}, \mathbf{p}) = s(q_0, p_0) \dots s(q_n, p_n)$; $((q)_p = q!/(q-p)!, s(q, p)$ 1 kind Stirling $\#$).

Ex. : $(n, q) = (1, 3)$ and $\mathbf{r} = (1, 1)$ and $|\mathbf{f}| = (2, 2)$:



$$\Rightarrow \Delta_{n,q}^{\mathbf{f}}(F) = \int \eta_0(dx^1) \eta_0(dx^2) \eta_0(dx^3) Q_1(x^1, dy^1) Q_1(x^1, dy^2) Q_1(x^3, dy^3) \\ Q_2(y^1, dz^1) Q_2(y^2, dz^2) Q_2(y^2, dz^3) F(z^1, z^2, z^3)$$

- $(\mathbb{Q}_{n,q}^N \rightsquigarrow \mathbb{P}_{n,q}^N) : \mathbb{E}((\gamma_n^N)^{\otimes q}(F)) \rightsquigarrow \mathbb{E}([(\gamma_0^N)^{\otimes q_0} \otimes \dots \otimes (\gamma_n^N)^{\otimes q_n}](F)) \oplus$ renormalisation techniques.