Particle approximation of multiple object filtering problems

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UNSW, School of Mathematics and Statistics

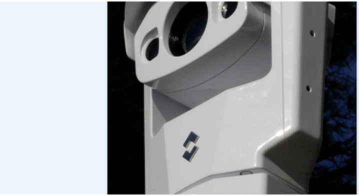
STM Workshop, IMS Tokyo 2014

Some hyper-refs

- **Particle approximations of a class of branching distribution flows arising in multi-target tracking**. SIAM Control. & Opt. (2011). *(joint work with Caron, Doucet, Pace)*
- **On the Conditional Distributions of Spatial Point Processes**. Advances in Applied Probability (2011). *(joint work with Caron, Doucet, Pace)*.
- **On the Stability & the Approximation of Branching Distribution Flows, with Applications to Nonlinear Multiple Target Filtering**. Stochastic Analysis and Applications (2011). *(joint work with Caron, Pace, Vo)*.
- **Comparison of implementations of Gaussian mixture PHD filters.** FUSION (2010). *(joint work with Caron, Pace, Vo)*.
- **Mean-field PHD filters based on generalized Feynman-Kac flow**.IEEE Journal of Selected Topics in Signal Processing. Special Issue on Multi-target tracking (2013). *(joint work with Pace)*
- **Mean field simulation for Monte Carlo integration.** Chapman & Hall, Series : Maths and Stat. (2013).
- **General multi-object filtering and association measure** Computational Advances in Multi-Sensor Adaptive Processing *joint work with Houssineau & Clark*

Contents

⊂ 2 Industrial research project



1. Defense industrial Contract : ALEA INRIA & DCNS Toulon (2009)
2. ⇝ `National Research project :  ANR PROPAGATION [2,3M€] (2009-2012):`

   Passive radar tracking and optronics liabilities for the protection of coastal infrastructures

   ALEA INRIA team ⊕ DCNS SIS, THALES, ECOMER, EXAVISION

   - Project members : + D. Arrivault, Fr. Caron, M. Pace.
   - Visiting researchers :
     D. Clark, A. Doucet, J. Houssineau, S.S. Sing, B.N. Vo.

Some basic notation

(μ, f) =(measure, function)$\in (\mathcal{M}(E) \times \mathcal{B}_b(E))$

$$\mu(f) = \int \mu(dx)\ f(x)$$

Delta-Dirac Measure at $a \in E$

$$\mu = \delta_a \quad \Rightarrow \quad \delta_a(f) = \int f(x)\ \delta_a(dx) = f(a)$$

Normalization of a positive measure $\mu \in \mathcal{M}_+(E)$ (when $\mu(1) \neq 0$)

$$\overline{\mu}(dx) := \mu(dx)/\mu(1) = \mathrm{probability} \in \mathcal{P}(E)$$

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Example

$$\mu = 10 \times \mathrm{Law}(X)$$

$$\Downarrow$$

$$\mu(1) = 10 \ \ \& \ \ \overline{\mu} = \mathrm{Law}(X) \ \ \& \ \ \overline{\mu}(f) = \mathbb{E}(f(X))$$

$Q(x, dy)$ integral operator from E into E'

Two operator actions :

$$f \in \mathcal{B}_b(E') \mapsto Q(f) \in \mathcal{B}_b(E) \quad \text{and} \quad \mu \in \mathcal{M}(E) \mapsto \mu Q \in \mathcal{M}(E')$$

with

$$\begin{aligned} Q(f)(x) &= \int Q(x, dx') f(x') \\ [\mu Q](dx') &= \int \mu(dx) Q(x, dx') \qquad (\Longleftrightarrow [\mu Q](f) := \mu[Q(f)]\ ) \end{aligned}$$

and the composition

$$(Q_1 Q_2)(x, dx'') = \int Q_1(x, dx') Q_2(x', dx'')$$

Q **Markov operator** $\Leftrightarrow$ $Q(1) = 1$ =unit function
$\rightsquigarrow$ notation : M or K (Markov transitions-kernel/Stochastic matrices)

Boltzmann-Gibbs transformation : $G \geq 0$ s.t. $\mu(G) > 0$

$$\Psi_G(\mu)(dx) = \frac{1}{\mu(G)} \; G(x) \; \mu(dx)$$

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Bayes' rule (with fixed observation y)

$$\mu(dx) = p(x)dx \quad \text{and} \quad G(x) = p(y|x)$$

$$\Downarrow$$

$$\Psi_G(\mu)(dx) = \frac{1}{p(y)}\ p(y|x)\ p(x)dx = p(x|y)\ dx$$

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Restriction

$$\mu(dx) = \mathbb{P}(X \in dx) = p(x)dx \quad \text{and} \quad G(x) = 1_A(x)$$

$$\Downarrow$$

$$\Psi_G(\mu)(dx) = \frac{1}{\mathbb{P}(X \in A)}\; 1_A(x)\; p(x)dx = \mathbb{P}(X \in dx \mid X \in A)$$

Boltzmann-Gibbs transformation : $G \geq 0$ s.t. $\mu(G) > 0$

$$\Psi_G(\mu)(dx) = \frac{1}{\mu(G)}\ G(x)\ \mu(dx)$$

$$\Downarrow$$

$\exists$ (non unique) Markov transport equation

$$\Psi_G(\mu)(dy) = \int \mu(dx) S_\mu(x, dy) \Longleftrightarrow \Psi_G(\mu) = \mu S_\mu$$

Example 1 : $(G \leq 1) \rightsquigarrow$ accept/reject/recycling/interacting jumps

$$S_\mu(x, dy) = G(x)\delta_x(dy) + (1 - G(x))\ \Psi_G(\mu)(dy)$$

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Note :

$$S_{\frac{1}{N}\sum_{1 \leq j \leq N} \delta_{X^j}}(X^i, dy)$$

$$= G(X^i)\delta_{X^i}(dy) + (1 - G(X^i))\ \sum_{1 \leq j \leq N} \frac{G(X^j)}{\sum_{1 \leq k \leq N} G(X^k)}\ \delta_{X^j}(dy)$$

Other examples of transport equations $\Psi_G(\mu) = \mu S_\mu$

Example 2 : $\forall \epsilon_\mu \quad \text{s.t.} \quad \epsilon_\mu G \leq 1 \quad \mu - \text{a.e.}$

$$S_\mu(x, dy) = \epsilon_\mu \; G(x) \; \delta_x(dy) + (1 - \epsilon_\mu G(x)) \; \Psi_G(\mu)(dy)$$

Example 3 : $\forall a$ **s.t.** $G > a$

$$S_\mu(x, dy) = \frac{a}{\mu(G)} \; \delta_x(dy) + \left(1 - \frac{a}{\mu(G)}\right) \; \Psi_{G-a}(\mu)(dy)$$

Example 4 : $\forall G$

$$S_\mu(x, dy) = \alpha(x) \; \delta_x(dy) + (1 - \alpha(x)) \; \Psi_{[G-G(x)]_+}(\mu)(dy)$$

with the acceptance rate

$$\alpha(x) \quad = \quad \mu[G \wedge G(x)]/\mu(G)$$

Spatial Branching models (time index $n \in \mathbb{N}$, state spaces E_n)

- **3 ingredients :** $G_n(x) \geq 1$, $\mu_n(dx) \geq 0$, and $M_n(x_{n-1}, dx_n)$ Markov.
  - **Branching rule (spawning)** :

Random mapping $x \rightsquigarrow g_n(x) \in \mathbb{N}$ offsprings, with $\mathbb{E}(g_n(x)) = G_n(x)$

$\supset$ *survival rates* $e_n(x)$ + *cemetery states :* $G_n \rightsquigarrow e_n(x)G_n(x)$

  - **Spontaneous births:** Spatial Poisson with intensity $\mu_n(dx)$
  - **Free motion between branching times :** M_n-evolutions

- $\rightsquigarrow$ **Random occupation measure (after the n-th evolution step)**

$$\mathcal{X}_n = \sum_{i=1}^{N_n} \delta_{X_n^i}$$

$E_n :=$ {types, locations, labels, excursions, paths,...}

Spatial Branching models (time index $n \in \mathbb{N}$, state spaces E_n)

- First moment recursion = branching intensity distribution

$$\gamma_{n+1}(f) \quad := \quad \mathbb{E}\left(\mathcal{X}_{n+1}(f)\right) = \gamma_n(Q_{n+1}(f)) + \mu_{n+1}(f)$$

with

$$Q_{n+1}(x, dy) = G_n(x) M_{n+1}(x, dy)$$

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Sketched proof ($\mu_n = 0$):

$$\mathcal{X}_{n+1} = \sum_{i=1}^{N_{n+1}} \delta_{X^i_{n+1}} = \sum_{i=1}^{N_n} \sum_{j=1}^{g^i_n(X^i_n)} \delta_{X^{i,j}_{n+1}}$$

$$\Downarrow$$

$$\mathbb{E}\left(\mathcal{X}_{n+1}(f) \mid \mathcal{X}_n, g_n(X_n)\right) = \sum_{i=1}^{N_n} g^i_n(X^i_n)\ M_{n+1}(f)(X^i_n)$$

$$\Downarrow$$

$$\mathbb{E}\left(\mathcal{X}_{n+1}(f) \mid \mathcal{X}_n\right) = \sum_{i=1}^{N_n} G_n(X^i_n)\ M_{n+1}(f)(X^i_n) = \mathcal{X}_n(Q_{n+1}(f))$$

Continuous time models

- **Geometric clocks $\rightsquigarrow$ exponential rates time mesh**
  $(t_n - t_{n-1}) \simeq 0$

$$
\begin{aligned}
X_n &= \mathcal{X}_{t_n} \\
G_n &= \text{survival} \times [\text{spawning} \times \text{mean} \sharp \text{ offsprings} + (1 - \text{spawning}) \times 1]
\end{aligned}
$$

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- $(t_n - t_{n-1}) \downarrow 0 \rightsquigarrow G = 1 + V\ dt$ and $M = Id + L\ dt$ and $t_n \to t$

$$\Downarrow$$

$$\frac{d}{dt}\gamma_t(f) = \gamma_t(L^V(f)) + \mu_t(f) \quad \text{with} \qquad L^V = L + V$$

Schrödinger operator

$\mu_n = 0 \Rightarrow$ Classical Feynman-Kac models

- Feynman-Kac representation ($\supset \uparrow$ Application domains)

$$\gamma_{n+1}(f) \quad = \quad \gamma_0(1)\ \mathbb{E}_{\eta_0}\left(f(X_{n+1}) \prod_{0\leq p\leq n} G_p(X_p)\right)$$

- Particle approximations $=$ Genetic type algo $=$ Particle filters $= \ldots$

$$Q_{n+1}(x,dy) = \underbrace{G_n(x)}_{\text{Selection potential}} \times \underbrace{M_{n+1}(x,y)}_{\text{Mutation transition}}$$

More general spatial Branching models (hyp. $\gamma_0 = \mu_0$)

$$\gamma_n = \gamma_{n-1} Q_n + \mu_n \quad \text{and} \quad \eta_n := \gamma_n / \gamma_n(1)$$

$$\Updownarrow$$

$$(\gamma_n(1), \eta_n) := \Gamma_{p,n}\left(\gamma_p(1), \eta_p\right)$$

Some problems

- Problem 1: Mass $\gamma_n(1)$ "unstable" $\gamma_n(1) \uparrow \infty$ or $\gamma_n(1) \downarrow 0$ as $n \uparrow \infty$
- Problem 2: $\mathcal{X}_n = \sum_{i=1}^{N_n} \delta_{X_n^i}$ generally NOT POISSON random field.
- Problem 3: $\exists$ non degenerate numerical sampling method?
- Problem 4: $\exists$ non degenerate approximation of γ_n?

Three scenarios $M_n = M$, $G_n = G \in [g_-, g_+]$, $\mu_n = \mu$

1. $G = 1 \Rightarrow \eta_\infty := \eta_\infty M$ (independent of μ)

$$\gamma_n(1) = \gamma_0(1) + n\mu(1) \quad \text{and} \quad \|\eta_n - \eta_\infty\|_{\mathrm{tv}} = O(1/n)$$

2. $g_+ < 1 \Rightarrow \eta_\infty := \gamma_\infty/\gamma_\infty(1)$ with γ_∞ given by Neumann series

$$\gamma_\infty := \sum_{n \geq 0} \mu Q^n \Longleftrightarrow \text{Poisson equation} \;\; \gamma_\infty(Id - Q) = \mu$$

and

$$|\gamma_n(f) - \gamma_\infty(f)| \vee |\eta_n(f) - \eta_\infty(f)| \leq c\; g_+^n \; \|f\|$$

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Continuous time models $G = e^{-V\Delta t}$ **&** $M = L\ \Delta t + Id$

$$\gamma_t(f) = \int_0^t \mathbb{E}_\mu\left(f(X_s)\ \exp\left(-\int_0^s V(X_r)dr\right)\right) ds$$

$t \to \infty \rightsquigarrow$ Poisson equation $\gamma_\infty L^V = \mu$, with $L^V = L + V$

The 3-rd scenario ($M_n = M$, $G_n = G \in [g_-, g_+]$, $\mu_n = \mu$)

$$g_- > 1 \Rightarrow \eta_\infty(f) := \eta_\infty Q(f)/\eta_\infty Q(1) \quad \text{(independent of } \mu\text{)}$$

$$\Downarrow$$

$$\lim_{n\to\infty} \frac{1}{n} \log \gamma_n(1) = \log \eta_\infty(G) \quad \text{and} \quad \|\eta_n - \eta_\infty\|_{\rm tv} \leq c\ e^{-\lambda n}$$

$\eta_\infty =$ [quasi-invariant meas., Yaglom meas., ground states, Feynman-Kac sg fixed points, infinite population stationary measure, ...]

Hyper-refs :

- `On the stability of interacting processes with applications to filtering and genetic algorithms.` (joint work with A. Guionnet) Annales IHP **(2001)**.
- `Particle approximations of Lyapunov exponents connected to Schrödinger operators and Feynman-Kac semigroups.` (joint work with L. Miclo) ESAIM: P&S **(2003)**.
- `Particle Motions in Absorbing Medium with Hard and Soft Obstacles.`

  (joint work with **A. Doucet**) Stochastic Analysis and Applications **(2004).**

Nonlinear equations

$$\eta_{n+1} \propto \gamma_n(1)\ \eta_n Q_{n+1} + \mu_{n+1}(1)\ \overline{\mu}_{n+1}$$

$$\Updownarrow$$

Nonlinear & interacting mass + proba measures equations

$$\begin{cases} \gamma_{n+1}(1) & = & \gamma_n(1)\ \eta_n(G_n) + \mu_{n+1}(1) \\ \\ \eta_{n+1} & = & \Psi_{G_n}(\eta_n) M_{n+1,(\gamma_n(1),\eta_n)} \end{cases}$$

with the Markov transitions:

$$M_{n+1,(m,\eta)}(x,dy) := \alpha_n\left(m,\eta\right) M_{n+1}(x,dy) + \left(1-\alpha_n\left(m,\eta\right)\right)\ \overline{\mu}_{n+1}(dy)$$

and the collection of $[0,1]$-parameters

$$\alpha_n\left(m,\eta\right) = \frac{m\ \eta(G_n)}{m\ \eta(G_n) + \mu_{n+1}(1)}$$

An extended Feynman-Kac model

$$\eta_n \xrightarrow{\text{updating}} \widehat{\eta}_n := \Psi_{G_n}(\eta_n) = \eta_n S_{n,\eta_n} \xrightarrow{\text{prediction}} \eta_{n+1} := \widehat{\eta}_n M_{n+1,(\gamma_n(1),\eta_n)}$$

$$\Downarrow$$

A couple of equations:

- The total mass evolution

$$\gamma_{n+1}(1) = \gamma_n(1)\ \eta_n(G_n) + \mu_{n+1}(1)$$

- The "nonlinear filtering/Feynman-Kac type" conservative equations

$$\eta_{n+1} = \eta_n S_{n,\eta_n} M_{n+1,(\gamma_n(1),\eta_n)} := \eta_n \underbrace{K_{n+1,(\gamma_n(1),\eta_n)}}_{\textbf{Markov transition}}$$

Mean field interacting particle models

$$\eta_n^N = \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi_n^i} \simeq_{N\uparrow\infty} \eta_n \quad \text{and} \quad \gamma_n^N(1) \simeq_{N\uparrow\infty} \gamma_n(1)$$

- the total mass evolution **["deterministic"]**

$$\gamma_{n+1}^N(1) := \gamma_n^N(1)\ \eta_n^N(G_n) + \mu_{n+1}(1)$$

- Mean field particle model

$$\xi_{n+1}^i = \text{r.v. with distribution} \quad K_{n+1,(\gamma_n^N(1),\eta_n^N)}(\xi_n^i, dx_{n+1})$$

$$\Downarrow$$

(Local) Stochastic perturbation model:

$$\eta_{n+1}^N := \eta_n^N K_{n+1,(\gamma_n^N(1),\eta_n^N)} + \frac{1}{\sqrt{N}}\ W_{n+1}^N$$

Theoretical convergence results

- **Independent local sampling error fluctuations**

$$(W_n^N)_{n\geq 0} \simeq_{N\uparrow\infty} \quad \text{iid centered Gaussian fields } (W_n)_{n\geq 0}$$

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- **Functional CLT(s) (with $\left[\gamma_n^N := \gamma_n^N(1) \times \eta_n^N\right]$)**

$$V_n^{\gamma,N} := \sqrt{N}\,(\gamma_n^N - \gamma_n) \quad \& \quad V_n^{\eta,N} := \sqrt{N}\,(\eta_n^N - \eta_n) \;\to_N\; V_n^{\gamma} \;\&\; V_n^{\eta}$$

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- **Uniform cv results (under some mixing conditions on M_n)**

$$\sup_{n\geq 0} \mathbb{E}\left(\left|\left[\eta_n^N - \eta_n\right](f)\right|^p\right) \leq c(p)/N^{p/2} \quad (\oplus \text{ uniform concentration})$$

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- **Unbiased particle total mass with variance $(N \geq n)$**

$$\mathbb{E}\left(\left[1-\gamma_n^N(1)/\gamma_n(1)\right]^2\right) \leq c\ n/N$$

Conditioning principles for marked point processes

- **Poisson point process** $\mathcal{X}$ with intensity $\gamma(dx_1)\ Q(x_1, dx_2)$ on $E = (E_1 \times E_2)$

$$\mathcal{X} := m_N(X_1, X_2) = \sum_{1 \leq i \leq N} \delta_{(X_1^i, X_2^i)} \quad \text{and} \quad \mathcal{X}_j := m_N(X_j) = \sum_{1 \leq i \leq N} \delta_{X_j^i}$$

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- **2 Bayes' rules:** Normalization $p(x_2|x_1) \oplus$ Markov operator $p(x_1|x_2)$

$$\overline{Q}(x_1, dx_2) = \frac{Q(x_1, dx_2)}{Q(x_1, E_2)} \quad \text{and} \quad \gamma(dx_1)\ Q(x_1, dx_2) = (\gamma Q)(dx_2)\ Q_\gamma(x_2, dx_1)$$

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- $\Rightarrow$ **2 conditional distributions formulae:**

$$\mathbb{E}\left(F_1(\mathcal{X}_1)\ \mid \mathcal{X}_2\right) = \int F_1\left(m_N(x_1)\right) \prod_{1\leq i\leq N} Q_\gamma(X_2^i, dx_1^i)$$

$$\mathbb{E}\left(F_2(\mathcal{X}_2)\ \mid \mathcal{X}_1\right) = \int F_2\left(m_N(x_2)\right) \prod_{1\leq i\leq N} \overline{Q}(X_1^i, dx_2^i)$$

Conditioning principles for marked point processes

- $(\mathcal{X}_1, \mathcal{X}_2) = (\mathcal{X}, \mathcal{Y})$, $\mathcal{X}$ **Poisson Signal** $\gamma(dx) \rightsquigarrow \mathcal{Y}$ **Poisson Obs.**

$$\left\{ \begin{array}{l} (X^i = x) \rightsquigarrow (Y^i = y) \sim \alpha(x)\ g(x,y)\ \lambda(dy) + (1-\alpha(x))\ \ \delta_c(dy) \\ \\ \oplus \quad \textbf{Clutter} \quad \mathcal{Y}' \quad \text{Poisson with intensity } \nu(dy) = h(y)\ \lambda(dy) \end{array} \right.$$

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- Observables $\mathcal{Y}^0 = \mathcal{Y} \times 1_{\neq c}$ ($\Leftrightarrow \alpha =$ detection rate)

$$\begin{aligned} \widehat{\gamma}(f) \quad &:= \quad \mathbb{E}\left(\mathcal{X}(f) \mid \mathcal{Y}^{\mathrm{o}}\right) \\ &= \quad \gamma((1-\alpha)f) + \int \mathcal{Y}^{\mathrm{o}}(dy)\,(1 - \beta_\gamma(y))\ \ \Psi_{\alpha g(y,\,.\,)}(\gamma)(f) \end{aligned}$$

with "the conditional clutter probability density"

$$\beta_\gamma(y) = h(y)/[h(y) + \gamma(\alpha g(y,\,.\,))]$$

Ex.: full detect and no clutter $\alpha = 1$ & $h = 0 \rightsquigarrow \mathcal{Y}^{\mathrm{o}} = \mathcal{Y}$

Conditional mean number of targets and "their distributions"

$$\widehat{\gamma}(1) = \mathcal{Y}(1)$$

and

$$\widehat{\eta}(f) := \frac{\widehat{\gamma}(f)}{\widehat{\gamma}(1)} = \int \underbrace{\overline{\mathcal{Y}}(dy)}_{=\mathcal{Y}/\mathcal{Y}(1)} \quad \underbrace{\Psi_{g(y,\cdot)}(\eta)(f)}_{\text{Bayes' rule}} \quad \text{with} \quad \eta := \gamma/\gamma(1)$$

Single target $\Leftrightarrow \mathcal{Y}^{\mathrm{o}} = \delta_Y \Leftrightarrow$ **Classical filtering updating equations**

$$\widehat{\eta} = \Psi_{g(Y,\cdot)}(\eta)$$

PHD filtering equation [Signal branching model (Q_n, μ_n)]

Hyp.: $\mathcal{X}_{n+1}$ **Poisson** $\gamma_{n+1} = \widehat{\gamma}_n Q_n + \mu_n \oplus$ **with obs.** $\mathcal{Y}^0_{n+1}$ **as before**

$$\Downarrow$$

$\Longrightarrow$ **PHD filtering equations:**

$$\begin{aligned}
\gamma_{n+1} &:= \widehat{\gamma}_n Q_n + \mu_n \\
\widehat{\gamma}_n(f) &:= \gamma_n((1-\alpha_n)f) + \int \mathcal{Y}^o_n(dy)\,(1-\beta_{\gamma_n}(y))\;\Psi_{\alpha_n g_n(y,\,.\,)}(\gamma_n)(f)
\end{aligned}$$

$$\Downarrow$$

$\subset$ **A class of measure valued equations** $\supset$ PHD; Bernoulli filters, etc.

$$\gamma_{n+1} = \gamma_n Q_{n+1,\gamma_n}$$

Stability properties of meas. valued equations

$\eta_n = \gamma_n/\gamma_n(1) \rightsquigarrow$ Nonlinear semigroup $\quad (\gamma_n(1), \eta_n) = \Gamma_{p,n}(\gamma_p(1), \eta_p)$

Stability Theorem :

$$\|\Gamma_{p,n}(m', \eta') - \Gamma_{p,n}(m, \eta)\| \leq c \; e^{-\lambda(n-p)}$$

$$\Downarrow$$

Regularity prop. $\rightsquigarrow$ 3 natural conditions on the PHD filter/model

1. **small clutter intensities**
2. **high detection probability**
3. **high spontaneous birth rates**

Nonlinear equations

$$\gamma_{n+1} = \gamma_n Q_{n+1,\gamma_n} \quad \rightsquigarrow \quad \eta_n := \gamma_n/\gamma_n(1) \quad \text{and} \quad G_{n,\gamma_n} = Q_{n+1,\gamma_n}(1)$$

$$\Downarrow$$

- **The total mass evolution**

$$\gamma_{n+1}(1) = \gamma_n(1)\ \eta_n(G_{n,\gamma_n(1)\eta_n})$$

- **The "nonlinear filtering" conservative equations**

$$\eta_{n+1}(f) = \frac{\eta_n Q_{n,\gamma_n(1)\eta_n}(f)}{\eta_n Q_{n,\gamma_n(1)\eta_n}(1)} := \eta_n K_{n,\gamma_n(1)\eta_n}(f)$$

Mean field particle models

$$\eta_n^N = \frac{1}{N}\sum_{1\leq i\leq N} \delta_{\xi_n^i} \simeq_{N\uparrow\infty} \eta_n \quad \text{and} \quad \gamma_n^N(1) \simeq_{N\uparrow\infty} \gamma_n(1)$$

with

$$\begin{array}{rcl} \gamma_{n+1}^N(1) & = & \gamma_n^N(1) \times \eta_n^N(G_{n,\gamma_n^N(1)\eta_n^N}) \\ \xi_{n+1}^i & = & \text{random var. with law} \quad K_{n+1,(\gamma_n^N(1)\eta_n^N)}(\xi_n^i, dx) \end{array}$$

$$\Downarrow$$

Same theorems as before with uniform convergence estimates

Mean field particle models

$$\eta_n^N = \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi_n^i} \simeq_{N\uparrow\infty} \eta_n \quad \text{and} \quad \gamma_n^N(1) \simeq_{N\uparrow\infty} \gamma_n(1)$$

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$$\begin{array}{rcl}\gamma_{n+1}^N(1) & = & \gamma_n^N(1)\times\eta_n^N\left(G_{n,\gamma_n^N(1)\eta_n^N}\right)\\ \xi_{n+1}^i & = & \text{random var. with law} \quad K_{n+1,(\gamma_n^N(1)\eta_n^N)}(\xi_n^i, dx)\end{array}$$

$$\Downarrow$$

Same theorems as before with uniform convergence estimates

$\oplus$ **Abstract general models**

- $\supset \forall$ numerical scheme with local errors
- $\supset$ Interacting Kalman type filters

  $\rightsquigarrow$ particle associations measures ($\simeq$ GM-PHD)

Association measures [$\alpha = 1$ & $h = 0$ & $Q_n = M_n$]

Ex. : Computable (exact or approximate) filters

$$\text{The mappings} \quad \eta \mapsto \Phi_{n+1}^{y_n}(\eta) := \Psi_{g_n(y_n,\cdot)}(\eta) M_{n+1}$$

$$\subset \{\text{Kalman, EKF, Ensemble Kalman filters, particle filters,} \ldots\}$$

Initial association measure

$$\eta_1 := \int \overline{\mathcal{Y}}_0(dy_0)\ \Phi_1^{y_0}(\eta_0) \simeq \eta_1^N := \int \overline{\mathcal{Y}}_0^N(dy_0)\ \Phi_1^{y_0}(\eta_0)$$

for instance

$$\overline{\mathcal{Y}}_0^N = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{Y_0^i} \text{ i.i.d. samples from } \overline{\mathcal{Y}}_0 \quad \text{or (if possible)} \quad \overline{\mathcal{Y}}_0^N = \overline{\mathcal{Y}}_0$$

Particle association measures $[\alpha = 1 \ \& \ h = 0 \ \& \ Q_n = M_n]$

$$\begin{aligned}
\eta_2 &\simeq \int \overline{\mathcal{Y}}_1(dy_1)\ \Phi_2^{y_1}(\eta_1^N) \\
&= \int \underbrace{\overline{\mathcal{Y}}_1(dy_1)\ \overline{\mathcal{Y}}_0^N(dy_0) \frac{\Phi_1^{y_0}(\eta_0)(g_1(y_1,.))}{\int \overline{\mathcal{Y}}_0^N(dy_0)\ \Phi_1^{y_0}(\eta_0)(g_1(y_1,.))}} \quad [\Phi_2^{y_1} \circ \ \Phi_1^{y_0}]\,(\eta_0) \\
&\simeq \int \qquad \overline{\mathcal{Y}}_{0,1}^N(d(y_0,y_1)) \qquad [\Phi_2^{y_1} \circ \ \Phi_1^{y_0}]\,(\eta_0)
\end{aligned}$$

for instance

$$\overline{\mathcal{Y}}_{0,1}^N = \frac{1}{N} \sum_{1 \le i \le N} \delta_{(Y_{0,1}^i, Y_{1,1}^i)}$$

i.i.d. samples from the $N \times \mathcal{Y}_1(1)$ supported measures

$$\overline{\mathcal{Y}}_1(dy_1)\ \overline{\mathcal{Y}}_0^N(dy_0) \frac{\Phi_1^{y_0}(\eta_0)(g_1(y_1,.))}{\int \overline{\mathcal{Y}}_0^N(dy_0)\ \Phi_1^{y_0}(\eta_0)(g_1(y_1,.))}\ \delta_{(y_0,y_1)}$$

and so on ...

Particle association measures - Track management

Association particle tree genealogies

$$\eta^N_{n+1} := \int \mathcal{Y}^N_{0,n}(d(y_0,\ldots,y_n)) \quad \left[\Phi^{y_n}_{n+1}\circ\ldots\circ\Phi^{y_0}_1\right](\eta_0)$$

with

$$\mathcal{Y}^N_{0,n} := \frac{1}{N}\sum_{1\leq i\leq N}\delta_{(Y^i_{0,n},Y^i_{1,n},\ldots,Y^i_{n,n})}$$

Stochastic models and cv analysis :

- *General case :*

(miss-detect, survival, spontaneous birth) $=$ as before *virtual obs.*

- $\subset$ Abstract models of the form $\gamma_{n+1}=\gamma_n Q_{n+1,\gamma_n}$.
- Mean field particle models $\Leftrightarrow$ Association particle measures.
- $\rightsquigarrow$ $\mathbb{L}_p$-bounds $\oplus$ Concentration sub-Gaussian inequalities.