Strong Propagations of Chaos in Moran's type Particle interpretations of Feynman-Kac measures

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Abstract

This article is concerned with strong propagations of chaos properties in Moran's type particle interpretations of continuous time Feynman-Kac formulae. These particle schemes can also be seen as approximating models of simple generalized spatially homogeneous Boltzmann equations. We provide a simple, and original semigroup analysis based on empirical tensor measures combinatorics properties, martingales techniques, and coupling arguments. We also design a general and abstract framework, without any topological assumption on the state space. This yields a natural way to analyze the propagations of chaos properties for interacting particle models on path space. Applications to genealogical type particle algorithms for the nonlinear filtering and smoothing problem are also discussed.

1 Introduction

Let $X = (X_t)_{t\geq 0}$ be a progressive and time inhomogeneous Markov process on some measurable state space $(E, \mathcal{E})$, equipped with a σ-algebra $\mathcal{E}$. We denote by $\mathcal{B}_{\rm b}(E)$, and $\mathcal{P}(E)$ the set of all bounded measurable functions (equipped with the supremum norm $\|\cdot\|$), and the set of all probabilities on E. We also consider a measurable, and locally bounded mapping $U : \mathbb{R}_+ \times E \ni (t, x) \mapsto U_t(x) \in \mathbb{R}_+$, in the sense that for any $T \geq 0$, we have

$$u_T \overset{\text{def.}}{=} \sup_{0\leq t\leq T} \|U_t\| \quad < \quad +\infty$$

We associate with these objects the Feynman-Kac distributions flows $(\gamma_t, \eta_t)_{t\geq 0}$ defined for any $t \geq 0$, and any bounded measurable function $\varphi : E \to \mathbb{R}$ by the following formulae

$$\eta_t(\varphi) \overset{\text{def.}}{=} \gamma_t(\varphi)/\gamma_t(\mathbf{1}) \quad \text{with} \quad \gamma_t(\varphi) \overset{\text{def.}}{=} \mathbb{E}\left[\varphi(X_t) \exp\left(\int_0^t U_s(X_s)\, ds\right)\right] \tag{1}$$

A simple calculation (cf. for instance [6]) shows that the normalizing constants $\gamma_t(1)$ can be expressed in terms of the normalized distribution flow $(\eta_s)_{s\leq t}$ with the formulae

$$\mathbb{E}_{\eta_0}\left[\exp\left(\int_0^t U_s(X_s)\, ds\right)\right] \quad = \quad \exp\left(\int_0^t \eta_s(U_s)\, ds\right)$$

This readily yields that

$$\gamma_t(\varphi) = \exp\left(\int_0^t \eta_s(U_s)\,ds\right) \times \eta_t(\varphi) \tag{2}$$

These Feynman-Kac models are at the corner of diverse scientific disciplines. In engineering science, and more particularly in signal processing, they provide a functional representation of the conditional distribution of a partially observed signal, with respect to noisy observation paths. In physics, they represent the distributions of a particle evolving in an absorbing medium with obstacles related to some potential functions. In quantum chemistry, these probabilistic models are rather thought as weak solutions of Schrödinger type equations. In biology, these Feyman-Kac models also occur in a variety of topics, such as chemical polymerizations, and genetic and genealogical population models. Of course, a full description of all these applications model aeras would be too much disgression. The interested reader is referred to the review article [5], and the research monograph of the first author [3].

Even if they look innocent, these Feynman-Kac measures are very complex mathematical objects, and they can rarely be solved explicitly. In [5], we introduced an original Moran's type interacting particle interpretation, based on genetic interacting jumps particle models. The idea is to associate to the Feynman-Kac model (1) a sequence of E^N-valued and interacting Markov processes $\xi^{(N)} = (\xi_t^{(N)})_{t\geq 0} = ((\xi_t^{(N,1)}, \xi_t^{(N,2)}, \cdots, \xi_t^{(N,N)}))_{t\geq 0}$ such that the N-empirical measures of the configurations

$$\eta_t^{(N)} \stackrel{\text{def.}}{=} \frac{1}{N} \sum_{1\leq i\leq N} \delta_{\xi_t^{(N,i)}} \tag{3}$$

converge, as N tends to infinity, to the desired distribution η_t. Mimicking formulae (2), we also construct an unbiased estimate for the unnormalized distribution flow

$$\gamma_t^N(\varphi) \stackrel{\text{def.}}{=} \exp\left(\int_0^t \eta_s^N(U_s)\,ds\right) \times \eta_t^N(\varphi) \quad \stackrel{N\uparrow\infty}{\longrightarrow} \quad \gamma_t(\varphi) = \exp\left(\int_0^t \eta_s(U_s)\,ds\right) \times \eta_t(\varphi)$$

The parameter N represents both the precision parameter of the approximation scheme, and the size of the particle population model. The precise mathematical description of the particle model is given in section 2.4. Loosely speaking, the motion of the N-particle Moran's type model is decomposed into two separate mechanisms. Between the interacting jumps, each particle $\xi_t^{(N,i)}$ evolves randomly according to the same random motion as the reference Markov process X_t. At rate $\eta_t^{(N)}(U_t)$, a randomly chosen particle $\xi_t^{(N,i)}$ jumps to a new location $\xi_t^{(N,j)}$, randomly chosen with a probability proportional to its fitness $U_t(\xi_t^{(N,j)})$. In this sense, this particle interpretation model can be seen as a natural interacting selection-rejection population Monte Carlo sampling methodology. This interacting stochastic process can also be alternatively interpreted as a Moran or a Nanbu particle model. For a general description of these two kinds of genetic type algorithms, we refer the reader to the pair of articles [2, 9].

A natural and traditional problem on mean field particle models is the analysis of the strong propagation of chaos properties. Namely, we are wondering if the law of a fixed particle, say $(\xi_t^{(N,1)})_{t\geq 0}$, converges in total variation sense toward the law of some natural time inhomogeneous Markovian process $\bar{X} = (\bar{X}_t)_{t\geq 0}$, with $\mathrm{Law}(\bar{X}_t) = \eta_t$, for any $t \geq 0$. The latter process is often referred as the nonlinear, or the target process associated with the limiting deterministic flow $(\eta_t)_{t\geq 0}$. Roughly speaking, it evolves as a single interacting particle evolving in an infinite population model. In other word, between the jumps, $\bar{X}_t$ evolves randomly according to the same random motion as the reference Markov process X_t. At rate $\eta_t(U_t)$, it jumps to a new location $\bar{X}_t$, randomly chosen with the Boltzmann-Gibbs distribution $U_t(x)\ \eta_t(dx)/\eta_t(U_t)$.

Our aim is to show that the first n particles, with $n \ll \sqrt{N}$, behave asymptotically as independent copies of the target process. To describe with some precision our main result, we let T, be a fixed time horizon. We also fix a particle block size $1 \leq n \leq N$, and we let $\bar{\mathbb{P}}_{\eta_0,[0,T]}$ be the distribution of target process $(\bar{X}_t)_{0\leq t\leq T}$, and $\mathbb{P}^{(N,\{1,\ldots,n\})}_{\eta_0,[0,T]}$ be the distribution of the first n particles $(\xi_t^{(N,i)})_{1\leq i\leq n,\, 0\leq t\leq T}$. We are now in position to state our result.

Theorem 1.1 *For any time horizon T, we have*

$$\limsup_{n^2/N\to 0} \frac{N}{n^2} \left\| \mathbb{P}^{(N,\{1,\ldots,n\})}_{\eta_0,[0,T]} - \bar{\mathbb{P}}^{\otimes n}_{\eta_0,[0,T]} \right\|_{\mathrm{tv}} \quad \leq \quad C_T$$

with the constant $C_T = 4(\exp(u_T T) - 1) + (14 + 28u_T T[1 + \exp(u_T T)])u_T T(u_T T + 1)$.

One important motivation for introducing a general and abstract set-up is that it applies without further work to particle models on path spaces, and genealogical tree based filtering models. These practical issues are discussed in the last short section. Taking into account the simple mechanism of selection in our models, we have a nice explicit expression (1) for the deterministic limiting objects, making them appear as ratios of linear terms with respect to η_0, which is hidden in $\mathbb{E}$ as the initial distribution. This semigroup structure is more tractable than the information one would get by merely looking at the nonlinear equation of evolution satisfied by the family $(\eta_t)_{t\geq 0}$. Using this natural semigroup technique, we also obtain without much difficulty the weak propagation of chaos for empirical tensor measures stated in the following theorem.

Theorem 1.2 *For any time horizon $T > 0$, any particle block size $n \in \mathbb{N}^*$, any sequence of times $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T$, and any function $\varphi \in \mathcal{B}_{\mathrm{b}}(E^n)$, with $\|\varphi\|$, we have that*

$$\begin{aligned}
\left|\mathbb{E}[(\eta_{t_1}^{(N)} \otimes \cdots \otimes \eta_{t_n}^{(N)})(\varphi)] - (\eta_{t_1} \otimes \cdots \otimes \eta_{t_n})(\varphi)\right| &\leq \epsilon_T\left(\frac{n^2}{N}\right) \\
\left|\mathbb{E}[(\gamma_{t_1}^{(N)} \otimes \cdots \otimes \gamma_{t_n}^{(N)})(\varphi)] - (\gamma_{t_1} \otimes \cdots \otimes \gamma_{t_n})(\varphi)\right| &\leq \widehat{\epsilon}_T\left(\frac{n(n-1)}{N}\right)\ (\gamma_{t_1} \otimes \cdots \otimes \gamma_{t_n})(1)
\end{aligned}$$

with the pair of mappings $(\epsilon_T, \widehat{\epsilon}_T)$ from $\mathbb{R}_+$ into itself given by

$$\begin{aligned}
\widehat{\epsilon}_T(a) &= a + 4(1 - \exp[-a u_T T]) + 2\left(\exp([\exp(u_T T) - 1] a u_T T) - 1\right) \\
\epsilon_T(a) &= 2[(\widehat{\epsilon}_T(a) + \widehat{\epsilon}_T(4a) + 2\widehat{\epsilon}_T(a)) \wedge 1]
\end{aligned}$$

Notice that

$$\lim_{a\to 0_+} \frac{\widehat{\epsilon}_T\,(a)}{a} = 1 + 2u_T T(1 + \exp(u_T T)) \quad \text{and} \quad \lim_{a\to 0_+} \frac{\epsilon_T(a)}{a} = 14 + 28u_T T[1 + \exp(u_T T)]$$

The proof of these two theorems are respectively housed in section 3, and in section 4, following the spirit prevailing in [5] and [6], for the elementary particle density profiles $\eta_t^{(N)}$. The propagations of chaos properties presented in theorem 1.1, will be deduced from the estimates stated in theorem 1.2, combined with two elementary coupling arguments. These pair of coupling techniques are presented in sections 4.1, and section 4.2.

We end by noting that our main task will be to find out nice martingales. This will hopefully prepare the way for central limit theorems, exponential estimates or other similar developments. Finally, we think that the study of the tensorized empirical measures could be developed further. At least it illustrates the flexibility of the semigroup approach and makes clear some links with the general theory of measure valued processes (cf. [2]).

2 Description of the model

As promised, we try in this paper to work under very minimal assumptions, in order to fix a robust framework and to see which unnecessary structures (mainly the topological ones) can be removed from our previous works. To fulfill this side goal, our approach will have to differ in some aspects from the one presented in [6], but it will be adapted to the proof of the strong propagation of chaos, which in the end will (almost) always be satisfied. Doing so, we will show that even the weak condition considered in [6] was in fact useless to get the weak propagation of chaos. The principal difference is that the Markovian process X will be very general, and in particular we will make no explicit reference to the "carrés du champ". At the present stage, we are still wondering if this set-up is sufficient to obtain the central limit theorem shown in [6], but this question will not be investigated here.

2.1 General conditions

We begin with presenting the rigorous definition of the objects entering into the composition of the r.h.s. of (1). The measurable process X appearing in (1) will be defined as a canonical coordinate process, under an appropriate "inhomogeneous" family of probabilities, ensuring that it satisfies the Markov property. So, the first problem to be tackled is the definition of the space of "canonical trajectories": A priori, one would just consider $\mathbf{M}(\mathbb{R}_+, E)$ the set of all measurable paths from $\mathbb{R}_+$ to E, but as we will try to explain it later, this space is too large to be handled efficiently. Nevertheless $X = (X_t)_{t\geq 0}$ will denote the related process of canonical coordinates or its restriction to any of the subset of $\mathbf{M}(\mathbb{R}_+, E)$. We make the assumption that we are given a nonempty set $\mathbb{M}(\mathbb{R}_+, E) \subset \mathbf{M}(\mathbb{R}_+, E)$ satisfying the condition (H1) which consists in the next two properties.

- *If we are given a sequence $(\omega_i)_{i\geq 0}$ of elements of $\mathbb{M}(\mathbb{R}_+, E)$ and an increasing sequence $(t_i)_{i\geq 0}$ of nonnegative real numbers, satisfying $t_0 = 0$ and $\lim_{i\to\infty} t_i = +\infty$, then the element $\omega \in \mathbf{M}(\mathbb{R}_+, E)$ defined by*

$$\forall\ i \geq 0, \forall\ t_i \leq s < t_{i+1}, \qquad X_s(\omega) \quad = \quad X_s(\omega_i)$$

  belongs to $\mathbb{M}(\mathbb{R}_+, E)$.

- *Let $\mathcal{M}(\mathbb{R}_+, E)$ be the σ-field generated by the coordinates $(X_t)_{t\geq 0}$ on $\mathbb{M}(\mathbb{R}_+, E)$. Then the mapping*

$$\mathbb{R}_+ \times \mathbb{M}(\mathbb{R}_+, E) \ni (t, \omega) \quad \mapsto \quad X_t(\omega) \ \in \ E$$

  is $\mathcal{R}_+ \otimes \mathcal{M}(\mathbb{R}_+, E)$-measurable, where $\mathcal{R}_+$ denotes the usual Borelian σ-field on $\mathbb{R}_+$ (quite similarly, for any Borelian set $I \subset \mathbb{R}_+$, $\mathcal{R}_I$ will stand for the trace of $\mathcal{R}_+$ on I).

We discuss condition (H1) at the end of this section, but in the whole subsequent development, we assume that a particular element $\diamond \in \mathbb{M}(\mathbb{R}_+, E)$ has been chosen (it often plays the role of a cemetery point). For $t \geq 0$, let $\mathbb{M}([t, +\infty[, E) \subset \mathbf{M}([t, +\infty[, E)$ be the image of $\mathbb{M}(\mathbb{R}_+, E)$ under the mapping $(X_s)_{s\geq t}$; it is the set of all "admissible" paths after time t. We endow it naturally with the σ-field $\mathcal{M}([t, +\infty[, E)$ generated by the variables $\{X_s \,:\, s \geq t\}$. As usually, we also need to consider on $\mathbb{M}([t, +\infty[, E)$ the filtration $(\mathcal{M}([t, s], E))_{s\geq t}$, for any interval I of $\mathbb{R}_+$, $\mathcal{M}(I, E)$ will designate $\sigma(X_u\,;\, u \in I)$ Note that for $0 \leq t \leq s$, the mapping

$$[t, s] \times \mathbb{M}([t, +\infty[, E) \ni (u, \omega) \quad \mapsto \quad X_u(\omega) \ \in \ E$$

is $\mathcal{R}_{[t,s]} \otimes \mathcal{M}([t, s], E)$-measurable. Our main object is a given family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$ of probabilities respectively defined on

$(\mathbb{M}([t,+\infty[,E),\mathcal{M}([t,+\infty[,E))$ and satisfying an initial condition parametrization property: for all $t\geq 0$ and $x\in E$, we have $X_t\circ\mathbb{P}_{t,x}=\delta_x$, the Dirac mass at x. We also use a regularity property: for all $t\geq 0$ and $A\in\mathcal{M}([t,+\infty[,E)$, the mapping

$$E\ni x\quad\mapsto\quad\mathbb{P}_{t,x}[A]$$

is $\mathcal{E}$-measurable.Finally, it useful to have the following Markovian compatibility property: for all $0\leq t\leq s$, all $x\in E$ and all $A\in\mathcal{M}(]s,+\infty[,E)$, we have $\mathbb{P}_{t,x}$-a.s. the following equality for the conditional expectation:

$$\mathbb{P}_{t,x}[A|\mathcal{M}([t,s],E)]\quad=\quad\mathbb{P}_{s,X_s}[A]$$

Taking into account the initial condition parametrization, it appears that this equality is in fact true for all $A\in\mathcal{M}([s,+\infty[,E)$, but reciprocally, note that this "extended" assumption does not imply the initial condition parametrization property. From now on, such a family will be called Markovian.

Thus for all fixed $(t,x)\in\mathbb{R}_+\times E$, the process $(X_s)_{s\geq t}$ is Markovian under $\mathbb{P}_{t,x}$. More generally, using the measurability assumption above, for any distribution $\eta_0\in\mathcal{P}(E)$, we can define a probability $\mathbb{P}_{\eta_0}$ on $(\mathbb{M}([0,+\infty[,E),\mathcal{M}([0,+\infty[,E))$, by stating that

$$\forall\ A\in\mathcal{M}([0,+\infty[,E),\qquad\mathbb{P}_{\eta_0}[A]\quad=\quad\int_E\mathbb{P}_{0,x}[A]\,\eta_0(dx)$$

($\mathbb{E}_{\eta_0}$ will stand for the expectation relative to $\mathbb{P}_{\eta_0}$, the probability $\eta_0\in\mathcal{P}(E)$ being fixed, and in (1) we should now replace $\mathbb{E}$ by $\mathbb{E}_{\eta_0}$). Then $X=(X_s)_{s\geq 0}$ is also easily seen to be Markovian under $\mathbb{P}_{\eta_0}$, and the distribution of X_0 is η_0. As $t\geq 0$ varies, the probabilities $\mathbb{P}_{t,x}$, for $x\in E$, are defined on different measurable spaces, and this fact can be annoying for the formulation of some properties. So for any fixed $t\geq 0$, we introduce the injection

$$I_t\,:\,\mathbb{M}([t,+\infty[,E)\quad\to\quad\mathbb{M}(\mathbb{R}_+,E)$$

defined by

$$\forall\ s\geq 0,\,\forall\ \omega\in\mathbb{M}([t,+\infty[,E),\qquad X_s(I_t(\omega))\quad=\quad\begin{cases}X_s(\diamond) & ,\text{ for } s<t\\ X_s(\omega) & ,\text{ for } s\geq t\end{cases}$$

This mapping is clearly measurable, so it enables us to see $\mathbb{P}_{t,x}$ as a probability on $(\mathbb{M}(\mathbb{R}_+,E)$, $\mathcal{M}(\mathbb{R}_+,E))$, for all $x\in E$, and we will keep abusing of the same notation (i.e. "$\mathbb{P}_{t,x}=I_t\circ\mathbb{P}_{t,x}$").

Our second and principal hypothesis just says that the Markovian family has some "time regularity":

(H2) *For all $A\in\mathcal{M}(\mathbb{R}_+,E)$, the mapping*

$$\mathbb{R}_+\times E\ni(t,x)\mapsto\mathbb{P}_{t,x}[A]$$

is $\mathcal{R}_+\otimes\mathcal{E}$-measurable.

As a consequence of monotonous class theorem, it appears that for all bounded measurable functions $f\,:\,\mathbb{R}_+\times E\times\mathbb{M}(\mathbb{R}_+,E)$, the mapping

$$\mathbb{R}_+\times E\ni(t,x)\mapsto\int f(t,x,\omega)\,\mathbb{P}_{t,x}(d\omega)$$

is measurable. Let us now introduce some functions which will be very interesting in the subsequent development. For all fixed $T\geq 0$, $V\in\mathcal{B}_{\mathrm{b}}([0,T]\times E)$ and $\varphi\in\mathcal{B}_{\mathrm{b}}(E)$, we define the mapping

$$F_{T,V,\varphi}\,:\,[0,T]\times E\ni(t,x)\quad\mapsto\quad\mathbb{E}_{t,x}\left[\exp\left(\int_t^T V_s(X_s)\,ds\right)\varphi(X_T)\right]\tag{4}$$

The consideration of the assumptions (H1) and (H2) and the measurability part of Fubini theorem enable us to see that $F_{T,V,\varphi}$ is indeed $\mathcal{R}_{[0,T]} \otimes \mathcal{E}$-measurable. But this mapping has some more interesting properties. We associate to the family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$ the following general time-space martingale problems: for every fixed $T \geq 0$, we denote by $\mathcal{A}_T$ the vector space of functions $f \in \mathcal{B}_{\mathrm{b}}([0,T]\times E)$ for which there exists another function $L_T(f) \in \mathcal{B}_{\mathrm{b}}([0,T]\times E)$ such that for every fixed $(t,x) \in [0,T]\times E$, the process $(M_{t,s}(f))_{t\leq s\leq T}$ defined by

$$\begin{aligned} \forall\, t \leq s \leq T, \qquad M_{t,s}(f) &= f(s,X_s) - f(t,X_t) - \int_t^s L_T(f)(u,X_u)\,du \qquad (5)\\ (&= M_{0,s}(f) - M_{0,t}(f)\,) \end{aligned}$$

is a $(\mathcal{M}([t,s],E))_{t\leq s\leq T}$-martingale under $\mathbb{P}_{t,x}$.

In this article the martingales will not be implicitly supposed to be càdlàg (a.s.), because this is not relevant to our setting. More accurately, noncàdlàg martingales appear naturally in our calculations, even if we had made more restrictive assumptions (cf. [6]), and in fact these occurrences contributed to our choice of an extended set-up. But when we will need elementary stochastic calculus, we will have to consider a càdlàg version of the martingales at hand, and each time we have carefully verified that one can carry out the classical modification via an extension of the filtration (e.g. see [7], note also that all our martingales will be bounded). An example of the kind of the manipulations we have to resort to will be developed in the proof of lemma 2.1 below. Notice that $L_T(f)$ may not be uniquely defined by $f \in \mathcal{A}_T$ (but again we will keep abusing of these notations), nevertheless this is not really important, since for martingale problems, one can consider multivalued operators, cf [8]. Here is the only preliminary result we will need, and which is somewhat well known in the theory of Feynman-Kac formulae:

Lemma 2.1 *For all fixed $T > 0$, $V \in \mathcal{B}_{\mathrm{b}}([0,T]\times E)$ and $\varphi \in \mathcal{B}_{\mathrm{b}}(E)$, the mapping $F_{T,V,\varphi}$ belongs to $\mathcal{A}_T$, and we can (and will) take*

$$\forall\, 0 \leq t \leq T,\, \forall\, x \in E, \qquad L_T(F_{T,V,\varphi})(t,x) \;=\; -V_t(x)F_{T,V,\varphi}(t,x)$$

Proof: We have already seen above that $F_{T,V,\varphi} \in \mathcal{B}_{\mathrm{b}}([0,T]\times E)$. Now let us denote for any fixed $T > 0$, $V \in \mathcal{B}_{\mathrm{b}}([0,T]\times E)$ and $\varphi \in \mathcal{B}_{\mathrm{b}}(E)$,

$$\forall\, 0 \leq t \leq T, \qquad N_t \;=\; F_{T,V,\varphi}(t,X_t)$$

we will show that $(N_t - N_0 + \int_0^t V_s(X_s)N_s\,ds)_{0\leq t\leq T}$ is a (a priori not necessarily càdlàg) martingale under $\mathbb{P}_{\eta_0}$, for any given $\eta_0 \in \mathcal{P}(E)$. The more general requirement (for all initial conditions $(t,x) \in [0,T]\times E \ldots$) is proved in the same way, and the announced results follow. The Markov property of X implies that the process $(M_t)_{0\leq t\leq T}$ defined by

$$\begin{aligned} \forall\, 0 \leq t \leq T, \qquad M_t &= \exp\left(\int_0^t V_s(X_s)\,ds\right) N_t \\ &= \mathbb{E}_{\eta_0}\left[\exp\left(\int_0^T V_s(X_s)\,ds\right)\varphi(X_T)\middle|\,\mathcal{M}([0,t],E)\right] \end{aligned}$$

is a martingale. As we have no information about its time regularity (except the measurability), we will go into all the details of the calculations, which otherwise would be immediate (just remove the subscripts $^+$ from (6)). Let $\mathcal{N}$ be the set of all $\mathbb{P}_{\eta_0}$-negligeable subsets, we denote for $t \geq 0$, $\mathcal{M}_t^+ = \mathcal{N} \vee \bigcap_{s>t}\mathcal{M}([0,s],E)$ and $M_t^+ = \limsup_{s\in\mathbb{Q}\cap]t,+\infty[,\, s\to t} M_s$. It is well known (see for instance [7]) that $(M_t^+)_{t\geq 0}$ is a $(\mathcal{M}_t^+)_{t\geq 0}$ càdlàg martingale such that for all $t \geq 0$, a.s.,

$$M_t \;=\; \mathbb{E}_{\eta_0}[M_t^+|\mathcal{M}([0,t],E)]$$

With some obvious notations, we have

$$\begin{aligned} N_t^+ &= \exp\left(-\int_0^t V_s(X_s)\,ds\right) M_t^+ \\ &= N_0 + \int_0^t \exp\left(-\int_0^s V_u(X_u)\,du\right)\,dM_s^+ - \int_0^t V_s(X_s)N_s^+\,ds \end{aligned} \tag{6}$$

So let $s \geq 0$ and $A \in \mathcal{M}([0,s],E)$ be given, from the previous equality we get that

$$\mathbb{E}\left[\left(N_t^+ - N_s^+ + \int_s^t V_u(X_u)N_u^+\,du\right)\mathbf{I}_A\right] = 0$$

nevertheless what we do want to show is

$$\mathbb{E}\left[\left(N_t - N_s + \int_s^t V_u(X_u)N_u\,du\right)\mathbf{I}_A\right] = 0$$

Now, it is quite clear that

$$\mathbb{E}[N_t^+\mathbf{I}_A] = \mathbb{E}[N_t\mathbf{I}_A] \quad \text{and} \quad \mathbb{E}[N_s^+\mathbf{I}_A] = \mathbb{E}[N_s\mathbf{I}_A]$$

Using the fact that the mapping

$$[s,t]\times\mathbb{M}(\mathbb{R}_+,E) \ni (u,\omega) \mapsto V_u(X_u(\omega))N_u(\omega)$$

is measurable and Fubini's theorem, we obtain that

$$\begin{aligned} \mathbb{E}\left[\left(\int_s^t V_u(X_u)N_u^+\,du\right)\mathbf{I}_A\right] &= \int_s^t \mathbb{E}\left[V_u(X_u)N_u^+\mathbf{I}_A\right]\,du \\ &= \int_s^t \mathbb{E}\left[V_u(X_u)N_u\mathbf{I}_A\right]\,du = \mathbb{E}\left[\left(\int_s^t V_u(X_u)N_u\,du\right)\mathbf{I}_A\right] \end{aligned}$$

from which our above assertion follows.

■

Remarks 2.2:

a) As here we will mainly work with a finite horizon $T \geq 0$, i.e. we will only consider the restriction of the Markovian family to the path space $\mathbb{M}([0,T],E)$, we could have replaced the first point of (H1) by the simplest following one:

• *If ω_0 and ω_1 are elements of $\mathbb{M}(\mathbb{R}_+,E)$ and $t > 0$ is given, then the element $\omega \in \mathbf{M}(\mathbb{R}_+,E)$ defined by*

$$\forall\ s \geq 0, \qquad X_s(\omega) = \begin{cases} X_s(\omega_0), & \text{if } s < t \\ X_s(\omega_1), & \text{if } s \geq t \end{cases}$$

belongs to $\mathbb{M}(\mathbb{R}_+,E)$.

Then by induction, the first point of (H1) is true, but for finite sequences of times, and that is the only thing we need on a bounded interval $[0,T]$.

b) The hypothesis (H1) can be seen as an ersatz for the lack of regularity of the trajectories, and from this point of view, its important condition is the second point, which corresponds to the traditional notion of progressive process. In fact, as soon as $\mathcal{E}$ is not a trivial σ-algebra, $\mathbf{M}(\mathbb{R}_+,E)$ does not satisfy (H1): just note that for any given $A \in \sigma(X_t\,;\,t\geq 0)$, there exist a sequence $(t_i)_{i\geq 0}$ of nonnegative real numbers and a measurable set $A' \in \mathcal{E}^{\otimes\mathbb{N}}$ such that

$$A = \{\omega \in \mathbf{M}(\mathbb{R}_+,E)\,:\,(X_{t_i}(\omega))_{i\geq 0} \in A'\}$$

But let $\varphi \in \mathcal{B}_{\mathrm{b}}(E)$ taking at least the two values 0 and 1. Then the previous characterisation shows that the set

$$\{\omega \in \mathbf{M}(\mathbb{R}_+, E) \,:\, \int_0^1 \varphi(X_u(\omega))\,du \quad > \quad 0\}$$

cannot belong to $\sigma(X_t\,;\, t \geq 0)$, whereas it should if $\mathbf{M}(\mathbb{R}_+, E)$ was to verify (H1).

c) To the Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$, we can associate the semigroup $(P_{s,t})_{0\leq s\leq t}$, whose elements act on $\mathcal{B}_{\mathrm{b}}(E)$ via the formulae:

$$\forall\; 0 \leq s \leq t,\, \forall\; \varphi \in \mathcal{B}_{\mathrm{b}}(E),\, \forall\; x \in E, \qquad P_{s,t}(\varphi)(x) \quad = \quad F_{t,0,\varphi}(s, x)$$

Due to our definition of the σ-algebras $\mathcal{M}([t, +\infty[, E)$, for $t \geq 0$, it is classical to see that the semigroup determines the Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$.

For all $T \geq 0$, we will denote the vector space

$$\mathcal{B}_T \quad \stackrel{\text{def.}}{=} \quad \{F_{T,0,\varphi}\;;\; \varphi \in \mathcal{B}_{\mathrm{b}}(E)\} \quad \subset \quad \mathcal{A}_T$$

Now let $(\widetilde{\mathbb{P}}_{t,x})_{t\geq 0,\, x\in E}$ be another Markovian family whose time-space generators are the $(\widetilde{\mathcal{A}}_T,\, \widetilde{L}_T)$, for $T \geq 0$. As a consequence of the above discussion, if we assume that for all $T \geq 0$, $(\widetilde{\mathcal{A}}_T, \widetilde{L}_T)$ is an extension of $(\mathcal{B}_T, L_T)$, i.e. $\mathcal{B}_T \subset \widetilde{\mathcal{A}}_T$ and $\widetilde{L}_{T|\mathcal{B}_T} = L_{T|\mathcal{B}_T}$, then we have $(\widetilde{\mathbb{P}}_{t,x})_{t\geq 0,\, x\in E} = (\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$.

In particular, the family $(\mathcal{A}_T, L_T)_{T\geq 0}$ is characteristic of $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$, i.e. the abstract martingale problems uniquely determine the Markovian family.

d) The definition of a Markovian family and the condition (H2) could also be considered with respect to other σ-fields verifying the second point of (H1) on our sets of paths (in which cases one has in addition to assume the measurability of the mapping appearing in remark (a) above). But taking into account our particular choice of $\mathcal{M}(\mathbb{R}_+, E)$ and monotonous class arguments, they can be simplified and expressed through the associated semigroup: for instance the condition (H2) is equivalent to

(H′2) *For all $T \geq 0$ and all $A \in \mathcal{E}$, the mapping*

$$[0, T] \times E \ni (t, x) \mapsto P_{t,T}(\mathbb{1}_A)(x)$$

is $\mathcal{R}_{[0,T]} \otimes \mathcal{E}$-measurable.

Thus in this situation it appears that the role of the particular path $\diamond$ entering in the definition of the injections I_t, for $t \geq 0$, is not very important: for instance (H2) would not have been affected if we had chosen to let $\diamond$ depend in a measurable way on $X_t(\omega)$ (e.g. we could have rather considered for any $t \geq 0$ the injection defined by $X_s \circ I_t = X_{s\vee t}$ for every $s \geq 0$, except that it is not so natural to assume that $\mathbb{M}(\mathbb{R}_+, E)$ contains all constant paths, as we will see it in section 5).

e) The role of $T > 0$ in the definition of the generator $(\mathcal{A}_T, L_T)$ is not innocent: in the same way, we could have considered $(\mathcal{A}, L)$ the generator acting on measurable and locally bounded functions defined on $\mathbb{R}_+ \times E$ for which the martingale problems are satisfied, but it can be shown (for instance in the case of the real Brownian motion) that $(\mathcal{A}_T, L_T)$ can be a strict extension of the natural restriction of $(\mathcal{A}, L)$ on $\mathcal{B}_{\mathrm{b}}([0, T] \times E)$. Also note that there are some links between $(\mathcal{A}, L)$ and the full generators defined in [8], but they are not strictly the same, in particular due to the inhomogeneity in time.

f) A traditional object in related set-ups is the family $(\theta_t)_{t\geq 0}$ of the time shifts acting on $\mathbf{M}(\mathbb{R}_+, E)$, which are defined by

$$\forall\; t, s \geq 0,\, \forall\; \omega \in \mathbf{M}(\mathbb{R}_+, E), \qquad X_s(\theta_t(\omega)) \quad = \quad X_{t+s}(\omega)$$

and more precisely, for $t \geq 0$ given, θ_t is a measurable map from $(\mathbf{M}([t,+\infty[,E),\sigma(X_s\,;\,s \geq t))$ to $(\mathbf{M}(\mathbb{R}_+,E),\sigma(X_s\,;\,s\geq 0))$. But with our definition of the $\mathbb{M}([t,+\infty[,E)$, for $t\geq 0$, it is not clear that the image of $\mathbb{M}([t,+\infty[,E)$ under θ_t is included into $\mathbb{M}(\mathbb{R}_+,E)$ (e.g. if the random variables X_t naturally take values in different subsets of E as $t\geq 0$ varies, see for instance the end of this remark).

Nevertheless, if we assume in addition that for all $t\geq 0$, $\theta_t(\mathbb{M}([t,+\infty[,E))\subset \mathbb{M}(\mathbb{R}_+,E)$, then (H2) is easily seen to be equivalent to

(H$''$2) *For all $A\in\mathcal{M}(\mathbb{R}_+,E)$ the mapping*

$$\mathbb{R}_+\times E\ni (t,x)\mapsto \mathbb{P}_{t,x}[\theta_t^{-1}(A)]$$

is $\mathcal{R}_+\otimes\mathcal{E}$-measurable.

Note that this hypothesis is just asking for the time-homogeneous Markov process $(t,X_t)_{t\geq 0}$ "with sufficiently regular trajectories" to admit a measurable kernel of transition probabilities from $\mathbb{R}_+\times E$ to $\mathbb{M}(\mathbb{R}_+,\mathbb{R}_+\times E)\stackrel{\text{def.}}{=}\mathbb{D}(\mathbb{R}_+,\mathbb{R}_+)\times\mathbb{M}(\mathbb{R}_+,E)$. So under the condition (H$''$2), there is no lost of generality to restrict ourself to the time-homogeneous case, for which (H2) is automatically fulfilled. This may seem as a very mild assumption, but one has sometimes to be careful about conditioning in just measurable settings, because of the lack of "regular" version (in fact our hypothesis on the Markov family consists in assuming the existence of regular conditional expectations, as we cannot deduce it from properties of the state space, and we will be able to construct every other conditional distributions we will need in terms of these ones). In the same spirit, recall that every stochastic process can be seen as an homogeneous Markov process, if the state space is sufficiently enlarged, so one can extend our setting to more general situations if one is able to check the existence of a measurable version of conditional probabilities (but in general this regularity property requires more structure on the new state space which is now a set of paths), see the example of development presented in section 5.

g) Finally let us note that the corresponding discrete time problem can be imbedded in our setting: there everything starts with a time-inhomogeneous family of transition probabilities $(P_n)_{n\geq 0}$ on a measurable space $(E,\mathcal{E})$ and a family $(g_n)_{n\geq 0}$ of functions belonging to $\mathcal{B}_{\mathrm{b}}(E)$ and satisfying $g_n\geq 1$ for all $n\geq 0$. Then one is interested in estimating the probability defined for any $n\geq 0$ by

$$\eta_n(\varphi)\quad\stackrel{\text{def.}}{=}\quad\frac{\mathbb{E}_{\eta_0}\left[\varphi(X_n)\prod_{0\leq m\leq n-1}g_m(X_m)\right]}{\mathbb{E}_{\eta_0}\left[\prod_{0\leq m\leq n-1}g_m(X_m)\right]}$$

where $(X_m)_{m\geq 0}$ is a Markov chain whose transition are given by the family $(P_m)_{m\geq 0}$ and whose initial law is a chosen probability η_0 on $(E,\mathcal{E})$, and where $\varphi\in\mathcal{B}_{\mathrm{b}}(E)$ is just a test function. Let the set $\mathbb{M}(\mathbb{R}_+,E)$ consists of trajectories $\omega\in\mathbf{M}(\mathbb{R}_+,E)$ for which there exist an increasing sequence $(t_i)_{i\geq 0}$ of elements of $\bar{\mathbb{R}}_+$, satisfying $t_0=0$ and $\lim_{i\to\infty}t_i=+\infty$, and a sequence $(x_i)_{i\geq 0}$ of elements of E, such that

$$\forall\ i\geq 0,\ \forall\ t_i\leq s<t_{i+1},\qquad X_s(\omega)\quad=\quad x_i$$

There is no problem in constructing a Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,\,x\in E}$ on $\mathbb{M}(\mathbb{R}_+,E)$ associated to the previous discrete time model, through the operation

$$(X_n)_{n\geq 0}\in E^{\mathbb{N}}\quad\mapsto\quad (X_{\lfloor t\rfloor})_{t\geq 0}\in\mathbb{M}(\mathbb{R}_+,E)$$

where $\lfloor\cdot\rfloor$ denote the integer part. Let us also introduce the function $U(t,x)=\ln(g_{\lfloor t\rfloor}(x))$, then it appears that for $n\in\mathbb{N}$, the measure η_n is also given by (1), so we can just use the following considerations to device an efficient algorithm and to derive estimates on it. But in [4] we have presented a related direct discrete time approach.

2.2 Bounded perturbations of generators

In the last subsection, we have presented a way to associate to any Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$ an abstract generator $(\mathcal{A}_T, L_T)$, for all given $T\geq 0$. Here we will show how one can add some bounded operators to this generator, and we will study the perturbations induced by this kind of manipulations. If it was not for the generality of our setting, these would be standard results (cf. for instance [1] or [8]), but in our situation we have to be a little more careful. There are two main motivations for these considerations:

- They give another family of simple and useful examples of functions belonging to $\mathcal{A}_T$.
- They will enable us to construct the approximating interacting particle systems in subsection 2.4 and to deduce some of their interesting features.

So again we consider a Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$ and for $T\geq 0$, let $(\mathcal{A}_T, L_T)$ be its associated generator on $[0,T]\times E$. Let $\widehat{R}$ be a locally bounded nonnegative kernel from $\mathbb{R}_+\times E$ to E, which is a mapping $(\mathbb{R}_+\times E)\times\mathcal{E}\to\mathbb{R}_+$ such that for any $(t,x)\in\mathbb{R}_+\times E$, the mapping $\mathcal{E}\ni A\mapsto \widehat{R}((t,x),A)$ is a nonnegative measure, for any $A\in\mathcal{E}$, the function $\mathbb{R}_+\times E\ni(t,x)\mapsto\widehat{R}((t,x),A)$ is measurable, and finally for all $T\geq 0$, we have

$$\sup_{(t,x)\in[0,T]\times E} r(t,x) \quad < \quad +\infty$$

where for every $(t,x)\in[0,T]\times E$, we took $r(t,x) = \widehat{R}((t,x),E) = \max_{A\in\mathcal{E}}\widehat{R}((t,x),A)$. Sometimes, we will write $\widehat{R}(t,x)$ for the measure $\mathcal{E}\ni A\mapsto\widehat{R}(t,x,A)\in\mathbb{R}_+$. To such a kernel we can associate the operator R on $\mathcal{B}_{\mathrm{b}}(\mathbb{R}_+\times E)$ (which should be seen as a locally bounded time-inhomogeneous family of generators on $\mathcal{B}_{\mathrm{b}}(E)$, under the interpretation of $\widehat{R}(t,x,A)$ as the intensity of the occurrence of a jump from $x\in E$ to $A\in\mathcal{E}$ at time $t\geq 0$, at least if $x\notin A$) defined for any $f\in\mathcal{B}_{\mathrm{b}}(\mathbb{R}_+\times E)$, and $(t,x)\in(\mathbb{R}_+\times E)$, by

$$R(f)(t,x) \quad = \quad \int f(t,y)\,\widehat{R}((t,x),dy) - r(t,x)f(t,x)$$

For $T\geq 0$, we also denote by R_T the natural restriction of R on $\mathcal{B}_{\mathrm{b}}([0,T]\times E)$.

Our first objective is to construct a Markovian family $(\widehat{\mathbb{P}}_{t,x})_{t\geq 0,\, x\in E}$ such that for all $T\geq 0$, its generator $(\widehat{\mathcal{A}}_T,\widehat{L}_T)$ is an extension of $(\mathcal{A}_T, L_T+R_T)$ To this end, we begin by considering homogeneous Markov chains on $\bar{\mathbb{R}}_+\times\mathbb{M}(\mathbb{R}_+,E)$, endowed with its natural σ-algebra, whose transition probability kernel $\check{P}$ is defined for any $(t,\omega)\in(\mathbb{R}_+\times\mathbb{M}(\mathbb{R}_+,E))$, $I\in\mathcal{R}_+$, and $A\in\mathcal{M}(\mathbb{R}_+,E)$ by

$$\check{P}((t,\omega),I\times A) \;=\; \int_{\mathbb{R}_+\times E}\mathbb{I}_I(t+s)\exp\left(-\int_0^s r(t+u,X_{t+u}(\omega))\,du\right)\widehat{R}((t+s,X_{t+s}(\omega)),dy)\mathbb{P}_{t+s,y}[A]\,ds$$

The missing mass of $\check{P}((t,\omega),\mathbb{R}_+\times\mathbb{M}(\mathbb{R}_+,E))$ is reported to $(+\infty,\diamond)$, by setting

$$\check{P}((+\infty,\diamond),\,\cdot\,) \quad = \quad \delta_{(+\infty,\diamond)}(\,\cdot\,)$$

Due to our hypotheses, especially the second point of (H1) and (H2), there is no real problem in verifying the measurability properties traditionally assumed for a probability kernel. Then according to theorem of Ionescu Tulcea (cf. for instance [12]), for all $(t,\omega)\in\mathbb{R}_+\times\mathbb{M}(\mathbb{R}_+,E)$ there exists a unique probability $\check{\mathbb{P}}_{t,\omega}$ on $(\bar{\mathbb{R}}_+\times\mathbb{M}(\mathbb{R}_+,E))^{\mathbb{N}}$ (endowed with its natural product σ-field) under which the canonical coordinate chain is Markovian with $\check{P}$ as transition probability kernel and starts from the initial distribution $\delta_{(t,\omega)}$. Furthermore, for all measurable subset $A\subset(\bar{\mathbb{R}}_+\times\mathbb{M}(\mathbb{R}_+,E))^{\mathbb{N}}$, the next mapping is measurable

$$\mathbb{R}_+\times\mathbb{M}(\mathbb{R}_+,E)\ni(t,\omega)\quad\mapsto\quad\check{\mathbb{P}}_{t,\omega}[A]$$

Let $\check{E}$ be the set of elements $x = (t_i, y_i)_{i\geq 0} \in (\bar{\mathbb{R}}_+ \times \mathbb{M}(\mathbb{R}_+, E))^{\mathbb{N}}$ such that there exists an index $0 < i_\infty \leq \infty$ satisfying

$$\forall\ i \geq 0, \qquad i < i_\infty \Rightarrow t_i < t_{i+1} \quad \text{and} \quad i \geq i_\infty \Rightarrow (t_i, y_i) = (+\infty, \diamond)$$

We equip this set with the σ-algebra $\check{\mathcal{E}}$ inherited from $(\mathcal{R}_{[0,+\infty]} \otimes \mathcal{M}(\mathbb{R}_+, E))^{\otimes \mathbb{N}}$. Notice that each $\check{\mathbb{P}}_{t,\omega}$, for $t \in \mathbb{R}_+$ and $\omega \in \mathbb{M}(\mathbb{R}_+, E)$, is in fact a probability on $(\check{E}, \check{\mathcal{E}})$, and from this point of view, the family $(\check{\mathbb{P}}_{t,\omega})_{t\geq 0,\, \omega\in\mathbb{M}(\mathbb{R}_+,E)}$ obviously retains the same measurability regularity. On the other hand, we can define on this domain the mapping $\Phi : \check{E} \to \mathbb{M}(\mathbb{R}_+, E)$ given by

$$\forall\ x \in \check{E}, \forall\ i \geq -1, \forall\ t_i \leq s < t_{i+1}, \qquad X_s(\Phi(x)) \;=\; X_s(y_i)$$

where we have used the same notations as before for elements of $\check{E}$, and with the convention that $t_{-1} = 0$ and x_{-1} is the a priori fixed path $\diamond$. Due to the first point of hypothesis (H1), $\Phi(x)$ really lives in $\mathbb{M}(\mathbb{R}_+, E)$. This mapping Φ is also clearly measurable. Now let us define for all $(t, x) \in \mathbb{R}_+ \times E$, $\widehat{\mathbb{P}}_{t,x} = \Phi(\check{\mathbb{P}}_{t,x})$, with the probability $\check{\mathbb{P}}_{t,x}$ on $\check{E}$ given by

$$\forall\ A \in \check{\mathcal{E}}, \qquad \check{\mathbb{P}}_{t,x}(A) \;=\; \int \check{\mathbb{P}}_{t,\omega}(A)\, \mathbb{P}_{t,x}(d\omega)$$

Notice that for $t \geq 0$ and in the sense of the injection I_t, $\widehat{\mathbb{P}}_{t,x}$ can be seen as a probability on $\mathbb{M}([t, +\infty[, E)$. It is time to check that the family obtained by putting together these probabilities will do the desired job.

Proposition 2.3 *The family $(\widehat{\mathbb{P}}_{t,x})_{t\geq 0,\, x\in E}$ is Markov, and it satisfies (H2). Furthermore, for any fixed $T \geq 0$, we have $\mathcal{A}_T \subset \widehat{\mathcal{A}}_T$ and for all $f \in \mathcal{A}_T$, $\widehat{L}_T(f) = L_T(f) + R_T(f)$.*

Proof: The measurability requirements (in particular (H2)) follow from the above considerations. In contrast with the approach followed by Ethier and Kurtz in [8], here it is not sufficient to consider the underlying martingale problems (i.e. to merely prove the second part of the proposition) to insure the validity of Markov property in the general way we have defined it; we are only allowed to play with the very basic objects we have just introduced. That is why the subsequent proof is quite too long and should be admitted at a first reading. Thus, we are wondering if for all $0 \leq t < s$, all $x \in E$, all $A \in \mathcal{M}([t, s], E)$ and all $B \in \mathcal{M}(]s, +\infty[, E)$, we have

$$\widehat{\mathbb{P}}_{t,x}[A \cap B] \;=\; \widehat{\mathbb{E}}_{t,x}[\mathbf{1}_A \widehat{\mathbb{P}}_{s,X_s}[B]]$$

Let us denote $\check{A} = \Phi^{-1}(A)$ and $\check{B} = \Phi^{-1}(B)$. Clearly, it is equivalent to show that

$$\check{\mathbb{P}}_{t,x}[\check{A} \cap \check{B}] \;=\; \check{\mathbb{E}}_{t,x}[\mathbf{1}_{\check{A}}\, \widehat{\mathbb{P}}_{s,X_s\circ\Phi}[B]] \tag{7}$$

To show these equalities are true, we consider for fixed $s \geq 0$, the function

$$\begin{aligned} H_s : \bar{\mathbb{R}}_+ \times \mathbb{M}(\mathbb{R}_+, E) &\to [0, s] \times \mathbb{M}([0, s], E) \times [s, +\infty] \times \mathbb{M}([s, +\infty[, E) \\ (u, \omega) &\mapsto (s \wedge u, (X_t(\omega))_{0\leq t\leq s}, s \vee u, (X_t(\omega))_{s\leq t}) \end{aligned}$$

We denote by $(\check{Z}_n)_{n\geq 0}$ the canonical coordinates on $\check{E}$ and next we naturally write for $n \geq 0$,

$$(T_n, Z_n, T'_n, Z'_n) \;\stackrel{\text{def.}}{=}\; H_s(\check{Z}_n)$$

Let us also define the integer variable

$$N \;=\; \inf\{n \geq 0 : T_n = s\}$$

which is $\check{\mathbb{P}}_{t,x}$-a.s. finite under our local boundedness condition on U. Then, at least on $\{N \geq 1\}$, we notice (e.g. by applying monotonous class theorem) that

$$\begin{array}{rcl} \check{A} & \in & \sigma(\check{Z}_{n\wedge(N-1)}\,;\, n\geq 0) \\ \check{B} & \in & \sigma(\check{Z}_{n+N-1}\,;\, n\geq 0) \end{array}$$

More precisely, there exist $\breve{A} \in (\mathcal{R}_{[0,s]} \otimes \mathcal{M}([0,s],E))^{\otimes\mathbb{N}}$ and $\breve{B} \in (\mathcal{R}_{[s,+\infty[}\otimes \mathcal{M}(]s,+\infty[,E))^{\otimes\mathbb{N}}$ such that

$$\begin{array}{rcl} \check{A} & = & \{(T_{n\wedge(N-1)}, Z_{n\wedge(N-1)})_{n\geq 0} \in \breve{A}\} \\ \check{B} & = & \{(T'_{n+N-1}, Z'_{n+N-1})_{n\geq 0} \in \breve{B}\} \end{array}$$

Unfortunately, the fact that $N-1$ is not a stopping time prevents us from applying directly strong Markov property for $(\check{Z}_n)_{n\geq 0}$ under $\check{\mathbb{P}}_{t,x}$.

But the interest of the previous objects is that the chain $(Z'_{n-1}, T'_n, Z_n, T_{n+1})_{n\geq 0}$ is also Markovian under $\check{\mathbb{P}}_{t,x}$, with the convention that $Z'_{-1} = \diamond$ (also identified with its restriction to the interval $[s,+\infty[$). More precisely, for fixed $0 \leq t < s$, its initial distribution is $\delta_\diamond \otimes \delta_s \otimes m_{t,x}$, where $m_{t,x}$ is the probability defined on $\mathbb{M}([0,s],E)\times[0,s]$ by

$\forall\, C \in \mathcal{M}([0,s],E)$, $\forall\, I \in \mathcal{R}_{[0,s]}$,

$$\begin{array}{rcl} m_{t,x}(C\times I) & = & \mathbb{P}_{t,x}\Big[\mathbf{1}_C((X_v)_{0\leq v\leq s})\Big\{\displaystyle\int_t^s \mathbf{1}_I(w) r(w,X_w)\exp\Big(-\int_t^w r(w',X_{w'})\,dw'\Big)\,dw \\ & & +\mathbf{1}_I(s)\exp\Big(-\displaystyle\int_t^s r(w,X_w)\,dw\Big)\Big\}\Big] \end{array}$$

We check that its probability transition kernel $\check{P}$ satisfies

$$\begin{array}{l} \check{P}((z',u',z,u), C'\times I'\times C\times I) \quad = \\ \quad \mathbf{1}_{\{u<s\}}\Big[\mathbb{P}_{s,X_s(z)}[C']\mathbf{1}_{I'}(s)\displaystyle\int_E \widetilde{R}(u,X_u(z),dy)\mathbb{E}_{u,y}\Big[\mathbf{1}_C((X_v)_{0\leq v\leq s})\Big\{\mathbf{1}_I(s)\exp\Big(-\int_u^s r(w,X_w)\,dw\Big) \\ \quad +\displaystyle\int_{[u,s[}\mathbf{1}_I(w) r(w,X_w)\exp\Big(-\int_u^w r(w',X_{w'})\,dw'\Big)\,dw\Big\}\Big]\Big] \\ \quad +\mathbf{1}_{\{u=s=u'\}}\Big[\mathbb{E}_{s,X_s(z)}\Big[\mathbf{1}_{C'}((X_{v'})_{v'\geq s})\displaystyle\int_{]s,+\infty[\cap I'} r(v,X_v)\exp\Big(-\int_s^v r(w,X_w)\,dw\Big)\,dv\Big]\mathbf{1}_C(\diamond)\mathbf{1}_I(s)\Big] \\ \quad +\mathbf{1}_{\{u=s,u'>s\}}\Big[\displaystyle\int_E \widetilde{R}(u',X_{u'}(z'),dy)\mathbb{E}_{u',y}\Big[\mathbf{1}_{C'}((X_{v'})_{v'\geq s})\int_{]u',+\infty[\cap I'} r(v,X_v) \\ \quad \exp\Big(-\displaystyle\int_{u'}^v r(w,X_w)\,dw\Big)\,dv\Big]\mathbf{1}_C(\diamond)\mathbf{1}_I(s)\Big] \end{array}$$

for any $(z',u',z,u)\in\mathbb{M}([s,+\infty[,E)\times[s,+\infty[\times\mathbb{M}([0,s],E)\times[0,s]$, and any $(C',I',C,I)\in \mathcal{M}([s,+\infty[,E)\times\mathcal{R}_{[s,+\infty[}\times\mathcal{M}([0,s],E)\times\mathcal{R}_{[0,s]}$. In the above displayed formulae, the new probability kernel $\widetilde{R}$ from $\mathbb{R}_+\times E$ to E is given by the renormalization

$$\forall\, u\geq 0,\,\forall\, x\in E,\qquad \widetilde{R}(u,x) \;=\; \begin{cases} \widehat{R}(u,x)/r(u,x) & \text{, if } r(u,x)>0 \\ \delta_\diamond & \text{, otherwise} \end{cases}$$

Notice that $\check{\mathbb{P}}_{t,x}$-a.s. and for any $n\geq 0$, either $T_{n+1}=+\infty$ or $r(Z_n,T_{n+1})>0$, and where as usual the possible missing mass is put on $(\diamond,+\infty,\diamond,s)$, which is also assumed to be a cemetery point.

Let us denote $\check{\mathbb{P}}_{z',u',z,u}$ the law of a Markov chain $(\check{Z}'_n,\check{T}'_n,\check{Z}_n,\check{T}_n)_{n\geq 0}$ starting from (z',u',z,u), with kernel $\check{P}$. Then we are in position to apply strong Markov property to the stopping time $N-1$ with respect to the chain $(Z'_{n-1},T'_n,Z_n,T_{n+1})_{n\geq 0}$, and we get for $x\in E$ and $0\leq t<s$ (which also insures that $\breve{\mathbb{P}}_{t,x}$-a.s. $N\geq 1$),

$$\begin{aligned}\breve{\mathbb{P}}_{t,x}[\breve{A}\cap\breve{B}] &= \breve{\mathbb{E}}_{t,x}[\mathbf{1}_{\breve{A}}((T_{n\wedge(N-1)},Z_{n\wedge(N-1)})_{n\geq 0})\mathbf{1}_{\breve{B}}((T'_{n+N-1},Z'_{n+N-1})_{n\geq 0})]\\ &= \breve{\mathbb{E}}_{t,x}[\mathbf{1}_{\breve{A}}((T_{n\wedge(N-1)},Z_{n\wedge(N-1)})_{n\geq 0})\breve{\mathbb{P}}_{Z'_{N-2},s,Z_{N-1},s}[\mathbf{1}_{\breve{B}}((\breve{T}'_n,\breve{Z}'_{n+1})_{n\geq 0})]]\end{aligned}$$

We observe that for any $(z,z')\in\mathbb{M}([0,s],E)\times\mathbb{M}([s,+\infty[,E)$, the law of $(\breve{T}'_n,\breve{Z}'_{n+1})_{n\geq 0}$ under $\breve{\mathbb{P}}_{z',s,z,s}$ is $\breve{\mathbb{P}}_{s,X_s(z)}$. Thus, (7) follows from the fact that $\breve{\mathbb{P}}_{s,X_s(z)}[\breve{B}]=\widehat{\mathbb{P}}_{s,X_s(z)}[B]$, and from the $\breve{\mathbb{P}}_{t,x}$-a.s. equality $X_s\circ\Phi=X_s(Z_{N-1})$. We have shown that $(\widehat{\mathbb{P}}_{t,x})_{t\geq 0,x\in E}$ is a Markovian family, it thus remains to verify the affirmation about the generators. Firstly, we notice that that it suffices to check that for any $0\leq t\leq s\leq T$, $x\in E$, and any $f\in\mathcal{A}_T$,

$$\widehat{\mathbb{E}}_{t,x}\left[f(s,X_s)-f(t,X_t)-\int_t^s\widehat{L}_T(f)(u,X_u)\,du\right] = 0 \tag{8}$$

If in addition we knew that the mapping $r:\mathbb{R}_+\times E\to\mathbb{R}_+$ is constant, we could transpose the usual arguments given by Ethier and Kurtz in the proof of proposition 10.2 p. 256 of [8] to verify this equality (their processes are assumed to be càdlàg so they are allowed to use the notation X_{u-}, and in our setting this has to be interpreted in the following sense: let us come back to notations introduced before proposition 2.3 and consider $x\in\breve{E}$, if there exists $1\leq i<i_\infty$ such that $u=t_i$, then we denote $X_{u-}(\Phi(x))=X_u(y_{i-1})$, otherwise we take $X_{u-}(\Phi(x))=X_u(\Phi(x))=X_u(y_i)\ \ldots$). Nevertheless, it is well-known that the general situation can be reduced to the previous case via an acceptation/rejection procedure: at each of more frequently selected times, there is more probability that the process stay at the present position, so these instants are only proposed jump times. We begin by noting that the law of $(X_u)_{t\leq u\leq s}$ under $\widehat{\mathbb{P}}_{t,x}$ and the values of $\widehat{L}_T(f)(u,y)$, for $t\leq u\leq s$ and $y\in E$, only depend on the restriction of $\widehat{R}$ on $[t,s]\times E\times\mathcal{E}$, so to prove (8), we can assume that

$$r \stackrel{\text{def.}}{=} \sup_{(u,y)\in\mathbb{R}_+\times E} r(u,y) < +\infty$$

Under this extra assumption, we construct a new bounded kernel $\widehat{R}'$ from $\mathbb{R}_+\times E$ to E via the formula

$$\forall\ (u,y)\in\mathbb{R}_+\times E,\qquad \widehat{R}'(u,y) = \widehat{R}(u,y)+(r-r(u,y))\delta_y$$

This kernel admits the required regularity conditions, and $r'(u,y)=r$, but $R'_T=R_T$. Arguing as above, we construct from $\widehat{R}'$ and $(\mathbb{P}_{u,y})_{(u,y)\in\mathbb{R}_+\times E}$ the Markovian family $(\widehat{\mathbb{P}}'_{u,y})_{(u,y)\in\mathbb{R}_+\times E}$. According to the previous case, we have for any $f\in\mathcal{A}'_T$

$$\mathcal{A}_T\subset\widehat{\mathcal{A}}'_T \quad\text{and}\quad \widehat{L}'_T(f)=L_T(f)+R_T(f)$$

Therefore, to prove (8) it suffices to check that $\widehat{\mathbb{P}}'_{t,x}=\widehat{\mathbb{P}}_{t,x}$. This is a classical computation based on one hand on the fact that for the construction of the $\widehat{\mathbb{P}}_{u,y}$, for $t\leq u\leq s$ and $y\in E$, the difference of the proposed jump times are mutually independent, independent of the trajectories between these proposed times and distributed as exponential variables of parameter r, and on the other hand on the following elementary observation.

Lemma 2.4 *Let $(\tau_n)_{n\geq 1}$ be a sequence of independent exponential random variables of parameter r and let $(V_n)_{n\geq 1}$ be a sequence of independent uniform random variables on [0,1], both families are furthermore assumed to be independent of each other. Let $g:\mathbb{R}_+\to[0,1]$ be a given measurable mapping. If we set*

$$N=\inf\{n\geq 1\,:\,V_n\leq g(\tau_1+\cdots+\tau_n)\}\ \leq\ +\infty \quad\text{and}\quad T=\sum_{1\leq n<N+1}\tau_n\ \leq\ +\infty$$

then the distribution of T is defined by

$$\forall\ u \geq 0, \qquad \mathbb{P}[T > u] \;=\; \exp\left(-r\int_0^u g(v)\,dv\right)$$

This result is applied, for any fixed $t \leq u \leq s$ and any trajectory $\omega \in \mathbb{M}([u, +\infty[, E)$, with $g : \mathbb{R}_+ \ni v \mapsto r(u+v, X_{u+v}(\omega))/r$, but the easy proofs are left to the reader (one has just to take into account the fact that at the proposed jump times "corresponding" to the mass $r - r(u, y)$, the process remains at the same place, so one is able to use Markov property at these times for the family $(\mathbb{P}_{u,y})_{u\geq 0,\, y\in E}$). We have seen that $(\widehat{\mathcal{A}}_T, \widehat{L}_T)$ is an extension of $(\mathcal{A}_T, L_T + R_T)$, for $T \geq 0$. We can go even further, because the converse is also true.

Proposition 2.5 *For ny $T \geq 0$, we have $(\widehat{\mathcal{A}}_T, \widehat{L}_T) = (\mathcal{A}_T, L_T + R_T)$.*

Proof: Let $T > 0$ be fixed, and $f \in \widehat{\mathcal{A}}_T$ be given. We have to show that for all fixed $0 \leq t \leq s \leq T$ and all fixed $x \in E$,

$$\mathbb{E}_{t,x}\left[f(s, X_s) - f(t, X_t) - \int_t^s (\widehat{L}_T - R_T)(f)(u, X_u)\,du\right] \;=\; 0 \tag{9}$$

since it will then follow that $f \in \mathcal{A}_T$ and that $L_T(f) = \widehat{L}_T(f) - R_T(f)$. This is a "local" result, so once again, we can assume that the mapping $r(\cdot, \cdot)$ is constant.

We will work under the probability $\check{\mathbb{P}}_{t,x}$, the random variable T_1 stands for the first proposed jump time appearing ($T_1 - t$ follows an exponential law of parameter r), X_{T_1-} is defined as it was at the end of the proof of proposition 2.3, and $\mathbb{P}_{t,x}$ is seen as the law of $X \stackrel{\text{def.}}{=} (X_u)_{u\geq t}$. With these notations, we can write

$$\begin{aligned}(f(s,X_s) - f(t,X_t))\mathbf{1}_{T_1\leq s} \;&=\; (f(s,X_s) - f(T_1,X_{T_1}))\mathbf{1}_{T_1\leq s} + (f(T_1,X_{T_1}) - f(T_1,X_{T_1-}))\mathbf{1}_{T_1\leq s}\\ &\quad + (f(T_1,X_{T_1-}) - f(t,X_t))\mathbf{1}_{T_1\leq s}\end{aligned}$$

But we notice that by construction, X admits a strong Markov property with respect to the time T_1, so using the fact that $f \in \mathcal{A}_T$, we get

$$\begin{aligned}\check{\mathbb{E}}_{t,x}[(f(s,X_s) - f(T_1,X_{T_1}))\mathbf{1}_{T_1\leq s}] \;&=\; \check{\mathbb{E}}_{t,x}[\mathbf{1}_{T_1\leq s}\check{\mathbb{E}}_{T_1,X_1}[f(s,X_s) - f(T_1,X_{T_1})]]\\ &=\; \check{\mathbb{E}}_{t,x}\left[\mathbf{1}_{T_1\leq s}\check{\mathbb{E}}_{T_1,X_1}\left[\int_{T_1}^s \widehat{L}_T(f)(u,X_u)\,du\right]\right]\\ &=\; \check{\mathbb{E}}_{t,x}\left[\int_{s\wedge T_1}^s \widehat{L}_T(f)(u,X_u)\,du\right]\end{aligned}$$

On the other hand, it is quite clear that by the properties of T_1,

$$\begin{aligned}\check{\mathbb{E}}_{t,x}[(f(T_1,X_{T_1}) - f(T_1,X_{T_1-}))\mathbf{1}_{T_1\leq s}] \;&=\; \check{\mathbb{E}}_{t,x}[R_T(f)(T_1,X_{T_1-})\mathbf{1}_{T_1\leq s}]/r\\ &=\; \int_t^s \mathbb{E}_{t,x}[R_T(f)(u,X_u)]\exp(-r(u-t))\,du\end{aligned}$$

and

$$\check{\mathbb{E}}_{t,x}[(f(T_1,X_{T_1-}) - f(t,X_t))\mathbf{1}_{T_1\leq s}] \;=\; r\int_t^s \mathbb{E}_{t,x}[f(u,X_u) - f(t,X_t)]\exp(-r(u-t))\,du$$

Let us denote for $t \leq s \leq T$, $g(s) \stackrel{\text{def.}}{=} \mathbb{E}_{t,x}[f(s,X_s) - f(t,X_t)]$, we have

$$\begin{aligned}g(s) \;&=\; \frac{\check{\mathbb{E}}_{t,x}[(f(s,X_s) - f(t,X_t))\mathbf{1}_{T_1>s}]}{\check{\mathbb{P}}_{t,x}[T_1 > s]}\\ &=\; \exp(r(s-t))(\check{\mathbb{E}}_{t,x}[f(s,X_s) - f(t,X_t)] - \check{\mathbb{E}}_{t,x}[(f(s,X_s) - f(t,X_t))\mathbf{1}_{T_1\leq s}])\end{aligned}$$

$$\begin{aligned}
&= \exp(r(s-t))\Big(\check{\mathbb{E}}_{t,x}\left[\int_t^s \widehat{L}_T(f)(u,X_u)\,du\right] - \check{\mathbb{E}}_{t,x}\left[\int_{s\wedge T_1}^s \widehat{L}_T(f)(u,X_u)\,du\right] \\
&\quad - \int_t^s \mathbb{E}_{t,x}[R_T(f)(u,X_u)]\exp(-r(u-t))\,du - r\int_t^s g(u)\exp(-r(u-t))\,du\Big) \\
&= \exp(r(s-t))\Big(\check{\mathbb{E}}_{t,x}\left[\int_t^s \widehat{L}_T(f)(u,X_u)\mathbf{I}_{u\leq T_1}\,du\right] \\
&\quad - \int_t^s \mathbb{E}_{t,x}[R_T(f)(u,X_u)]\exp(-r(u-t))\,du - r\int_t^s g(u)\exp(-r(u-t))\,du\Big) \\
&= \exp(r(s-t))\Big(\int_t^s \mathbb{E}_{t,x}\left[\widehat{L}_T(f)(u,X_u)\right]\exp(-r(u-t))\,du \\
&\quad - \int_t^s \mathbb{E}_{t,x}[R_T(f)(u,X_u)]\exp(-r(u-t))\,du - r\int_t^s g(u)\exp(-r(u-t))\,du\Big) \\
&= \exp(r(s-t))\Big(\int_t^s \mathbb{E}_{t,x}\left[(\widehat{L}_T - R_T)(f)(u,X_u)\right]\exp(-r(u-t))\,du \\
&\quad -r\int_t^s g(u)\exp(-r(u-t))\,du\Big)
\end{aligned}$$

This differential equation satisfied by $\int_t^s g(u)\exp(-r(u-t))\,du$, for $t \leq s \leq T$, has a unique continuous solution, which is

$$\begin{aligned}
&\int_t^s g(u)\exp(-r(u-t))\,du \\
&\quad = \int_t^s \exp(r(u-s))\int_t^u \mathbb{E}_{t,x}\left[(\widehat{L}_T - R_T)(f)(v,X_v)\right]\exp(-r(v-t))\,dv\,du
\end{aligned}$$

■

Let us give a first consequence of this identity, mentioned at the beginning of this subsection and which will be a powerful tool in the subsequent development (because it is the one which will enable us to remove all regularity assumptions). More precisely, as we will mainly work with martingales (and not directly with their increasing processes), we need to know a lot of them, and the following result is a good way to construct some interesting ones, via the description of new elements of $\mathcal{A}_T$. The proof of the next result readily follows from an elementary combination of lemma 2.1 and proposition 2.5.

Corollary 2.6 *Let $T > 0$ be fixed, and let $V \in \mathcal{B}_{\mathrm{b}}([0,T]\times E)$ and $\varphi \in \mathcal{B}_{\mathrm{b}}(E)$ be given. We consider the function defined for any $0 \leq t \leq T$, and $x \in E$ by*

$$G_{T,V,\varphi,\widehat{R}}(t,x) \quad = \quad \widehat{\mathbb{E}}_{t,x}\left[\exp\left(\int_t^T V_s(X_s)\,ds\right)\varphi(X_T)\right]$$

The mapping $G_{T,V,\varphi,\widehat{R}}$ belongs to $\mathcal{A}_T$, and we have

$$L_T(G_{T,V,\varphi,\widehat{R}})(t,x) \quad = \quad -V_t(x)G_{T,V,\varphi,\widehat{R}}(t,x) - R(G_{T,V,\varphi,\widehat{R}})(t,x)$$

2.3 Coupling techniques

As we shall see in the subsequent development, it is sometimes useful to compare the initial Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,\,x\in E}$ with its just constructed modification $(\widehat{\mathbb{P}}_{t,x})_{t\geq 0,\,x\in E}$; at least in the cases where the perturbation $\widehat{R}$ is small, and one seemingly nice way to do it would be to couple them. But once again our general and abstract setting does not enable us to work it out in the traditional way. For instance, even if $\widehat{R} \equiv 0$ there may not exist the usual Markov coupling of $(\mathbb{P}_{t,x})_{t\geq 0,\,x\in E}$ with himself (which would satisfy that when the two coordinates coincide they stay together and so evolve identically, and when they are different they evolve independently, up to the (contingently) time when they would be equal). The main difficulty is that in general it is not clear that the diagonal set $\triangle(E) = \{(x,x) \in E^2 \,:\, x \in E\}$ belongs to $\mathcal{E} \otimes \mathcal{E}$. Recall that we have even not assumed that $(E, \mathcal{E})$ is separated, but to consider its natural diagonal

$$\bar{\triangle}(E) \quad = \quad \{(x,y) \in E^2 \,:\, \delta_x = \delta_y\}$$

would not have improved the situation. In order to overcome this difficulty, we will only couple probabilities (i.e. we look for probabilities on a product space with specified marginals, e.g. $\mathbb{P}_{0,x}$ and $\widehat{\mathbb{P}}_{0,x}$, for a fixed $x \in E$) and not Markovian families; in fact our couplings will not be Markovian processes. This will not be important, because our purpose is to try to make processes issued from a same position to stay together the longest possible time, and not (as it is more customary) to attempt to make them come back together if they are separated. We also have resort to another trick, as we will really be interested in the diagonal of $E \times E$ for general measurable space $(E, \mathcal{E})$. If m is a nonnegative finite measure on $(E^2, \mathcal{E}^{\otimes 2})$, we will make the convention that

$$m(\triangle(E)) \quad \stackrel{\text{def.}}{=} \quad \sup m_1(E)$$

where the supremum (which is in fact a maximum) is taken over all nonnegative measures m_1 defined on $(E, \mathcal{E})$ such that $m \geq m_2$, where m_2 is the image of m_1 under the mapping $E \ni x \mapsto (x,x) \in E^2$. This notion is quite natural, because the classical proof shows that if μ_1 and μ_2 are probabilities on $(E, \mathcal{E})$, then there exists a coupling m of them on $(E^2, \mathcal{E}^{\otimes 2})$ verifying

$$m(E^2) - m(\triangle(E)) \quad = \quad \frac{1}{2} \left\| \mu_1 - \mu_2 \right\|_{\text{tv}} \tag{10}$$

As we already mentioned it, we will consider here couplings of probabilities on path spaces, so we have to be a little more precise about path spaces associated to product state space: for $N \geq 2$, we will always take $\mathbb{M}(\mathbb{R}_+, E^N) = \mathbb{M}(\mathbb{R}_+, E)^N$. This definition implies that $\mathcal{M}(\mathbb{R}_+, E^N) = \mathcal{M}(\mathbb{R}_+, E)^{\otimes N}$, so (H1) is clearly satisfied. A typical example of the kind of results we are looking for is the following one, where the Markovian family $(\widehat{\mathbb{P}}_{t,x})_{t \geq 0, x \in E}$ is constructed as in the previous section, starting from $(\mathbb{P}_{t,x})_{t \geq 0,\, x \in E}$ and $\widehat{R}$.

Proposition 2.7 *Let $T \geq 0$ and $x \in E$ be given. Then there exists a coupling $\mathbb{P}^{\dagger}_{0,(x,x)}$ of $\mathbb{P}_{0,x}$ and $\widehat{\mathbb{P}}_{0,x}$ such that*

$$\mathbb{P}^{\dagger}_{0,(x,x)}[(X_t)_{0 \leq t \leq T} \neq (\widehat{X}_t)_{0 \leq t \leq T}] \quad \leq \quad 1 - \exp\left(-T \sup_{0 \leq t \leq T,\, x \in E} \widehat{R}(t,x,E)\right)$$

(where $(X_t, \widehat{X}_t)_{t \geq 0}$ stands for the canonical coordinate process on $\mathbb{M}(\mathbb{R}_+, E^2)$, and by convention, we have taken $\mathbb{P}^{\dagger}_{0,(x,x)}[(X_t)_{0 \leq t \leq T} \neq (\widehat{X}_t)_{0 \leq t \leq T}] = 1 - \mathbb{P}_{0,(x,x)}[\triangle(\mathbb{M}([0,T], E))]$).

Proof: The horizon $T \geq 0$ being fixed, it is sufficient to construct a coupling $\mathbb{P}^{\dagger}_{0,(x,x),[0,T]}$ on $\mathbb{M}([0,T], E^2)$ of the restrictions to $\mathbb{M}([0,T], E)$ of $\mathbb{P}_{0,x}$ and $\widehat{\mathbb{P}}_{0,x}$, satisfying the required condition, because it is then immediate to extend it to a coupling over the whole $\mathbb{M}(\mathbb{R}_+, E^2)$, by letting, after time T, the coordinates evolve independently and respectively according to $(\mathbb{P}_{t,x})_{t \geq T, x \in E}$ and $(\widehat{\mathbb{P}}_{t,x})_{t \geq T, x \in E}$. This remark make it clear that there is no lost of generality to come down to the situation where the quantity $R(t,x,E)$ does not depend on $t \geq 0$ and $x \in E$, and where its common value is $r = \sup_{0 \leq t \leq T,\, x \in E} R(t,x,E)$, in terms of the initial kernel. Thus, under this hypothesis we consider a "generalized" Markov family on $E^2 \times \{0,1\}$, $(\mathbb{P}^{\ddagger}_{t,x})_{t \geq 0,\, x \in E^2 \times \{0,1\}}$, in the sense that it will not verify the first assumption of initial parametrization property: For $(x,y) \in E^2$, the probability $\mathbb{P}^\ddagger_{t,(x,y,0)}$ is $(\mathbb{P}^\ddagger_{t,(x,x)} \otimes \delta_{0_{[t,+\infty[}})$, The first factor is the image of the probability $\mathbb{P}_{t,x}$ by the mapping

$$\mathbb{M}([t,+\infty[,E) \ni \omega \quad \mapsto \quad (\omega,\omega) \in \mathbb{M}([t,+\infty[,E^2)$$

Furthermore, for any $t \geq 0$, and any a, $a_{[t,+\infty[}$ stands for the constant path defined over the time interval $[t,+\infty[$ and always taking the value a. For $(x,y) \in E^2$, the probability $\mathbb{P}^\ddagger_{t,(x,y,1)}$ is just the tensor product $(\mathbb{P}_{t,x} \otimes \widehat{\mathbb{P}}_{t,y} \otimes \delta_{1_{[t,+\infty[}})$.

To obtain a new generalized Markovian family $(\widehat{\mathbb{P}}^\ddagger_{t,x})_{t\geq 0,\, x\in E^2\times\{0,1\}}$, we construct a perturbation of this family by the nonnegative kernel $\widehat{R}^\ddagger$ from $\mathbb{R}_+ \times E^2 \times \{0,1\}$ to $E^2 \times \{0,1\}$, defined for any $t \geq 0$, and $(x,y,z) \in (E^2 \times \{0,1\})$, by

$$\widehat{R}^\ddagger(t,(x,y,z)) \quad = \quad \delta_x \otimes \widehat{R}(t,y) \otimes \delta_1$$

Then let $\mathbb{P}^\dagger_{0,(x,x)}$ be the image of $\mathbb{P}^\ddagger_{0,(x,x,0)}$ under the natural projection of $\mathbb{M}(\mathbb{R}_+, E^2 \times \{0,1\})$ on $\mathbb{M}(\mathbb{R}_+, E^2)$. It is not difficult to convince oneself that it is indeed a coupling of $\mathbb{P}_{0,x}$ with $\widehat{\mathbb{P}}_{0,x}$. Furthermore, we have, denoting by $(Z_t)_{t\geq 0}$ the canonical coordinates on $\{0,1\}$,

$$\mathbb{P}^\dagger_{0,(x,x)}[(X_t)_{0\leq t\leq T} \neq (\widehat{X}_t)_{0\leq t\leq T}] \quad \leq \quad \mathbb{P}^\ddagger_{0,(x,x,0)}[Z_T = 1] = 1 - \exp(-rT)$$

because under $\mathbb{P}^\ddagger_{0,(x,x,0)}$, Z_T is distributed as $\mathbf{1}_{[0,T]}(S)$, where S is an exponential random variable of parameter r.

■

The latter coupling is rather crude. Roughly speaking, up to any given time we are considering only two possibilities: either the trajectories of the two processes coincide, either they are different. But we will need to be a little more precise, by quantifying the distance between the positions of the two processes, more specifically in the case of a system of particles, we would like to know how many particles are different. As there is no a priori metric on the state space in our setting, this is the only natural comparison we can consider.

So, let us give a general definition of a particle system with interactions changing one particle at each time. First we still assume that we are given a Markov family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$ on E. Then let $N \in \mathbb{N}^*$ be a number of particles. As underlying "unperturbed" Markovian family on E^N, we consider the one, again written $(\mathbb{P}_{t,x})_{t\geq 0, x\in E^N}$, which corresponds to a Markov process on E^N whose coordinates evolve independently and according to $(\mathbb{P}_{t,x})_{t\geq 0, x\in E}$, and which is rigorously defined by

$$\forall\ t \geq 0,\ \forall\ x = (x_1,\ldots,x_N) \in E^N, \qquad \mathbb{P}_{t,x} = \bigotimes_{1\leq i\leq N} \mathbb{P}_{t,x_i}$$

clearly it also satisfies (H2). For each $1 \leq i \leq N$, we consider a locally bounded nonnegative kernel $\widehat{R}_i$ from $\mathbb{R}_+ \times E^N$ to E. In order to simplify the presentation, we will work under the hypothesis that the quantity $r \stackrel{\text{def.}}{=} N\widehat{R}_i(t,x,E)$ does not depend on $1 \leq i \leq N$, $t \geq 0$ and $x \in E$. From these kernels, we define a new one $\widehat{R}$ from $\mathbb{R}_+ \times E^N$ to E^N, via the formulae

$$\forall\ t \geq 0,\ \forall\ x = (x_i)_{1\leq i\leq N} \in E^N,$$

$$\widehat{R}(t,x) \quad = \quad \sum_{1\leq i\leq N} \delta_{x_1} \otimes \cdots \otimes \delta_{x_{i-1}} \otimes R_i(t,x) \otimes \delta_{x_{i+1}} \otimes \cdots \otimes \delta_{x_N}$$

Let us denote by $(\widehat{\mathbb{P}}_{t,x})_{t\geq 0, x\in E^N}$ the perturbation of $(\mathbb{P}_{t,x})_{t\geq 0, x\in E^N}$ by this kernel $\widehat{R}$. The mechanism of its interactions at any selected time $t \geq 0$ can be interpreted in the following way: one choose uniformly an index $1 \leq i \leq N$, and then the coordinate x_i of a position $x \in E^N$ is replaced by the value obtained from a sampling according to the law $\widehat{R}_i(t,x,\cdot)/\widehat{R}_i(t,x,E)$, the other coordinates remain unchanged. Between the random interacting times, the particles evolve independently. We also denote by $\nu^{(r)}$ the law of the usual Poisson process of parameter r on the set of "càdlàg" paths $\mathbb{M}(\mathbb{R}_+,\mathbb{N})$ from $\mathbb{R}_+$ into $\mathbb{N}$. We get the following result which in this situation is more precise than proposition 2.7:

Proposition 2.8 *For any $x\in E^N$, there exists a coupling $\mathbb{P}^\dagger_{0,(x,x)}$ of $\mathbb{P}_{0,x}\otimes\nu^{(r)}$ with $\widehat{\mathbb{P}}_{0,x}$, such that if we denote by $((X_t^{(i)})_{1\leq i\leq N},K_t,(\widehat{X}_t^{(i)})_{1\leq i\leq N})_{t\geq 0}$ the canonical coordinates on $\mathbb{M}(\mathbb{R}_+,E^N\times\mathbb{N}\times E^N)$, then for any $T\geq 0$ we have*

$$\mathbb{P}^\dagger_{0,(x,x)}\left[\sum_{1\leq i\leq N}\mathbf{1}_{(X_t^{(i)})_{0\leq t\leq T}\neq(\widehat{X}_t^{(i)})_{0\leq t\leq T}}\geq K_T\right] = 0$$

In particular, for any $k\geq 0$ we get that

$$\mathbb{P}^\dagger_{0,(x,x)}\left[\sum_{1\leq i\leq N}\mathbf{1}_{(X_t^{(i)})_{0\leq t\leq T}\neq(\widehat{X}_t^{(i)})_{0\leq t\leq T}}\geq k\right] \leq \sum_{l\geq k}\frac{(rT)^l}{l!}\exp(-rT)$$

Here again, the above results have to be interpreted in a special way: let $(F,\mathcal{F})$ be a measurable space, we endow $F^N\times F^N$ with its canonical coordinates $((X_i)_{1\leq i\leq N},(\widehat{X}_i)_{1\leq i\leq N})$. Let m be a nonnegative finite measure on $(F^N\times F^N,\mathcal{F}^{\otimes N}\otimes\mathcal{F}^{\otimes N})$, for any $0\leq k\leq N$, we define

$$m\left(\sum_{1\leq i\leq N}\mathbf{1}_{X_i\neq\widehat{X}_i}\geq k\right) = \sup m_1(F^N\times F^N)$$

where the supremum is taken over all nonnegative measure $m_1\leq m$ on $(F^N\times F^N,\mathcal{F}^{\otimes N}\otimes\mathcal{F}^{\otimes N})$ which can be decomposed into

$$m_1 = \sum_{A\subset\{1,\ldots,N\},\,\mathrm{card}(A)=k} m_{1,A}$$

where $m_{1,A}$ satisfies that its image $\widetilde{m}_{1,A}$ by the natural projection from $F^N\times F^N$ to $F^A\times F^A$ verifies $\widetilde{m}_{1,A}(\triangle(F^A))=m_{1,A}(F^A\times F^A)$ (in this case also the supremum is a maximum, but except for $k=N$, the optimal above decomposition is not unique in general). For the first equality, it means that when, for $k\geq 0$ given, we look at the restriction of $\mathbb{P}^\dagger_{0,(x,x)}$ to the set $\{K_T=k\}$ and consider its projection $\mathbb{P}^{\dagger,k}_{0,(x,x)}$ to $\mathbb{M}([0,T],E^N\times E^N)$, then it satisfies

$$\mathbb{P}^{\dagger,k}_{0,(x,x)}\left[\sum_{1\leq i\leq N}\mathbf{1}_{(X_t^{(i)})_{0\leq t\leq T}\neq(\widehat{X}_t^{(i)})_{0\leq t\leq T}}\geq k\right] = 0$$

We could go further and give a meaning to the affirmation that

$$\mathbb{P}^\dagger_{0,(x,x)}\left[\exists\, T\geq 0\,:\,\sum_{1\leq i\leq N}\mathbf{1}_{(X_t^{(i)})_{0\leq t\leq T}\neq(\widehat{X}_t^{(i)})_{0\leq t\leq T}}\geq K_T\right] = 0$$

but we will not need it (be careful, $\sum_{1\leq i\leq N}\mathbf{1}_{(X_t^{(i)})_{0\leq t\leq T}\neq(\widehat{X}_t^{(i)})_{0\leq t\leq T}}$ is not a random variable, so one cannot use its monotonicity with respect to $t\geq 0$, rather one has to use a measurable conditioning by $(K_t)_{t\geq 0}$, which can be well-defined here, if one consider only the increasing trajectories of $\mathbb{M}(\mathbb{R}_+,\mathbb{N})$ with jumps of height 1). **Proof:** It is quite similar to the proof of proposition 2.7, we begin by considering a generalized Markov family on $E^{2N}\times\mathcal{P}_N\times\mathbb{N}$, where $\mathcal{P}_N$ is the set of the subsets of $\{1,\ldots,N\}$, defined by

$\forall\ t\geq 0,\ \forall\ (x,y,z,k)\in E^{2N}\times\mathcal{P}_N\times\mathbb{N}$,

$$\mathbb{P}^{\ddagger}_{t,(x,y,z,k)} = \left(\bigotimes_{i\not\in z}\mathbb{P}^{\ddagger}_{t,x_i,x_i}\bigotimes_{i\in z}(\mathbb{P}_{t,x_i}\otimes\mathbb{P}_{t,y_i})\right)\otimes\delta_{z_{[t,+\infty[}}\otimes\delta_{k_{[t,+\infty[}}$$

We introduce a new nonnegative kernel $\widehat{R}^{\ddagger}$ from $\mathbb{R}_+\times E^{2N}\times\mathcal{P}_N\times\mathbb{N}$ to $E^{2N}\times\mathcal{P}_N\times\mathbb{N}$, by taking, for all $t\geq 0$ and all $(x,y,z,k)\in E^{2N}\times\mathcal{P}_N\times\mathbb{N}$,

$$\widehat{R}^{\ddagger}(t,(x,y,z,k)) = \sum_{1\leq i\leq N}\delta_x\otimes\widehat{R}(t,y)\otimes\delta_{z\cup\{i\}}\otimes\delta_{k+1}$$

and we make a perturbation of the family $(\mathbb{P}^{\ddagger}_{t,(x,y,z,k)})_{t\geq 0,(x,y,z,k)\in E^{2N}\times\mathcal{P}_N\times\mathbb{N}}$ by this kernel to get a new family $(\widehat{\mathbb{P}}^{\ddagger}_{t,(x,y,z,k)})_{t\geq 0,(x,y,z,k)\in E^{2N}\times\mathcal{P}_N\times\mathbb{N}}$. For $x\in E^N$, let $\mathbb{P}^{\dagger}_{0,(x,x)}$ be the image of $\widehat{\mathbb{P}}^{\ddagger}_{0,(x,x,\emptyset,0)}$ under the projection

$$\mathbb{M}(\mathbb{R}_+,E^{2N}\times\mathcal{P}_N\times\mathbb{N})\ni(\omega_1,\omega_2,\omega_3,\omega_4) \mapsto (\omega_1,\omega_4,\omega_2)\in\mathbb{M}(\mathbb{R}_+,E^N\times\mathbb{N}\times E^N)$$

The proposition now follows quite easily from this construction. For instance, under $\widehat{\mathbb{P}}^{\ddagger}_{0,(x,x,\emptyset,0)}$, K_t ($\geq\mathrm{card}(Z_t)$) counts the number of interaction jump(s) proposed during the time interval $[0,t]$, so $(K_t)_{t\geq 0}$ is distributed as a Poisson process of parameter r.

■

2.4 The interacting particle system

For any given number of particles $N\in\mathbb{N}^*$, the Markov process $\xi^{(N)}$ presented in the introduction is constructed directly from the family $(\mathbb{P}_{t,x})_{t\geq 0,\,x\in E}$, and not merely defined by a martingale problem, as it was the case in [6]. There are three reasons for this choice: first we believe that it emphasizes the close links between the object under study, $(\eta_t)_{t\geq 0}$, and the approximating scheme $(\xi^{(N)})_{N\geq 1}$, which are both deduced directly from the same basic family $(\mathbb{P}_{t,x})_{t\geq 0,\,x\in E}$, and it gives a way to sample the interacting particle processes in practice, at least under the assumption that one knows how to do it w.r.t. $\mathbb{P}_{t,x}$, for any $t\geq 0$ and $x\in E$. Secondly, the direct construction is nicely adapted to coupling arguments. The last reason is even more technical: if one wants to start from the martingale problems, one will have to consider a priori a set of functions on $[0,T]\times E^N$, for $T\geq 0$ given, which in some sense is a (space) tensorization of $\mathcal{A}_T$ (cf. [6]). But in general this domain is too small for our purposes, because it is strictly included in the domain of functions giving rise to natural martingales relatively to $\xi^{(N)}$, and one would have to extend it via some closures. As we will see, it is more convenient to first tensorize the family $(\mathbb{P}_{t,x})_{t\geq 0,\,x\in E}$, to perturb it in a bounded way, and then to consider the general associated martingale problem.

Quite obviously, we will use the above sections to construct the interactions between the coordinates of $\xi^{(N)}$. The underlying "unperturbed" Markovian family is the one previously defined, $(\mathbb{P}_{t,x})_{t\geq 0,x\in E^N}$, corresponding to independent evolutions of coordinates according to $(\mathbb{P}_{t,x})_{t\geq 0,x\in E}$. For any given horizon $T\geq 0$, let us denote by $(\mathcal{A}_{T,N},\widetilde{\mathcal{L}}^{(N)}_T)$ the generator associated as before (recall that it is contingently multi-valued) to this Markovian family. The third point alluded to in the above introductory paragraph just amounts to the observation that in general, $\mathcal{A}_{T,N}$ is strictly larger than the set of functions $f\,:\,[0,T]\times E^N\to\mathbb{R}$ which can be written as

$$f(t,x) = \prod_{1\leq i\leq N}f_i(t,x_i)$$

with $f_1, \cdots, f_N \in \mathcal{A}_T$, and any $t \in [0, T]$, and $x = (x_1, \cdots, x_N) \in E^N$. In order to define the interaction we want to add to the $\widetilde{\mathcal{L}}_T^{(N)}$, for $T \geq 0$, let us introduce the following notation: for all $1 \leq i, j \leq N$ and all $x = (x_1, \cdots, x_N) \in E^N$, $x^{i,j}$ is the element of E^N given by

$$\forall\ 1 \leq k \leq N, \qquad x_k^{i,j} \ = \ \begin{cases} x_k & , \text{ if } k \neq i \\ x_j & , \text{ if } k = i \end{cases}$$

Then we consider the locally bounded nonnegative kernel $\widehat{R}$ from $\mathbb{R}_+ \times E^N$ to E^N defined for any $t \geq 0$, and $x = (x_1, \cdots, x_N) \in E^N$ by

$$\widehat{R}(t, x) \ = \ \frac{1}{N} \sum_{1 \leq i,j \leq N} U_t(x_j) \delta_{x^{i,j}}$$

The corresponding generator on $[0, T] \times E^N$ is defined for any $f \in \mathcal{B}_{\mathrm{b}}([0, T] \times E^N)$, and $(t, x) \in [0, T] \times E^N$ by the following formula

$$\widehat{\mathcal{L}}_T^{(N)}(f)(t, x) \ = \ \frac{1}{N} \sum_{1 \leq i,j \leq N} U_t(x_j)(f(t, x^{i,j}) - f(t, x))$$

Using the results of section 2.2, we are in position to construct the Markov family $(\widehat{\mathbb{P}}_{t,x})_{t \geq 0, x \in E^N}$ with the generators $(\mathcal{A}_{T,N}, \mathcal{L}_T^{(N)})$ given by

$$\mathcal{L}_T^{(N)} \ = \ \widetilde{\mathcal{L}}_T^{(N)} + \widehat{\mathcal{L}}_T^{(N)}$$

It appears that at time $0 \leq t \leq T$, the generator $\widehat{\mathcal{L}}_T^{(N)}(t, \cdot)$ tends to select coordinates x_j with large potential value $U_t(x_j)$, and to replace another coordinate by this one. This is the typical Moran selection step with cost function U_t. The operator $(\mathcal{A}_{T,N}, \mathcal{L}_T^{(N)})$ can be seen as a genetic type generator based on the mutation (or a priori) generator $\widetilde{\mathcal{L}}_T^{(N)}$ which makes the coordinates explore independently the space E.

If η_0 is the initial law which has been seen in the introduction, we are particularly interested in the interacting particle system $(\xi_t^{(N,i)})_{t \geq 0, 1 \leq i \leq N}$, whose law is the probability, denoted by $\mathbb{P}$ for simplicity, defined on $\mathbb{M}(\mathbb{R}_+, E^N)$ by

$$\forall\ A \in \mathcal{M}(\mathbb{R}_+, E^N), \qquad \mathbb{P}(A) \ = \ \int_{E^N} \eta_0^{\otimes N}(dx) \widehat{\mathbb{P}}_{0,x}(A)$$

i.e. we will assume that initially the coordinates of $\xi_0^{(N)}$ are independent and identically distributed according to η_0, but a careful study of the following proofs would indicate how much this assumption can be weakened.

3 Evolution of empirical tensor measures

The purpose of this section is to revisit some weak convergence results given in [6] in order to improve and extend them. In the classical approach (cf. for instance [13] or [9]), one deduces the weak propagation from the strong one, but we will proceed in the other way round, getting the strong property in section 4 from a generalization of the weak form presented here. More precisely, our main goal is to prove theorem 1.2. The case $n = 1$ could easily be deduced from the estimations proved in [6], nevertheless, in order to deal with the general situation, we have to develop a new approach, which will also enable us to recover this case $n = 1$, but under the less restrictive hypotheses considered here.

The basic idea is to adopt a “dynamical point of view”, in some sense interpreting a quantity closely related to $\eta_{t_1}^{(N)} \otimes \cdots \otimes \eta_{t_n}^{(N)}(\varphi)$ as a terminal value, so that we can find nice martingales to calculate its expectation. Unfortunately its cautious development is as long as its principle is simple. For $T>0$, $N,n\in\mathbb{N}^*$ and $0\leq t_1\leq t_2\leq\cdots\leq t_n\leq T$ fixed, let us denote by $\widetilde{\eta}^{(N)}_{t_1,\cdots,t_n}$ the "integrated" law on E^n defined for any $\varphi\in\mathcal{B}_{\mathrm{b}}(E^n)$ by

$$\widetilde{\eta}^{(N)}_{t_1,\cdots,t_n}(\varphi) \quad = \quad \mathbb{E}[\eta^{(N)}_{t_1}\otimes\cdots\otimes\eta^{(N)}_{t_n}(\varphi)]$$

The theorem 1.2 stated in the introduction just says that in the total variation sense, we have

$$\left\|\widetilde{\eta}^{(N)}_{t_1,\cdots,t_n}-\eta_{t_1}\otimes\cdots\otimes\eta_{t_n}\right\|_{\mathrm{tv}} \quad \leq \quad \epsilon_T\left(\frac{n^2}{N}\right)$$

As we will explain it latter on, the dependence of the upper bound in n^2/N, with $T>0$ fixed, implies (through the second coupling presented in section 4.2) the same type of convergence as that obtained by Graham and Méléard [9] for the strong propagation of chaos. But if we were less exacting on this point, it could be possible to give a little more straightforward proof of a weaker upper bound with respect to n.

3.1 Actions of the generators

As we are interested in getting results on empirical measures, we will attempt to understand more particularly the action of the generators $\widetilde{\mathcal{L}}^{(N)}_T$ and $\widehat{\mathcal{L}}^{(N)}_T$, for $T\geq 0$ and $n\in\mathbb{N}^*$, on functions of $\mathcal{A}_{T,N}$ whose dependence on the space parameter goes more or less naturally through the mapping

$$\begin{array}{rcl} m^{(N)}\,:\,E^N & \to & \mathcal{P}(E) \\ x=(x_i)_{1\leq i\leq N} & \mapsto & m^{(N)}(x)=\dfrac{1}{N}\displaystyle\sum_{1\leq i\leq N}\delta_{x_i} \end{array}$$

This is what we have already done in [6] for the case $n=1$. Here we will have to consider probabilities on E^n, and one could think that the natural object replacing $m^{(N)}(x)$, for $x\in E^N$, is $(m^{(N)}(x))^{\otimes n}$, but it seems that (for $1\leq n\leq N$) it is preferable to first look at

$$m^{\odot(N,n)}(x) \quad \stackrel{\text{def.}}{=} \quad \frac{1}{N^n}\sum_{(i_1,i_2,\cdots,i_n)\in I(N,n)}\delta_{(x_{i_1},x_{i_2},\cdots,x_{i_n})} \quad \in \quad \mathcal{P}(E^n)$$

where $I(N,n)$ is the set of $(i_1,i_2,\cdots,i_n)\in\{1,\cdots,N\}^n$ such that all i_l and i_k are different for $1\leq l\neq k\leq n$. More precisely, we will concentrate our study on mappings of the following form

$$\begin{array}{rcl} F_f\,:\,[0,T]\times E^N & \to & \mathbb{R} \\ (t,x) & \mapsto & m^{\odot(N,n)}(x)[f(t,\cdot)] \end{array}$$

where $T\geq 0$ and $f\in\mathcal{A}_{T,n}$ are fixed. The time dependence appearing above will be important in what follows. We observe that the definition of $m^{\odot(N,n)}$ (and the assumed regularity of f) implies that $F_f\in\mathcal{A}_{T,N}$. Notice that this would not have been so if we had considered $(m^{(N)})^{\otimes n}$ instead of $m^{\odot(N,n)}$. Indeed, for $(i_1,i_2,\cdots,i_n)\in I(N,n)$ and $f\in\mathcal{A}_{T,n}$, let us designate by $f^{(i_1,i_2,\cdots,i_n)}$ the function belonging to $\mathcal{A}_{T,N}$ and defined by

$$\forall\, 0\leq t\leq T,\,\forall\, x=(x_1,\cdots,x_N)\in E^N, \qquad f^{(i_1,i_2,\cdots,i_n)}(t,x) \quad = \quad f(t,x_{i_1},x_{i_2},\cdots,x_{i_n})$$

(if $(i_1,i_2,\cdots,i_n)\in\{1,\cdots,N\}^n\setminus I(N,n)$, it is not clear that the above mapping belongs to $\mathcal{A}_{T,N}$), then we have

$$F_f \quad = \quad \frac{1}{N^n}\sum_{(i_1,i_2,\cdots,i_n)\in I(N,n)} f^{(i_1,i_2,\cdots,i_n)} \;\in\; \mathcal{A}_{T,N}$$

Taking into account the obvious observation from the martingale problems that

$$\widetilde{\mathcal{L}}_T^{(N)}(f^{(i_1,i_2,\cdots,i_n)}) \quad = \quad (\widetilde{\mathcal{L}}_T^{(n)}(f))^{(i_1,i_2,\cdots,i_n)}$$

we get the following result:

Lemma 3.1 *Assume that $1 \leq n \leq N$ and let $f \in \mathcal{A}_{T,n}$. Then the next commutation relation holds*

$$\widetilde{\mathcal{L}}_T^{(N)}(F_f) \quad = \quad F_{\widetilde{\mathcal{L}}_T^{(n)}(f)}$$

In fact this lemma is true for all $n \geq 1$, since for $n > N$, the usual conventions give $m^{\odot n}(x) \equiv 0$ In order to describe the action of $\widehat{\mathcal{L}}_T^{(N)}$ on functions of type F_f, we take into consideration a renormalisation of the Moran kernel from $\mathbb{R}_+ \times E^n$ to E^n, and more accurately we will need the restriction to $\mathcal{B}_{\mathrm{b}}([0,T] \times E^n)$ of its associated generator, which is just given by $l_{T,N,n} \stackrel{\text{def.}}{=} \frac{n}{N}\widehat{\mathcal{L}}_T^{(n)}$. Its interest comes from the following formula.

Lemma 3.2 *For $1 \leq n \leq N$ and $T \geq 0$ fixed, we consider any function $f \in \mathcal{B}_{\mathrm{b}}([0,T \times E^n)$. Then we get that*

$$\widehat{\mathcal{L}}_T^{(N)}(F_f) \quad = \quad F_{\bar{U}^{(n)}f + l_{T,N,n}(f)} - F_{\bar{U}^{(n)}}F_f$$

where $\bar{U}^{(n)}$ stands for the restriction on $[0,T] \times E^n$ of mapping defined by

$$\forall\ t \geq 0,\ \forall\ y = (y_i)_{1 \leq i \leq n} \in E^n, \qquad \bar{U}(n)(t,y) \quad = \quad \sum_{1 \leq i \leq n} U(t,y_i)$$

Proof: The formula results from elementary combinatorial computations. For any $0 \leq t \leq T$ and any $x = (x_i)_{1 \leq i \leq N} \in E^N$, we have

$$\begin{aligned}
&\widehat{\mathcal{L}}_T^{(N)}(F_f)(t,x) \\
&= \frac{1}{N} \sum_{1 \leq i,j \leq N} (m^{\odot(N,n)}(x^{i,j})[f_t] - m^{\odot(N,n)}(x)[f_t])U_t(x_j) \\
&= \frac{1}{N^{n+1}} \sum_{1 \leq i,j \leq N} \sum_{(i_1,\ldots,i_n) \in I(N,n)} (f_t(x_{i_1}^{i,j},\ldots,x_{i_n}^{i,j}) - f_t(x_{i_1},\ldots,x_{i_n}))U_t(x_j) \\
&= \frac{1}{N^{n+1}} \sum_{(i_1,\ldots,i_n) \in I(N,n)} \sum_{1 \leq k \leq n} \sum_{1 \leq j \leq N} (f_t(x_{i_1}^{i_k,j},\ldots,x_{i_n}^{i_k,j}) - f_t(x_{i_1},\ldots,x_{i_n}))U_t(x_j) \\
&= \frac{1}{N^{n+1}} \sum_{(i_1,\ldots,i_n) \in I(N,n)} \sum_{1 \leq k \leq n} \sum_{1 \leq j \leq N} f_t(x_{i_1},\ldots,x_{i_{k-1}},x_j,x_{i_{k+1}},\ldots,x_{i_n})U_t(x_j) \\
&\quad -nm^{(N)}(x)[U_t]m^{\odot(N,n)}(x)[f_t]
\end{aligned}$$

This yields that

$$\begin{aligned}
\widehat{\mathcal{L}}_T^{(N)}(F_f)(t,x) \quad = \quad & \frac{N-n+1}{N^{n+1}} \sum_{1 \leq k \leq n} \sum_{l \in \{1,\ldots,n\} \setminus \{k\}} \\
& \qquad \sum_{(i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n) \in I(N,n-1)} f_t(x_{i_1},\ldots,x_{i_{k-1}},x_{i_l},x_{i_{k+1}},\ldots,x_{i_n})U_t(x_j) \\
& + \frac{N-n+1}{N^{n+1}} \sum_{(i_1,\ldots,i_n) \in I(N,n)} f_t(x_{i_1},\ldots,x_{i_n}) \sum_{1 \leq k \leq n} U_t(x_{i_k}) \\
& -nm^{(N)}(x)[U_t]m^{\odot(N,n)}(x)[f_t]
\end{aligned}$$

Notice that the intermediate term in the last expression is given $\frac{N-n+1}{N} m^{\odot(N,n)}(f_t \bar{U}_t^{(n)})$. Thus, it just remains to treat the first term which we decompose into the two quantities:

$$
\begin{aligned}
&\tfrac{1}{N^{n+1}} \sum_{1\le k\le n} \sum_{l\neq k} \sum_{(i_1,\ldots,i_n)\in I(N,n-1)} (f_t(x_{i_1},\ldots,x_{i_{k-1}},x_{i_l},x_{i_{k+1}},\ldots,x_{i_n}) - f_t(x_{i_1},\ldots,x_{i_n}))U_t(x_{i_l}) \\
&+\tfrac{1}{N^{n+1}} \sum_{1\le k\le n} \sum_{l\neq k} \sum_{(i_1,\ldots,i_n)\in I(N,n-1)} f_t(x_{i_1},\ldots,x_{i_n})U_t(x_{i_l}) \\
&= \tfrac{1}{N^n} \sum_{(i_1,\ldots,i_n)\in I(N,n)} l_{T,N,n}(f)(t,x_{i_1},\ldots,x_{i_n}) \\
&+\tfrac{n-1}{N^{n+1}} \sum_{(i_1,\ldots,i_n)\in I(N,n)} f(t,x_{i_1},\ldots,x_{i_n})\bar{U}_t^{(n)}(x_{i_1},\ldots,x_{i_n}) \\
&= m^{\odot(N,n)}(x)[l_{T,N,n}(f)(t,\cdot)] + \tfrac{n-1}{N} m^{\odot(N,n)}(x)(f_t\bar{U}_t^{(n)})
\end{aligned}
$$

This implies that for any $0 \le t \le T$, and any $x \in E^N$, we have

$$
\begin{aligned}
\widehat{\mathcal{L}}_T^{(N)}(F_f)(t,x) &= m^{\odot(N,n)}(x)[\bar{U}_t^{(n)} f_t + l_{T,N,n}(f)(t,\cdot)] - nm^{(N)}(x)[U_t] m^{\odot(N,n)}(x)[f_t] \\
&= m^{\odot(N,n)}(x)[\bar{U}_t^{(n)} f_t + l_{T,N,n}(f)(t,\cdot)] - m^{\odot(N,n)}(x)[\bar{U}_t^{(n)}] m^{\odot(N,n)}(x)[f_t]
\end{aligned}
$$

■

We let $(\mathbb{P}_{t,x}^{\odot(N,n)})_{t\ge 0, x\in E^n}$ be the Markov family on E^n constructed as in section 2.4, by perturbing with the bounded operators $l_{T,N,n}$, for $T \ge 0$, the generators of the n-product of independent coordinates evolving according to $(\mathbb{P}_{t,x})_{t\ge 0, x\in E}$. We will also denote by $Y = (Y_t)_{t\ge 0}$ the canonical coordinate process on $\mathbb{M}(\mathbb{R}_+, E^n)$. Besides, the horizon $T \ge 0$ and a function $\varphi \in \mathcal{B}_{\mathrm{b}}(E^n)$ being fixed, we introduce the mapping defined on $[0,T]\times E^n$ by

$$
G_{T,\varphi}^{\odot(N,n)}(t,y) = \mathbb{E}_{t,y}^{\odot(N,n)}\left[\exp\left(\int_t^T \bar{U}^{(n)}(s,Y_s)\,ds\right)\varphi(Y_T)\right]
$$

It is also convenient to consider the process $\Gamma_{T,\varphi} = (\Gamma_{T,\varphi}(t))_{0\le t\le T}$ given for any $t \le T$ by

$$
\Gamma_{T,\varphi}(t) = \exp\left(\int_0^t n\eta_s^{(N)}(U_s)\,ds\right) \times \eta_t^{\odot(N,n)}(G_{T,\varphi}^{\odot(N,n)}(t,\cdot))
$$

with $\eta_t^{\odot(N,n)} = m^{\odot(N,n)}(\xi_t^{(N)})$. Then we have

Proposition 3.3 *The process $\Gamma_{T,\varphi}$ is a martingale.*

Proof: Firstly, we examine the process given for any $t \le T$ by $R_t \stackrel{\text{def.}}{=} \eta_t^{\odot(N,n)}(G_{T,\varphi}^{\odot(N,n)}(t,\cdot))$. To show how it can give rise to a martingale, we need to check that $H_{T,\varphi}$ belongs to $\mathcal{A}_{T,N}$ and to compute $\mathcal{L}_T^{(N)}(H_{T,\varphi})$; with the mapping $H_{T,\varphi}$ defined on $[0,T]\times E^N$ by $H_{T,\varphi} = F_{G_{T,\varphi}^{\odot(N,n)}}$. According to corollary 2.6, we know that $G_{T,\varphi}^{\odot(N,n)} \in \mathcal{A}_{T,n}$, and

$$
\widetilde{\mathcal{L}}_T^{(n)}(G_{T,\varphi}^{\odot(N,n)})(t,y) = -\bar{U}_t^{(n)}(y) G_{T,\varphi}^{\odot(N,n)}(t,y) - l_{T,N,n}(G_{T,\varphi}^{\odot(N,n)})(t,y)
$$

Taking into account lemma 3.1, and lemma 3.2, it appears that

$$
\forall\ 0 \le t \le T,\ \forall\ x \in E^N, \qquad \mathcal{L}_T^{(N)}(H_{T,\varphi})(t,x) = -nm^{(N)}(x)[U_t] H_{T,\varphi}(t,x)
$$

Thus, $\left(R_t + n\int_0^t \eta_s^{(N)}[U_s] R_s\,ds\right)_{0\le t\le T}$ is a martingale. Now, the proposition can be deduced without difficulty, via standard manipulations, under the precautions already presented in the proof of lemma 2.1.

■

More generally, the same arguments show that for all $0 \leq t \leq T$ and for all $x \in E^N$, the process

$$\left(\exp\left(\int_t^s n\eta_u^{(N)}(U_u)\,du\right)\eta_s^{\odot(N,n)}(G_{T,\varphi}^{\odot(N,n)}(s,\cdot))\right)_{t\leq s\leq T}$$

is a martingale under $\widehat{\mathbb{P}}_{t,x}$, with respect to the usual filtration. The previous constructions enable us to approximate the quantity

$$\mathbb{E}[\gamma_{t_1}^{(N)}\otimes\gamma_{t_2}^{(N)}\otimes\cdots\otimes\gamma_{t_n}^{(N)}(\varphi)]$$

where $n \in \mathbb{N}^*$, $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T$ and $\varphi \in \mathcal{B}_{\mathrm{b}}(E^n)$. But once again, we have to introduce a new object, looking like a generalization/composition of the $G_{T,\varphi}^{\odot(N,n)}$. It is a family of operators, the $K_{t_0,t_1,\ldots,t_n}^{\odot(N,n)}$, indexed by $n \in \mathbb{N}^*$ and $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$, and acting respectively on the $\mathcal{B}_{\mathrm{b}}(E^n)$. They are defined by induction on $n \in \mathbb{N}^*$. For $n = 1$, we are only considering the Feynman-Kac semigroup associated with our initial Markov family $(\mathbb{P}_{t,x})_{t\geq 0,x\in E}$. These models are defined for any $0 \leq t_0 \leq t_1$, $\varphi \in \mathcal{B}_{\mathrm{b}}(E)$,and $y \in E$ by the following formulae

$$K_{t_0,t_1}^{\odot(N,1)}(\varphi)(y) \quad=\quad \mathbb{E}_{t_0,y}\left[\exp\left(\int_{t_0}^{t_1} U_s(X_s)\,ds\right)\varphi(X_{t_1})\right]$$

Assuming that all the operators $K_{t_0,t_1,\ldots,t_n}^{\odot(N,n)}$ have been constructed, for a given $n \geq 1$, we define $K_{t_0,t_1,\ldots,t_{n+1}}^{\odot(N,n+1)}(\varphi)(y)$, by setting for any $t_0 \leq \cdots \leq t_{n+1}$, $\varphi \in \mathcal{B}_{\mathrm{b}}(E^{n+1})$, and $y \in E^{n+1}$

$$K_{t_0,t_1,\ldots,t_{n+1}}^{\odot(N,n+1)}(\varphi)(y) \;=\; G_{t_1,\Psi_{t_1,t_2,\ldots,t_{n+1}}}^{\odot(N,n+1)}(t_0,y)$$

In the above displayed formulae $\Psi_{t_1,t_2,\ldots,t_n}$ is the mapping given for any $z = (z_1,\ldots,z_{n+1}) \in E^{n+1}$ by

$$\Psi_{t_1,t_2,\ldots,t_{n+1}}(z) \quad=\quad K_{t_1,t_2,\ldots,t_{n+1}}^{\odot(N,n)}(\varphi_{z_1})(z_2,\ldots,z_n)$$

When some variables appear in the subscript of a function, it means that we are considering the function where these variables are fixed, e.g. φ_{z_1} is $\varphi(z_1,\cdot)$, for any given $z_1 \in E$. We give an interpretation of the above operators in next section, nevertheless to justify their study, we begin by presenting why they are natural in our context.

Proposition 3.4 *For all $n \in \mathbb{N}^*$, all $0 \leq t_1 \leq \cdots \leq t_n$ and all $\varphi \in \mathcal{B}_{\mathrm{b}}(E^n)$, we have the estimation*

$$\left|\mathbb{E}[\gamma_{t_1}^{(N)}\otimes\cdots\otimes\gamma_{t_n}^{(N)}(\varphi)] - \eta_0^{\otimes n}[K_{0,t_1,\ldots,t_n}^{\odot(N,n)}(\varphi)]\right| \quad\leq\quad \frac{i(N,n)}{1+i(N,n)}\,\|\varphi\|\,\eta_0^{\otimes n}[K_{0,t_1,\ldots,t_n}^{\odot(N,n)}(\mathbb{1})]$$

where

$$i(N,n) \quad=\quad 1-\prod_{1\leq i\leq n-1}\left(1-\frac{i}{N}\right) \quad=\quad \frac{N^n-\mathrm{card}(I(N,n))}{N^n} \leq \frac{(n-1)^2}{N} \tag{11}$$

Proof: We will look at the l.h.s. as a telescopic sum. The basic computation comes directly from the note after proposition 3.3, via an application of Markov property at time t_p, and says that for any $1 \leq p \leq n-1$, we have

$$\begin{aligned}
&\mathbb{E}\left[\int \mathcal{E}_{t_p,t_{p+1}}^{n-p}\eta_{t_{p+1}}^{\odot(N,n-p)}[K_{t_{p+1},\ldots,t_n}^{\odot(N,n-p-1)}(\varphi_{z_1,\ldots,z_p,\cdot})(\cdot)]\,\mathcal{E}_{0,t_p}^{n-p}\gamma_{t_1}^{(N)}(dz_1)\cdots\gamma_{t_p}^{(N)}(dz_p)\right] \\
&\quad=\quad \mathbb{E}\left[\int \eta_{t_p}^{\odot(N,n-p)}[G_{t_{p+1},K_{t_{p+1},\ldots,t_n}^{\odot(N,n-p-1)}(\varphi_{z_1,\ldots,z_p,\cdot})(\cdot)}^{\odot(N,n-p)}(t_p,\cdot)]\,\mathcal{E}_{0,t_p}^{n-p}\gamma_{t_1}^{(N)}(dz_1)\cdots\gamma_{t_p}^{(N)}(dz_p)\right] \\
&\quad=\quad \mathbb{E}\left[\int \eta_{t_p}^{\odot(N,n-p)}[K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\varphi_{z_1,\ldots,z_{p-1}})]\,\mathcal{E}_{0,t_p}^{n-p}\gamma_{t_1}^{(N)}(dz_1)\cdots\gamma_{t_p}^{(N)}(dz_p)\right]
\end{aligned}$$

We have used the convention that $K_{t_n}^{\odot(N,0)}$ is the identity operator, for any $t_n \geq 0$, and

$$\forall\ 0 \leq s \leq t, \qquad \mathcal{E}_{s,t} \;=\; \exp\left(\int_s^t \eta_u^{(N)}(U_u)\,du\right)$$

Let us remark that we can write

$$\eta_{t_p}^{(N)} \otimes \eta_{t_p}^{\odot(N,n-p)} - \eta_{t_p}^{\odot(N,n-p+1)} \;=\; \frac{1}{N^{n-p+1}} \sum_{(i_p,\ldots,i_n)\in I(N,n-p)} \sum_{i\in\{i_p,\ldots,i_n\}} \delta_{(\xi_{t_p}^{(N,i)},\xi_{t_p}^{(N,i_p)},\ldots,\xi_{t_p}^{(N,i_n)})}$$

so at any fixed $(z_1,\ldots,z_{p-1}) \in E^{p-1}$, we get

$$\begin{aligned}
&\Big|\mathbb{E}\left[\int \eta_{t_p}^{\odot(N,n-p)}[K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\varphi_{z_1,\ldots,z_{p-1}})]\,\mathcal{E}_{0,t_p}^{n-p}\gamma_{t_1}^{(N)}(dz_1)\cdots\gamma_{t_p}^{(N)}(dz_p)\right]\\
&\quad -\mathbb{E}\left[\int \mathcal{E}_{t_{p-1},t_p}^{n-p+1}\eta_{t_p}^{\odot(N,n-p+1)}[K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\varphi_{z_1,\ldots,z_{p-1},\cdot})(\cdot)]\,\mathcal{E}_{0,t_{p-1}}^{n-p+1}\gamma_{t_1}^{(N)}(dz_1)\cdots\gamma_{t_{p-1}}^{(N)}(dz_{p-1})\right]\Big|\\
&= \frac{1}{N}\left|\mathbb{E}\left[\sum_{p\leq k\leq n}\int \eta_{t_p}^{\odot(N,n-p)}[K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\varphi_{z_1,\ldots,z_{p-1},\xi_{t_p}^{(N,i_k)}})]\,\mathcal{E}_{0,t_p}^{n-p+1}\gamma_{t_1}^{(N)}(dz_1)\cdots\gamma_{t_p}^{(N)}(dz_{p-1})\right]\right|\\
&\leq \|\varphi\|\,\frac{1}{N}\mathbb{E}\left[\sum_{p\leq k\leq n}\int \eta_{t_p}^{\odot(N,n-p)}[K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\mathbf{1}_{z_1,\ldots,z_{p-1},\xi_{t_p}^{(N,i_k)}})]\,\mathcal{E}_{0,t_p}^{n-p+1}\gamma_{t_1}^{(N)}(dz_1)\cdots\gamma_{t_p}^{(N)}(dz_{p-1})\right]
\end{aligned}$$

Now summing these estimations for $1 \leq p \leq n-1$, and taking into account that for $p = 0$ we also have

$$\mathbb{E}\left[\mathcal{E}_{0,t_1}^n\eta_{t_1}^{\odot(N,n)}[K_{t_1,\ldots,t_n}^{\odot(N,n-1)}(\varphi_\cdot)(\cdot)]\right] \;=\; \mathbb{E}\left[\eta_0^{\odot(N,n)}[K_{t_0,\ldots,t_n}^{\odot(N,n)}(\varphi)]\right]$$

we obtain that

$$\begin{aligned}
&\left|\mathbb{E}[\gamma_{t_1}^{(N)}\otimes\cdots\otimes\gamma_{t_n}^{(N)}(\varphi)] - \mathbb{E}[\eta_0^{\odot(N,n)}[K_{0,t_1,\ldots,t_n}^{\odot(N,n)}(\varphi)]\right|\\
&\leq \sum_{1\leq p\leq n-1}\Big|\mathbb{E}\left[\int \mathcal{E}_{t_p,t_{p+1}}^{n-p}\eta_{t_{p+1}}^{\odot(N,n-p)}[K_{t_{p+1},\ldots,t_n}^{\odot(N,n-p-1)}(\varphi_{z_1,\ldots,z_p,\cdot})(\cdot)]\,\mathcal{E}_{0,t_p}^{n-p}\gamma_{t_1}^{(N)}(dz_1)\cdots\gamma_{t_p}^{(N)}(dz_p)\right]\\
&\quad -\mathbb{E}\left[\int \mathcal{E}_{t_{p-1},t_p}^{n-p+1}\eta_{t_p}^{\odot(N,n-p+1)}[K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\varphi_{z_1,\ldots,z_{p-1},\cdot})(\cdot)]\,\mathcal{E}_{0,t_{p-1}}^{n-p+1}\gamma_{t_1}^{(N)}(dz_1)\cdots\gamma_{t_{p-1}}^{(N)}(dz_{p-1})\right]\Big|\\
&\leq \frac{\|\varphi\|}{N}\sum_{1\leq p\leq n-1}\\
&\quad \mathbb{E}\left[\sum_{p\leq k\leq n}\int \eta_{t_p}^{\odot(N,n-p)}[K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\mathbf{1}_{z_1,\ldots,z_{p-1},\xi_{t_p}^{(N,i_k)}})]\,\mathcal{E}_{0,t_p}^{n-p+1}\gamma_{t_1}^{(N)}(dz_1)\cdots\gamma_{t_p}^{(N)}(dz_{p-1})\right]\\
&= \|\varphi\|\left|\mathbb{E}[\gamma_{t_1}^{(N)}\otimes\cdots\otimes\gamma_{t_n}^{(N)}(\mathbf{1})] - \mathbb{E}[\eta_0^{\odot(N,n)}[K_{0,t_1,\ldots,t_n}^{\odot(N,n)}(\mathbf{1})]\right|
\end{aligned}$$

Let us come back to the above intermediate step in the case $\varphi = \mathbf{1}$. Using the fact that the quantity $K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\mathbf{1}_{z_1,\ldots,z_{p-1},\xi_{t_p}^{(N,i)}})$ does not depend on the choice of $1 \leq i \leq N$, we find that

$$\begin{aligned}
&\eta_{t_p}^{(N)}\otimes\eta_{t_p}^{\odot(N,n-p)}[K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\mathbf{1}_{z_1,\ldots,z_{p-1},\cdot})(\cdot)]\\
&= \left(1-\tfrac{n-p}{N}\right)\eta_{t_p}^{\odot(N,n-p+1)}[K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\varphi_{z_1,\ldots,z_{p-1},\cdot})(\cdot)]
\end{aligned}$$

Considering all the previous steps, we get the equality

$$\mathbb{E}[\gamma_{t_1}^{(N)}\otimes\cdots\otimes\gamma_{t_n}^{(N)}(\mathbf{1})] \;=\; \prod_{1\leq i\leq n-1}\left(1-\frac{i}{N}\right)\mathbb{E}[\eta_0^{\odot(N,n)}[K_{0,t_1,\ldots,t_n}^{\odot(N,n)}(\mathbf{1})]$$

from which we deduce that

$$\left|\mathbb{E}[\gamma_{t_1}^{(N)}\otimes\cdots\otimes\gamma_{t_n}^{(N)}(\mathbf{1})]-\mathbb{E}[\eta_0^{\odot(N,n)}[K_{0,t_1,\ldots,t_n}^{\odot(N,n)}(\mathbf{1})]\right| \leq i(N,n)\mathbb{E}[\eta_0^{\odot(N,n)}[K_{0,t_1,\ldots,t_n}^{\odot(N,n)}(\mathbf{1})]$$

We also notice, due to the initial independence of the particles, that for any $\varphi\in\mathcal{B}_{\mathrm{b}}(E^n)$,

$$\mathbb{E}[\eta_0^{\odot(N,n)}[\varphi]] = \prod_{1\leq i\leq n}\left(1-\frac{i}{N}\right)\eta_0^{\otimes n}[\varphi]$$

■

3.2 Estimates on Moran semigroups

The aim of this section is to analyze the operators $K_{0,t_1,\ldots,t_n}^{\odot(N,n)}$. We will use here a preliminary coupling argument to give an upper bound on the difference between $K_{0,t_1,\ldots,t_n}^{\odot(N,n)}$ and $K_{0,t_1,\ldots,t_n}^{\otimes(N,n)}$, for $n\in\mathbb{N}^*$ and $0\leq t_1\leq\cdots\leq t_n\leq T$, where the last operator is constructed in the same way as the former, but assuming that the coordinates evolves independently. More precisely, for fixed $1\leq n\leq N$, and $\varphi\in\mathcal{B}_{\mathrm{b}}(E^n)$, and any $0\leq t_0\leq t_1$, $\forall\ y\in E^n$, we define

$$G_{t_1,\varphi}^{\otimes(N,n)}(t_0,y) \stackrel{\text{def.}}{=} \mathbb{E}_{t,y}\left[\exp\left(\int_t^T\bar{U}^{(n)}(Y_s)\,ds\right)\varphi(Y_T)\right]$$

We define the operators $K_{t_0,t_1,\ldots,t_n}^{\otimes(N,n)}$, also indexed by $n\in\mathbb{N}^*$ and $0\leq t_0\leq t_1\leq\cdots\leq t_n$, and acting respectively on the $\mathcal{B}_{\mathrm{b}}(E^n)$, by induction on the parameter $n\in\mathbb{N}^*$. For $n=1$, $t_0\leq t_1$ and we set $K_{t_0,t_1}^{\otimes(N,1)}=K_{t_0,t_1}^{\odot(N,1)}$. Assuming that all the operators $K_{t_0,t_1,\ldots,t_n}^{\otimes(N,n)}$ have been constructed, for a given $n\geq 1$, for any $0\leq t_0\leq\cdots\leq t_{n+1}$, and $\varphi\in\mathcal{B}_{\mathrm{b}}(E^{n+1})$, and $y\in E^{n+1}$, we define

$$K_{t_0,t_1,\ldots,t_{n+1}}^{\otimes(N,n+1)}(\varphi)(y) = G_{t_1,K_{t_1,t_2\ldots,t_n}^{\otimes(N,n)}(\varphi\cdot(\cdot))}^{\otimes(N,n+1)}(t_0,y)$$

In order to take advantage of the considerations of section 2.3, we need to interpret the above operators as something looking as the semigroups associated to some Markov processes, one being seen as a bounded perturbation of the other. We start with the "tensorized" operators, which will play the role of the "unperturbed" ones. We assume that $n\in\mathbb{N}^*$ and $0\leq t_1\leq\cdots\leq t_{n-1}$ are fixed. We will construct a locally bounded function $V:\mathbb{R}_+\times E^n\to\mathbb{R}_+$ and for any given $y\in E^n$, a probability $\check{\mathbb{P}}_{0,y}^{\otimes(N,n)}$ on $(\mathbb{M}(\mathbb{R}_+,E^n),\mathcal{M}(\mathbb{R}_+,E^n))$ such that for all $t_n\geq t_{n-1}$, all $y\in E^n$ and all $\varphi\in\mathcal{B}_{\mathrm{b}}(E^n)$,

$$K_{0,t_1,\ldots,t_n}^{\otimes(N,n)}(\varphi)(y) = \check{\mathbb{E}}_{0,y}^{\otimes(N,n)}\left[\exp\left(\int_0^{t_n}V(s,Y_s)\,ds\right)\varphi(Y_{t_n})\right] \tag{12}$$

As usual, Y stands for the canonical process. The latter probability will in fact be a product probability, each coordinate evolving independently (but not according to the same law): if $y=(y_i)_{1\leq i\leq n}$,

$$\check{\mathbb{P}}_{0,y}^{\otimes(N,n)} = \bigotimes_{1\leq i\leq n-1}\mathbb{P}_{0,y_i}^{(t_i)}\bigotimes\mathbb{P}_{0,y_n}$$

where for $t\in\mathbb{R}_+$ and $z\in E$, $\mathbb{P}_{0,z}^{(t)}$ is just the image of $\mathbb{P}_{0,z}$ under the mapping J_t : $\mathbb{M}(\mathbb{R}_+,E)\to\mathbb{M}(\mathbb{R}_+,E)$ defined for any $\omega\in\mathbb{M}(\mathbb{R}_+,E)$, and $s\geq 0$ by

$$X_s(J_t(\omega)) = X_{s\wedge t}(\omega)$$

Clearly, these probabilities can be embedded into a Markov family $(\check{\mathbb{P}}^{\otimes(N,n)}_{t,y})_{t\geq 0, y\in E^n}$, by taking for any $t\geq 0$, and $y=(y_i)_{1\leq i\leq n}\in E^n$

$$\check{\mathbb{P}}^{\otimes(N,n)}_{t,y} = \bigotimes_{1\leq i\leq n-1} J_{t\vee t_i}(\mathbb{P}_{t,y_i})\bigotimes \mathbb{P}_{t,y_n}$$

where the J_s, for $s\geq t$, are rather seen as acting on $\mathbb{M}([t,+\infty[,E)$, and note that $J_t(\mathbb{P}_{t,z})$ is the Dirac mass at the trajectory of a nonmoving particle starting from $z\in E$ at time $t\geq 0$. The definition of V is also very simple. For any $t\geq 0$, and $y=(y_i)_{1\leq i\leq n}\in E^n$, we set

$$V(t,y) = \begin{cases} \sum_{i\leq j\leq n} U(t,y_j) & \text{, if there exists } 1\leq i\leq n-1 \text{ such that } t_{i-1}\leq t<t_i \\ U(t,y_n) & \text{, if } t\geq t_{n-1}\end{cases}$$

Immediate computations shows that (12) is fulfilled. With the parameters $n\in\mathbb{N}^*$ and $0\leq t_1\leq\cdots\leq t_{n-1}$ still fixed, we want to construct for any given $y\in E^n$, a probability $\check{\mathbb{P}}^{\odot(N,n)}_{0,y}$ on $(\mathbb{M}(\mathbb{R}_+,E^n),\mathcal{M}(\mathbb{R}_+,E^n))$ such that for any $t_n\geq t_{n-1}$, $y\in E^n$, and any $\varphi\in\mathcal{B}_{\mathrm{b}}(E^n)$ we have that

$$K^{\odot(N,n)}_{0,t_1,\ldots,t_n}(\varphi)(y) = \check{\mathbb{E}}^{\odot(N,n)}_{0,y}\left[\exp\left(\int_0^{t_n} V(s,Y_s)\,ds\right)\varphi(Y_{t_n})\right] \tag{13}$$

and it is possible to do it via the perturbation techniques of section 2.2. So we just have to describe the corresponding nonnegative kernel $\widehat{R}$ from $\mathbb{R}_+\times E^n$ into E^n given by

$$R(t,x,\cdot) = \begin{cases} \frac{1}{N}\sum_{i\leq j\neq k\leq n} U(t,y_k)\delta_{y^{j,k}} & \text{, if there exists } 0\leq i\leq n-1 \text{ s.t. } t\in[t_{i-1},t_i[\\ 0 & \text{, if } t\geq t_{n-1}\end{cases}$$

Again, direct and not very stimulating computations show that (13) is satisfied, where the Markovian family $(\check{\mathbb{P}}^{\odot(N,n)}_{t,y})_{t\geq 0,y\in E^n}$ is the perturbation of $(\check{\mathbb{P}}^{\otimes(N,n)}_{t,y})_{t\geq 0,y\in E^n}$ by $\widehat{R}$. Now, we are in position to use the results of section 2.3.

Proposition 3.5 *For all $T\geq 0$, all $n\in\mathbb{N}^*$, all $0\leq t_1\leq\cdots\leq t_n\leq T$, all $y\in E^n$ and all $\varphi\in\mathcal{B}_{\mathrm{b}}(E^n)$, we are assured of the bound*

$$\left|K^{\odot(N,n)}_{0,t_1,\ldots,t_n}[\varphi](y)-K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}[\varphi](y)\right| \leq \widetilde{\epsilon}_T\left(\frac{(n-1)n}{N}\right)\|\varphi\|\,K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}[\mathbb{1}](y)$$

where for any $a\geq 0$,

$$\widetilde{\epsilon}_T\,(a) = 2(1-\exp[-au_TT])+\exp([\exp(u_TT)-1]au_TT)-1$$

which is equivalent to $u_TT(1+\exp(u_TT))a$, for small $a>0$.

Proof: As usual, we start by fixing the horizon $T\geq 0$ and we work on the interval $[0,T]$. Let $\mathbb{P}$ be a coupling of $\check{\mathbb{E}}^{\otimes(N,n)}_{0,y}\otimes\nu^{(r)}$ and $\check{\mathbb{E}}^{\odot(N,n)}_{0,y}$ satisfying property of proposition 2.8, with $r=\frac{(n-1)n}{N}u_T$. Then we can write, with notations introduced there (but replacing X by Y):

$$\begin{aligned}
&\left|K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}[\varphi](y)-K^{\odot(N,n)}_{0,t_1,\ldots,t_n}[\varphi](y)\right| \\
&= \left|\mathbb{E}\left[\exp\left(\int_0^{t_n}V(s,Y_s)\,ds\right)\varphi(Y_{t_n})-\exp\left(\int_0^{t_n}V(s,\widehat{Y}_s)\,ds\right)\varphi(\widehat{Y}_{t_n})\right]\right| \\
&= \left|\sum_{k\geq 1}\mathbb{E}\left[\left(\exp\left(\int_0^{t_n}V(s,Y_s)\,ds\right)\varphi(Y_{t_n})-\exp\left(\int_0^{t_n}V(s,\widehat{Y}_s)\,ds\right)\varphi(\widehat{Y}_{t_n})\right)\mathbb{1}_{\{K_T=k\}}\right]\right| \\
&\leq \left|\sum_{k\geq 1}\mathbb{E}\left[\exp\left(\int_0^{t_n}V(s,Y_s)\,ds\right)\left(\varphi(Y_{t_n})-\varphi(\widehat{Y}_{t_n})\right)\mathbb{1}_{\{K_T=k\}}\right]\right|
\end{aligned}$$

$$
\begin{aligned}
& +\left|\sum_{k\geq 1}\mathbb{E}\left[\left(\exp\left(\int_0^{t_n} V(s,Y_s)\,ds\right)-\exp\left(\int_0^{t_n} V(s,\widehat{Y}_s)\,ds\right)\right)\varphi(\widehat{Y}_{t_n})\mathbf{1}_{\{K_T=k\}}\right]\right| \\
\leq \quad & 2\,\|\varphi\|\,\mathbb{E}\left[\exp\left(\int_0^{t_n} V(s,Y_s)\,ds\right)\mathbf{1}_{\{K_T\geq 1\}}\right] \\
& +\|\varphi\|\sum_{k\geq 1}\mathbb{E}\left[\left|\exp\left(\int_0^{t_n} V(s,Y_s)\,ds\right)-\exp\left(\int_0^{t_n} V(s,\widehat{Y}_s)\,ds\right)\right|\mathbf{1}_{\{K_T=k\}}\right]
\end{aligned}
$$

This implies that for any φ, with $\|\varphi\|\leq 1$

$$
\begin{aligned}
\left|K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}[\varphi](y)-K^{\odot(N,n)}_{0,t_1,\ldots,t_n}[\varphi](y)\right| \quad \leq \quad & 2\mathbb{E}\left[\exp\left(\int_0^{t_n} V(s,Y_s)\,ds\right)\right]\mathbb{P}[K_T\geq 1] \\
+ \quad & \sum_{k\geq 1}\mathbb{E}\left[\exp\left(\int_0^{t_n} V(s,Y_s)\,ds\right)(\exp(kTu_T)-1)\mathbf{1}_{\{K_T=k\}}\right] \\
\leq \quad & 2K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}[\mathbf{1}](y)(1-\exp(-rT)) \\
& +K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}[\mathbf{1}](y)\sum_{k\geq 1}(\exp(kTu_T)-1)\frac{(rT)^k}{k!}\exp(-rT) \\
\leq \quad & \epsilon_T\left(\frac{(n-1)n}{N}\right)K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}[\mathbf{1}](y)
\end{aligned}
$$

In the sense described in section 2.3, we have used the fact that on the set $\{K_T=k\}$, we have $\int_0^{t_n}\left|V(s,Y_s)-V(s,\widehat{Y}_s)\right|\,ds\leq Tku_T$.

■

3.3 Proof of theorem 1.2

The deterministic measure-valued flow $(\gamma_t)_{t\geq 0}$ is obtained from η_0 by the application of the semigroup $K^{\otimes(N,1)}$:

$$
\forall\ t\geq 0,\qquad \gamma_t \;=\; \eta_0 K^{\otimes(N,1)}_{0,t} \tag{14}
$$

It was these simple acknowledgments which lead us to believe that the γ_t should be easy to compare with the $\gamma_t^{(N)}$, for $t\geq 0$, and in fact the latter are estimations without bias of the formers (cf. [6], or proposition 3.4 with $n=1$). For the higher tensor products ($n\geq 2$) this property is lost but the above considerations enable to bound the error.

Proposition 3.6 *For all $n\in\mathbb{N}^*$, all $0\leq t_1\leq\cdots\leq t_n$ and all $\varphi\in\mathcal{B}_{\mathrm{b}}(E^n)$, we have the following bound on the bias:*

$$
\left|\mathbb{E}[\gamma^{(N)}_{t_1}\otimes\cdots\otimes\gamma^{(N)}_{t_n}(\varphi)]-\gamma_{t_1}\otimes\cdots\otimes\gamma_{t_n}(\varphi)\right| \;\leq\; \widehat{\epsilon}_T\left(\frac{n(n-1)}{N}\right)\|\varphi\|\,\gamma_{t_1}\otimes\cdots\otimes\gamma_{t_n}(\mathbf{1})
$$

where $\widehat{\epsilon}_T(a)=2\widetilde{\epsilon}_T(a)+a$, for any $a\geq 0$.

Proof: Combining proposition 3.4, proposition 3.5, and the upper bound (11), we obtain

$$
\begin{aligned}
& \left|\mathbb{E}[\gamma^{(N)}_{t_1}\otimes\cdots\otimes\gamma^{(N)}_{t_n}(\varphi)]-\eta_0^{\otimes n}[K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}(\varphi)]\right| \\
\leq \quad & \left|\eta_0^{\otimes n}[K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}(\varphi)-K^{\odot(N,n)}_{0,t_1,\ldots,t_n}(\varphi)]\right|+i(N,n)\,\|\varphi\|\,\eta_0^{\otimes n}[K^{\odot(N,n)}_{0,t_1,\ldots,t_n}(\mathbf{1})] \\
\leq \quad & (1+i(N,n))\widetilde{\epsilon}_T\left(\frac{(n-1)n}{N}\right)\|\varphi\|\,\eta_0^{\otimes n}[K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}(\mathbf{1})]+i(N,n)\,\|\varphi\|\,\eta_0^{\otimes n}[K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}(\mathbf{1})] \\
\leq \quad & \widehat{\epsilon}_T\left(\frac{n(n-1)}{N}\right)\|\varphi\|\,\eta_0^{\otimes n}[K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}(\mathbf{1})]
\end{aligned}
$$

Thus, the result now follows from the equality $\eta_0^{\otimes n}[K_{0,t_1,\ldots,t_n}^{\otimes(N,n)}(\varphi)] = (\gamma_{t_1}\otimes\cdots\otimes\gamma_{t_n})(\varphi)$, which in turn comes from (14) and the product structure.

■

The above approximation has the interesting property to be "self-improving":

Proposition 3.7 *For all $n\in\mathbb{N}^*$, all $0\leq t_1\leq\cdots\leq t_n$ and all $\varphi\in\mathcal{B}_{\mathrm{b}}(E^n)$, we have the following bound on the square mean error:*

$$\left|\mathbb{E}[(\gamma_{t_1}^{(N)}\otimes\cdots\otimes\gamma_{t_n}^{(N)}(\varphi)-\gamma_{t_1}\otimes\cdots\otimes\gamma_{t_n}(\varphi))^2]\right| \quad\leq\quad \bar{\epsilon}_T\left(\frac{n^2}{N}\right)\|\varphi\|^2\,(\gamma_{t_1}\otimes\cdots\otimes\gamma_{t_n}(\mathbf{1}))^2$$

where $\bar{\epsilon}_T(a)=\widehat{\epsilon}_T(4a)+2\widehat{\epsilon}_T(a)$, for any $a\geq 0$.

Proof: To simplify the notation, we write $\gamma_{t_1,\ldots,t_n}^{(N)\otimes}=\gamma_{t_1}^{(N)}\otimes\cdots\otimes\gamma_{t_n}^{(N)}$, and $\gamma_{t_1,\ldots,t_n}^{\otimes}=\gamma_{t_1}\otimes\cdots\otimes\gamma_{t_n}$. We consider the expansion, for $\varphi\in\mathcal{B}_{\mathrm{b}}(E^n)$,

$$\begin{aligned}
&\mathbb{E}[(\gamma_{t_1,\ldots,t_n}^{(N)\otimes}(\varphi)-\gamma_{t_1,\ldots,t_n}^{\otimes}(\varphi))^2]\\
&= \mathbb{E}[(\gamma_{t_1,\ldots,t_n}^{(N)\otimes}(\varphi))^2]-(\gamma_{t_1,\ldots,t_n}^{\otimes}(\varphi))^2-2\gamma_{t_1,\ldots,t_n}^{\otimes}(\varphi)(\mathbb{E}[\gamma_{t_1,\ldots,t_n}^{(N)\otimes}(\varphi)]-\gamma_{t_1,\ldots,t_n}^{\otimes}(\varphi))\\
&\leq \left|\mathbb{E}[\gamma_{t_1,t_1,t_2,\ldots,t_n}^{(N)\otimes}(\varphi\otimes\varphi)]-\gamma_{t_1,t_1,t_2,\ldots,t_n}^{\otimes}(\varphi\otimes\varphi)\right|\\
&\qquad +2\,\|\varphi\|\,\gamma_{t_1,\ldots,t_n}^{\otimes}(\mathbf{1})\left|\mathbb{E}[\gamma_{t_1,\ldots,t_n}^{(N)\otimes}(\varphi)]-\gamma_{t_1,\ldots,t_n}^{\otimes}(\varphi)\right|\\
&\leq \widehat{\epsilon}_T\left(\frac{4n^2}{N}\right)\|\varphi\otimes\varphi\|\,\gamma_{t_1,t_1,t_2,\ldots,t_n}^{\otimes}(\mathbf{1})+2\,\|\varphi\|\,\gamma_{t_1,\ldots,t_n}^{\otimes}(\mathbf{1})\widehat{\epsilon}_T\left(\frac{n^2}{N}\right)\|\varphi\|\,\gamma_{t_1,\ldots,t_n}^{\otimes}(\mathbf{1})\\
&= \bar{\epsilon}_T\left(\frac{n^2}{N}\right)\|\varphi\|^2\,(\gamma_{t_1,\ldots,t_n}^{\otimes}(\mathbf{1}))^2
\end{aligned}$$

■

More generally, one can found in the same way explicit bounds of any given moment of an integer order $p\geq 1$ (which will always be asymptotically equivalent to a factor times n^2/N, when this quantity is small, as $p\geq 1$ and $T>0$ are fixed). The above proposition also emphasizes the basic principle underlying this article: usually in order to study martingales associated to Markov processes, one looks at their increasing processes, which are given by the integration along the trajectories of the famous carrés du champs. But that approach leads to difficulties relative to domains of pregenerators which should be algebras (cf. for instance [6]). Here in order to avoid these kinds of embarrassing problems, in some sense we have straightly worked with the squares of the martingales: they are related to the squares of the functionals we are interested in and since the latter are empirical probabilities acting on some mappings, their squares can be seen as 2-tensorized empirical measures applied on 2-tensorized functions, which we study directly (or at least their closely related $\odot$-product).

Now the proof of theorem 1.2 is quite a standard task: first we write that

$$\begin{aligned}
&\eta_{t_1}^{(N)}\otimes\cdots\otimes\eta_{t_n}^{(N)}(\varphi)-\eta_{t_1}\otimes\cdots\otimes\eta_{t_n}(\varphi)\\
&= \frac{1}{\gamma_{t_1,\ldots,t_n}^{\otimes}(\mathbf{1})}\left[\gamma_{t_1,\ldots,t_n}^{(N)\otimes}(\varphi)-\gamma_{t_1,\ldots,t_n}^{\otimes}(\varphi)+\eta_{t_1}^{(N)}\otimes\cdots\otimes\eta_{t_n}^{(N)}(\varphi)(\gamma_{t_1,\ldots,t_n}^{\otimes}(\mathbf{1})-\gamma_{t_1,\ldots,t_n}^{(N)\otimes}(\mathbf{1}))\right]
\end{aligned}\tag{15}$$

This enables us to get a preliminary bound on the second moment

$$\begin{aligned}
&\mathbb{E}[(\eta_{t_1}^{(N)}\otimes\cdots\otimes\eta_{t_n}^{(N)}(\varphi)-\eta_{t_1}\otimes\cdots\otimes\eta_{t_n}(\varphi))^2]\\
&\leq 2\frac{1}{(\gamma_{t_1,\ldots,t_n}^{\otimes}(\mathbf{1}))^2}\Big(\mathbb{E}[(\gamma_{t_1,\ldots,t_n}^{(N)\otimes}(\varphi)-\gamma_{t_1,\ldots,t_n}^{\otimes}(\varphi))^2]+\|\varphi\|^2\,\mathbb{E}[(\gamma_{t_1,\ldots,t_n}^{\otimes}(\mathbf{1})-\gamma_{t_1,\ldots,t_n}^{(N)\otimes}(\mathbf{1}))^2]\Big)\\
&\leq 3\bar{\epsilon}_T\left(\frac{n^2}{N}\right)\|\varphi\|^2
\end{aligned}$$

To conclude, we again integrate (15)

$$
\begin{aligned}
&\mathbb{E}[\eta_{t_1}^{(N)}\otimes\cdots\otimes\eta_{t_n}^{(N)}(\varphi)-\eta_{t_1}\otimes\cdots\otimes\eta_{t_n}(\varphi)]\\
&= \frac{1}{\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbf{1})}\Big[\mathbb{E}[\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\varphi)-\gamma^{\otimes}_{t_1,\ldots,t_n}(\varphi)]+\eta_{t_1}\otimes\cdots\otimes\eta_{t_n}(\varphi)\mathbb{E}[\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbf{1})-\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\mathbf{1})]\\
&\quad +\mathbb{E}[(\eta_{t_1}^{(N)}\otimes\cdots\otimes\eta_{t_n}^{(N)}(\varphi)-\eta_{t_1}\otimes\cdots\otimes\eta_{t_n}(\varphi))(\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbf{1})-\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\mathbf{1}))]\Big]\\
&\leq \frac{1}{\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbf{1})}\Big[\mathbb{E}[\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\varphi)-\gamma^{\otimes}_{t_1,\ldots,t_n}(\varphi)]+\|\varphi\|\,\Big|\mathbb{E}[\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbf{1})-\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\mathbf{1})]\Big|\\
&\quad +\sqrt{\mathbb{E}[(\eta_{t_1}^{(N)}\otimes\cdots\otimes\eta_{t_n}^{(N)}(\varphi)-\eta_{t_1}\otimes\cdots\otimes\eta_{t_n}(\varphi))^2]}\sqrt{\mathbb{E}[(\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbf{1})-\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\mathbf{1}))^2]}\\
&\leq \epsilon_T\left(\frac{n^2}{N}\right)\|\varphi\|
\end{aligned}
$$

with $\epsilon_T(a)=2[(\widehat{\epsilon}_T(a)+\bar{\epsilon}_T(a))\wedge 1]$.

4 Strong propagation of chaos estimates

The propagation of chaos in our interacting particle approximation model explains the behavior of the n^{th} first coordinates of the interacting particle system $\xi^{(N)}$ as n^2/N is small. In particular it shows that they are asymptotically independent. Tthis property accounts for the name "propagation of chaos" due to Kac [10], see for instance [13] in a different context: if initially the coordinates are independent, then in the limit of a large number of particles, any fixed finite number of them end up to be still independent over bounded time interval, despite the interactions. Let the horizon $T\geq 0$, and the numbers of particles $1\leq n\leq N$ be fixed parameters. The object under study is $\mathbb{P}^{(N,\{1,\ldots,n\})}_{\eta_0,[0,T]}$ the law of $(\xi_t^{(N,i)})_{1\leq i\leq n,\,0\leq t\leq T}$ under $\mathbb{P}$. In order to describe its limit, we need more notations. Recall that η_0 being supposed given, we have at our disposal a flow $(\eta_t)_{t\geq 0}$ of probabilities defined by (1). Starting from them, we introduce the non-negative kernel $\bar{R}$ from $\mathbb{R}_+\times E$ to E given for any $t\geq 0$, $x\in E$, and $A\in\mathcal{E}$ by

$$
\bar{R}(t,x,A) \;=\; \int_A U_t(y)\,\eta_t(dy)
$$

We consider the time-inhomogeneous Markov family $(\bar{\mathbb{P}}_{t,x})_{t\geq 0,\,x\in E}$, which is the perturbation of $(\mathbb{P}_{t,x})_{t\geq 0,\,x\in E}$ by $\bar{R}$. Let $\bar{X}\stackrel{\text{def.}}{=}(\bar{X}_t)_{t\geq 0}$ be the canonical coordinate process on $\mathbb{M}(\mathbb{R}_+,\mathbb{E})$ under the law $\bar{\mathbb{P}}_{\eta_0}\stackrel{\text{def.}}{=}\int\eta_0(dx)\mathbb{P}_{0,x}$. For $T\geq 0$, we will also write $\bar{\mathbb{P}}_{\eta_0,[0,T]}$ for the law of $(\bar{X}_t)_{0\leq t\leq T}$ on $\mathbb{M}([0,T],E)$. The initial law η_0 being always the same one considered everywhere.

Proposition 4.1 *For any $T\geq 0$, the law of $\bar{X}_T$ under $\bar{\mathbb{P}}_{\eta_0}$ is η_T.*

Proof: We need to check that for any fixed horizon $T\geq 0$ and function $\varphi\in\mathcal{B}_{\text{b}}(E)$, we have that

$$
\mathbb{E}[\varphi(\bar{X}_T)] \;=\; \eta_T(\varphi)\stackrel{\text{def.}}{=}\mathbb{E}\left[\varphi(X_T)\exp\left(\int_0^T U_s(X_s)-\eta_s(U_s)\,ds\right)\right]
$$

We consider the mapping defined by

$$
\begin{aligned}
F\;:\;[0,T]\times E &\;\to\; \mathbb{R}\\
(t,x) &\;\mapsto\; \mathbb{E}_{t,x}\left[\varphi(X_T)\exp\left(\int_t^T U_s(X_s)-\eta_s(U_s)\,ds\right)\right]
\end{aligned}
$$

According to corollary 2.6, the time-space generator of $(\bar{X}_t)_{0\leq t\leq T}$ is such that

$$\begin{aligned} L_T(F)(t,x) &= L_T(F)(t,x) + \int (F(t,y)-F(t,x))U(t,y)\,\eta_t(dy) \\ &= -U(t,x)F(t,x) + \int F(t,y)U(t,y)\,\eta_t(dy) \end{aligned}$$

from where we get that

$$\mathbb{E}[F(T,\bar{X}_T)] = \mathbb{E}[F(0,\bar{X}_0)] - \int_0^T \mathbb{E}[U(t,\bar{X}_t)F(t,\bar{X}_t)] - \eta_t(U_tF_t)\,dt$$

which can be expressed as

$$m_T(\varphi) = \eta_T(\varphi) - \int_0^T m_t(U_tF_t) - \eta_t(U_tF_t)\,dt \tag{16}$$

where m_t is the law of $\bar{X}_t$, for any $t\geq 0$. This easily implies that

$$\|m_T-\eta_T\|_{\mathrm{tv}} \leq u_T T\exp(u_T T)\sup_{0\leq t\leq T}\|m_t-\eta_t\|_{\mathrm{tv}}$$

and more generally in the same way we obtain

$$\forall\ 0\leq t\leq T, \qquad \|m_t-\eta_t\|_{\mathrm{tv}} \leq u_T t\exp(u_T t)\sup_{0\leq s\leq t}\|m_s-\eta_s\|_{\mathrm{tv}}$$

Therefore, if $t_0>0$ is such that $u_Tt_0\exp(u_Tt_0)=1/2$, it appears that $m_t=\eta_t$ for all $0\leq t\leq t_0$. Now, rather considering $(\bar{X}_{t-t_0})_{t\geq t_0}$, and replacing η_0 by η_{t_0}, we obtain that for all $t_0\leq t\leq 2t_0$, $m_t=\eta_t$. Thus in a finite number of steps, we can conclude that $\eta_T=m_T$.

■

Notice that we cannot deduce from (16) that

$$\|m_T-\eta_T\|_{\mathrm{tv}} \leq \int_0^T \|m_t-\eta_t\|_{\mathrm{tv}}\,dt$$

just because we have no measurability results for $[0,T]\ni t\mapsto \|m_t-\eta_t\|_{\mathrm{tv}}$. We are now in position to prove the theorem 1.1. The proof is based on the next two direct coupling arguments, the crucial ingredient being theorem 1.2. Nevertheless, let us mentioned that the dependence of the constant C_T in $T\geq 0$ is very bad, except for the small ones, and we are wondering if it would not be possible to improve it by using this behavior for small $T>0$.

4.1 A first coupling

We will present in this section another very simple interacting system on E^N, whose n^{th} first coordinates have a special behavior (they take information from the other particles but do not have influence on them, so globally the system will no longer be exchangeable) but are close enough to the n^{th} first particles of our previous algorithm (at least for n^2/N small).

So we begin by describing this auxiliary interacting particle system which is also of the general type considered in section 2.3: more precisely, with usual notations, we make a perturbation of the Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,x\in E^N}$ by the kernel $\check{R}^{(N)}$ defined for any $t\geq 0$, and $x=(x_i)_{1\leq i\leq N}\in E^N$ by

$$\check{R}^{(N)}(t,x) = \frac{1}{N}\sum_{i=1}^N\sum_{j=n+1}^N U(t,x_j)\delta_{x^{i,j}}$$

where $1 \leq n \leq N$ are fixed, to get a new Markovian family $(\breve{\mathbb{P}}_{t,x})_{t\geq 0,\, x\in E^N}$. Its associated generator can be written $(\mathcal{A}_{T,N}, \breve{\mathcal{L}}_T)$, where

$$\breve{\mathcal{L}}_T^{(N)} \quad = \quad \widetilde{\mathcal{L}}_T^{(N)} + \check{\mathcal{L}}_T^{(N)}$$

with the selection generator given for any $\phi \in \mathcal{B}_{\rm b}([0,T] \times E^N)$ by

$$\check{\mathcal{L}}_T^{(N)}(\phi)(t,x) \quad = \quad \frac{1}{N} \sum_{i=1}^{N} \sum_{j=n+1}^{N} (\phi(t, x^{i,j}) - \phi(t,x))\, U_t(x_j)$$

In order to avoid confusion with the canonical process on $\mathbb{M}(\mathbb{R}_+, E^N)$, we will denote by $(\xi_t^{(N,i)})_{1\leq i\leq n,\, t\geq 0}$ and $(\breve{\xi}_t^{(N,i)})_{1\leq i\leq n,\, t\geq 0}$ the processes appearing in the "explicit" construction of the families $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E^N}$ and $(\breve{\mathbb{P}}_{t,x})_{t\geq 0,\, x\in E^N}$. The generator $\check{\mathcal{L}}_T^{(N)}$ is quite similar to the selection operator $\widehat{\mathcal{L}}_T^{(N)}$, except that the particles $\breve{\xi}_t^{(N,i)}$, for $n \leq i \leq N$ and $0 \leq t \leq T$, are not permitted to inherit by a selection step the values of $\breve{\xi}_t^{(N,j)}$, for $1 \leq j \leq n$. Furthermore, the restriction of $\check{\mathcal{L}}_T^{(N)}$ on functions depending only on the coordinates whose indices belong to $\{n+1, \ldots, N\}$ is equal to $\widehat{\mathcal{L}}_T^{(N-n)}$, up to a factor $(N-n)/N$ and to the reindexing of these indices obtained by adding n.

Observe that if we make a perturbation of $(\breve{\mathbb{P}}_{t,x})_{t\geq 0,\, x\in E^N}$ by the kernel given at time $t \geq 0$ and point $x = (x_i)_{1\leq i\leq N} \in E^N$ by

$$\frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{n} U(t, x_j) \delta_{x^{i,j}}$$

then we end up with the Markovian family of the algorithm we considered before (due to the uniqueness of the associated generators). We will use this important feature to show our main result here:

Proposition 4.2 *Let $\breve{\mathbb{P}}_{\eta_0,[0,T]}^{(N,\{1,\ldots,n\})}$ be the law of $(\breve{\xi}_t^{(N,i)})_{1\leq i\leq n,\, 0\leq t\leq T}$ "under" the reference measure $\int \eta_0^{\otimes N}(dx)\, \breve{\mathbb{P}}_{0,x}$. Then we have that*

$$\left\| \mathbb{P}_{\eta_0,[0,T]}^{(N,\{1,\ldots,n\})} - \breve{\mathbb{P}}_{\eta_0,[0,T]}^{(N,\{1,\ldots,n\})} \right\|_{\rm tv} \quad \leq \quad 4\frac{n^2}{N}\, (\exp(u_T T) - 1)$$

Proof: As we will use a coupling argument, let us come back to the construction of $(\xi_t^{(N)})_{0\leq t\leq T}$ (the horizon $T \geq 0$ is assumed to be fixed) which follows from the considerations of section 2.2 and 2.4: we denote by $(S_p)_{p\geq 1}$ the proposed selection times (such that the differences $(S_p - S_{p-1})_{p\geq 1}$ are independent and identically distributed according to exponential laws of parameter Nu_T, with the convention that $S_0 = 0$) and by $(Z_t)_{t\geq 0}$ the corresponding Poisson process. Let us also consider the following independent objects: $(I_p, J_p)_{p\geq 1}$ a family of independent uniformly distributed random variables in $\{1, \cdots, N\}^2$ and $(V_p)_{p\geq 1}$ a family of independent uniformly distributed random variables in $[0,1]$. We can assume that for any $p \geq 1$, the sampling of $\xi_{S_p}^{(N)}$ knowing "$\xi_{S_p-}^{(N)}$" is done according to the next mechanism: we replace the I_p-th coordinate of $\xi_{S_p-}^{(N)}$ by its J_p-th coordinate, if $V_p \leq U_{S_p}(\xi_{S_p}^{(N,J_p)})/u_T$, otherwise we take $\xi_{S_p}^{(N)} = \xi_{T_p-}^{(N)}$ (classical acceptation/rejection procedure).

Meanwhile, from the sequence $(I_p, J_p)_{p\geq 1}$ we can define a family $(A_p)_{p\geq 1}$ of random variables taking values in the subsets of $\{1, \ldots, N\}$: we start with $A_0 = \{1, \ldots, n\}$ and if A_p has been defined, we put

$$A_{p+1} \quad = \quad \begin{cases} A_p \cup \{I_{p+1}\} & \text{, if } J_{p+1} \in A_p \\ A_p & \text{, otherwise} \end{cases}$$

To see its interest, let us remark that using the same intuitive ideas and technical precautions as those presented in section 2.3, we can construct a process $(\breve{\xi}_t^{(N)})_{0\leq t\leq T}$ whose law will be the restriction to $\mathbb{M}([0,T],E^N)$ of $\int \eta_0^{\otimes N}(dx)\breve{\mathbb{P}}_{0,x}$ and which is coupled to $(\xi_t^{(N)})_{0\leq t\leq T}$ in the sense that the next property is satisfied:

$$\forall\, p\geq 1,\, \forall\, T\wedge S_p\leq t< T\wedge S_{p+1},\, \forall\, n+1\leq i\leq N,\qquad \breve{\xi}_t^{(N,i)}\neq \xi_t^{(N,i)} \quad\Longrightarrow\quad i\in A_p$$

(in particular we are starting with $\breve{\xi}_0^{(N)}=\xi_0^{(N)}$). Heuristically, for $p\geq 0$, $A_p\setminus\{1,\ldots,n\}$ is the set of subscripts $n+1\leq i\leq N$ such that $\breve{\xi}_{S_p}^{(N,i)}$ has a good chance to be different from $\xi_{S_p}^{(N,i)}$. Then, it appears that on the set

$$\{Z_T=0\}\sqcup\bigsqcup_{p\geq 1}\{Z_T=p,\,\forall\, 1\leq q\leq p,\, I_q\not\in\{1,\ldots,n\}\text{ or } J_q\not\in A_p\}$$

we are assured that $(\breve{\xi}_t^{(N,i)})_{1\leq i\leq n,\,0\leq t\leq T}=(\xi_t^{(N,i)})_{1\leq i\leq n,\,0\leq t\leq T}$. With the usual conventions enforced, we get that

$$\begin{aligned}
\mathbb{P}[(\breve{\xi}_t^{(N)})_{0\leq t\leq T}\neq(\xi_t^{(N)})_{0\leq t\leq T}] &\leq \sum_{p\geq 1}\mathbb{P}[Z_T=p,\,\exists\, 1\leq q\leq p,\, I_q\in\{1,\ldots,n\}\text{ and } J_q\in A_q]\\
&= \sum_{p\geq 1}\exp(Nu_TT)\frac{(Nu_TT)^p}{p!}\frac{n}{N^2}\sum_{q=1}^{p}\mathbb{E}[\mathrm{card}(A_q)]
\end{aligned}$$

Here $\mathbb{P}$ denotes the underlying probability and not the law of the interacting particle system. This leads us to consider the sequence $(B_p)_{p\geq 0}\stackrel{\text{def.}}{=}(\mathrm{card}(A_p))_{p\geq 0}$. It is quite clear that it is an increasing inhomogeneous Markov chain taking values in $\{n,\ldots,N\}$, whose probabilities of transition are given by

$$\forall\, p\geq 0,\forall\, n\leq k,l\leq N,\qquad \mathbb{P}[B_{p+1}=l|B_p=k] \;=\; \begin{cases}\frac{(N-k)k}{N^2} & ,\text{ if } l=k+1\\ 1-\frac{(N-k)k}{N^2} & ,\text{ if } l=k\\ 0 & ,\text{ otherwise}\end{cases}$$

This yields that for $p\geq 0$,

$$\begin{aligned}
\mathbb{E}[B_{p+1}] &= \mathbb{E}[\mathbb{E}[B_{p+1}|B_p]]\\
&= \mathbb{E}\left[B_p+\frac{(N-B_p)B_p}{N^2}\right]\\
&\leq \left(1+\frac{1}{N}\right)\mathbb{E}[B_p]\leq\left(1+\frac{1}{N}\right)^{p+1}\mathbb{E}[B_0] \;=\; \left(1+\frac{1}{N}\right)^{p+1}n
\end{aligned}$$

and we deduce from this inequality that

$$\begin{aligned}
\mathbb{P}[(\breve{\xi}_t^{(N)})_{0\leq t\leq T}\neq(\xi_t^{(N)})_{0\leq t\leq T}] &\leq \sum_{p\geq 1}\exp(-Nu_TT)\frac{(Nu_TT)^p}{p!}\frac{n}{N^2}\sum_{q=1}^{p}\left(1+\frac{1}{N}\right)^q n\\
&= \left(1+\frac{1}{N}\right)\sum_{p\geq 1}\exp(-Nu_TT)\frac{(Nu_TT)^p}{p!}\frac{n^2}{N}\left(\left(1+\frac{1}{N}\right)^p-1\right)\\
&\leq 2\frac{n^2}{N}\left[\sum_{p\geq 0}\exp(-Nu_TT)\frac{(u_TT(N+1))^p}{p!}-1\right]\\
&= 2\frac{n^2}{N}(\exp(u_TT)-1)
\end{aligned}$$

which implies the upper bound of the proposition.

■

If we want to analyze the asymptotic behavior of $\mathbb{P}^{(N,\{1,\ldots,n\})}_{\eta_0,[0,T]}$, as n^2/N tends to zero, we need to understand the one of $\breve{\mathbb{P}}^{(N,\{1,\ldots,n\})}_{\eta_0,[0,T]}$. For that purpose, we note that under this probability, $(\breve{\xi}^{(N,i)}_t)_{n+1\leq i\leq N,\, 0\leq t\leq T}$ is a Markov process with same law as $(\xi^{(N-n,i)}_t)_{1\leq i\leq N-n,\, 0\leq t\leq T}$, if we replace U by $\frac{N-n}{N}U$ in the working-out of the latter. We will use this property to construct our second coupling in next section, via the estimation of theorem 1.2.

4.2 A second coupling

The aim of this section is to find a judicious way to couple $\breve{\mathbb{P}}^{(N,\{1,\ldots,n\})}_{\eta_0,[0,T]}$ with $\bar{\mathbb{P}}^{\otimes n}_{\eta_0,[0,T]}$. We begin with analyzing more precisely the structure of the former probability. As we noticed before, we can first construct $(\breve{\xi}^{(N,i)}_t)_{n+1\leq i\leq N,\, 0\leq t\leq T}$ because this process is Markovian by himself. Then, let us define for $0\leq t\leq T$, the random probability

$$\breve{\eta}^{(N,\{n+1,\ldots,N\})}_t \quad = \quad \frac{1}{N-n}\sum_{n+1\leq i\leq N}\delta_{\breve{\xi}^{(N,i)}_t}$$

It is quite standard to design a probability space $(\Omega,\mathcal{F},\mathbb{P})$ on which are defined random variables Z, $(T_i)_{i\geq 1}$ and $(Y_i)_{i\geq 1}$ satisfying the following properties: Z is distributed according to a Poisson law of parameter nu_TT, knowing that $Z=k\in\mathbb{N}$, $T_1<T_2<\cdots<T_k$ are the ordering of k independent and uniformly distributed random variables on $]0,T[$ and $T_p=T$ for $p>k$, finally knowing that $Z=k$ and that $(T_i)_{i\geq 1}=(t_i)_{i\geq 1}$, $(Y_i)_{1\leq i\leq k}$ is distributed on E^k according to the "integrated" law given by

$$\mathbb{E}[\breve{\eta}^{(N,\{n+1,\ldots,N\})}_{t_1}\otimes\breve{\eta}^{(N,\{n+1,\ldots,N\})}_{t_2}\otimes\cdots\otimes\breve{\eta}^{(N,\{n+1,\ldots,N\})}_{t_k}]$$

while we put $Y_p=\diamond\not\in E$, for $p>k$.

Next assuming that $Z=k$, $(T_i)_{i\geq 1}=(t_i)_{i\geq 1}$ and $(Y_i)_{i\geq 1}=(y_i)_{i\geq 1}$ have been sampled according to the previous distribution, we construct a path of $\mathbb{M}([0,T],E^n)$ in the way described below; we start by considering in addition the two following independent objects: a sequence $(V_i)_{i\geq 1}$ of independent random variables uniformly distributed on $[0,1]$ and $(\breve{\xi}^{(N,i)}_0)_{1\leq i\leq n}$ whose law on E^n is $\eta_0^{\otimes n}$. Knowing $(\breve{\xi}^{(N,i)}_0)_{1\leq i\leq n}$, we sample $(\breve{\xi}^{(n,i)}_t)_{1\leq i\leq n,\, 0\leq t\leq t_1}$ according to $\otimes_{1\leq i\leq n}\mathbb{P}_{0,\breve{\xi}^{(N,i)}_0}$ (at least its restriction to $\mathbb{M}([0,t_1],E^n)$). Then we choose $1\leq i_1\leq n$ uniformly and take for $0\leq t\leq t_1$ and $1\leq i\leq n$,

$$\breve{\xi}^{(n,i)}_t \quad = \quad \left\{\begin{array}{ll} Y_1 & \text{, if } i=i_1,\ t=t_1 \text{ and } V_1\leq\frac{N-n}{N}\ U(t_1,Y_1) \\ \breve{\xi}^{(n,i)}_t & \text{, otherwise}\end{array}\right.$$

Now we let $(\breve{\xi}^{(n)}_t)_{t\geq t_1}$ be distributed according to $\otimes_{1\leq i\leq n}\mathbb{P}_{t_1,\breve{\xi}^{(n,i)}_{t_1}}$, then at time t_2 we contingently proceed at the replacement of $\breve{\xi}^{(n,i_2)}_{t_2}$ by Y_2, where again $1\leq i_2\leq n$ is independently and uniformly chosen, and so on. In a formalized way (using the hypothesis (H2)), this construction leads to a kernel Q from $\mathbb{N}\times[0,T]^{\mathbb{N}}\times(E\sqcup\{\diamond\})^{\mathbb{N}}$ to $\mathbb{M}([0,T],E^n)$ such that

$$\breve{\mathbb{P}}^{(N,\{1,\ldots,n\})}_{\eta_0,[0,T]}(\cdot) \quad = \quad \mathbb{E}_{(\Omega,\mathcal{F})}[Q(Z,(T_i)_{i\geq 1},(Y_i)_{i\geq 1},\cdot)]$$

The interest of this representation is that if above we replace $(Y_i)_{i\geq 1}$ by a family of random variables $(\bar{Y}_i)_{i\geq 1}$ which satisfies that knowing that $Z=k$ and that $(T_i)_{i\geq 1}=(t_i)_{i\geq 1}$, $(\bar{Y}_i)_{1\leq i\leq k}$ is distributed on E^k according to $\eta_{t_1}\otimes\cdots\otimes\eta_{t_k}$, while $\bar{Y}_p=\diamond$ for $p>k$, then

$$\bar{\mathbb{P}}^{\otimes n}_{\eta_0,[0,T]} \quad = \quad \mathbb{E}_{(\Omega,\mathcal{F})}[Q(Z,(T_i)_{i\geq 1},(\bar{Y}_i)_{i\geq 1},\cdot)]$$

Thus, theorem 1.1 will be implied by the next result:

Proposition 4.3 *There exists a construction of the random variables* $(Z,(T_i)_{i\geq 1},(Y_i)_{i\geq 1},(\bar{Y}_i)_{i\geq 1})$ *with the above prescribed distribution such that*

$$\mathbb{P}[(Y_i)_{i\geq 1}\neq(\bar{Y}_i)_{i\geq 1}] \quad\leq\quad \frac{1}{2}\widetilde{\epsilon}(N,n)$$

where the l.h.s. is understood in the sense of section 2.3 and where the r.h.s. satisfies the condition

$$\limsup_{n^2/N\to 0}\frac{N}{n^2}\,\widetilde{\epsilon}(N,n) \quad\leq\quad (14+28u_TT[1+\exp(u_TT)])u_TT(u_TT+1)$$

Proof: We begin by constructing $(\bar{Y}_i)_{i\geq 1}$ independently from $(Y_i)_{i\geq 1}$, knowing Z and $(T_i)_{i\geq 1}$. According to formula (10), there exists a smarter coupling if we can show that for any mapping $f:\mathbb{N}\times[0,T]^{\mathbb{N}}\times(E\sqcup\{\diamond\})^{\mathbb{N}}\to[-1,1]$ which is measurable, we have

$$\mathbb{E}[f(Z,(T_i)_{i\geq 1},(Y_i)_{i\geq 1})]-\mathbb{E}[f(Z,(T_i)_{i\geq 1},(\bar{Y}_i)_{i\geq 1})] \quad\leq\quad \widetilde{\epsilon}(N,n)$$

Let us define for all $k\geq 1$ and $0<t_1<\cdots<t_k<T$, a function $f_{k,t_1,\ldots,t_k}$ on E^k by

$$f_{k,t_1,\ldots,t_k}(y_1,\ldots,y_k) \quad=\quad f(k,(t_1,\ldots,t_k,T,T,\ldots),(y_1,\ldots,y_k,\diamond,\diamond,\ldots))$$

The l.h.s. can also be written as

$$\begin{aligned}\exp(-nu_TT)\sum_{k\geq 1}\frac{(nu_TT)^k}{k\,!}\int_{]0,T[^k}\mathbb{I}_{t_1<t_2<\cdots<t_k}\Big(\mathbb{E}[\breve{\eta}_{t_1}^{(N)}\otimes\cdots\otimes\breve{\eta}_{t_k}^{(N)}(f_{k,t_1,\ldots,t_k})]\\ -\eta_{t_1}\otimes\cdots\otimes\eta_{t_k}(f_{k,t_1,\ldots,t_k})\Big)\,dt_1\cdots dt_k\\ \leq\quad \exp(-nu_TT)\sum_{k\geq 1}\frac{(nu_TT)^k}{k\,!}\epsilon\left(\frac{k^2}{N}\right)\end{aligned}$$

where $\epsilon\left(\frac{k^2}{N}\right)$ is the quantity appearing in theorem 1.2 (note that this constant is increasing in u_T so it was harmless to replace the latter by $\frac{N-n}{N}u_T$). So we can define $\widetilde{\epsilon}(N,n)$ as the above r.h.s. and let us verify that it satisfies the condition mentioned in the proposition. We divide the sum in two parts:

$$\begin{aligned}\widetilde{\epsilon}(N,n) &\quad=\quad \epsilon_1(N,n)+\epsilon_2(N,n)\\ &\quad\overset{\text{def.}}{=}\quad \exp(-nu_TT)\sum_{1\leq k\leq k_0(N,n)}\frac{(nu_TT)^k}{k\,!}\epsilon\left(\frac{k^2}{N}\right)\\ &\qquad\qquad +\exp(-nu_TT)\sum_{k\geq k_0(N,n)+1}\frac{(nu_TT)^k}{k\,!}\epsilon\left(\frac{k^2}{N}\right)\end{aligned}$$

where $k_0(N,n)\;=\;\min\left\{k\geq 2nu_TT\;:\;\exp(-nu_TT)\frac{(nu_TT)^k}{k!}\;\leq\;\frac{n^4}{N^2}\right\}$. This number admits the following interesting property, which comes from the very fast decreasing of $\frac{(nu_TT)^k}{k!}$ to zero for large k.

Lemma 4.4 *The quantity* $k_0^2(N,n)/N$ *goes to zero with* n^2/N*.*

Proof: In order to get this result, it is sufficient to see that for all $\alpha>0$, all $\mathbb{N}^*$-valued sequences $(N_p)_{p\geq 1}$ and $(n_p)_{p\geq 1}$ satisfying $\lim_{p\to\infty}N_p=\infty$ and $\lim_{p\to\infty}n_p^2/N_p=0$, if we take $k_p=\alpha\sqrt{N_p}$ for $p\geq 1$ (so $k_p\geq 2n_pu_TT$ for p large enough), then

$$\lim_{p\to\infty}\exp(-n_pu_TT)\frac{N_p^2(n_pu_TT)^{k_p}}{n_p^4(k_p!)} \quad=\quad 0$$

because this easily leads to a contradiction. Using a Sterling's expansion, this convergence follows at once.

■

Thus, we deduce the next estimate for $\epsilon_2(N,n)$: noting that for $k \geq 2u_T Tn$,

$$\frac{(nu_T T)^{k+1}}{(k+1)!} \leq \frac{1}{2}\frac{(nu_T T)^k}{k!}$$

we get the bound

$$\epsilon_2(N,n) \leq \exp(-nu_T T)\frac{(nu_T T)^{k_0(N,n)}}{k_0(N,n)!}\sum_{p\geq 0}\frac{1}{2^p} = 2\exp(-nu_T T)\frac{(nu_T T)^{k_0(N,n)}}{k_0(N,n)!}$$

This implies that

$$\epsilon_2(N,n) \leq \sqrt{2}\left(\frac{n^2}{N}\right) \quad \text{and} \quad \lim_{n^2/N\to 0}\frac{N}{n^2}\epsilon_2(N,n) = 0$$

We now consider $\epsilon_1(N,n)$: let $\alpha > 0$ be given, according to theorem 1.2 and lemma 4.4, we can find $\beta > 0$ such that for all n and N verifying $n^2/N \leq \beta$, the quantity $k_0^2(N,n)/N$ is small enough to ensure that for all $1 \leq k \leq k_0(N,n)$,

$$\epsilon\left(\frac{k^2}{N}\right) \leq (1+\alpha)(14+28u_T T[1+\exp(u_T T)])\frac{k^2}{N}$$

Then it appears that for such n and N,

$$\begin{aligned}
\epsilon_1(N,n) &\leq (1+\alpha)(14+28u_T T[1+\exp(u_T T)])\exp(-nu_T T)\sum_{1\leq k\leq k_0(N,n)}\frac{(nu_T T)^k}{k\,!}\frac{k^2}{N} \\
&\leq (1+\alpha)(14+28u_T T[1+\exp(u_T T)])\exp(-nu_T T)\sum_{k\geq 1}\frac{(nu_T T)^k}{k\,!}\frac{1}{N}(k(k-1)+k) \\
&\leq (1+\alpha)(14+28u_T T[1+\exp(u_T T)])\frac{u_T Tn(u_T Tn+1)}{N} \\
&\leq (1+\alpha)(14+28u_T T[1+\exp(u_T T)])u_T T(u_T T+1)\frac{n^2}{N}
\end{aligned}$$

and the expected behavior of $\epsilon(N,n)$ follows.

5 Path spaces

The main purpose of this part is to motivate the abstract considerations of subsection 2.1 by presenting an interesting consequence for the so-called genealogical/historical processes associated to the particle systems. As we will indicate it, this application is strongly related to the practical smoothing problem in nonlinear filtering. The point is that we will now take advantage of our general setting in order to consider path sets for state spaces. This situation would have been especially uneasy to deal with in the usual setting of pregenerators defined on algebras (cf. for instance [6]), even if one could keep a Polish state space assumption, via the standard (but not trivial in contrast with what follows) use of the Skorokhod topology.

We begin by looking at the "new" object we want to numerically approximate: by analogy with formula (1) of the introduction, we define for any $T \geq 0$ a probability $\eta_{[0,T]}$ on $\mathbb{M}([0,T],E)$ by the formulae

$$\eta_{[0,T]}(\varphi) \stackrel{\text{def.}}{=} \frac{\mathbb{E}_{\eta_0}\left[\varphi((X_t)_{0\leq t\leq T})\exp\left(\int_0^T U_s(X_s)\,ds\right)\right]}{\mathbb{E}_{\eta_0}\left[\exp\left(\int_0^T U_s(X_s)\,ds\right)\right]} \tag{17}$$

valid for all bounded and measurable test functions $\varphi : \mathbb{M}([0,T],E) \to \mathbb{R}$.

One of the interest of these measures is that they are the theoretical solutions to some basic problems in nonlinear filtering. Without entering into the details, let us give a few heuristics about this subject: assume that a signal $(S_t)_{t\geq 0}$ taking values in E is only seen through an observation process $(Y_t)_{t\geq 0}$ (in a nonlinear and noisy way). Then under some hypotheses on the evolution of the Markovian couple $(S_t, Y_t)_{t\geq 0}$ and via some changes of probabilities, one can obtain for any given time $t \geq 0$, a representation of the law of S_t knowing $(Y_s)_{0\leq s\leq t}$ in the form of (1), where at any instant $s \geq 0$, the generator of X and the function U_s depend in fact on Y_s (see for instance [6], a more abstract characterization of the general nonlinear problems whose solutions can be written as quotients of Feynman-Kac integrals should be the object of a forthcoming article). This is the classical nonlinear filtering question. Now if we are interested in law of the whole $(S_t)_{0\leq t\leq T}$ knowing $(Y_t)_{0\leq t\leq T}$, it can be expressed as (17). Then one can deduce for instance the law of X_0 knowing $(Y_t)_{0\leq t\leq T}$ and this is a particularly important example of smoothing problem: after some observations, to estimate from where the signal has started (i.e. its conditional distribution).

Nevertheless, one can think of other justifications for (17), as it is also possible to treat cases where $U_s(X_s)$ is replaced by $U_s((X_u)_{0\leq u\leq s})$ under some measurability assumptions.

Indeed the basic principle is to consider $((X_s)_{0\leq s\leq t})_{t\geq 0}$ as a Markov process whose state space consists of paths. This idea is very old in the theory of stochastic processes, but we are now able to use it in order to device natural "particle" algorithms approximating (17) for which we get the relatively explicit and general bounds presented in the previous sections. More precisely, for $t \geq 0$, the measure $\eta_{[0,t]}$ will again be approximated by empirical distributions

$$\eta^{(N)}_{[0,t]} \stackrel{\text{def.}}{=} \frac{1}{N} \sum_{1\leq i\leq N} \delta_{(\breve{\xi}^{(N,i)}_t(s))_{0\leq s\leq t}}$$

where for $1 \leq i \leq N$, the path $(\breve{\xi}^{(N,i)}_t(s))_{0\leq s\leq t}$ belongs to $\mathbb{M}([0,t],E)$. The whole evolution $\mathbb{R}_+ \ni t \mapsto (\breve{\xi}^{(N,i)}_t)_{1\leq i\leq N}$ is called the genealogical process associated to our previous particle system. Heuristically it is constructed in the following way: between selection times, the paths are prolonged according to the underlying Markov process and at a selection time, say when in our algorithm above the i^{th} particle was replaced by the j^{th} one, now all the trajectory associated to the i^{th} particle is replaced by that of the j^{th} one.

But in order to be more precise in this direction, we have to verify that our setting is in some sense "stable" when we go from points to trajectories. Thus let us develop the corresponding preliminaries.

As before, we start from a measurable space $(E, \mathcal{E})$ and a given set of paths $\mathbb{M}(\mathbb{R}_+, E)$ satisfying the condition (H1). As new state space, we consider $\bar{E} = E \times \mathbb{M}(\mathbb{R}_+, E)$ endowed with its natural coordinates $(Y, (X_t)_{t\geq 0})$ and the σ-field they generate.

If $0 \leq s \leq t$ and $\omega, w \in \mathbb{M}(\mathbb{R}_+, E)$ are given, we define a new path $I_{s,t}(\omega, w)$ belonging to $\mathbb{M}(\mathbb{R}_+, E)$ by

$$\forall\ u \geq 0, \qquad X_u(I_{s,t}(\omega,w)) \;=\; \begin{cases} X_u(\omega) & \text{, if } 0 \leq u < s \text{ or } u \geq t \\ X_u(w) & \text{, otherwise} \end{cases}$$

We also introduce the following related object: for $t \geq 0$, $\omega \in \mathbb{M}(\mathbb{R}_+, E)$ and $w \in \mathbb{M}([t, +\infty[, E)$, $W_t(\omega, w)$ is the path of $\mathbf{M}([t, +\infty[, \bar{E})$ such that

$$\forall\ s \geq t, \qquad \bar{X}_s(W_t(\omega,w)) \;=\; (X_s(w), I_{t,s}(\omega,w))$$

where $(\bar{X}_s)_{s\geq 0}$ will denote the canonical coordinate process on $\mathbf{M}(\mathbb{R}_+, \bar{E})$.

This kind of trajectories will in some sense be generating: $\mathbb{M}(\mathbb{R}_+, \bar{E})$ will stand for the subset of $\mathbf{M}(\mathbb{R}_+, \bar{E})$ obtained by stabilization of the set of trajectories $\{I_0(\omega, w) : \omega, w \in$

$\mathbb{M}(\mathbb{R}_+, E)\}$ with respect to the operation described in the first point of (H1), i.e. it consists of the paths $W \in \mathbf{M}(\mathbb{R}_+, \bar{E})$ for which there exist an increasing sequence $(t_i)_{i\geq 1}$ of positive reals satisfying $\lim_{i\to\infty} t_i = +\infty$, and a sequence $(\omega_i, w_i)_{i\geq 0}$ of elements of $(\mathbb{M}(\mathbb{R}_+, E))^2$ such that

$$\forall\ i \geq 0,\ \forall\ t_i \leq s < t_{i+1}, \qquad \bar{X}_s(W) \;=\; \bar{X}_s(W_0(\omega_i, w_i))$$

(where the traditional convention $t_0 = 0$ is enforced).

Remark that the assumption that $\mathbb{M}(\mathbb{R}_+, E)$ should contains all constant paths is not very natural, because under our construction it would not have been preserved at the $\bar{E}$-level.

Lemma 5.1 *The set of paths $\mathbb{M}(\mathbb{R}_+, \bar{E})$ satisfies the required condition (H1).*

Proof: The first point of this hypothesis is quite immediate, so it is sufficient to look at the second one. Let $\mathbf{Y}$ be the mapping defined by

$$\mathbb{M}(\mathbb{R}_+, \bar{E}) \ni W \mapsto (Y(\bar{X}_t(W)))_{t\geq 0}$$

it appears that in fact it takes values in $\mathbb{M}(\mathbb{R}_+, E)$ and is $\mathcal{M}(\mathbb{R}_+, E)/\mathcal{M}(\mathbb{R}_+, \bar{E})$-measurable. Thus we obtain that

$$\mathbb{R}_+ \times \mathbb{M}(\mathbb{R}_+, \bar{E}) \ni (t, W) \;\;\mapsto\;\; Y(\bar{X}_t(W)) \in E$$

is $\mathcal{R}_+ \otimes \mathcal{M}(\mathbb{R}_+, \bar{E})$-measurable, because it can be decomposed into

$$\mathbb{R}_+ \times \mathbb{M}(\mathbb{R}_+, \bar{E}) \ni (t, W) \;\;\mapsto\;\; (t, \mathbf{Y}(W)) \;\;\mapsto\;\; X_t(\mathbf{Y}(W))$$

Besides, for $s \geq 0$ and $W \in \mathbb{M}(\mathbb{R}_+, \bar{E})$ fixed, we have that the mapping

$$\mathbb{R}_+ \ni t \;\;\mapsto\;\; X_s(\bar{X}_t(W)) \in E$$

is piecewise constant and the corresponding intervals are closed at the left end and open at the right end (i.e. this path is càdlàg if one puts on E the total topology generated by the singletons). So it makes it clear that for $s \geq 0$ fixed, the mapping

$$\mathbb{R}_+ \times \mathbb{M}(\mathbb{R}_+, \bar{E}) \ni (t, W) \;\;\mapsto\;\; X_s(\bar{X}_t(W)) \in E$$

is $\mathcal{R}_+ \otimes \mathcal{M}(\mathbb{R}_+, \bar{E})$-measurable and by definition of $\mathcal{M}(\mathbb{R}_+, E)$, it follows that the same is true for

$$\mathbb{R}_+ \times \mathbb{M}(\mathbb{R}_+, \bar{E}) \ni (t, W) \;\;\mapsto\;\; (X_s(\bar{X}_t(W)))_{s\geq 0} \in \mathbb{M}(\mathbb{R}_+, E)$$

■

Now it is time to lift a given Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$ to the $\bar{E}$-level: so let $t \geq 0$ and $\bar{x} = (x, \omega) \in \bar{E}$ be given, we define the probability $\bar{\mathbb{P}}_{t,\bar{x}}$ on $\mathbb{M}([t, +\infty[, \bar{E})$ as the image of $\mathbb{P}_{t,x}$ under the mapping

$$\mathbb{M}([t, +\infty[, E) \ni w \;\;\mapsto\;\; W_t(\omega, w)$$

Via extensive use of monotonous class theorem, there is no difficulty in verifying that $(\bar{\mathbb{P}}_{t,\bar{x}})_{t\geq 0,\, \bar{x}\in\bar{E}}$ is indeed a Markovian family satisfying (H2). The simplicity of this procedure underlines once again the advantage one has to work directly with laws and not with pregenerators (at least theoretically).

Thus we can apply all the results of the previous sections with the function $\mathbf{U}$ defined by

$$\forall\ (t, \bar{x}) \in \mathbb{R}_+ \times \bar{E}, \qquad \mathbf{U}(t, \bar{x}) \;=\; U(t, Y(\bar{x}))$$

In particular, let us describe the evolution of the associated N-particles system in this case: we denote by P_0 the image of $\mathbb{P}_{0,\eta_0}$ under the mapping $\mathbb{M}(\mathbb{R}_+, E) \ni \omega \mapsto (X_0(\omega), \omega) \in \bar{E}$. Then we sample $(X_0^{(N,1)}, \omega_0^{(N,1)})$, ..., $(X_0^{(N,N)}, \omega_0^{(N,N)})$ independently according to P_0.

To simplify the presentation, we fix a horizon $T > 0$, and we only consider the time interval $[0, T]$. So let $(T_i)_{i\geq 1}$ be a sequence of $\mathbb{R}_+^*$ valued random variables such that the $T_i - T_{i-1}$, for $i \geq 1$ and with $T_0 = 0$, are independent and distributed according to exponential laws of parameter Nu_T. At any instant $0 \leq t < T \wedge T_1$, the particle system is given by

$$\forall\ 1 \leq i \leq N, \qquad \bar{\xi}_t^{(N,i)} \;=\; (X_t(\omega_0^{(N,i)}), \omega_0^{(N,i)})$$

.

At time T_1, we choose two indices $1 \leq I_1, J_1 \leq N$, in a equidistributed way for I_1 and according to the probability

$$\frac{1}{Nu_T} \sum_{1\leq j\leq N} U(T_1, X_{T_1}(\omega_0^{(N,j)})\delta_j$$

for J_1. Let also V_1 be uniformly distributed on $[0, 1]$. Then if $T_1 \leq T$, the particle system at this time T_1 is

$$\forall\ 1 \leq i \leq N, \qquad \bar{\xi}_{T_1}^{(N,i)} \;=\; \begin{cases} (X_{T_1}(\omega_0^{(N,J_1)}), \omega_0^{(N,J_1)}) & \text{, if } i = I_1 \text{ and } V_1 \leq U(T_1, X_{T_1}(\omega_0^{(N,j)}))/u_T \\ (X_{T_1}(\omega_0^{(N,i)}), \omega_0^{(N,i)}) & \text{,otherwise} \end{cases}$$

The next step consists in sampling $(\omega_1^{(N,1)}, \ldots, \omega_1^{(N,N)})$ according to

$$\mathbb{P}_{T_1, Y(\bar{\xi}_{T_1}^{(N,1)})} \otimes \cdots \otimes \mathbb{P}_{T_1, Y(\bar{\xi}_{T_1}^{(N,N)})}$$

and then at any instant $T \wedge T_1 \leq t < T \wedge T_2$ and for any index $1 \leq i \leq N$, we put the i^{th} particle at the "position"

$$\bar{\xi}_t^{(N)} \;\stackrel{\text{def.}}{=}\; (X_t(\omega_1^{(N,i)}), I_{T_1,t}((X_s(\bar{\xi}_{T_1}^{(N,i)}))_{s\geq 0}, \omega_1^{(N,i)})) \;\in\; \bar{E}$$

and so on, in a Poissonian random number of steps we end up with $(\bar{\xi}_t^{(N)})_{0\leq t\leq T}$.

In order to recover a more usual object, let us denote for $t \geq 0$, $\breve{\xi}_t^{(N)}$ the path of $\mathbb{M}([0, t], E^N)$ defined by

$$\forall\ 1 \leq i \leq N, \forall\ 0 \leq s \leq t, \qquad \breve{\xi}_t^{(N,i)}(s) \;=\; \begin{cases} (X_s(\bar{\xi}_t^{(N,i)}))_{1\leq i\leq N} & \text{, if } s < t \\ (Y(\bar{\xi}_t^{(N,i)}))_{1\leq i\leq N} & \text{, if } s = t \end{cases}$$

It appears that $(\breve{\xi}_t^{(N)}(t))_{t\geq 0}$ has the same law as our previous algorithm $\xi^{(N)}$ (or equivalently, the N-product version of the mapping $\mathbf{Y}$ defined in the proof of lemma 5.1 could enable us to recover $\xi^{(N)}$ from $\bar{\xi}^{(N)}$, and consequently the results on E from their $\bar{E}$-counterparts), but furthermore $(\breve{\xi}_t^{(N)})_{t\geq 0}$ gives its genealogy, in the sense that for $0 \leq s \leq t$ and $1 \leq i \leq N$, $\breve{\xi}_t^{(N,i)}(s)$ is the "ancestor" of $\breve{\xi}_t^{(N,i)}(t)$ at time s, that is why $\breve{\xi}^{(N)}$ is sometimes called the historical process associated to the particle system $\xi^{(N)}$. We have not been able to use it directly in our definitions above, because rigorously its state space is varying with time, peculiarity which is not allowed in our setting (one can try to develop such a theory, but this leads to more far-fetched considerations than the a priori strange introduction of $\bar{E}$, one of the main difficulties comes from the initial parametrization property in our definition of Markovian families which has an innocent touch at first glance but is especially important in the proof of proposition 3.4).

Thus we are lead to consider for $T \geq 0$,

$$\eta_{[0,T]}^{(N)} \quad \stackrel{\text{def.}}{=} \quad \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\breve{\xi}_T^{(N,i)}} \ \in \ \mathcal{P}(\mathbb{M}([0,T],E))$$

since it is a good estimator of $\eta_{[0,T]}$: there exists a constant $C_T \geq 0$ such that for any $\varphi \in \mathcal{B}_{\rm b}(\mathbb{M}([0,T],E))$, we are assured of

$$\left|\mathbb{E}[\eta_{[0,T]}^{(N)}(\varphi)] - \eta_{[0,T]}(\varphi)\right| \quad \leq \quad C_T \frac{\|\varphi\|}{N} \tag{18}$$

or alternatively

$$\mathbb{E}\left[\left|\eta_{[0,T]}^{(N)}(\varphi) - \eta_{[0,T]}(\varphi)\right|\right] \quad \leq \quad C_T \frac{\|\varphi\|}{\sqrt{N}} \tag{19}$$

Up to our knowledge, these estimations are the first ones in that direction.

In particular, if we are only interested in the smoothing problem mentioned before (i.e. we are only considering mapping φ of the form $\varphi = \psi \circ X_0$, with $\psi \in \mathcal{B}_{\rm b}(E)$) it appears that we should rather look at the approximating empirical probabilities

$$\eta_{0,T}^{(N)} \quad \stackrel{\text{def.}}{=} \quad \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\breve{\xi}_T^{(N,i)}(0)} \ \in \ \mathcal{P}(E)$$

only putting mass on the initial particles $\breve{\xi}_0^{(N,i)}(0)$, for $1 \leq i \leq N$, which can be identified with the $\xi_0^{(N,i)}$.

More precisely, let us notice that the process $((\breve{\xi}_t^{(N,i)}(0), \breve{\xi}_t^{(N,i)}(t))_{1 \leq i \leq N})_{t \geq 0}$ taking values in $(E \times E)^N$ is indeed Markovian and can be constructed in a way similar to the one above, so it is not necessary to keep track of the whole process $(\breve{\xi}_t^{(N)})_{t \geq 0}$, which would ask for too much memory if we wanted to implement the previous algorithm as practical code on a computer.

We also recall that our estimates are good asymptotically as the number N of particles is very large, but we are not saying anything about the behavior for long time $T \geq 0$. In fact, if $N \geq 1$ is fixed and if the cost function is bounded away from zero (ie there exists $\alpha > 0$ such that for all $t \geq 0$ and $x \in E$, $U(t,x) \geq \alpha$), then for large T, the probability $\eta_{0,T}^{(N)}$ is a.s. converging to a Dirac measure (this corresponds to the fact that asymptotically in time there is an unique initial ancestor, because of too much selection procedures), i.e. we are choosing only one of the initial particles as an estimator of the distribution $\eta_{0,T}$ defined by

$$\mathcal{E} \ni A \quad \mapsto \quad \frac{\mathbb{E}_{\eta_0}\left[\mathbf{1}_A(X_0) \exp\left(\int_0^t U_s(X_s)\, ds\right)\right]}{\mathbb{E}_{\eta_0}\left[\exp\left(\int_0^t U_s(X_s)\, ds\right)\right]}$$

which may not be a smart choice: for instance consider the case where $E = \{-1, 1\}$, X is the nonmoving Markov process, $\eta_0 = (\delta_{-1} + \delta_1)/2$, $U \equiv 1$ and $\varphi = \mathrm{id}$, then we have

$$\sup_{T \geq 0} \mathbb{E}\left[\left|\eta_0^{(N)}(\varphi) - \eta_{0,T}(\varphi)\right|\right] \leq \frac{1}{\sqrt{N}} \quad \text{whereas} \quad \lim_{T \to +\infty} \mathbb{E}\left[\left|\eta_{0,T}^{(N)}(\varphi) - \eta_{0,T}(\varphi)\right|\right] = 1$$

This is also the occasion for us to mention that while representation (1) does not uniquely determine U (for instance one can add to it a locally bounded, measurable and nonnegative function depending only on time without changing the flow $(\eta_t)_{t \geq 0}$), it is always in our interest to work with the smaller one possible, either for the theoretical bounds or for the number of selection procedures needed algorithmically. In the trivial example above, this corresponds to the choice of $U \equiv 0$, for which $\eta_{0,T}^{(N)} = \eta_0^{(N)}$ for any $T \geq 0$.

Remark 5.2: Bounds (18) and (19) merely express quantitative weak propagation of chaos in $\mathbb{L}^1$ and $\mathbb{L}^2$. For them it is not necessary to go through our whole development, because the only ingredient needed is the estimate of theorem 1.2 with $n = 2$ and $t_1 = t_2 = T$, result which can be obtained quite directly through proposition 3.3 (let us recall that the main difficulty of section 3 was the n^2/N-dependence).

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