Some theoretical aspects of Particle Filters and Ensemble Kalman Filters

P. Del Moral

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  and much more since

  Filtering/Bayes' rule $\subset$ **Feynman-Kac sg** $\supset$ ground states Schrödinger sg, rare events, molecular chemistry, polymers,...

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1. **Find some possibly nonlinear process**

$$\overline{X}_t \sim \mathrm{Law}(X_t \mid \mathbf{Y_0}, \ldots, \mathbf{Y_t})$$

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1. **Find some possibly nonlinear process**

$$\overline{X}_t \sim \mathrm{Law}(X_t \mid \mathbf{Y_0}, \ldots, \mathbf{Y_t}) \quad \textbf{(Time varying} \oplus \textbf{Possibly/Often unstable...)}$$

2. **Then apply mean field particle methodology**

Say N = precision (nb of samples/particles, time steps $\Delta t = 1/N, \ldots$)

Maths literature abounds with fancy bounds/theo of the type:

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Impossible to run such particle algorithm

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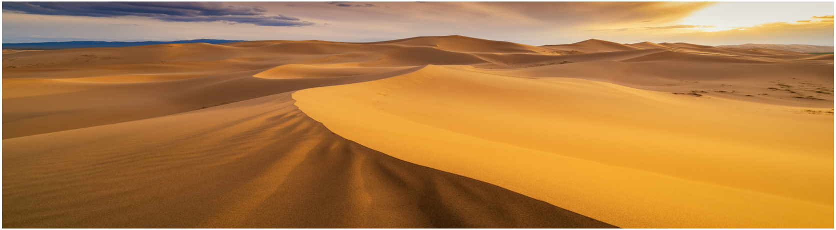
BUT FOR $t \leqslant 6 \Longrightarrow$
Eventually use all sand grains on earth



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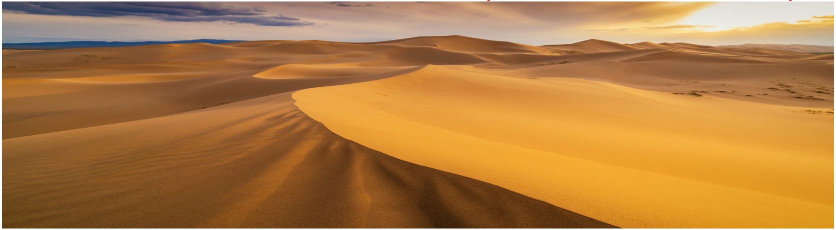
BUT FOR $t \leqslant 6 \Longrightarrow$
Eventually use all sand grains on earth (some care with N^2-costs)



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- Feynman-Kac/particle filters (+ Miclo Sem Proba 2000)+...
- $\supset$ **New approach: stochastic perturbation $\sim$ stability limiting process**

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- Lyapunov sg. & illustrations + Arnaudon, Ouhabaz SAA 23 RECOMMENDED
- Stability positive sg. + Horton AAP23

Brief review on stoch. perturbation

$$\text{Sample mean} \quad m_t := \frac{1}{N} \sum_{1 \leqslant i \leqslant N} X_t^i \quad \text{with } \textbf{iid} \text{ copies } X_t^i \text{ of } X_t$$

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Measure valued sg/Mean field particle sampler $(s \leqslant t)$

$$\eta_t = \Phi_{s,t}(\eta_s) \overset{N\to\infty}{\simeq} \Phi^{\mathbf{N}}_{s,t}(m_s) := m_t := \frac{1}{N}\sum_{1\leqslant i\leqslant N} \delta_{\xi^i_t}$$

Discrete time $\rightsquigarrow$ nonlinear Markov transitions $K_{t,\eta}$

$$\left\{\begin{array}{rcl} \eta_t & = & \Phi_{t-1,t}\left(\eta_{t-1}\right) \\ \\ m_t & = & \Phi_{t-1,t}\left(m_{t-1}\right) + \frac{1}{\sqrt{N}}\, M_t \end{array}\right.$$

Continuous time $\rightsquigarrow$ generators $L_{t,\eta}$

$$\left\{\begin{array}{rcl} d\eta_t & = & \eta_t L_{t,\eta_t}\, dt \\ \\ dm_t & = & m_t L_{t,m_t}\, dt + \frac{1}{\sqrt{N}}\, dM_t \end{array}\right.$$

Propagation of local perturbations/Stochastic interpolation

$$\Phi^N_{0,t}(m_0) - \Phi_{0,t}(m_0) = \int_0^t \, d_s \left(\Phi_{s,t} \circ \Phi^N_{0,s}\right)(m_0) = \int_0^t \, d_s \Phi_{s,t}(m_s)$$
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$\rightsquigarrow$ **Stochastic Alekseev-Gröbner interpolation**

Discrete time (+Guionnet CRAS+IHP 1998/99) (+Miclo Sem Prob. 2000)

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Nice but also particle blocks/prop.chaos, LDP,CLT,. . . ,

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- **Path space/Feynman-Kac/Smoothing/conditional paths/. . .**

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- **Empirical proc. (+Ledoux JTP-2000,+ Miclo Sem Proba 2000. . .)**

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- **More interesting $\rightsquigarrow$ Uniform exponential concentration**
  (+Rio AAP-2011, +Wu (FTML-2011), +Hu-Wu, Stat Sinica-2015**)**, Mean field Simulation CRC -2013,...
  Some examples of events with probability $\geqslant 1 - e^{-x}$

$$\sup_{f\in\mathcal{F}} |m_t(f) - \eta_t(f)| \leqslant \frac{c(\mathcal{F})}{\sqrt{N}} \sqrt{1+x} \quad \text{and} \quad \sup_{0\leqslant s\leqslant t} \sup_{f\in\mathcal{F}} |m_s(f) - \eta_s(f)| \leqslant \frac{c(\mathcal{F})}{\sqrt{N}} \sqrt{(1+x)(1+\log(t))}$$

Stochastic perturbation $\sim$ stability limiting process

More recent applications to Measure-valued proc. & diffusions

- Stoch. Riccati matrix diff + Bishop-Niclas IHP-20
- Interacting jumps + Arnaudon EJP-20.
- Interacting diffusions + Arnaudon AAP-20 + SAA-19.
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⇝ **Linear-Gaussian world - possibly unstable/transient "signals"**

- Stability Kalman-Bucy filters + Bishop SIAM-17.
- Inflation/localisation + Bishop, S. Pathiraja SPA-17.
- Harmonic Oscillator ($Y = 0$)+ Horton CIMP-23/Arxiv21.

KEY OBS:
Innovation/Weights/Penalties/likelihoods... stabilizing effects !!

Uniform estimates for EnKF continuous time models

For stable signals

- AAP-18/Arxiv-16 (Unif. EnKBF)+ Tugaut.
- SIAM-17/Arxiv-16 (Unif. Extended EnKBF)+ Kurtzmann, Tugaut.
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Ensemble KB filters for possibly unstable/transient signals

- AAP-19 (1d-case)+ Bishop, Kamatani, Rémillard.
- IHP-19 (Perturbation Stoch. Riccati)+ Bishop, A. Niclas.
- EJP-19 (Stability Stoch. Riccati)+ Bishop.
- $\rightsquigarrow$ MCSS-23 Review article (+ Bishop, Arxiv 20)

Continuous time **Linear+Gaussian filtering problem**

$$\begin{cases} dX_t &= A\ X_t\ dt\ +\ R^{1/2}\ dW_t\ \in \mathbb{R}^r \\ dY_t &= C\ X_t\ dt\ +\ \Sigma^{1/2}\ dV_t \qquad \rightsquigarrow \mathcal{Y}_t := \sigma(Y_s,\ s \leqslant t). \end{cases}$$

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Optimal $\mathbb{L}_2$-estimate = Kalman-Bucy filter

$$\widehat{X}_t := \mathbb{E}(X_t \mid \mathcal{Y}_t) \quad \text{and} \quad P_t := \mathbb{E}\left(\left(X_t - \widehat{X}_t\right)\left(X_t - \widehat{X}_t\right)'\right)$$

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with the gain given by the matrix Riccati equation

$$\partial_t P_t = \mathrm{Ricc}(P_t) := AP_t + P_tA' - P_t\mathbf{S}P_t + R \quad \text{with} \quad \mathbf{S} := \mathbf{C}'\mathbf{\Sigma}^{-1}\mathbf{C}$$

Reformulation $\rightsquigarrow$ Nonlinear Kalman-Bucy diffusion

$\Longleftrightarrow$ **McKean-Vlasov type diffusions $\overline{X}_t$ such that (given $\mathcal{Y}_t$)**

$$\eta_t := \mathrm{Law}(\overline{X}_t) = \mathcal{N}\left[\widehat{X}_t, P_t\right]$$

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$$\eta_t := \mathrm{Law}(\overline{X}_t) = \mathcal{N}\left[\widehat{X}_t, P_t\right]$$ **May be highly unstable !**

$\rightsquigarrow$ **Interacting with their conditional mean and covariance matrices**

$$\mathcal{P}_{\eta_t} = \eta_t\left[(e - \eta_t(e))(e - \eta_t(e))'\right] \quad \text{with} \quad e(x) := x.$$

2 classes of McKean-Vlasov type diffusions

1) "Vanilla EnKF" ($\rightsquigarrow$ (corrected) discrete time - Evensen 94)

$$d\overline{X}_t = A\,\overline{X}_t dt + R^{1/2}\,d\overline{W}_t + \mathcal{P}_{\eta_t} C'\Sigma^{-1}\left[dY_t - \left(C\,\overline{X}_t dt + \Sigma^{1/2}\,d\overline{V}_t\right)\right]$$

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Nonlinear models:

Tempting to replace "A x" and "C x" by $\mathbf{A}(\mathbf{x}), \mathbf{C}(\mathbf{x})$ (often done)

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BUT NOT CONSISTENT WITH THE OPTIMAL FILTER

Example: Extended Kalman-Bucy Ensemble filter $(A(x))$

$$\begin{cases} d\widehat{X}_t = A(\widehat{X}_t)\ dt + P_t C' \Sigma^{-1} \left(dY_t - C\widehat{X}_t\ dt\right) \\ \partial_t P_t = \partial A(\widehat{X}_t)\ P_t + P_t\ \partial A(\widehat{X}_t)' - P_t S P_t + R \end{cases}$$

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The Ensemble Kalman-Bucy filter

(Ex. Case 1) Mean field sampler $\rightsquigarrow$ $N+1$ interacting diffusions

$$d\xi_t^i = A\,\xi_t^i dt + R^{1/2} d\overline{W}_t^i + p_t C' \Sigma^{-1} \left[dY_t - \left(C\xi_t^i dt + \Sigma^{1/2}\, d\overline{V}_t^i \right) \right]$$

with the rescaled particle covariance matrices

$$p_t := \frac{1}{N} \sum_{1 \leqslant i \leqslant N+1} \left(\xi_t^i - m_t\right) \left(\xi_t^i - m_t\right)'$$

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where are the Kalman-Bucy filter and the Riccati equations ?

Perturbation theorem(s):

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\begin{aligned}
dm_t &= A\ m_t\ dt + p_t\ C'\Sigma^{-1}\ (dY_t - Cm_t\ dt) + \frac{1}{\sqrt{N+1}}\ dM_t \\
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  Vanilla EnKF with $S > 0$ p_t **heavy tailed invariant measure**.
  Deterministic EnKF p_t **Gaussian tailed invariant measure**.
- **Stoch. perturbation** $\Rightarrow$ **time-uniform estimates** $\forall(A, R, S)$.

In terms of random matrices with $\epsilon := \frac{2}{\sqrt{N}}$

$$\mathcal{W}_t = (\mathcal{W}_t(i,j))_{1 \leqslant i,j \leqslant r} \quad \text{independent Brownian motions}$$

$$\downarrow$$

$$dp_t \overset{law}{=} \left[Ap_t + p_t A' + R - p_t S p_t\right] dt + \epsilon \left(p_t^{1/2} d\mathcal{W}_t \left(U + p_t V p_t\right)^{1/2}\right)_{sym}$$

- $(U, V) \overset{case}{=}{}^{1} (R, S)$ and $(U, V) \overset{case}{=}{}^{2} (R, 0)$.
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$p_t \rightsquigarrow$ *non colliding eigenvalues* $\quad \lambda_r(t) < \ldots < \lambda_2(t) < \lambda_1(t)$

$$d\lambda_i(t) = \left[2\alpha\lambda_i(t) + \beta - \lambda_i(t)^2\gamma + \frac{\epsilon^2}{4}\sum_{j\neq i}\frac{\lambda_i(t)+\lambda_j(t)}{\lambda_i(t)-\lambda_j(t)}\right] dt + \epsilon\,\sqrt{\lambda_i(t)}\; dW_t^i$$

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This talk = answer these 2 questions for 1D linear/Gaussian

- `1d-discrete time/EnKF (+ Horton, Arxiv 21, AAP 23)`
- $\rightsquigarrow$ `EnKF Review article (+ Bishop, Arxiv 20, MCSS 23)`

Linear+Gaussian+discrete 1d-filtering problem

$$\left\{\begin{array}{rclcrcl} X_{n+1} & = & A\,X_n + B\,W_{n+1} & \quad & X_0 & \sim & \mathcal{N}(\widehat{X}_0^-, P_0) \\ Y_n & = & C\,X_n + D\,V_n & & n & \in & \mathbb{N} := \{0, 1, 2\ldots\} \end{array}\right.$$

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$$\Downarrow \quad \mathcal{Y}_n := (Y_0, \ldots, Y_n)$$

One-step predictor & Optimal filter = Gaussian

$$\mathrm{Law}(X_n \mid \mathcal{Y}_{n-1}) = \mathcal{N}(\widehat{X}_n^-, P_n) \quad \& \quad \mathrm{Law}(X_n \mid \mathcal{Y}_n) = \mathcal{N}(\widehat{X}_n, \widehat{P}_n)$$

Linear+Gaussian+discrete $1d$-*filtering problem*

$$\begin{cases} X_{n+1} &= A\,X_n + B\,W_{n+1} \qquad X_0 \sim \mathcal{N}(\widehat{X}_0^-, P_0) \\ Y_n &= C\,X_n + D\,V_n \qquad n \in \mathbb{N} := \{0, 1, 2\ldots\} \end{cases}$$

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$\rightsquigarrow$ **Kalman filter (1960s') = Gauss-Legendre regression (1800s')**

$$(\widehat{X}_n^-, P_n) \xrightarrow{\textit{updating}} (\widehat{X}_n, \widehat{P}_n) \xrightarrow{\textit{prediction}} (\widehat{X}_{n+1}^-, P_{n+1})$$

Particle filters = GA = SMC = DMC = ...

$$\left(\xi_n^{i-}\right)_{1\leqslant i\leqslant N} \in \mathbb{R}^N \xrightarrow{\textbf{Selection}} \left(\boldsymbol{\xi_n^{j}}\right)_{\mathbf{1\leqslant j\leqslant N}} \xrightarrow{\textbf{Mutation}} \left(\boldsymbol{\xi_{n+1}^{i-}}\right)_{\mathbf{1\leqslant i\leqslant N}}$$

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Selection/ **(Vanilla)** **Mutation**:

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and for $|A| < 1$**?**

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and for $|A| < 1$? $\rightsquigarrow$ **Time-uniform estimates (new coming article)**

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and for $|A| < 1$?$\rightsquigarrow$ Time-uniform estimates (new coming article)
$\rightsquigarrow$ find stable twisted guiding mutations (up to change of weights/probab)

(Conditional) Mean equations

$$\left\{\begin{array}{rcl} \widehat{X}_n & = & \widehat{X}_n^- + Gain_n \left(Y_n - C\widehat{X}_n^-\right) \quad \text{with} \quad Gain_n := CP_n/(C^2 P_n + D^2) \\ \widehat{X}_{n+1}^- & = & A\,\widehat{X}_n \end{array}\right.$$

Offline Riccati equations

$$\left\{\begin{array}{rcl} \widehat{P}_n & = & (1 - G_n C)P_n = P_n/(1 + SP_n) \quad \text{with} \quad S := (C/D)^2 \\ \\ P_{n+1} & = & A^2 \widehat{P}_n + R \quad \text{with} \quad R = B^2 \end{array}\right.$$

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$$\rightsquigarrow P_{n+1} = \phi(P_n) := \frac{aP_n + b}{cP_n + d} \quad \text{with} \quad (a, b, c, d) := \left(A^2 + RS, R, S, 1\right)$$

Reminder:

P_n and $(\widehat{X}_n - X_n)$ are stable for any A (Kalman/Bucy-Stab Theory) !

Conditional-Nonlinear Markov chain ($\mathfrak{Perfect\ Sampler}$)

$$\left\{\begin{array}{rcl} \widehat{\mathfrak{X}}_n & = & \mathfrak{X}_n + \mathfrak{gain}_n\,(Y_n - (C\,\mathfrak{X}_n + D\,\mathfrak{V}_n)) \quad \text{with} \quad \mathfrak{gain}_n := C\mathfrak{P}_n/(C^2\mathfrak{P}_n + D^2) \\ \mathfrak{X}_{n+1} & = & A\,\widehat{\mathfrak{X}}_n + B\,\mathfrak{W}_{n+1}. \end{array}\right.$$

$(\mathfrak{V}_n, \mathfrak{W}_n)$ copies of (V_n, W_n) and $\mathfrak{P}_n$ variance of the state $\mathfrak{X}_n$.

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$$\Downarrow$$

Consistency property:

$$\mathfrak{X}_n \sim \mathcal{N}\left(\widehat{X}_n^-, P_n = \mathfrak{P}_n\right) \quad \text{and} \quad \widehat{\mathfrak{X}}_n \sim \mathcal{N}\left(\widehat{X}_n, \widehat{P}_n\right).$$

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Nb: *The Sakov & Oke (a.k.a. deterministic EnKF) discrete time versions are* **NOT consistent**.

EnKF = Mean field interacting particle sampler

$$\begin{cases} \widehat{\xi}_n^i &= \xi_n^i + g_n \left(Y_n - (C\xi_n^i + D\mathcal{V}_n^i)\right) \quad \text{with} \quad g_n := C p_n/(C^2 p_n + D^2) \\ \xi_{n+1}^i &= A\,\widehat{\xi}_n^i + B\,\mathcal{W}_{n+1}^i \qquad i \in \{1, \ldots, N+1\} \end{cases}$$

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$(\mathcal{V}_n^i, \mathcal{W}_n^i)$ *copies of* (V_n, W_n) *and re-scaled sample variance*

$$p_n := \frac{1}{N} \sum_{1 \leqslant i \leqslant N+1} (\xi_n^i - m_n)^2$$

with the sample mean

$$m_n := \frac{1}{N+1} \sum_{1 \leqslant i \leqslant N+1} \xi_n^i$$

Perturbation theo. *(Up to a change of probability space)*

$$\left\{\begin{array}{rcl} \widehat{m}_n & = & m_n + g_n\,(Y_n - Cm_n) + \dfrac{1}{\sqrt{N+1}}\,\widehat{\upsilon}_n \\ \\ \widehat{p}_n & = & (1 - g_n C)\; p_n + \dfrac{1}{\sqrt{N}}\,\widehat{\nu}_n \end{array}\right. \quad \left\{\begin{array}{rcl} m_{n+1} & = & A\;\widehat{m}_n + \dfrac{1}{\sqrt{N+1}}\,\upsilon_{n+1} \\ \\ p_{n+1} & = & A^2\;\widehat{p}_n + R + \dfrac{1}{\sqrt{N}}\,\nu_{n+1}. \end{array}\right.$$

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local perturbations υ_n, ν_n and $\widehat{\nu}_n, \widehat{\upsilon}_n$ in terms of χ^2

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local perturbations υ_n, ν_n and $\widehat{\nu}_n, \widehat{\upsilon}_n$ in terms of χ^2

Corollary: p_n Markov chain $\mathcal{K}(p, dq)$ = Stochastic Riccati eq.

$$p_{n+1} = \phi(p_n) + \frac{1}{\sqrt{N}}\,\delta_{n+1} \quad \text{with} \quad \delta_{n+1} := A^2\,\widehat{\nu}_n + \nu_{n+1}.$$

Stab. theo. (cf. 10 pages section 2 in (+Arnaudon, Ouhabaz SAA 23))

Theo 1: $\exists$ Lyapunov fct. $\mathcal{U}(p) = 1 + u(p + 1/p)$ s.t. $\beta_{\mathcal{U}}(\mathcal{K}) < 1$

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Key/difficulty: Estimates/Expo. decays $\forall A$ of random products

$$\mathcal{E}_{l,n} := \prod_{l \leqslant k \leqslant n} \frac{A}{1 + Sp_k}$$

OK for $|A| < 1$ but also for $|A| \geqslant 1$ unstable/effective dimension/...

Time uniform estimates for any A

Theo 1 [(Under) Bias]: $\forall k \geqslant 1\ \exists \iota_k < \infty$ s.t. $\forall N \geqslant 1\ \forall n \geqslant 0$

$$0 \leqslant P_n - \mathbb{E}(p_n) \leqslant \iota_1 / N$$

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& Uniform control of the bias $(N \geqslant N_k)$

$$\mathbb{E}\left(|\mathbb{E}(\widehat{m}_n \mid \mathcal{Y}_n) - \widehat{X}_n|^k \right)^{1/k} \leqslant \iota_k / N$$

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(state estimates for $N \geqslant N_k$)

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Many other results/fairly complete analysis: multivariate central limit theorems, exponential decays random products,...