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**APPLICATION OF A PARTICLE FILTER-BASED SUBSET SIMULATION METHOD
TO THE RELIABILITY ASSESSMENT OF A MARINE STRUCTURE**

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ABSTRACT

The present study aims at investigating advanced subset simulation techniques, which are based on the theory of particle filter, for the assessment of the failure probability of a marine structure under extreme loading conditions. Three approaches are considered, namely the classical particle filter method, the subset simulation with a branching process and one using the minimum values of the samples as levels. They are, first, intensively applied on a simple example for which a known analytical solution is available, in order to investigate their parameter settings. Then, they are applied, with good performance, using their respective best parameter settings, to the assessment of failure probability of a FPSO subjected to extreme roll motion.

INTRODUCTION

The design of a marine structure requires to estimate the probability that the structure would fail under extreme load conditions. That issue is of great importance for the roll motion, which can significantly impacts the safety and the performance in operational conditions. It leads to the computation of the probability that the structural response, exceeds a given critical threshold for a given reference time (*e.g.* the timescale of a sea state, one year, ...):

$$P_F = \text{Prob}(\Phi(X) > L_c) \quad (1)$$

where $\Phi(X)$ denotes the structural response and L_c the critical threshold.

Under some restrictive assumptions, namely when the loading is a random Gaussian process and the structural response, Φ , is linear or quadratic, closed-form solutions or good analytical approximation are available. But in general, the structural response results from complex non-linear dynamic equations involving time consuming time-domain computations and the representation of the dynamic loadings as random processes involves a high number of random variables. In that case, there is no simple analytical solution and, for an accurate assessment of the failure probability, simulation-based reliability methods are more appropriate, but become unpractical as the number of simulations increases dramatically when low failure probabilities are estimated.

An appealing way to reduce the number of simulations, required for small probabilities, is to use the subset simulation method. That method expresses the small failure probability, in virtue of the Bayes theorem, as the product of conditional probabilities, which are not so small and can be estimated with a reasonable number of simulations.

$$P_F = \prod_{k=1}^{M-1} \text{Prob}(\Phi(X) > L_{k+1} | \Phi(X) > L_k) \quad (2)$$

where the L_k 's are intermediate levels such that $-\infty = L_1 < L_2 < \dots < L_M = L_c$. Yet, the efficiency of this method is not systematic but depends on the choice of the intermediate levels and on the techniques used to simulate samples conditional to the intermediate failure events. The first approaches of that method use a Markov-Chain Monte-Carlo

(MCMC), such as the Metropolis-Hastings algorithm or the Gibbs sampling, to sample the conditional distributions with a reversible Markov chain which has the conditional distribution under consideration as its stationary distribution (Robert & Casella, 2004). However, they need sometimes large Markov chain sizes to ensure convergence to the stationary distribution leading to long computation times. The introduction of particle filtering techniques in the subset simulations strategy yields some improvements as it allows almost independent and identically distributed samples and is therefore suitable for a parallel computation architecture. Moreover, it leads to a rigorous mathematical convergence analysis of the subset method (Del Moral, 2004). The particle filter based subset simulation is often applied in a wide range of areas including stochastic model for economy and insurance, DNA sequencing, electrical engineering,... (Dembo & Zeitouni, 2010), but are much less used in offshore structural reliability problems.

In the present study, two new approaches, namely one using a branching process (Del Moral & Lezaud, 2006) and one using the minimum values of the samples as levels (Guyader, *et. al.*, 2011), are considered and compared to the classical technique. They are investigated for solving structural reliability problems encountered in offshore engineering. Various algorithms, corresponding to different sampling strategies, are considered. The effects in terms of accuracy of the parameter settings of those algorithms is also analysed.

The algorithms are first intensively applied on a simple example for which a closed-form solution for the failure probability can be derived. The performances of those algorithms are then compared in order to identify the best sampling strategies and the proper parameter settings. Finally, the method is illustrated on the assessment of the failure probability of an FPSO (Floating Production Storage and Offloading Platform) subjected to roll motion.

PARTICLE FILTER BASED SUBSET SIMULATION

Global framework

The objective is to compute the small probability of the failure event given by $\{\Phi(X) > L_c\}$, where X is a random element on \mathbb{R}^d , for d , a positive integer that represents the dimension of the random space, and Φ a response function of X that yields real numbers. The subset method introduces a sequence of intermediate levels $-\infty = L_1 < L_2 < \dots < L_M = L_c$ and gives a convenient way to sample the distribution of X restricted to the set of the elements, the response of which are larger than a given intermediate level.

It is quite difficult to find the proper intermediate levels a priori, since it requires a good knowledge of the structural response behavior in order to avoid that the samples of the conditional distributions die (*i.e.* $\text{Prob}(\Phi(X) > L_{k+1} | \Phi(X) > L_k) = 0$).

It is easier to adaptively choose them with respect to a prescribed value p_0 of the conditional probabilities. In that case, a level is given by the $(1-p_0)$ empirical quantile of the conditional distribution corresponding to the previous level. The number M of levels is reached when one finally obtains an empirical quantile larger than the critical threshold, which stops the sequence of levels. Thus, the number of levels is random. However, when the size of samples at each step is large enough that number is almost fixed and converges to the ratio of logarithms (*i.e.* the ratio of the power of ten of P_F and p_0):

$$M = \frac{\log(P_F)}{\log(p_0)} \quad (3)$$

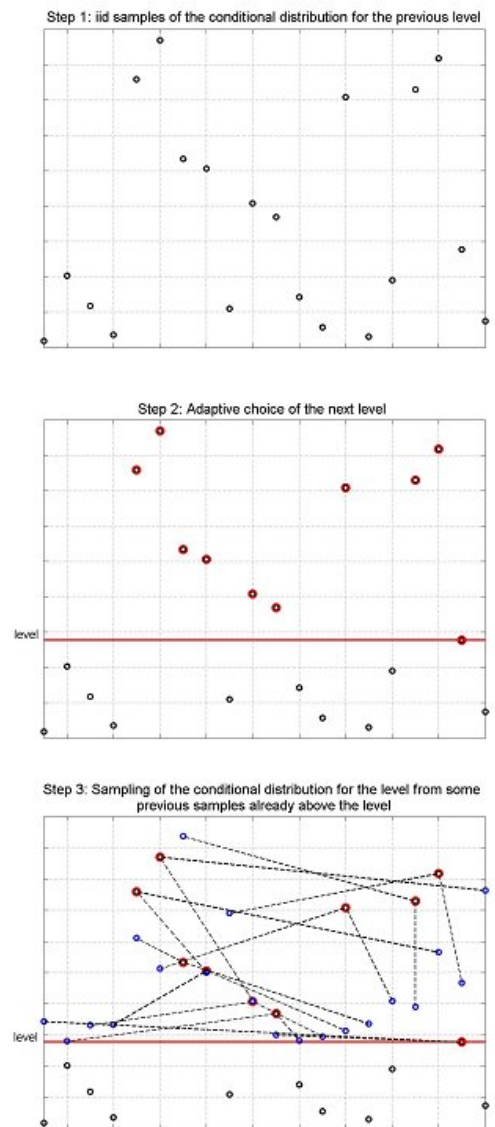


Fig.1: Global framework of the sampling of an intermediate condition distribution by particle filter-based subset method.

The general framework to generate simulations with respect to a distribution of X conditional to a given level is schematically explained on Fig.1. In the first step, one assumes that one has N iid samples ($X_i, i=1..N$) of the conditional distribution for the level L_k . Those samples appear as a cloud of particles, and the objective is to randomly propagate those particles with respect to the conditional distribution for the next level.

In the second step, the next level L_{k+1} is set to the $(1-p_0)$ empirical quantile (*i.e.* the minimum of the N_0 particles with the higher response function values, where $N_0/N=p_0$). Then, N particles are selected according to the empirical distribution of the N_0 particles that have already reached the next level L_{k+1} . That distribution reads:

$$\hat{\eta}(x) = \frac{\sum_{i=1}^N \text{Ind}\{\Phi(x) > L_{k+1}\}(x) \delta[X_i](x)}{\sum_{j=1}^N \text{Ind}\{\Phi(x) > L_{k+1}\}(X_j)} \quad (4)$$

where $\text{Ind}\{\Phi(x) > L_{k+1}\}$ stands for the indicator function and $\delta[X_i]$ is the kroeneker operator. It means that the selected particles are copies of those which have already reached the next level. To select those copies with respect to the empirical distribution, one can pick them uniformly in the set of the N_0 particles. Another kind of selection, quite similar to the uniform one, is that based on accept-reject concept, which ensures that all the element of the set of the N_0 particles are selected at least once.

In the third step, N new particles are generated with a transformation of the selected ones in such a way that the new particles keep the distribution of those from which they are obtained, namely the conditional distribution for the next level L_{k+1} . This can be done by Markov chain Monte-Carlo (MCMC) simulation using a Metropolis-Hastings algorithm (Brooks, et. al., 2010). That algorithm applies a random perturbation that preserves the distribution to a selected sample. Then, it assigns the obtained result to the new sample if this one has reached the next level, otherwise the new sample is set to the selected one. That is written as follows:

$$\tilde{X} = \begin{cases} F(\hat{X}, W) & \text{if } \Phi(F(\hat{X}, W)) > L_{k+1} \\ \hat{X} & \text{otherwise} \end{cases} \quad (5)$$

where F is a function and W is a random element on \mathbb{R}^d . In offshore applications, the random input is generally a random vector of iid standard Gaussian variables (*i.e.* Gaussian with zero mean value and unit standard deviation). An interesting random perturbation function is given by:

$$F(X, W) = \sqrt{1-\alpha} X + \alpha W \quad (6)$$

where α is a scatter parameter. As one can see, that random perturbation preserves the standard Gaussian distribution. One can notice that, when α is small the resulting transformation stays close to the sample from which it is obtained and therefore has more chances to reach the next level, however it does not explore properly the local random space. On the contrary, when α is larger, the local random space exploration is better but the risk is even higher that the transformation does not reach the next level and is not accepted. So, the scatter parameter has to be chosen in order to make a balance between the quality of the local random space exploration and the acceptance rate of the random perturbation. A study has shown that for Gaussian perturbation in Metropolis-Hasting algorithm an optimal scatter parameter exists and is of the order of $\alpha \propto 1/\sqrt[3]{d}$ (Roberts & Rosenthal, 2001).

One can also notice that the new samples are not as iid as the samples from which they are obtained, since each new sample depends by the random perturbation on the one from which it is obtained. A way to improve the iid property of the new samples is to apply many times the random transformation so as to make the resulting sample forget the sample it is generated from and to increase the mixing property of the transformation.

In summary, the global framework of the subset method based on particle filter involves at each step two operations, namely a selection of samples from which new samples of the same conditional distribution has to be generated, and a mutation process consisting in propagating or randomly transforming the selected samples keeping the distribution unchanged.

As mentioned above, some asymptotic convergence results are derived for the subset methods that involve particle filter techniques. It has been demonstrated that under the assumption of continuity of the cumulative distribution of the response function, the probability estimated by the adaptive subset based on particle filter has the following convergence properties (C  rou, et. al., 2011):

$$\begin{aligned} \sqrt{N}(\hat{P}_F - P_F) &\xrightarrow{N \rightarrow \infty} \text{Gaussian}(0, \sigma^2) \\ \text{where} & \\ \sigma^2 &= P_F^2 \left(N_0 \frac{1-p_0}{p_0} + \frac{1-r_0}{r_0} \right) \\ \text{and} & \\ N \frac{E[\hat{P}_F] - P_F}{P_F} &\xrightarrow{N \rightarrow \infty} N_0 \frac{1-p_0}{p_0} \end{aligned} \quad (7)$$

In the variance expression, r_0 denotes the proportion of samples that have reached the last level which is the critical threshold. Those convergence results allows to give confidence intervals of the estimate of the probability. Note that the estimate of the

probability is biased, nevertheless the bias is of order $1/N$ and is therefore negligible compared to the standard deviation.

Tables 1~2 shows respectively the coefficient of variation of the relative error on the estimate of the probability and the relative bias for various values of p_0 the prescribed value of the conditional probabilities and various sample sizes. Table 3 shows the corresponding expected total number of simulations to get those accuracies. All those results are obtained for a reference probability of 10^{-5} . One can see that the bias are relatively small. Yet it seems difficult to get very accurate results for the probability since the coefficient of variation of the error can hardly get lower than 10%, but this is not a problem, since the error on probability are usually drastically reduced on the corresponding extreme quantile. For instance, a 50% coefficient of variation on the estimate of a probability of 10^{-5} results in a coefficient of variation of the corresponding extreme quantile lower than 5%. Therefore one can still accept even a 50% error on the probability estimates. One can consequently choose a reasonably small sample size N of a few hundred. In the following applications, N is set to 300 and p_0 is set to 0.75, which leads to a 43% coefficient of variation and a 4% relative bias of the estimate of $P_F = 10^{-5}$ for an expected total number of simulations equal to 12000. That number of simulations is extremely low compared to what is needed by crude Monte-Carlo simulations (e.g. at least 10^6).

Table 1. Coefficient of variation of $P_F = 10^{-5}$ estimates [%]

$p_0 \backslash N$	100	200	300	400	500	600	700	800	900	1000
0.1	147%	104%	85%	73%	66%	60%	56%	52%	49%	46%
0.15	126%	89%	73%	63%	56%	51%	48%	45%	42%	40%
0.2	113%	80%	65%	57%	51%	46%	43%	40%	38%	36%
0.25	104%	73%	60%	52%	46%	42%	39%	37%	35%	33%
0.3	97%	68%	56%	48%	43%	39%	37%	34%	32%	31%
0.35	90%	64%	52%	45%	40%	37%	34%	32%	30%	29%
0.4	88%	62%	51%	44%	39%	36%	33%	31%	29%	28%
0.45	86%	61%	49%	43%	38%	35%	32%	30%	29%	27%
0.5	82%	58%	48%	41%	37%	34%	31%	29%	27%	26%
0.55	81%	57%	47%	40%	36%	33%	31%	29%	27%	26%
0.6	78%	55%	45%	39%	35%	32%	30%	28%	26%	25%
0.65	76%	54%	44%	38%	34%	31%	29%	27%	25%	24%
0.7	75%	53%	43%	38%	34%	31%	28%	27%	25%	24%
0.75	74%	52%	43%	37%	33%	30%	28%	26%	25%	23%
0.8	72%	51%	42%	36%	32%	29%	27%	25%	24%	23%
0.85	71%	50%	41%	35%	32%	29%	27%	25%	24%	22%
0.9	70%	49%	40%	35%	31%	29%	26%	25%	23%	22%

Table 2. Relative bias of $P_F = 10^{-5}$ estimates [%]

$p_0 \backslash N$	100	200	300	400	500	600	700	800	900	1000
0.1	45%	23%	15%	11%	9%	8%	6%	6%	5%	5%
0.15	34%	17%	11%	9%	7%	6%	5%	4%	4%	3%
0.2	28%	14%	9%	7%	6%	5%	4%	4%	3%	3%
0.25	24%	12%	8%	6%	5%	4%	3%	3%	3%	2%
0.3	21%	11%	7%	5%	4%	4%	3%	3%	2%	2%
0.35	19%	9%	6%	5%	4%	3%	3%	2%	2%	2%
0.4	18%	9%	6%	5%	4%	3%	3%	2%	2%	2%
0.45	17%	9%	6%	4%	3%	3%	2%	2%	2%	2%
0.5	16%	8%	5%	4%	3%	3%	2%	2%	2%	2%
0.55	16%	8%	5%	4%	3%	3%	2%	2%	2%	2%
0.6	15%	7%	5%	4%	3%	2%	2%	2%	2%	1%
0.65	14%	7%	5%	4%	3%	2%	2%	2%	2%	1%
0.7	14%	7%	5%	3%	3%	2%	2%	2%	2%	1%
0.75	13%	7%	4%	3%	3%	2%	2%	2%	1%	1%
0.8	13%	6%	4%	3%	3%	2%	2%	2%	1%	1%
0.85	12%	6%	4%	3%	2%	2%	2%	2%	1%	1%
0.9	12%	6%	4%	3%	2%	2%	2%	2%	1%	1%

Table 3. Expected number of simulation for $P_F = 10^{-5}$

$p_0 \backslash N$	100	200	300	400	500	600	700	800	900	1000
0.1	500	1000	1500	2000	2500	3000	3500	4000	4500	5000
0.15	600	1200	1800	2400	3000	3600	4200	4800	5400	6000
0.2	700	1400	2100	2800	3500	4200	4900	5600	6300	7000
0.25	800	1600	2400	3200	4000	4800	5600	6400	7200	8000
0.3	900	1800	2700	3600	4500	5400	6300	7200	8100	9000
0.35	1000	2000	3000	4000	5000	6000	7000	8000	9000	10000
0.4	1200	2400	3600	4800	6000	7200	8400	9600	10800	12000
0.45	1400	2800	4200	5600	7000	8400	9800	11200	12600	14000
0.5	1800	3200	4800	6400	8000	9600	11200	12800	14400	16000
0.55	1900	3800	5700	7600	9500	11400	13300	15200	17100	19000
0.6	2200	4400	6600	8800	11000	13200	15400	17600	19800	22000
0.65	2800	5200	7800	10400	13000	15600	18200	20800	23400	26000
0.7	3200	6400	9600	12800	16000	19200	22400	25600	28800	32000
0.75	4000	8000	12000	16000	20000	24000	28000	32000	36000	40000
0.8	5100	10200	15300	20400	25500	30600	35700	40800	45900	51000
0.85	7000	14000	21000	28000	35000	42000	49000	56000	63000	70000
0.9	10900	21800	32700	43600	54500	65400	76300	87200	98100	1E+05

Some other approaches of the subset simulation method based on the general framework presented above has also been developed. They include some tricks to control the exploration of the random space within the simulations at each step of the subset method. In this study one has investigated two of them. The first one involves branching process techniques in the selection and the mutation process of the samples at each step (Del Moral & Lezaud, 2006). The second one simplifies the choice of the subset levels taking them as the minimum value of the response on the samples at each step (Guyader, *et. al.*, 2011).

Subset simulation with branching process

In the general framework of the subset method presented in the previous section, the selection gives the same weight to all the samples. In the subset approach including a branching process the samples do not have the same weight: they are selected with respect to their chances to be propagated by mutation to the next level.

Let us assume that one has N_1 iid samples of the conditional distribution for the level L_k . From each of those samples, one generates N_2 new samples using the mutation process. Thus one gets $N = N_1 \times N_2$ samples. At this stage, the next level L_{k+1} can be set adaptively to the $(1-p_0)$ empirical quantile of the $N=N_1 \times N_2$ samples. Then, the probability for the N_1 current samples to reach the next level after mutation can be empirically computed. Let us denote $G(X)$ that probability. The branching process-based subset proposes to select N_1 samples with respect to the following empirical distribution:

$$\hat{\eta}(x) = \sum_{i=1}^{N_1} \frac{G(x)}{\sum_{j=1}^{N_1} G(X_j)} \delta[X_i](x) \quad (8)$$

which is based on the chances they have to be propagated to the next level. Once the selection is done, one does not have to carry out another mutation of the selected sample. One has just, for each selected sample, to pick one element of its corresponding N_2 mutations that have reached the next level. So, in this approach the mutation precedes the selection operation or they are both carried out at the same time. Nevertheless, it as been shown that this approach is in theory equivalent to the global framework (Del Moral & Lezaud, 2006).

No asymptotic convergence property appears to have been explicitly derived for that approach in the literature. However, one can reasonably guess that its probability estimates have coefficients of variation of almost the same order as that of the global framework, in virtue of the equivalence of the two approaches.

Subset simulation with minimum sampled responses as levels

This approach is a refinement of the adaptive subset method. Let us first assume that an iid samples of size N is generated with respect to the distribution of X . The first level L_1 is set to the minimum value of the evaluated response function of that samples. Then, the particle that achieves that minimum is removed from the set of samples and replaced by a new particle obtained from a mutation with respect to the conditional distribution for L_1 . The next level L_2 is set, the same way, to the minimum value of the evaluated response function of the samples just modified by the replacement one particle. That operation is iteratively repeated till the minimum value of the response of the samples reaches the critical threshold (Fig.2).

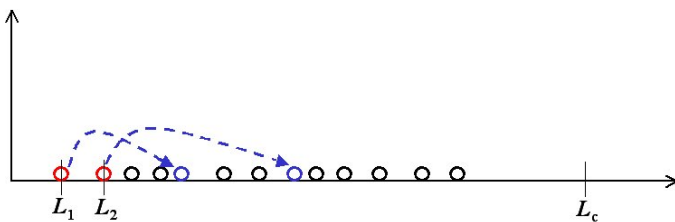


Fig. 2: Subset simulation with minimum sampled responses as levels

The number of levels, $M = \max\{k; L_k \leq L_c\}$ is a random variable shown to be distributed according to a Poisson law with parameter $-N \log(P_F)$. Since N is assumed to be large and P_F to be small, a natural estimate of the probability P_F is:

$$P_F = \left(1 - \frac{1}{N}\right)^M \quad (9)$$

It is also demonstrated that (9) is an unbiased estimator of P_F with variance:

$$\sigma^2 = P_F^2 \left(P_F \frac{1}{N} - 1 \right) \quad (10)$$

For this approach, confidence intervals for P_F are also derived. The reader can refer to the paper of Guyader, *et. al.* (2011) for the complete expressions of those confidence intervals.

Table 4 shows the values of the standard deviation and the coefficient of variation of the estimate of a reference probability of 10^{-5} . One can see that, quite accurate results are expected from that approach for a relatively low number of simulations. However, since only one particle is propagated at each iteration, that algorithm does not allow parallel computing.

Table 4. Standard deviation an total number of simulation for $P_F=10^{-5}$.

Sample size	Standard deviation	Coef. of variation	nb. of simulations
100	3,49 10^{-06}	35%	1145
200	2,43 10^{-06}	24%	2296
300	1,98 10^{-06}	19%	3448
400	1,71 10^{-06}	17%	4599
500	1,53 10^{-06}	15%	5750
600	1,39 10^{-06}	14%	6901
700	1,29 10^{-06}	13%	8053
800	1,20 10^{-06}	12%	9204
900	1,13 10^{-06}	11%	10355
1000	1,08 10^{-06}	10%	11507

NUMERICAL APPLICATION

Simple example

Let us consider $X = (u_1, \dots, u_d)$, a vector of independent standard Gaussian variables u_j . The response function Φ is given by: $x \mapsto \max_j u_j^2$. It is clear that $\Phi(X)$ is distributed as the maximum of independent random variables which follows a chi-square distribution with one degree of freedom. The cumulative distribution of $\Phi(X)$ reads thus:

$$F_{\Phi}(x) = \left[\frac{1}{\Gamma(1/2)} \int_0^{x/2} \frac{e^{-t}}{\sqrt{t}} dt \right]^d \quad (11)$$

In this example, the three approaches of the subset method are intensively applied, namely, the general subset method with the uniform selection and the accept-reject selection kind, the

subset method with branching and the one with minimum sampled responses as levels, and their results are compared to the exact solution. $N = 500$ response samples are generated for each intermediate level, in particular, for the method with branching $N_1=50$ and $N_2 = 10$. Two parameter settings, *e.g.* the scatter parameter α of the mutation and the number Q of iterations of that mutation, are respectively given the various values $\{1/2\sqrt[3]{d}; 1/\sqrt[3]{d}; 2/\sqrt[3]{d}\}$ and $\{2,5,10,20\}$. Two sizes of the random space are considered, *e.g.* $d = 128$ and $d = 1024$. One set the target P_F to 10^{-5} . Inversing (11) one obtains the corresponding critical thresholds equal to 28.85 for $d = 128$ and 32.89 for $d = 1024$. Now, for a given threshold, 50 runs of each subset algorithm are carried out to compute the failure probability. The mean values of the estimates are reported in Tables 5 & 6, while the coefficient of variation of the estimates are given in Tables 7 & 8.

Table 5. Mean value of the estimations of $P_F = 10^{-5}$ for $d = 128$.

	2	5	10	20
$1/2\sqrt[3]{d}$	$8.53 \cdot 10^{-6}$	$8.57 \cdot 10^{-6}$	$7.97 \cdot 10^{-6}$	$8.89 \cdot 10^{-6}$
	$8.72 \cdot 10^{-6}$	$8.72 \cdot 10^{-6}$	$8.22 \cdot 10^{-6}$	$8.75 \cdot 10^{-6}$
	$1.04 \cdot 10^{-5}$	$1.10 \cdot 10^{-5}$	$1.12 \cdot 10^{-5}$	$1.18 \cdot 10^{-5}$
	$1.06 \cdot 10^{-5}$	$9.21 \cdot 10^{-5}$	$9.97 \cdot 10^{-6}$	$9.95 \cdot 10^{-6}$
$1/\sqrt[3]{d}$	$9.29 \cdot 10^{-6}$	$9.01 \cdot 10^{-6}$	$8.80 \cdot 10^{-6}$	$8.41 \cdot 10^{-6}$
	$9.34 \cdot 10^{-6}$	$8.44 \cdot 10^{-6}$	$8.71 \cdot 10^{-6}$	$8.12 \cdot 10^{-6}$
	$8.46 \cdot 10^{-6}$	$9.27 \cdot 10^{-6}$	$1.04 \cdot 10^{-5}$	$1.13 \cdot 10^{-5}$
	$1.06 \cdot 10^{-5}$	$1.04 \cdot 10^{-5}$	$1.03 \cdot 10^{-5}$	$9.84 \cdot 10^{-6}$
$2/\sqrt[3]{d}$	$3.97 \cdot 10^{-5}$	$1.56 \cdot 10^{-5}$	$1.06 \cdot 10^{-5}$	$9.83 \cdot 10^{-6}$
	$2.12 \cdot 10^{-5}$	$1.11 \cdot 10^{-5}$	$9.94 \cdot 10^{-6}$	$9.07 \cdot 10^{-6}$
	$1.28 \cdot 10^{-7}$	$3.64 \cdot 10^{-6}$	$6.82 \cdot 10^{-6}$	$9.09 \cdot 10^{-6}$
	$7.75 \cdot 10^{-6}$	$9.03 \cdot 10^{-6}$	$1.00 \cdot 10^{-5}$	$1.00 \cdot 10^{-5}$

Uniform ; accept-reject ; branching ; min as level.

Table 6. Mean value of the estimations of $P_F = 10^{-5}$ for $d = 1024$.

	2	5	10	20
$1/2\sqrt[3]{d}$	$8.56 \cdot 10^{-6}$	$8.34 \cdot 10^{-6}$	$9.21 \cdot 10^{-6}$	$8.50 \cdot 10^{-6}$
	$8.67 \cdot 10^{-6}$	$8.75 \cdot 10^{-6}$	$8.62 \cdot 10^{-6}$	$8.58 \cdot 10^{-6}$
	$1.90 \cdot 10^{-5}$	$1.24 \cdot 10^{-5}$	$1.18 \cdot 10^{-5}$	$1.17 \cdot 10^{-5}$
	$1.07 \cdot 10^{-5}$	$1.08 \cdot 10^{-5}$	$1.00 \cdot 10^{-5}$	$1.06 \cdot 10^{-5}$
$1/\sqrt[3]{d}$	$8.88 \cdot 10^{-6}$	$8.55 \cdot 10^{-6}$	$8.57 \cdot 10^{-6}$	$8.17 \cdot 10^{-6}$
	$9.14 \cdot 10^{-6}$	$8.90 \cdot 10^{-6}$	$9.07 \cdot 10^{-6}$	$8.65 \cdot 10^{-6}$
	$1.23 \cdot 10^{-5}$	$1.07 \cdot 10^{-5}$	$1.07 \cdot 10^{-5}$	$1.13 \cdot 10^{-5}$
	$1.03 \cdot 10^{-5}$	$1.00 \cdot 10^{-5}$	$9.92 \cdot 10^{-6}$	$9.84 \cdot 10^{-6}$
$2/\sqrt[3]{d}$	$9.91 \cdot 10^{-6}$	$8.79 \cdot 10^{-6}$	$8.61 \cdot 10^{-6}$	$8.50 \cdot 10^{-6}$
	$8.58 \cdot 10^{-6}$	$9.06 \cdot 10^{-6}$	$9.30 \cdot 10^{-6}$	$9.03 \cdot 10^{-6}$
	$9.16 \cdot 10^{-6}$	$1.01 \cdot 10^{-5}$	$1.10 \cdot 10^{-5}$	$1.18 \cdot 10^{-5}$
	$1.04 \cdot 10^{-5}$	$9.98 \cdot 10^{-6}$	$9.93 \cdot 10^{-6}$	$9.84 \cdot 10^{-6}$

The unbiased estimates of the “min as level” algorithm is verified on those results. One observes that the “branching” algorithm seems also unbiased. “Uniform” and “accept-reject” algorithms are always biased and they tends to give slightly lower estimates of the probability.

The results shows that the accuracy of the algorithms depends mostly on the parameter settings. This is more visible for $d=128$. Larger value of α makes the estimate strongly deviate from the exact solution. This is certainly due to a too low acceptance rate during the mutation. In that case, large numbers of mutation iterations reduce that deviation and bring the estimate close to the exact solution. In fact, both parameters α and Q play the same role. Each of them increases the mixing property of the mutation process as they get larger. So, when the acceptance rate is very sensitive to the scatter of the mutation then the number of iteration have to be increased.

Table 7. Coef. of var. of the estimations of $P_F = 10^{-5}$ for $d = 128$.

	2	5	10	20
$1/2\sqrt[3]{d}$	25%	25%	21%	24%
	30%	20%	18%	26%
	31%	22%	26%	21%
	40%	24%	18%	23%
$1/\sqrt[3]{d}$	27%	26%	21%	21%
	32%	23%	26%	28%
	50%	31%	25%	20%
	43%	27%	23%	19%
$2/\sqrt[3]{d}$	47%	44%	32%	26%
	55%	28%	34%	21%
	370%	149%	59%	31%
	127%	66%	37%	33%

Table 8. Coef. of var. of the estimations of $P_F = 10^{-5}$ for $d = 1024$.

	2	5	10	20
$1/2\sqrt[3]{d}$	32%	20%	26%	26%
	28%	25%	23%	23%
	40%	29%	25%	25%
	67%	33%	20%	20%
$1/\sqrt[3]{d}$	29%	20%	22%	21%
	21%	25%	22%	25%
	35%	23%	23%	23%
	35%	24%	18%	18%
$2/\sqrt[3]{d}$	35%	25%	22%	21%
	32%	21%	20%	24%
	43%	30%	21%	24%
	27%	21%	20%	18%

From those results, one can conclude that $\alpha=1/\sqrt[3]{d}$ and $Q = 5$ is a good combination of the parameter settings. Therefore, one retained those value for the following offshore design application.

Offshore design application

This application deals with the risk of excessive roll motion of a FPSO. It is one of the key aspect in a FPSO design analysis. In fact, large roll motion amplitudes impact the comfort on board leading to bad crew performance, but it is also critical for some equipments or components of the structure, for instance the risers connectors could be subjected to high motion leading to extreme loads.

Let us consider roll motion with single degree of freedom described as:

$$I\ddot{\theta}(t) + T\dot{\theta}(t) + B\theta(t) + K\theta(t) = C(t) \quad (12)$$

where θ is the roll angle, I is the mass moment of inertia in roll, $T\dot{\theta}(t)$ is the nonlinear damping moment, $B\theta(t)$ is the linear damping moment, $K\theta(t)$ is the restoring moment (which is assumed linear) and $C(t)$ is the exciting moment by waves. We want to compute the probability that the roll angle exceeds a given threshold when the FPSO is subjected to a random sea state. Thus, $C(t)$ is generated by a transfer filter with a random process of the sea state condition: $C(t) = AC\eta(t)$, where $\eta(t)$, the sea surface elevation, is a Gaussian process, the spectral density of which follows a JONSWAP model with a peak period T_p and a significant wave height H_S . T_p and H_S are assumed to be random dependent variables. H_S follows a Gumbel distribution while T_p is a log-normal variable, the parameter of which are given in terms of H_S . Thus, the sea state condition is represented by a random process, which spectral parameters are also random. Note that no model uncertainty (e.g. related to the damping term in eq (12)) is introduced.

The motion differential equation (12) is solved in the time domain from time series of the sea surface elevation by a fourth-order Runge-Kutta method. The sea surface elevation is simulated in the time domain from its spectral representation given an instance of the random T_p and H_S . That simulation involves 1024 independent standard gaussian variables for a sea state duration of 1h30min. Note that samples of T_p and H_S can be obtained by transformation of standard Gaussian variables. Therefore the input random variables of the system is made of 1026 independent standard Gaussian variables. To compute the maximum roll motion from those random inputs, the two first variables are used to generate samples of H_S then T_p , and to obtain the power spectral density of the wave condition, from which time series are derived using the remaining 1024 inputs, finally the maximum of that signal is return as the response of the system.

A failure probability $P_F = 1.39 \cdot 10^{-5}$, which is the probability that the roll angle for a sea state duration exceeds a critical roll angle $\theta_c = 11^\circ$ is obtained by a Monte-Carlo simulation method with 107 samples. That value of θ_c was chosen in order to have a failure probability of about 10^{-5} . Then, one run of the three algorithms of the subset method is carried out to estimate the same P_F as the probability that the maximum roll motion exceeds L_c . The parameter settings are $\alpha = 1/\sqrt[3]{d}$ and $Q = 5$. The results are given in table 9 and are plotted with their respective 95% confidence intervals (except for "branching") in Fig.3.

All the algorithms give reasonably accurate results for the failure probability P_F . The estimate with uniform and accept/reject selection are accurate for a total number of call of

the system fifty times lower than that of the Monte-Carlo simulation. The result obtained with branching is however not very close to the reference solution. It could be due to the choice of the combination (N_1, N_2) . One has, therefore, to further analyze the effect of that combination on the accuracy of that method. The "min as level" algorithm gives accurate estimate for a very low number of simulations. That method appears as the best in terms of number of simulations. However, that method can not use parallel architecture compared to the other ones. It is interesting to point out that the real value of P_F is within all the 95% confidence intervals.

Table 9. Estimate of $P_F = 1.39 \cdot 10^{-5}$ by the subset simulation algorithms.

algorithm	estimate	nb. of call of the system
Uniform with $N=300$	$1.34 \cdot 10^{-5}$	$58500 \times 5 = 292500$
accept-reject with $N=300$	$1.79 \cdot 10^{-5}$	$57000 \times 5 = 285000$
Branching $N=300$	$5.65 \cdot 10^{-5}$	$52500 \times 5 = 262500$
min as level with $N=300$	$1.88 \cdot 10^{-5}$	$3559 \times 5 = 17795$
nin as level with $N=500$	$1.31 \cdot 10^{-5}$	$6115 \times 5 = 30575$

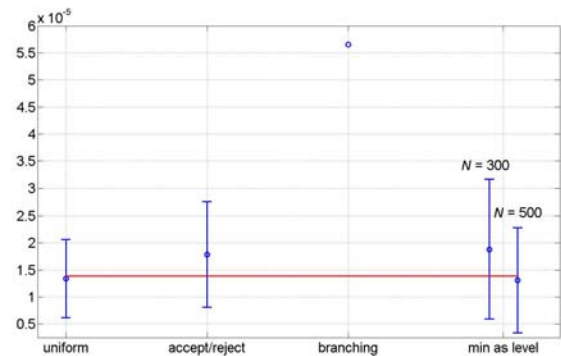


Fig.3. Confidence intervals of the estimates of P_F .

CONCLUSION

In this study, the subset method based on particle filter, which is an advance simulation-based reliability method has been investigated. That method is interesting as it allows almost accurate estimate for a relatively low number of simulations. This way, it makes affordable the cost, in term of CPU time, of the reliability assessment of an even time-consuming strongly nonlinear mechanical systems as those encountered in the offshore engineering. Many subset simulation strategies have been investigated on a simple application. The quality of the estimator of the probability appear to be slightly sensitive to the kind of algorithm. It shows that, given a sample size, the efficiency of the method depends on the good choice of the parameter settings, namely the scatter of the propagation of

particles within the random local space and on the number of iterations of that propagation. A good balance between those two parameters which play almost the same role is the key for an accurate estimate. However, the appropriate choice of those parameter settings may change from one application to another. Nevertheless, that method is an appealing mean to carry out reliability analysis. This is well illustrated by the reasonably accurate assessment of the probability of an extreme roll motion of a FPSO.

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