Concentration $\leq$ for Mean Field Particle Models

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- *Concentration Inequalities for Mean Field Particle Models* [HAL-INRIA RR-6901 (09)]. To appear in the Annals of Applied Probability (2010).
- *A Backward Particle Interpretation of Feynman-Kac Formulae.*
+ A. Doucet and S.S. Singh. RR-7019(09). M2AN, vol 44, no. 5 (2010)
- *An introduction to probabilistic methods with applications*
+ N. Hadjiconstantinou. M2AN, vol 44, no. 5 (2010)

Outline

Summary

Some notation

E measurable state space, $\mathcal{M}(E)$ & $\mathcal{P}(E)$ measures & probabilities on E
$\mathcal{B}(E)$ bounded meas. functions

- $(\mu, f) \in \mathcal{M}(E) \times \mathcal{B}(E) \quad \mapsto \quad \mu(f) = \int \mu(dx)\ f(x)$
- $M(x, dy)$ **integral operator over E**

$$\begin{aligned} M(f)(x) &= \int M(x,dy)f(y) \\ [\mu M](dy) &= \int \mu(dx)M(x,dy) \qquad (\Longrightarrow [\mu M](f) = \mu[M(f)]\ ) \end{aligned}$$

- **Composition:** $(M_1 M_2)(x, dz) = \int M_1(x, dy) M_2(y, dz)$
- **Boltzmann-Gibbs transformation :** $G : E \to [0, \infty[$ with $\mu(G) > 0$

$$\Psi_G(\mu)(dx) = \frac{1}{\mu(G)}\ G(x)\ \mu(dx)$$

Running example : Feynman-Kac integration models

- Markov chain X_n on some state spaces E_n, n=time index .
- Potential functions $G_n \ : \ x_n \in E_n \rightarrow G_n(x_n) \in [0,1]$

Feynman-Kac path measures:

$$d\mathbb{Q}_n := \frac{1}{\mathcal{Z}_n} \left\{ \prod_{0 \leq p < n} G_p(X_p) \right\} d\mathbb{P}_n \quad \text{with} \quad \mathbb{P}_n := \text{Law}(X_0, \ldots, X_n)$$

The n-time marginals: $\forall f_n \in \mathcal{B}(E_n)$

$$\eta_n(f_n) := \frac{\gamma_n(f_n)}{\gamma_n(1)} \quad \text{with} \quad \gamma_n(f_n) := \mathbb{E}\left(f_n(X_n) \prod_{0 \leq p < n} G_p(X_p) \right)$$

Updated measures $\longrightarrow$ $\widehat{\mathbb{Q}}_n, \widehat{\eta}_n$, and $\widehat{\gamma}_n$ w.r.t. the product $\prod_{0 \leq p \leq n}$ up to n.

Ex 1: Filtering, Hidden Markov Chains, Bayesian Inference

Signal-observation type model

$$\mathbb{P}\left((X_n, Y_n) \in d(x,y) | (X_{n-1}, Y_{n-1})\right) := M_n(X_{n-1}, dx)\; g_n(x,y)\; \lambda_n(dy)$$

- Given the observation sequence $Y = y$ with $G_n(x_n) = g_n(x_n, y_n)$

$$\eta_n = \mathrm{Law}(X_n \mid \forall 0 \leq p < n \quad Y_p = y_p) \quad \text{and} \quad \mathcal{Z}_{n+1} \propto p_n(y_0, \ldots, y_n)$$

- In path space settings

$$\mathbb{Q}_n = \mathrm{Law}((X_0, \ldots, X_n) \mid \forall 0 \leq p < n \quad Y_p = y_p)$$

Continuous time filtering models

$$(\mathrm{X,Y}) = (\text{Signal-Observation}) \quad X_n = X'_{[t_n,t_{n+1}]} \quad \text{and} \quad dY_t = H(X'_t)\,dt + \sigma\,dV_t$$

$$G_n(X_n) = e^{\int_{t_n}^{t_{n+1}} \left[H^\star(X'_n(s))dY_s - \frac{1}{2}H^\star(X'_n(s))H(X'_n(s))ds\right]}$$

$$\Downarrow$$

$$\mathbb{Q}_n = \mathrm{Law}((X'_{[0,t_{n+1}]}) \mid Y_s,\ s \leq t_n)$$

The t_n-marginals η_{t_n} $\rightsquigarrow$ Kushner-Stratonovich SPDE (X' inf. generator L')

$$\eta_{t_n}(f) = \nu_0(f) + \int_0^{t_n} \eta_s(L'(f))ds + \int_0^{t_n} \left[\eta_s(H^\star f) - \eta_s(H^\star)\eta_s(f)\right]\ (dY_s - \eta_s(H)ds)$$

Note: Euler time discretization $\rightsquigarrow$ full discrete time model

Ex 2 : Absorption model G : $x \in E^c := E \cup \{c\} \mapsto G(x) \in [0,1]$

$$\begin{cases} X_n^c \in E_n^c \xrightarrow{absorption\ \sim(1-G)} \widehat{X}_n^c \xrightarrow{free-motion\ \sim M} X_{n+1}^c \\ T = \inf\{n : \widehat{X}_n^c = c\} \Rightarrow \mathbb{Q}_n = \mathrm{Law}((X_0^c, \ldots, X_n^c) | T \geq n) \end{cases}$$

Spectral radius-Lyapunov exponents : $Q(x,dy) = G(x)M(x,dy)$ sub-Markov

$$\gamma_n(1) = \prod_{0 \leq p < n} \eta_p(G) = \mathbb{P}(T \geq n) \simeq e^{-\lambda n}$$

with $e^{-\lambda} \stackrel{M\ reg.}{=}$ **Q-top eigenvalue & eigenfunction** $Q(h) = e^{-\lambda} h$

$$\lambda = \lim_{n\to\infty} -\frac{1}{n} \log \|Q^n\| = \lim_{n\to\infty} -\frac{1}{n} \sum_{0\leq p\leq n} \log \eta_p(G) = -\log \eta_\infty(G)$$

Limiting FK measures η_∞=Yaglom limits = quasi-stationary measures :

$$M\ \mu-\mathrm{reversible} \Rightarrow \eta_n(f) = \mathbb{E}(f(X_n^c) \mid T \geq n) \simeq \eta_\infty(f) := \frac{\mu(h\ M(f))}{\mu(h)}$$

$\sim$ *Contraction of nonlinear FK sg* $\rightsquigarrow$ `Slides Prob & Analysis, Nice 2008`

Continuous time models

$$\frac{d}{dt}\,\gamma_t(f) := \gamma_t(L^V(f)) \quad \text{with the Schrodinger operator} \quad L^V = L - V$$

$$L(f) = \frac{1}{2}\sum_{i,j=1}^{d} a^{i,j}\partial_{x^i}\partial_{x^j}(f) + \sum_{i=1}^{d} b^i\partial_{x^i}(f)$$

Feynman-Kac representation $\gamma_t = \gamma_0 P_t^V$ with

$$\begin{aligned} P_t^V(f)(x) &= \mathbb{E}_x\left(f(X'_t)\ \exp\left\{-\int_0^t V(X'_s)\ ds\right\}\right) \\ &\stackrel{t=t_n}{=} \mathbb{E}_x\left(f_n(X'_{[t_{n-1},t_n]})\ \prod_{0\leq p<n} G_p(X_p)\right) \end{aligned}$$

with $X_n = X'_{[t_n,t_{n+1}]}$ and $G_n(X_n) = e^{-\int_{t_n}^{t_{n+1}} V(X'_s)ds}$ (alternative = Euler schemes).

Sensitivity analysis

$$\theta \mapsto \quad M_n(x_{n-1}, dx_n) = p_n^{\theta}(x_{n-1}, x_n)\ \lambda_n^X(dx_n) \quad \text{and/or} \quad G_n(x_n) = G_n^{\theta}(x_n)$$

$$\Downarrow$$

θ-Feynman-Kac models : $(G_n, \mathbb{Q}_n, \gamma_n, \eta_n, \lambda, \ldots) \rightsquigarrow (G_n^\theta, \mathbb{Q}_n^\theta, \gamma_n^\theta, \eta_n^\theta, \lambda^\theta, \ldots)$

Eigenvalues derivatives :

$$-\frac{\partial \lambda^\theta}{\partial \theta} \simeq \frac{1}{n+1}\ \frac{\partial}{\partial \theta} \log \widehat{\gamma}_n^\theta(1) = \widehat{\mathbb{Q}}_n^\theta(F_n^\theta)$$

with the additive functional

$$F_n^\theta(x_0, \ldots, x_n) \quad := \quad \frac{1}{n+1} \sum_{0 \le k \le n} \left(\frac{\partial \log G_k^\theta(x_k)}{\partial \theta} + \frac{\partial \log p_k^\theta(x_{k-1}, x_k)}{\partial \theta} \right)$$

$\oplus$ High order derivatives using the covariance formulae

$$\frac{\partial}{\partial \theta} \widehat{\mathbb{Q}}_n^\theta(\varphi_n) = (n+1)\ \widehat{\mathbb{Q}}_n^\theta \left(F_n^\theta\ [\varphi_n - \widehat{\mathbb{Q}}_n^\theta(\varphi_n)] \right)$$

4 Key observations

$$\eta_n(f_n) := \frac{\gamma_n(f_n)}{\gamma_n(1)} \quad \text{with} \quad \gamma_n(f_n) := \mathbb{E}\left(f_n(X_n) \prod_{0 \le p < n} G_p(X_p)\right)$$

- **1st important observation:**

$$[\ X_n := (X'_0, \ldots, X'_n)\ \ \&\ \ G_n(X_n) := G'_n(X'_n)] \implies \eta_n = \mathbb{Q}'_n$$

- **2nd important observation:**

$$\mathcal{Z}_n = \gamma_n(1) = \mathbb{E}\left(\prod_{0 \le p < n} G_p(X_p)\right) = \prod_{0 \le p < n} \eta_p(G_p)$$

Proof:

$$\gamma_n(1) = \gamma_{n-1}(G_{n-1}) = \eta_{n-1}(G_{n-1})\ \gamma_{n-1}(1)$$

- **3rd important observation:** (H) $M_n(x, dx') = H_n(x, x')\ \lambda_n(dx')$

$$\mathbb{Q}_n(d(x_0, \ldots, x_n)) \quad = \quad \eta_n(dx_n)\ M_{n,\eta_{n-1}}(x_n, dx_{n-1}) \ldots M_{1,\eta_0}(x_1, dx_0)$$

with the backward transitions :

$$M_{n,\eta}(x, dx') \propto \eta(dx') G_{n-1}(x')\ H_n(x', x)$$

Absorption models : $\rightsquigarrow$ h-process $X_n^{(h)} \sim M^{(h)}(x, dy) \propto M(x, dy)h(y)$

$$\mathbb{P}\left((X_0^{(h)}, \ldots, X_n^{(h)}) \in d(x_0, \ldots, x_n)\right)\ \frac{h^{-1}(x_n)}{\mathbb{E}(h^{-1}(X_n^{(h)}))} = \mathbb{Q}_n(d(x_0, \ldots, x_n))$$

Additive functionals

$$\mathbb{E}_{\mathbb{Q}_n}\left(\frac{1}{n}\sum_{0\leq p<n} f(X_p)\right) \rightarrow_{n\uparrow\infty} \eta_\infty^{(h)}(f) = \eta_\infty^{(h)} M^{(h)}(f) = \Psi_h\left(\eta_\infty\right)(f)$$

4th Key observation

- A two step correction prediction model

$$\eta_n \xrightarrow{\text{Updating-correction}} \widehat{\eta}_n = \Psi_{G_n}(\eta_n) \xrightarrow{\text{Prediction/Markov transport}} \eta_{n+1} = \widehat{\eta}_n M_{n+1}$$

- Selection nonlinear transport formulae

$$\Psi_{G_n}(\eta_n) = \eta_n S_{n,\eta_n}$$

with, for any $\epsilon_n \in [0,1]$

$$S_{n,\eta_n}(x,.) := \epsilon_n G_n(x)\ \delta_x + (1 - \epsilon_n G_n(x))\ \Psi_{G_n}(\eta_n)$$

$$\Downarrow$$

$$\eta_{n+1} = \eta_n \left(S_{n,\eta_n} M_{n+1}\right) := \eta_n K_{n+1,\eta_n}$$

Note : Continuous time models = nonlinear Moran type interacting jump proc.

Summary

Nonlinear Markov chains $\eta_n = \text{Law}(\overline{X}_n)$=Perfect sampling algorithm

- **Nonlinear transport formulae :**

$$\eta_{n+1} = \eta_n K_{n+1,\eta_n}$$

- **Local transitions :**

$$\mathbb{P}(\overline{X}_n \in dx_n \mid \overline{X}_{n-1}) = K_{n,\eta_{n-1}}(\overline{X}_{n-1}, dx_n) \quad \text{avec} \quad \eta_{n-1} = \text{Law}(\overline{X}_{n-1})$$

- **McKean measures (canonical process) :**

$$\mathbb{P}_n(d(x_0,\ldots,x_n)) = \eta_0(dx_0)\ K_{1,\eta_0}(x_0,dx_1)\ldots K_{n,\eta_{n-1}}(x_{n-1},dx_n)$$

Other examples : Gaussian-McKean Vlasov type transitions

$$\overline{X}_{n+1} := d_n\left(\overline{X}_n, \eta_n\right) + \mathcal{N}(0, Q_n)$$

Sampling pb ⇒ Mean field particle interpretations

- Markov Chain $\xi_n = (\xi_n^1, \ldots, \xi_n^N) \in E_n^N$ s.t.

$$\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq_{N \uparrow \infty} \eta_n$$

- Approximated local transitions ($\forall 1 \leq i \leq N$)

$$\xi_{n-1}^i \rightsquigarrow \xi_n^i \sim K_{n,\eta_{n-1}^N}(\xi_{n-1}^i, dx_n)$$

Feynman-Kac models $\Leftrightarrow$ Genetic type stochastic algo.

$$
\left.\begin{array}{c}
\xi_n^1 \\ \vdots \\ \xi_n^i \\ \vdots \\ \xi_n^N
\end{array}\right]
\xrightarrow{S_{n,\eta_n^N}}
\left[\begin{array}{ccc}
\widehat{\xi}_n^1 & \xrightarrow{M_{n+1}} & \xi_{n+1}^1 \\
\vdots & & \vdots \\
\widehat{\xi}_n^i & \longrightarrow & \xi_{n+1}^i \\
\vdots & & \vdots \\
\widehat{\xi}_n^N & \longrightarrow & \xi_{n+1}^N
\end{array}\right]
$$

Accept/Reject-Selection : [Geometric clocks] [Confinement ex. : $G_n = 1_A$]

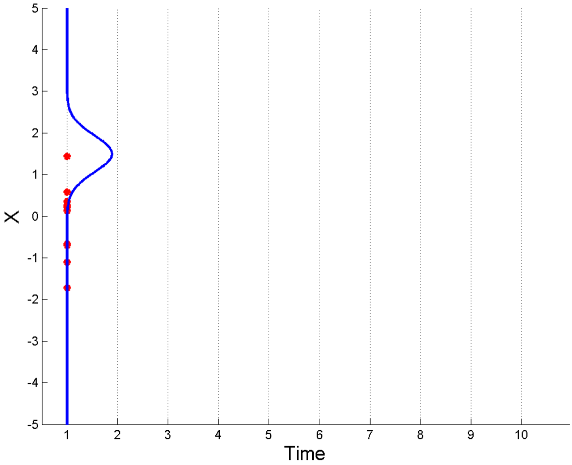
$$
S_{n,\eta_n^N}(\xi_n^i, .) := \epsilon_n G_n(\xi_n^i)\ \delta_{\xi_n^i} + \left(1 - \epsilon_n G_n(\xi_n^i)\right)\ \sum_{j=1}^N \frac{G_n(\xi_n^j)}{\sum_{k=1}^N G_n(\xi_n^k)} \delta_{\xi_n^j}
$$

$\oplus$ Unbias particle normalizing Cts

$$
\mathcal{Z}_n^N := \prod_{0 \le p < n} \eta_p^N(G_p) \simeq \mathcal{Z}_n
$$

$\supset$ *Particle filters, Diffusion Monte Carlo (DMC), Quantum Monte Carlo (QMC), Sequential Monte Carlo methods (SMC), ...*

Genealogical tree evolution in dimension 1



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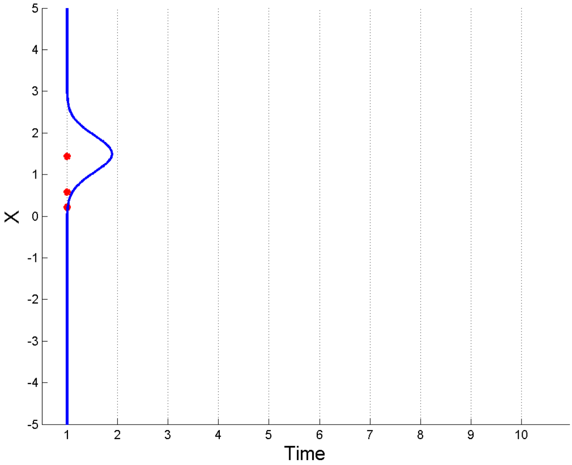

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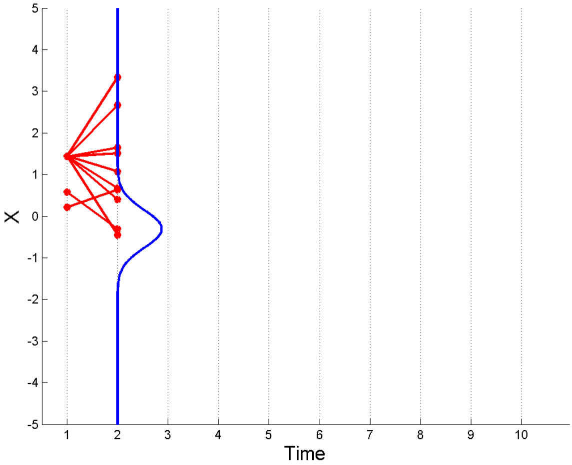

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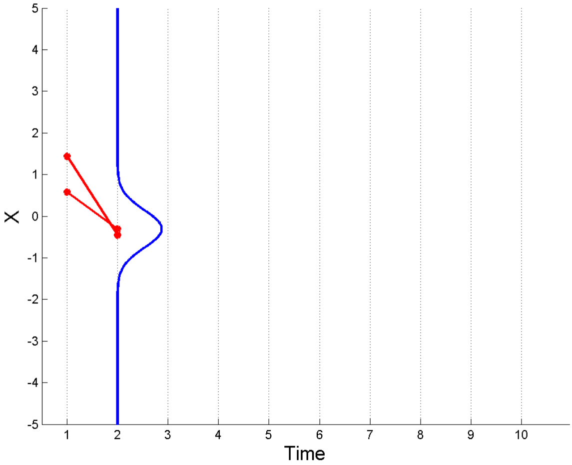

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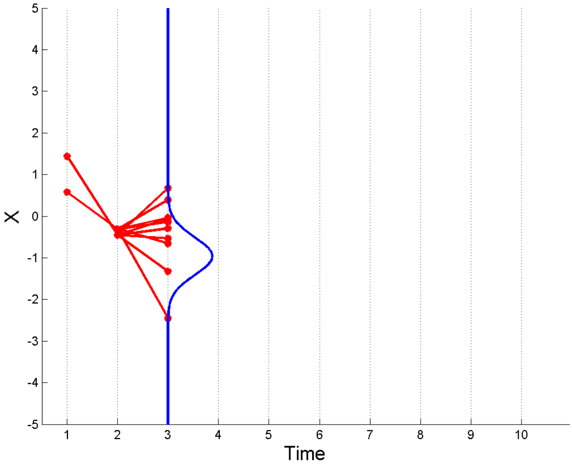

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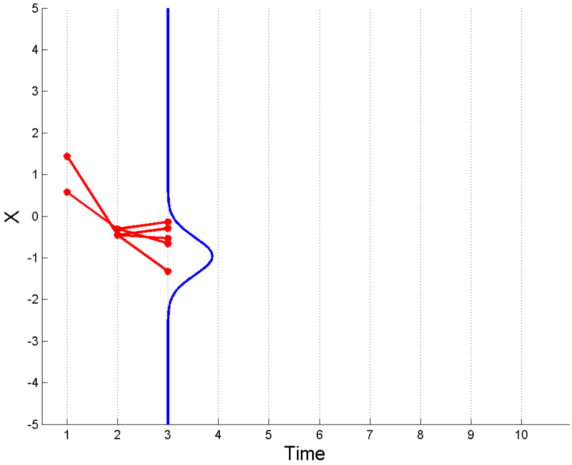

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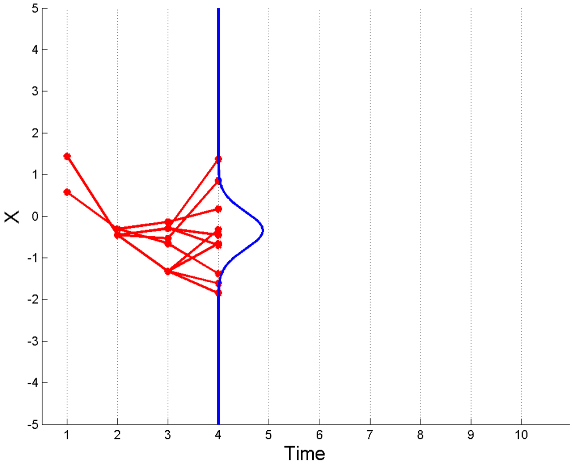

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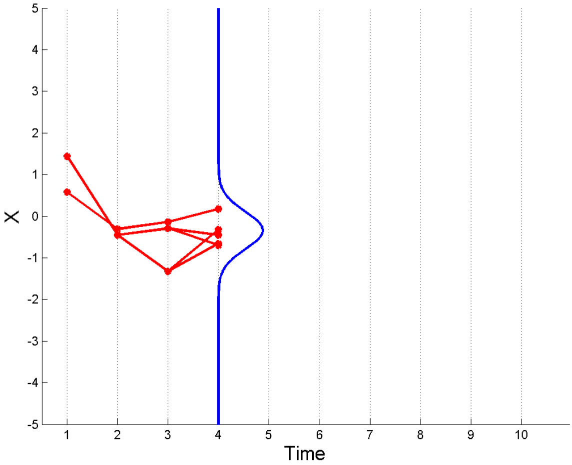

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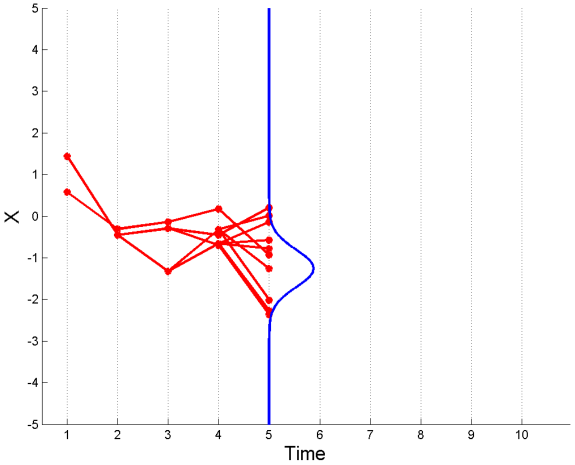

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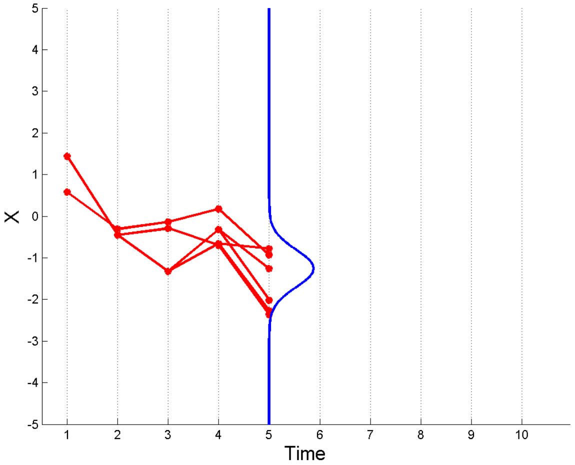

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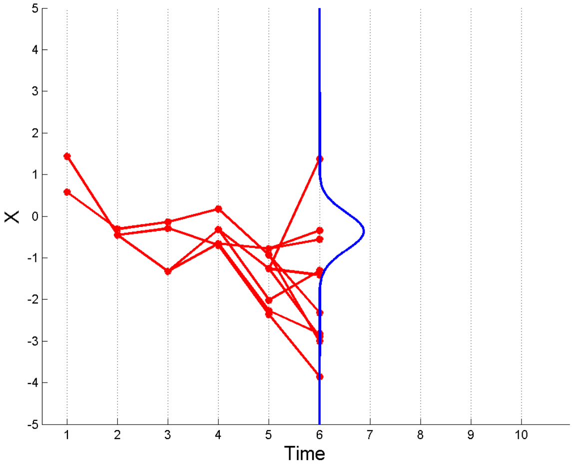

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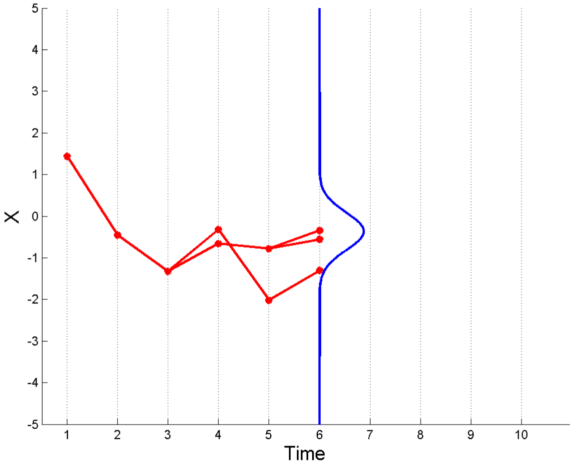

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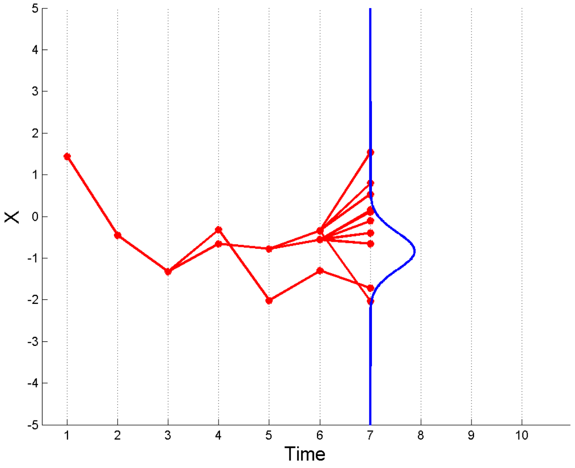

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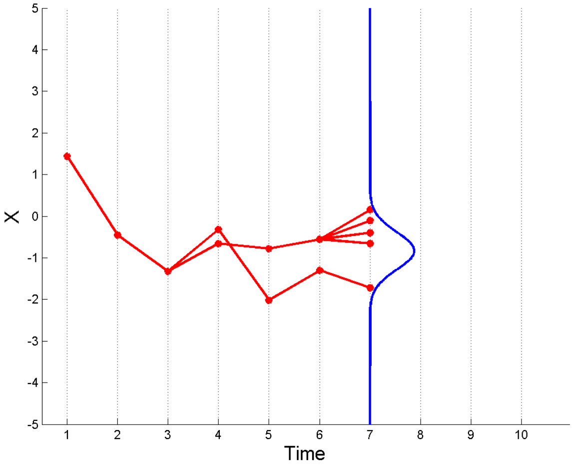

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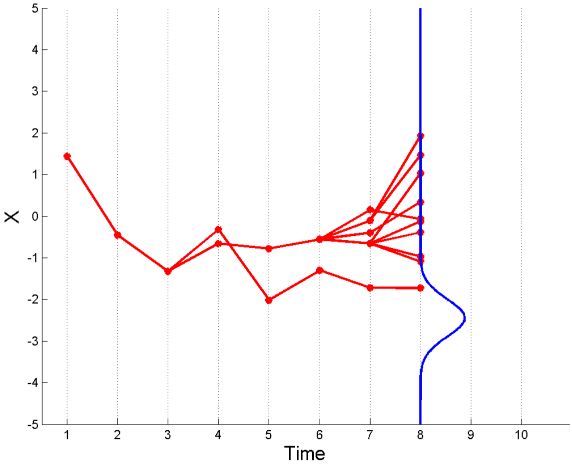

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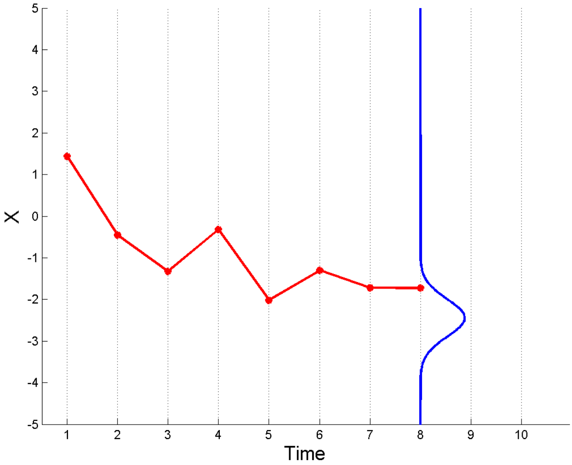

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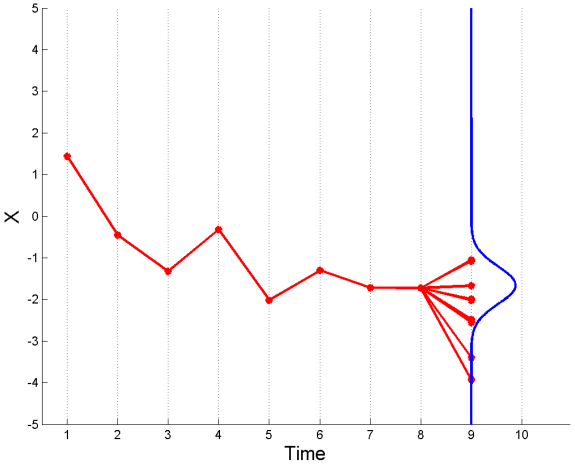

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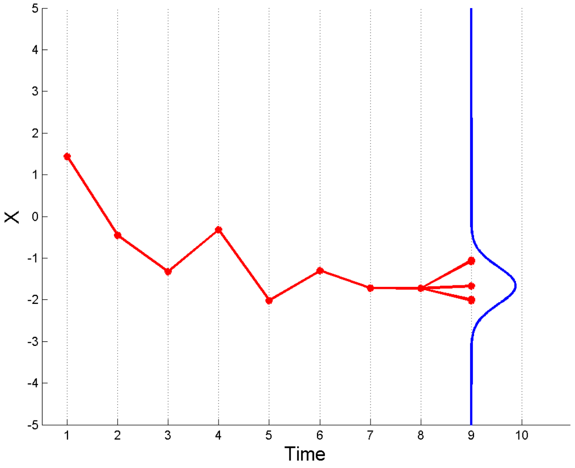

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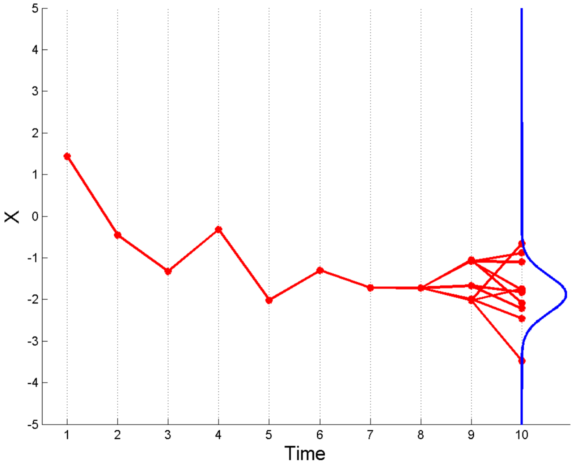
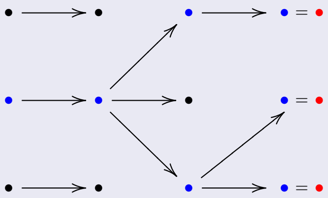


Interaction/branch. process $\hookrightarrow$ 4 types of occupation measures

$(N = 3)$



- Current population $\hookrightarrow \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_n^i \leftarrow i\text{-th individual at time } n} \simeq \eta_n$

- Genealogical tree $\hookrightarrow \frac{1}{N}\sum_{i=1}^{N} \delta_{(\xi_{0,n}^i,\xi_{1,n}^i,\ldots,\xi_{n,n}^i) \leftarrow i\text{-th ancestral line}} \simeq \mathbb{Q}_n$

- **Complete genealogical tree** $\hookrightarrow \frac{1}{N}\sum_{i=1}^{N} \delta_{(\xi_0^i,\xi_1^i,\ldots,\xi_n^i)} \simeq$ **McKean meas.**

- Forward particle approximation $\sim$ complete genealogical tree :

$$\mathbb{Q}_n^N(d(x_0,\ldots,x_n)) = \eta_n^N(dx_n)\; M_{n,\eta_{n-1}^N}(x_n, dx_{n-1})\ldots M_{1,\eta_0^N}(x_1, dx_0)$$

Some key advantages

- Mean field models=stochastic linearization/perturbation technique :

$$\eta_n^N = \eta_{n-1}^N K_{n,\eta_{n-1}^N} + \frac{1}{\sqrt{N}}\ W_n^N$$

with [theorem] $(W_n^N)_n \simeq (W_n)_n$ Indep. Centered Gaussian Fields.

- $\eta_n = \eta_{n-1} K_{n,\eta_{n-1}}$ stable $\Rightarrow$ No propagation of local sampling errors

$\Longrightarrow$ Uniform control w.r.t. the time horizon

- "No burning, no need to study the stability of MCMC type models".
- PDE viewpoint : Stochastic adaptive grid approximation
- Nonlinear system $\rightsquigarrow$ "positive-benefic interactions.
- Simple and natural sampling algorithm.

Summary

1. Introduction, motivations
2. Mean field particle models
3. Convergence analysis
    - Feynman-Kac type models
    - Concentration inequalities

Feynman-Kac models = rather complete analysis [LDP,CLT, Propagation of chaos] (FK Springer 2004 + refs)

- Example: Empirical processes $\mathbb{L}_p$-mean error estimates

$$\sup_{n\geq 0} \sup_{N\geq 1} \sqrt{N}\ \mathbb{E}\left(\sup_{f\in\mathcal{F}_n} |\eta_n^N(f)-\eta_n(f)|^p\right) < \infty$$

- (New) Propagations of chaos *(+Patras & Rubenthaler* (AAP 10)*)*

$$\begin{aligned}\mathbb{P}^N_{n,q} &:= \mathrm{Law}(\xi_n^1,\ldots,\xi_n^q)\\ &\simeq \eta_n^{\otimes q} + \frac{1}{N}\ \partial^1\mathbb{P}_{n,q} + \ldots + \frac{1}{N^k}\ \partial^k\mathbb{P}_{n,q} + \frac{1}{N^{k+1}}\ \partial^{k+1}\mathbb{P}^N_{n,q}\end{aligned}$$

with $\sup_{N\geq 1}\|\partial^{k+1}\mathbb{P}^N_{n,q}\|_{\mathrm{tv}} < \infty$ & $\sup_{n\geq 0}\|\partial^1\mathbb{P}_{n,q}\|_{\mathrm{tv}} \leq c\ q^2$.

- (New) Additive functional *(+ Doucet & Singh* M2AN 10*)* :

$$N\ \mathbb{E}\left([(\mathbb{Q}_n^N-\mathbb{Q}_n)(F_n)]^2\right) \leq c \times (1/n+1/N)$$

Concentration inequalities

- Large deviation principles

  [joint works with D. Dawson (Springer 05), A. Guionnet (SPA 98,IHP 01), T. Zajic (Bernoulli 03)]

$$\rightsquigarrow \sup_{n\geq 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(\sup_{p\leq n} \frac{1}{2} \sum_{1\leq k\leq d} |\eta_p^N(A_k) - \eta_p(A_k)| \geq \epsilon \right) \leq -\epsilon^2/\beta(\Phi)$$

  with $\beta(\Phi)$ related to the contraction properties of the sg $\Phi_{p,n}(\eta_p) = \eta_n$.

- Non asymptotic estimates

  [joint wortks with M. Ledoux (JTP 00), A. Doucet & S.Singh *(Feynman-Kac path space models & Additive functionals)* (M2AN-10)]

$$\rightsquigarrow \sup_{n\geq 0} \frac{1}{N} \log \ \mathbb{P}\left(\left|[\mathbb{Q}_n^N - \mathbb{Q}_n](F_n)\right| \geq \frac{b}{\sqrt{N}} + \epsilon \right) \leq -\epsilon^2/(2b^2)$$

Concentration inequalities $\sim$ stability prop. semigroups $\Phi_{p,n}(\eta_p) = \eta_n$

[+ E. Rio (HAL-INRIA 09 & AAP 10) $\supset$ FK and McKean-Vlasov type models]

$$(\mathbf{H\Phi})\ \forall \Phi_{p,n}\ s.t. \quad \Phi_{p,n}(\eta) - \Phi_{p,n}(\mu) = [\eta - \mu] \underbrace{D_\mu \Phi_{p,n}}_{\text{1st order op.}} + \underbrace{\mathcal{R}_{p,n}(\eta,\mu)}_{\text{2nd order } (\eta-\mu)^{\otimes 2}(\ldots)}$$

Example : FK semigroups

$$\Phi_{p,n}(\eta)(f) \propto \eta Q_{p,n}(f) \quad \text{with} \quad Q_{p,n}(f) = \mathbb{E}_{p,x_p}\left(f(X_n) \prod_{p \le q < n} G_q(X_q) \right)$$

Taylor expansion $\Rightarrow$ (HΦ)

$$\begin{aligned} \left[\Phi_{p,n}(\mu + \epsilon\nu) - \Phi_{p,n}(\mu)\right](f) &\propto (\not\mu + \epsilon\nu) Q_{p,n}\left[f - \Phi_{p,n}(\mu)(f)\right] \\ &\propto \epsilon \times \nu D_\mu \Phi_{p,n}(f) \end{aligned}$$

$$\text{with the operator} \qquad D_\mu \Phi_{p,n} \propto Q_{p,n}\left[Id - \Phi_{p,n}(\mu)\right]$$

Concentration inequalities $\sim$ stability prop. semigroups $\Phi_{p,n}(\eta_p) = \eta_n$

[Theo:] $(\mathbf{H\Phi}) \Rightarrow \forall x \geq 0$, the probability of the next events is $\geq 1 - e^{-x}$

$$(\eta_n^N - \eta_n)(f) \leq \frac{r_n}{N}\left(1 + \epsilon_0^{-1}(x)\right) + \overline{\sigma}_n^2\, b_n^\star\, \epsilon_1^{-1}\left(\frac{x}{N\overline{\sigma}_n^2}\right) \quad \text{[Bennett } (r=0)\text{]}$$

$$(\eta_n^N - \eta_n)(f) \leq \frac{r_n}{N}\left(1 + \epsilon_0^{-1}(x)\right) + \sqrt{\frac{2x}{N}}\,\beta_n \quad \text{[Hoeffding } (r=0)\text{]}$$

with

$$\epsilon_0(\lambda) = \frac{1}{2}\left(\lambda - \log(1+\lambda)\right), \quad \epsilon_1(\lambda) = (1+\lambda)\log(1+\lambda) - \lambda$$

Cts :

$r_n \sim$ bias second order & $\overline{\sigma}_n, \beta_n, b_n^\star \sim$ Dobrushin coef. 1st order operators.

Note : Crude bounds

$$\epsilon_0^{-1}(x) \leq 2x + 2\sqrt{x} \qquad \epsilon_1^{-1}(x) \leq x/3 + \sqrt{2x} \quad \rightsquigarrow \text{[Bernstein]}$$

Application : *Time homogeneous Feynman-Kac models* (M, G)

$$(H)\ \exists m\ :\quad M^m(x,\cdot) \geq \epsilon\ M^m(y,\cdot) \quad \text{and} \quad \delta := \sup \prod_{0\leq p<m} \left(G(x_p)/G(y_p)\right) < \infty$$

[Corollary:]

$\forall x \geq 0$, $\forall$ time horizon n, the probability of the next events is $\geq 1 - e^{-x}$

$$(\eta_n^N - \eta_n)(f) \quad \leq \quad \frac{4a}{N}\left(1 + \epsilon_0^{-1}(x)\right) + 8a\ \epsilon_1^{-1}\left(\frac{x}{4bN}\right)$$

$$(\eta_n^N - \eta_n)(f) \quad \leq \quad \frac{4a}{N}\left(1 + \epsilon_0^{-1}(x)\right) + 2\sqrt{\frac{2bx}{N}}$$

with $a \leq m\ (\delta/\epsilon)^5$ and $b \leq m\ (\delta/\epsilon)^4$.