On the Stability of Feynman-Kac semigroups

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$\sim$ **Joint works : A. Guionnet, L. Miclo.**

Outline

Notation

E measurable space, $\mathcal{P}(E)$ proba. on E, $\mathcal{B}(E)$ bounded meas. functions.

- $(\mu, f) \in \mathcal{P}(E) \times \mathcal{B}(E) \quad \longrightarrow \quad \mu(f) = \int f(x)\mu(dx)$
- $M(x, dy)$ **integral operator on E**

$$\begin{aligned} M(f)(x) &= \int M(x, dy) f(y) \\ [\mu M](dy) &= \int \mu(dx) M(x, dy) \Longleftrightarrow [\mu M](f) = \mu[M(f)] \end{aligned}$$

- **Boltzmann-Gibbs transformation :** $G : E \to [0, 1]$ with $\mu(G) > 0$

$$\Psi_G(\mu)(dx) = \frac{1}{\mu(G)}\; G(x)\; \mu(dx)$$

Updating-prediction transformations

Time parameter $n \in \mathbb{N}$, $M_n(x, dy)$ Markov transitions and $G_n : E \to [0,1]$

$$\eta_{n+1} = \Phi_{n+1}(\eta_n) := \Psi_{G_n}(\eta_n) M_{n+1} \qquad (1)$$

Markov chain X_n with transitions M_n and initial condition $X_0 \simeq \eta_0$:

$$(1) \Longleftrightarrow \eta_n(f) \propto \gamma_n(f) = \mathbb{E}\left(f(X_n) \prod_{0 \leq p < n} G_p(X_p) \right)$$

Observations

- $G_n = 1 \Rightarrow \eta_n =Law(X_n)$.
- Partition functions-Normalizing constants :

$$\gamma_n(1) = \mathbb{E}\left(\prod_{0 \leq p < n} G_p(X_p) \right) = \prod_{0 \leq p < n} \eta_p(G_p)$$

Nonlinear distribution flows

- **Nonlinear Markov models :** $\eta_n = \eta_{n-1} K_{n,\eta_{n-1}} = \mathrm{Law}\left(\overline{X}_n\right)$

$$K_{n+1,\eta_n}(x,dz) \quad = \quad \int S_{n,\eta_n}(x,dy)\ M_{n+1}(y,dz)$$

$$S_{n,\eta_n}(x,dy) := G_n(x)\ \delta_x(dy) + (1 - G_n(x))\ \ \Psi_{G_n}(\eta_n)(dy)$$

Mean field particle interpretation

- Markov chain $\xi_n = (\xi_n^1, \ldots, \xi_n^N) \in E_n^N$ s.t.

$$\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \ \simeq_{N\uparrow\infty} \ \eta_n$$

- Particle approximation transitions ($\forall 1 \leq i \leq N$)

$$\xi_{n-1}^i \rightsquigarrow \xi_n^i \sim K_{n,\eta_{n-1}^N}(\xi_{n-1}^i, dx_n)$$

Particle absorption models

- Unnormalized linear semigroup :

$$\gamma_n = \gamma_{n-1} Q_n$$

with the sub-Markov operator :

$$Q_n(x, dy) = G_{n-1}(x)\ M_n(x, dy)$$

- $\rightsquigarrow$ Markov on $E^c = E \cup \{c\}$ (with $f(c) = 0$).

$$X_n^c \in E_n^c \xrightarrow{\text{absorption } \sim (1-G_n)} \widehat{X}_n^c \xrightarrow{\text{free exploration } \sim M_{n+1}} X_{n+1}^c$$

With:

- **Absorption:** $\widehat{X}_n^c = X_n^c$, with proba $G_n(X_n^c)$; otherwise $\widehat{X}_n^c = c$.
- **Exploration:** elementary free explorations $X_n \rightsquigarrow X_{n+1}$

Log-Lyapunov exponents λ and Ground state energies H

- $T = \inf\{n : \widehat{X}_n^c = c\} \Rightarrow \gamma_n(f) = \mathbb{E}(f(X_n^c)\ ;\ (T \geq n)) = \mathbb{E}(f(X_n^c))$.
- **Time homogeneous models :** $Q(x, dy) = G(x)\ M(x, dy)$

$$\mathbb{P}(T \geq n) \simeq e^{-\lambda n}$$

with $e^{-\lambda} \overset{M\ reg.}{=}$ Q-top eigenvalue or

$$\begin{aligned} \lambda &= -\mathrm{LogLyap}(Q) = \lim_{n\to\infty} -\frac{1}{n} \log \|Q^n\| \\ &= \lim_{n\to\infty} -\frac{1}{n} \sum_{0\leq p\leq n} \log \eta_p(G) = -\log \eta_\infty(G) \end{aligned}$$

$M \quad \mu - \text{reversible :}$

$$\mathbb{E}(f(X_n^c) \mid T > n) \simeq \Psi_G(\eta_\infty)(f) = \frac{\mu(H\ f)}{\mu(H)} \quad \text{with} \quad Q(H) = e^{-\lambda} H$$

Open questions

- **Fixed point measures :** $\eta_\infty = \Phi(\eta_\infty) = \Psi_G(\eta_\infty)M$?
- **Stability properties :**

$$\eta_n = \Phi_{p,n}(\eta_p) \longrightarrow_{n\uparrow\infty} \eta_\infty \ ?$$

- **Related questions $\sim$ different application model areas.**
  - Filtering : $Y_n = H_n(X_n, V_n)$

$$G_n = \text{likelihood functions} \rightsquigarrow \eta_n = \text{Law}(X_n \mid Y_0, \ldots, Y_{n-1})$$

  - $\eta_0 = \text{Law}(X_0)$ unknown $\rightsquigarrow$ asympt. stability properties :

$$[\Phi_{0,n}(\eta_0) - \Phi_{0,n}(\eta_0')] \longrightarrow_{n\uparrow\infty} 0 \ ?$$

- **Mean field models = Particle filters =Quantum Monte Carlo.**
  Stab. prop. $\rightsquigarrow \exists$ Uniform control of the mean errors estimates ?

h convex, $h(ax,ay) = ah(x,y) \in \mathbb{R} \cup \{\infty\}$, $h(1,1) = 0$

$$H(\eta,\mu) = \int h(d\eta, d\mu) = \int g(d\eta/d\mu)\ d\mu \qquad \text{with} \quad g(x) = h(x,1)$$

Ex.:

- *Total variation and $\mathbb{L}_p$-norms:* $g(t) \propto |t-1|^p$
- *Boltzmann or Shannon-Kullback entropy:* $g(t) = t\log t$
- *Havrda-Charvat entropy order $p > 1$:* $g(t) = \frac{1}{p-1}(t^p - 1)$
- *Kakutani-Hellinger integrals order $\alpha \in (0,1)$:* $g(t) = t - t^\alpha$

Markov transition $M(x, dy)$

- **Definition :**

$$\beta(M) := \sup_{x,y} \|M(x,.) - M(y,.)\|_{tv}$$

- **Theorem** [L.Miclo, M. Ledoux, P.DM (PTRF 2003)]

$$H(\mu M, \eta M) \leq \beta(M)\ H(\mu, \eta)$$

Ex.: $M(x,.) \geq \epsilon M(y,.) \Longrightarrow \beta(M) \leq (1-\epsilon)$

$$\Downarrow$$

- **Corollary** *(Filtering with a wrong initial condition $\eta'_0 \rightsquigarrow \eta'_n$)*

$$\mathbb{E}(\mathrm{Ent}(\eta_n \mid \eta'_n)) \leq \left[\prod_{p=1}^{n} \beta(M_p)\right] \mathrm{Ent}(\eta_0 \mid \eta'_0)$$

- **Unnormalized semigroups :**

$$\gamma_p Q_{p,n} = \gamma_n \quad \text{with} \quad Q_{p,n}(f)(x) := \mathbb{E}_{p,x}\left(f(X_n) \prod_{p \leq k < n} G_k(X_k) \right)$$

- **Normalized semigroups :**

$$\Phi_{p,n}(\mu)(f) = \frac{\mu Q_{p,n}(f)}{\mu Q_{p,n}(1)} = \frac{\mu(G_{p,n} P_{p,n}(f))}{\mu(G_{p,n})}$$

with

$$G_{p,n} := Q_{p,n}(1) \quad \text{and} \quad P_{p,n}(f) := \frac{Q_{p,n}(f)}{Q_{p,n}(1)}$$

$$\Downarrow$$

Updating/Prediction form : $\Phi_{p,n}(\mu) = \Psi_{G_{p,n}}(\mu) P_{p,n}$

Theorem

- Dobrushin's contraction coef. of the Markov operator $P_{p,n}$

$$\beta(P_{p,n}) = \sup_{\mu,\eta} \|\Phi_{p,n}(\eta) - \Phi_{p,n}(\mu)\|_{tv}$$

- $h(x,y)$ suff. regular. $\Rightarrow\ \exists\ \alpha_h(t)\ \uparrow$ s.t.

$$H(\Phi_{p,n}(\mu), \Phi_{p,n}(\eta)) \ \leq\ \alpha_h(g_{p,n})\ \beta(P_{p,n})\ H(\mu,\eta)$$

  with

$$g_{p,n} := \sup_{x,y} G_{p,n}(x)/G_{p,n}(y)$$

- **Ex. :**
  $\alpha_h(t) = t$ (total variation norm and Boltzmann entropy),
  $\alpha_h(t) = t^{1+p}$ (Havrda-Charvat and Kakutani-Hellinger integrals of order p, $\alpha_h(t) = t^3$ ($\mathbb{L}_2$-norms),...

Contraction estimates

- **(H)$_m$:** $M^m(x,.) \geq \epsilon\, M^m(y,.)$ and $G_n(x) \leq r\, G_n(y)$
- **Lemma :**

$$(H)_m \Longrightarrow \sup_{n\geq p} g_{p,n} \leq r^m/\epsilon \quad \text{and} \quad \beta(P_{p,p+nm}) \leq (1-\epsilon^2/r^{m-1})^n$$

- $\Rightarrow$ **Corollary :**

$$\|\Phi_{p,p+nm}(\eta) - \Phi_{p,p+nm}(\mu)\|_{tv} \leq (1-\epsilon^2/r^{m-1})^n$$

and

$$H(\Phi_{p,p+nm}(\mu), \Phi_{p,p+nm}(\eta)) \leq \alpha_h(r^m/\epsilon)\,(1-\epsilon^2/r^{m-1})^n\, H(\mu,\eta)$$

- **Extensions:** *nonhomogeneous models, continuous time FK s.g., etc.*

A local transport formulation

$$W_n^N := \sqrt{N}\ \left[\eta_n^N - \Phi_n\left(\eta_{n-1}^N\right)\right] \simeq W_n \ \perp\ \text{Gaussian field}$$

$$
\begin{array}{ccccccccc}
\eta_0 & \rightarrow & \eta_1 = \Phi_1(\eta_0) & \rightarrow & \eta_2 = \Phi_{0,2}(\eta_0) & \rightarrow & \cdots & \rightarrow & \Phi_{0,n}(\eta_0) \\
\Downarrow & & & & & & & & \\
\eta_0^N & \rightarrow & \Phi_1(\eta_0^N) & \rightarrow & \Phi_{0,2}(\eta_0^N) & \rightarrow & \cdots & \rightarrow & \Phi_{0,n}(\eta_0^N) \\
 & & \Downarrow & & & & & & \\
 & & \eta_1^N & \rightarrow & \Phi_2(\eta_1^N) & \rightarrow & \cdots & \rightarrow & \Phi_{1,n}(\eta_1^N) \\
 & & & & \Downarrow & & & & \\
 & & & & \eta_2^N & \rightarrow & \cdots & \rightarrow & \Phi_{2,n}(\eta_2^N) \\
 & & & & & & & & \vdots \\
 & & & & & & \Downarrow & & \\
 & & & & & & \eta_{n-1}^N & \rightarrow & \Phi_n(\eta_{n-1}^N) \\
 & & & & & & & & \Downarrow \\
 & & & & & & & & \eta_n^N
\end{array}
$$

$\rightsquigarrow$ Key decomposition formula

$$\eta_n^N - \eta_n \quad = \quad \sum_{q=0}^{n} \left[\Phi_{q,n}(\eta_q^N) - \Phi_{q,n}(\Phi_q(\eta_{q-1}^N))\right]$$

Mean field particle models=DMC=QMC=Particle filters=SMC=...

- **Bias estimates :** $\mathrm{osc}(f) \leq 1$

$$N \left|\mathbb{E}\left(\left[\eta_n^N - \eta_n\right](f)\right)\right| \leq 4 \sum_{p=0}^{n} g_{p,n}^3 \; \beta(P_{p,n})$$

- **Crude $\mathbb{L}_p$-estimates :**

$$\sqrt{N}\; \mathbb{E}\left(\left|\left[\eta_n^N - \eta_n\right](f)\right|^p\right)^{\frac{1}{p}} \leq 2\; b(p) \sum_{p=0}^{n} g_{p,n} \; \beta(P_{p,n})$$

with $b(2p)^{2p} = (2p)_p \; 2^{-p}$ and $b(2p+1)^{2p+1} = \frac{(2p+1)_{(p+1)}}{\sqrt{p+1/2}} \; 2^{-(p+1/2)}$.

- **$(\mathbf{H})_m \Rightarrow$ Uniform estimates $\rightsquigarrow$ Example :**

$$\sup_{n\geq 0} \sup_{N\geq 1} \sqrt{N}\; \mathbb{E}\left(\left|\left[\eta_n^N - \eta_n\right](f)\right|^p\right)^{\frac{1}{p}} \leq 2\; b(p)\; m\; r^{2m-1}/\epsilon^3$$

Mean field particle models=DMC=QMC=Particle filters=SMC=...

- **A crude exponential estimates :** $\forall\ \mathrm{osc}(f) \leq 1,\ \delta \in [0, 1/2]$

$$\mathbb{P}\left(\left| \left[\eta_n^N - \eta_n\right](f)\right| \geq \ \delta\right) \leq 6\ \exp\left(-\frac{N\ \delta^2}{32 \sum_{p=0}^{n}\ g_{p,n}^3\ \beta(P_{p,n})}\right)$$

- **(H)$_m$ for some $(\epsilon, r) \Rightarrow$ Uniform concentration estimates :**

$$\sup_{n\geq 0} \mathbb{P}\left(\left| \left[\eta_n^N - \eta_n\right](f)\right| \geq \ \delta\right) \leq 6\ \exp\left(-N\ \delta^2\ \epsilon^5/(32 m r^{4m-1})\right)$$

- **Extensions to seminorms** $\|\eta_n^N - \eta_n\|_{\mathcal{F}} = \sup_{f\in\mathcal{F}} |[\eta_n^N - \eta_n](f)|$

- **Contraction and stability analysis :**
  - On the stability of interacting processes with applications to filtering and Genetic Models. Joint work with Guionnet A. *Ann. de l'Inst. H. Poincaré*, Vol. 37, No. 2, 155-194 (2001).
  - Asymptotic Stability of Non Linear Semigroup of Feynman-Kac Type. Joint work with Miclo L. *Ann. Fac. Sci. Toulouse Math.* (6) 11 no. 2, pp. 135-175, (2002).
  - Contraction properties of Markov kernels. Joint work withLedoux M. and Miclo L. *Probab. Theory and Related Fields*, vol. 126, pp. 395-420, (2003).
- **Mean field particle models**
  - `Feynman-Kac formulae. Genealogical and interacting particle systems`, P. Del Moral, Springer (2004) ⊕ Refs.
  - Sequential Monte Carlo Simulation: Methods and Theory. Book Project in progress with A. Doucet and A. Jasra.

Some references

- **Particle absorption models**
  - joint work with L. Miclo. `Particle Approximations of Lyapunov Exponents Connected to Schrodinger Operators and Feynman-Kac Semigroups`. ESAIM Probability & Statistics, vol. 7, pp. 169-207 (2003).
  - joint work with A. Doucet. Particle Motions in Absorbing Medium with Hard and Soft Obstacles. Stochastic Analysis and Applications, vol. 22 (2004).