Branching particle models in environmental studies

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Outline

Nonlinear finite state space models

Individuals with type $\in E = \{1, \ldots, d\}$ at time $t \in \mathbb{N}$ or $\mathbb{R}_+$

$$\eta_t = [\eta_t(1), \ldots, \eta_t(d)] \in \text{Proportion space} = \text{Probab. on E}$$

Types : predators-preys, species, colonies, etc.

- **Discrete time** $\eta_{t+1}(j) = \sum_{i=1}^{d} \eta_t(i)\ K_{t,\eta_t}(i,j)$
  with a nonlinear Markov chain $\overline{X}_t$:

$$\mathbb{P}\left(\overline{X}_{t+1} = j \mid \overline{X}_t = i\right) := K_{t,\eta_t}(i,j) \quad \text{with} \quad \eta_t = \text{Law}(\overline{X}_t)$$

- **Continuous time** (*Logistic, Lotka Volterra type, simplex syst.*)

$$\frac{d}{dt}\eta_t(j) = \sum_{i=1}^{d} \eta_t(i)\ (K_{t,\eta_t}(i,j) - \delta_j(i))$$

  with a nonlinear Markov process $\eta_t = \text{Law}(\overline{X}_t)$

Duality model

- $\eta := (\eta(1), \ldots, \eta(d))$

$$f := \begin{pmatrix} f(1) \\ \vdots \\ f(d) \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} K(1,1) & \cdots & K(1,d) \\ \vdots & \vdots & \vdots \\ K(d,1) & \cdots & K(d,d) \end{pmatrix}.$$

$$\Downarrow \ (f = 1_i)$$

$$\eta(f) = \sum_{j=1}^{d} \eta(j)\ f(j) = \eta(i)\ , \quad K(f)(j) = \sum_{k=1}^{d} K(j,k) f(k) = K(j,i)$$

$$\begin{aligned} \text{and} \quad \eta(K(f)) &= \sum_{i=1}^{d} \eta(i) \left[\sum_{j=1}^{d} K(i,j)\ f(j) \right] \\ &= \sum_{j=1}^{d} \left[\sum_{i=1}^{d} \eta(i)\ K(i,j) \right]\ f(j) := (\eta K)(f) \end{aligned}$$

Weak equations $\rightsquigarrow$ Individual based & mean field particle models

- **Weak sol. of nonlinear and discrete generation dyn. syst. :**

$$\forall j \quad \eta_{t+1}(j) = \sum_{i=1}^{d} \eta_t(i)\; K_{t,\eta_t}(i,j) \Longleftrightarrow \forall f \quad \eta_{t+1}(f) = \eta_t K_{t,\eta_t}(f)$$

- **Weak solutions of nonlinear PDE :**

$$\forall j \quad \frac{d}{dt}\eta_t(j) = \sum_{i=1}^{d} \eta_t(i)\; (K_{t,\eta_t}(i,j) - \delta_j(i))$$

$$\Longleftrightarrow \forall f \quad \frac{d}{dt}\eta_t(f) := \eta_t(K_{t,\eta_t} - Id)(f) = \eta_t L_{t,\eta_t}(f)$$

- Stochastic processes : Branching, birth and death, cloning, jumps, free evolution, historical processes , genealogical tree evolutions, etc.
- State space examples :

$$E = \mathbb{R}^d, \mathbb{N}^{d_1} \cup \mathbb{R}^{d_2}, \cup_{d \geq 0} S^d, \text{path spaces}, \text{excursion spaces}$$

- Duality models:

$$\begin{aligned} \eta(K(f)) &= \int \eta(dx) \left[\int K(x,dy)\ f(y) \right] \\ &= \int \left[\int \eta(dx)\ K(x,dy) \right] f(y) = (\eta K)(f) \end{aligned}$$

Weak equations $\rightsquigarrow$ Individual based & mean field particle models

- **Weak sol. of nonlinear and discrete generation dyn. syst. :**

$$\eta_{t+1}(f) = \eta_t K_{t,\eta_t}(f)$$

- **Weak solutions of nonlinear PDE :**

$$\frac{d}{dt}\eta_t(f) = \eta_t L_{t,\eta_t}(f)$$

Example : Feynman-Kac path models $\rightsquigarrow$ Genetic type particle models

$$\eta_t(f) \propto \mathbb{E}\left(f(X_t)\prod_{0\leq s\leq t} G_s(X_s)\right) \text{ or } \mathbb{E}\left(f(X_t)\exp\int_{0\leq s\leq t} V_s(X_s)ds\right)$$

Not excluded path space models $X_t = (X'_s)_{0\leq s\leq t}$

Tools $\rightsquigarrow$ **Dynamical Syst. $\cap$ Partial differential eq. $\cap$ Probability**

Individual based & mean field particle models

- **Discrete time $\rightsquigarrow$ Markov chain $\xi_t = (\xi_t^1, \ldots, \xi_t^N) \in E^N$ s.t.**

$$\eta_t^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_t^i} \simeq_{N \uparrow \infty} \eta_t$$

  Approximate local transitions :

$$(\forall 1 \leq i \leq N) \qquad \xi_{t-1}^i \rightsquigarrow \xi_t^i \sim K_{t,\eta_{t-1}^N}(\xi_{t-1}^i, dx_t)$$

- **Some advantages :**
  - Microscopic-individual interpretations, historical processes,...
  - Mean field particle=stoch. perturbation-linearization model:

$$\eta_t^N = \eta_{t-1}^N K_{t,\eta_{t-1}^N} + \frac{1}{\sqrt{N}} W_t^N$$

    with $W_t^N \simeq W_t$ i.i.d. centered gaussian fields.
  - $\eta_t = \eta_{t-1} K_{t,\eta_{t-1}}$ stable $\Rightarrow$ non propagation local errors.

Individual based & mean field particle models

- **Continuous time** $\rightsquigarrow$ **Markov process** $\xi_t = (\xi_t^1, \ldots, \xi_t^N) \in E^N$ **s.t.**

$$\eta_t^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_t^i} \simeq_{N\uparrow\infty} \eta_t$$

Infinitesimal generators :

$$\mathcal{L}_t(F)(x^1, \ldots, x^N) := \sum_{i=1}^N L^{(i)}_{t, \frac{1}{N} \sum_{i=1}^N \delta_{x^i}} F(x^1, \ldots, x^i, \ldots, x^N)$$

- **stoch. perturbation-linearization model**

$$d\eta_t^N(f) = \eta_t^N L_{t,\eta_t^N}(f) dt + \frac{1}{\sqrt{N}} \, dM_t^N(f)$$

with

$$\langle M^N(f) \rangle_t = \int_0^t \eta_s^N \Gamma_{L_{s,\eta_s^N}}(f, f) \, ds$$

Free N-diffusion models (between branching-death-jump events)

- 1-dim SDE :

$$dX_t = a_t(X_t)dt + b_t(X_t)dW_t$$

- 1-dim infinitesimal generator :

$$df(X_t) = L_t(f)(X_t)dt + dM_t(f)$$

- N independent free L_t-evolutions

$$\left\{ \begin{array}{rcl} d\xi_t^i & = & a_t(\xi_t^i)dt + b_t(\xi_t^i)dW_t^i \\ i & = & 1, \ldots, N \end{array} \right.$$

- N-dim infinitesimal generator :

$$\mathcal{L}_t^{free}(F)(x^1, \ldots, x^N) := \sum_{i=1}^{N} L_t^{(i)} F(x^1, \ldots, x^i, \ldots, x^N)$$

Free N-diffusion models

- $\xi_t = (\xi_t^1, \ldots, \xi_t^N) \rightsquigarrow dF(\xi_t) = \mathcal{L}_t^{free}(F)dt + d\mathcal{M}_t(F)$ with

$$\frac{d}{dt}\langle \mathcal{M}(F)\rangle_t = \Gamma_{\mathcal{L}_t^{free}}(F,F)(\xi_t) := \mathcal{L}_t^{free}\left([F - F(\xi_t)]^2\right)(\xi_t)$$

- Empirical processes : $F(\xi_t) = m(\xi_t)(f) := \frac{1}{N}\sum_{i=1}^N f(\xi_t^i)$

$$L_t^{(i)}F(\xi_t) = \frac{1}{N}L_t(f)(\xi_t^i) \quad \Rightarrow \quad dm(\xi_t)(f) = m(\xi_t)(L_t f)dt + dM_t^N(f)$$

and

$$\begin{aligned}
\mathcal{L}_t^{free}\left([F - F(\xi_t)]^2\right)(\xi_t) &= \frac{1}{N^2}\sum_{i=1}^N L_t\left(\left[f - f(\xi_t^i)\right]^2\right)(\xi_t^i) \\
&= \frac{1}{N}\, m(\xi_t)\left(\Gamma_{L_t}(f,f)\right) = \frac{d}{dt}\langle M^N(f)\rangle_t
\end{aligned}$$

- $\rightsquigarrow$ LLN : $m(\xi_t) \simeq_{N\uparrow\infty} \eta_t =Law(X_t)$ 1-diff. L_t-model.

Branching mean field particle models

Stochastic processes $\in \cup_{N\geq 0} E^N$, $E^0 := \{c\}$ cemetery-coffin state

Birth and death type infinitesimal generator (N_0=initial pop. size)

$$\forall x = (x^1, \ldots, x^N) \in \cup_{N\geq 1} E^N \rightsquigarrow m(x) := \frac{1}{N_0} \sum_{i=1}^{N} \delta_{x^i}$$

$$\mathcal{L}_t^{b,d}(F)(x^1, \ldots, x^N)$$

$$:= \sum_{i=1}^{N} L_{t,m(x)}^{(i)} F(x^1, \ldots, x^i, \ldots, x^N)$$

$$= \sum_{i=1}^{N} \lambda_t(m(x), x^i)$$

$$\times \int_{\cup_{d\geq 0} E^d} \left(F(x^1, \ldots, x^{i-1}, u, x^{i+1}, \ldots, x^N) - F(x^1, \ldots, x^N)\right) \; B_{t,m(x)}(x^i, du)$$

Death $u = c$, jump $x^i \rightsquigarrow u \in E$, d-birth $x^i \rightsquigarrow u = (u^1, \ldots, u^{d+1})$, cloning $x^i \rightsquigarrow u = (x^i, \ldots, x^i)$,...

Feynman-Kac particle models with $N_0 = N_t$

$$\eta_t(f) \propto \mathbb{E}\left(f(X_t)\exp\left\{-\int_{0\leq s\leq t} V_s(X_s)ds\right\}\right) \quad \text{with} \quad X_t = L_t - \text{motion}$$

$$\rightsquigarrow \mathcal{L}_t = \mathcal{L}_t^{\text{free}} + \mathcal{L}_t^{\text{jump}}$$

$$\mathcal{L}_t(F)(x^1,\ldots,x^N)$$

$$:= \sum_{i=1}^N L_t^{(i)} F(x^1,\ldots,x^i,\ldots,x^N)$$

$$+\sum_{i=1}^N V_t(x^i)$$

$$\times \int_E \left(F(x^1,\ldots,x^{i-1},u,x^{i+1},\ldots,x^N) - F(x^1,\ldots,x^N)\right) \; m(x)(du)$$

$\rightsquigarrow$ LLN-Propagation of chaos : $m(\xi_t) \simeq_{N\uparrow\infty} \eta_t$.

1-Birth $\rightsquigarrow u = (x^i, y)$ with rate $= \lambda_t^b(x^i)$

$$F(x) = m(x)(f) = \frac{1}{N_0} \sum_{i=1}^{N} f(x^i)$$

$$\Downarrow$$

$$\mathcal{L}_t^b(F)(x^1, \ldots, x^N)$$

$$= \sum_{i=1}^N \lambda_t^b(x^i)$$

$$\times \int_{\cup_{d\geq 0} E^d} \left(F(x^1, \ldots, x^{i-1}, u, x^{i+1}, \ldots, x^N) - F(x^1, \ldots, x^N)\right) \; B_{t,m(x)}(x^i, du)$$

$$= \frac{1}{N_0} \sum_{i=1}^N \lambda_t^b(x^i) \int f(y) \; B'_{t,m(x)}(x^i, dy) = m(x) \left(\lambda_t^b \; B'_{t,m(x)}(f)\right)$$

and

$$\Gamma_{\mathcal{L}_t^b}(F, F)(x) = \frac{1}{N_0} \; m(x) \left(\lambda_t^b \; B'_{t,m(x)}(f^2)\right)$$

1-Death $\rightsquigarrow u = c$ with rate $= \lambda_t^d(x^i)$

$$F(x) = m(x)(f) = \frac{1}{N_0}\sum_{i=1}^{N} f(x^i)$$

$$\mathcal{L}_t^d(F)(x^1,\ldots,x^N)$$

$$= \sum_{i=1}^{N} \lambda_t^d(x^i)$$

$$\times \int_{\cup_{d\geq 0} E^d} \left(F(x^1,\ldots,x^{i-1},u,x^{i+1},\ldots,x^N) - F(x^1,\ldots,x^N)\right)\ B_{t,m(x)}(x^i,du)$$

$$= -\frac{1}{N_0}\sum_{i=1}^{N} \lambda_t^d(x^i)\ f(x^i) = -m(x)\left(\lambda_t^d\ f\right)$$

and

$$\Gamma_{\mathcal{L}_t^d}(F,F)(x) = \frac{1}{N_0}\ m(x)\left(\lambda_t^d\ f^2\right)$$

Free evolution+1-birth+1-death

$$\mathcal{L}_t = \mathcal{L}_t^{free} + \mathcal{L}_t^b + \mathcal{L}_t^d \quad \text{and} \quad \Gamma_{\mathcal{L}_t} = \Gamma_{\mathcal{L}_t^{free}} + \Gamma_{\mathcal{L}_t^b} + \Gamma_{\mathcal{L}_t^d}$$

$$F(x) = m(x)(f) = \frac{1}{N_0}\sum_{i=1}^{N} f(x^i)$$

$$\Downarrow$$

$$\begin{aligned} \mathcal{L}_t(F)(x) &= m(x)(L_t(f)) + m(x)\left(\lambda_t^b \; B'_{t,m(x)}(f)\right) - m(x)\left(\lambda_t^d \; f\right) \\ &:= m(x)(L_{t,m(x)}(f)) \end{aligned}$$

$$\Downarrow$$

$$dm(\xi_t)(f) = m(\xi_t)(L_{t,m(\xi_t)}(f))dt + dM_t^{N_0}(f)$$

$$\frac{d}{dt}\langle M^{N_0}(f)\rangle_t = \frac{1}{N_0}\; m(\xi_t)\left[\Gamma_{L_t}(f,f) + \lambda_t^b \; B'_{t,m(x)}(f^2) + \lambda_t^d \; f^2\right]$$

$$f=1 \Rightarrow \frac{dN_t}{N_0} = m(\xi_t)\left(\left[\lambda_t^b - \lambda_t^d\right]\right) dt + dM_t^{N_0}(1)$$

and

$$\frac{d}{dt}\langle M^{N_0}(1)\rangle_t = \frac{1}{N_0}\ m(\xi_t)\left[\lambda_t^b + \lambda_t^d\right]$$

Observations :

- $\lambda_t^b - \lambda_t^d = \Delta = Cte \rightsquigarrow \mathbb{E}(N_t) = N_0\ e^{t\Delta}$ (explosion or extinction)
- $\lambda_t^b = \lambda_t^d = \lambda$ bounded $\rightsquigarrow N_t/N_0 = M_t^N(1)$ and

$$\mathbb{E}\left(\left[\frac{N_t}{N_0} - 1\right]^2\right) = \frac{2}{N_0}\int_0^t\ (m(\xi_s)(\lambda_s))\, ds \leq \frac{2\|\lambda\| t}{N_0}$$

$\rightsquigarrow$ Mean field interpretation model :

$$m(\xi_t) \sim_{N_0 \uparrow \infty} \eta_t \text{ s.t. } \frac{d}{dt}\eta_t(f) = \eta_t(L_{t,\eta_t}(f))$$

- Otherwise $\rightsquigarrow$ an avenue of open problems

- **Mean field and Feynman-Kac type particle models :**
  - **[Discrete time]** `Feynman-Kac formulae. Genealogical and interacting particle systems`, P. Del Moral, Springer (2004) ⊕ Refs.
  - **[Continuous time]** P. Del Moral & L. Miclo. A Moran particle system approximation of Feynman-Kac formulae. *Stochastic Processes and their Applications*, Vol. 86, 193-216 (2000).

- **Branching and genetic type models :**
  - P. Del Moral & L. Miclo Asymptotic Results for Genetic Algorithms with Applications to Non Linear Estimation. *Proceedings Second EvoNet Summer School on Theoretical Aspects of Evolutionary Computing*, Ed. B. Naudts, L. Kallel. Natural Computing Series, Springer (2000).